Ideal interpolation: Mourrain's condition *vs* D-invariance

C. de Boor

By definition (see [Bi]), **ideal interpolation** is provided by a linear projector whose kernel is an ideal in the ring Π of polynomials (in d real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) variables). The standard example is Lagrange interpolation; the most general example has been called 'Hermite interpolation' (in [M] and [citB05]) since that is what it reduces to in the univariate case.

Ideal projectors also occur in computer algebra, as the maps that associate a polynomial with its *normal form* with respect to an ideal; see, e.g., [CLO]. It is in this latter context that Mourrain [Mo] poses and solves the following problem. *Among all linear projectors N on*

$$\Pi_1(F) := \sum_{j=0}^{d} ()_j F$$

with range the linear space F, characterize those that are the restriction to $\Pi_1(F)$ of an ideal projector with range F. Here,

$$()_j := ()^{\varepsilon_j}, \quad \varepsilon_j := (\delta_{jk} : k = 1{:}d), \quad j = 0{:}d,$$

with

$$()^\alpha : \mathbb{F}^d \to \mathbb{F} : x \mapsto x^\alpha := \prod_{j=1}^{d} x(j)^{\alpha(j)}$$

a handy if nonstandard notation for the **monomial with exponent** $\alpha \in \mathbb{Z}_+^d$. I also use the corresponding notation

$$D_j$$

for the derivative with respect to the jth argument, and

$$D^\alpha := \prod_{j=1}^{d} D_j^{\alpha(j)}, \quad \alpha \in \mathbb{Z}_+^d.$$

To state Mourrain's result, I also need the following, standard, notations. The (total) **degree** of the polynomial $p \neq 0$ is the nonnegative integer

$$\deg p := \max\{|\alpha| : \widehat{p}(\alpha) \neq 0\},$$

with

$$p =: \sum_\alpha ()^\alpha \widehat{p}(\alpha),$$

and

$$|\alpha| := \sum_j \alpha(j),$$

while

$$\Pi_{<n} := \{p \in \Pi : \deg p < n\}.$$

Theorem 1 ([Mo]). *Let F be a finite-dimensional linear subspace of Π satisfying* **Mourrain's condition***:*

$$f \in F \quad \Longrightarrow \quad f \in \Pi_1(F \cap \Pi_{<\deg f}), \tag{2}$$

and let N be a linear projector on $\Pi_1(F)$ with range F. Then, the following are equivalent:
(a) N is the restriction to $\Pi_1(F)$ of an ideal projector with range F.
(b) The linear maps $M_j : F \to F : f \mapsto N(()_j f)$, $j = 1{:}d$, commute.

For a second proof of this theorem and some unexpected use of it in the setting of ideal interpolation, see [citB05].

Mourrain's condition (2) implies that, if F contains an element of degree k, it must also contain an element of degree $k-1$. In particular, if F is nontrivial, then it must contain a constant polynomial. This explains why Mourrain [Mo] calls a linear subspace satisfying his condition **connected to 1**. Since the same argument can be made in case F is D-invariant, this raises the question what connection if any there might be between these two properties.

In particular, for the special case $d = 1$, if F is a linear subspace of dimension n and either satisfying Mourrain's condition or being D-invariant, then, necessarily, $F = \Pi_{<n}$. More generally, if F is an n-dimensional subspace in the subring generated by

$$\langle \cdot, y \rangle : \mathbb{F}^d \to \mathbb{F} : x \mapsto \langle x, y \rangle := \sum_{j=1}^{d} x(j) y(j)$$

for some $y \neq 0$, then, either way,

$$F = \operatorname{ran}[\langle \cdot, y \rangle^{j-1} : j = 1{:}n] \; := \; \{\sum_{j=1}^{n} \langle \cdot, y \rangle^{j-1} a(j) : a \in \mathbb{F}^n\}.$$

As a next example, assume that F a **monomial** space (meaning that it is spanned by monomials). If such F is D-invariant, then, with each $()^\alpha$ for which $\alpha - \varepsilon_j \in \mathbb{Z}_+^d$, it also contains $()^{\alpha - \varepsilon_j}$ and therefore evidently satisfies Mourrain's condition.

Slightly more generally, assume that F is **dilation-invariant**, meaning that it contains $f(h\cdot)$ for every $h > 0$ if it contains f or, equivalently, F is spanned by homogeneous polynomials. Then every $f \in F$ is of the form

$$f =: f_\uparrow + f_0,$$

with $f_\uparrow$ the **leading** term of f, i.e., the unique homogeneous polynomial for which

$$\deg(f - f_\uparrow) < \deg f,$$

hence in F by dilation-invariance, therefore also

$$f_0 \in F_{<\deg f} := F \cap \Pi_{<\deg f},$$

while, by the homogeneity of $f_\uparrow$,

$$\sum_{j=1}^{d} ()_j D_j(f_\uparrow) = (\deg f) f_\uparrow$$

(this is **Euler's theorem for homogeneous functions**; see, e.g., [Encycl: p281] which gives the reference [E: §225 on p154]). If now F is also D-invariant, then $D_j(f_\uparrow) \in F_{<\deg f}$, hence, altogether,

$$f \in \Pi_1(F_{<\deg f}), \quad f \in F.$$

In other words, *if a dilation-invariant finite-dimensional subspace F of Π is D-invariant, then it also satisfies Mourrain's condition.*

On the other hand, the linear space

$$\mathrm{ran}[()^0, ()^{1,0}, ()^{1,1}]$$

fails to be D-invariant even though it satisfies Mourrain's condition and is monomial, hence dilation-invariant.

The final example, of a space that is D-invariant but does not satisfy Mourrain's condition, is slightly more complicated. In its discussion, I find it convenient to refer to

$$\mathrm{supp}\,\widehat{p}$$

as the **'support'** of the polynomial $p = \sum_\alpha ()^\alpha \widehat{p}(\alpha)$, with the quotation marks indicating that it isn't actually the support of p but, rather, the support of its coefficient sequence, $\widehat{p}$. The example is provided by the D-invariant space F generated by the polynomial

$$p = ()^{1,7} + ()^{3,3} + ()^{5,0},$$

hence the 'support' of p is

$$\mathrm{supp}\,\widehat{p} \;=\; \{(1,7), (3,3), (5,0)\}$$

(see (4) below). Here are a first few elements of F:

$$D_1 p = ()^{0,7} + 3()^{2,3} + 5()^{4,0}, \quad D_2 p = 7()^{1,6} + 3()^{3,2},$$

hence

$$D_1 D_2 p = 7()^{0,6} + 9()^{2,2}, \quad D_2^2 p = 42()^{1,5} + 6()^{3,1},$$

also

$$D_1^2 p = 6()^{1,3} + 20^{3,0}, \quad D_1 D_2^2 p = 42()^{0,5} + 18()^{2,1},$$

etc. This shows (see (4) below) that any $q \in \Pi_1(F_{<\deg p})$ having some 'support' in $\operatorname{supp}\widehat{p}$ is necessarily a weighted sum of $()_1 D_1 p$ and $()_2 D_2 p$ (and, perhaps, others not having any 'support' in $\operatorname{supp}\widehat{p}$), yet $(p, ()_1 D_1 p, ()_2 D_2 p)$ is linearly independent 'on' $\operatorname{supp}\widehat{p}$, as the matrix

$$\begin{bmatrix} 1 & 1 & 7 \\ 1 & 3 & 3 \\ 1 & 5 & 0 \end{bmatrix}$$

(of their coefficients indexed by $\alpha \in \operatorname{supp}\widehat{p}$) is evidently 1-1. Consequently, $p \notin \Pi_1(F_{<\deg p})$, i.e., this F does not satisfy Mourrain's condition.

This space also provides the proof that, in Theorem 1, one may not, in general, replace Mourrain's condition by D-invariance.

Proposition 3. *Let F be the D-invariant space spanned by*

$$p = ()^{1,7} + ()^{3,3} + ()^{5,0}.$$

Then there exists a linear projector, N, on $\Pi_1(F)$ with range F for which (b) but not (a) of Theorem 1 is satisfied.

Proof: For $\alpha, \beta \in \mathbb{Z}_+^d$, set

$$[\alpha \mathinner{..} \beta] := \{\gamma \in \mathbb{Z}_+^d : \alpha \le \gamma \le \beta\},$$

with

$$\alpha \le \gamma := \forall\, j \quad \alpha(j) \le \gamma(j).$$

With this, we determine a basis for F as follows.

Since $D^{0,4}p$ is a positive scalar multiple of $()^{1,3}$, we know, by the D-invariance of F, that

$$\{()^\zeta : \zeta \in [(0,0) \mathinner{..} (1,3)]\} \subset F.$$

This implies, considering $D^{2,0}p$, that $()^{3,0}$, hence also $()^{2,0}$, is in F. Hence, altogether,

$$F = \Pi_{\Xi_0} \oplus \operatorname{ran}[D^\alpha p : \alpha \in [(0,0) \mathinner{..} (1,3)]],$$

with

$$\Pi_\Gamma := \operatorname{ran}[()^\gamma : \gamma \in \Gamma]$$

and

$$\Xi_0 := [(0,0) \mathinner{..} (1,3)] \cup \{(2,0), (3,0)\}.$$

This provides the convenient basis

$$b_\Xi := [b_\xi : \xi \in \Xi]$$

for F, indexed by

$$\Xi := \Xi_0 \cup \Xi_1, \quad \Xi_1 := [(0,4) \mathinner{..} (1,7)],$$

namely

$$b_\xi := \begin{cases} ()^\xi, & \xi \in \Xi_0; \\ D^{(1,7)-\xi}p, & \xi \in \Xi_1. \end{cases}$$

The following schema indicates the sets $\operatorname{supp}\widehat{p}$, Ξ_0, and Ξ_1, as well as the sets $\partial\Xi_0$ and $\partial\Xi_1$ defined below:

$$\begin{array}{cccccc} \times & \times & & & & \\ 1 & \otimes\!\!\!\!\!1 & \times & & & \\ 1 & 1 & \times & & & \\ 1 & 1 & \times & & & \\ 1 & 1 & \times & & & \\ 0 & 0 & + & \otimes & & \\ 0 & 0 & + & & & \\ 0 & 0 & + & + & & \\ 0 & 0 & 0 & 0 & + & \otimes \end{array} \qquad\qquad \begin{array}{rl} \otimes: & \operatorname{supp}\widehat{p} \\ 0: & \Xi_0 \\ 1: & \Xi_1 \\ +: & \partial\Xi_0 \\ \times: & \partial\Xi_1 \end{array} \tag{4}$$

Now, let N be the linear projector on $\Pi_1(F)$ with range F and kernel $\operatorname{ran}[b_Z]$, with b_Z obtained by thinning

$$[b_\Xi, ()_1 b_\Xi, ()_2 b_\Xi]$$

to a basis $[b_\Xi, b_Z]$ for $\Pi_1(F)$. This keeps the maps $M_j : F \to F : f \mapsto N(()_j f)$ very simple since, as we shall see, for many of the $\xi \in \Xi$, $()_j b_\xi$ is an element of the extended basis $[b_\Xi, b_Z]$, hence N either reproduces it or annihilates it.

Specifically, it is evident that the following are in F, hence not part of b_Z:

$$()_1 b_\xi, \quad \xi \in [(0,0) \mathinner{\ldotp\ldotp} (0,2)],$$

$$()_2 b_\xi, \quad \xi \in [(0,0) \mathinner{\ldotp\ldotp} (1,3)].$$

Further, for each

$$\zeta \in \partial\Xi_0 \cup \partial\Xi_1,$$

with

$$\partial\Xi_0 := \{(2,3),(2,2),(2,1),(3,1),(4,0)\}, \quad \partial\Xi_1 = \{[(2,4) \mathinner{\ldotp\ldotp} (2,7)],(1,8),(0,8)\},$$

there is $\xi \in \Xi$ so that, for some j,

$$\zeta - \xi = \varepsilon_j := \begin{cases} (1,0), & j=1; \\ (0,1), & j=2. \end{cases}$$

Set, correspondingly,

$$b_\zeta := ()_j b_\xi.$$

Then, none of these is in F, and, among them, each b_ζ is the only one having some 'support' at ζ, hence they form a linearly independent sequence. Therefore, each such b_ζ is in b_Z.

The remaining candidates for membership in b_{Z} require a more detailed analysis. We start from the 'top', showing also along the way that (b) of Theorem 1 holds for this F and N by verifying that

$$M_1 M_2 = M_2 M_1 \text{ on } b_\xi \tag{5}$$

for every $\xi \in \Xi$.

$\xi = (1,7)$: As already pointed out, both $()_1 b_{1,7}$ and $()_2 b_{1,7}$ are in b_{Z}, hence (5) holds trivially for $\xi = (1,7)$.

$\xi = (0,7), (1,6)$: Both $()_1 b_{0,7} = ()^{1,7} + 3()^{3,3} + 5()^{5,0}$ and $()_2 b_{1,6} = 7()^{1,7} + 3()^{3,3}$ have their 'support' in that of $p = b_{1,7} = ()^{1,7} + ()^{3,3} + ()^{5,0}$, while, as pointed out and used earlier, the three are independent. Hence $()_1 b_{0,7}, ()_2 b_{1,6} \in b_{\mathrm{Z}}$, while we already pointed out that $()_2 b_{0,7}, ()_1 b_{1,6} \in b_{\mathrm{Z}}$, therefore (5) holds trivially.

$\xi = (0,6), (1,5)$: Both $()_1 b_{0,6} = 7()^{1,6} + 9()^{3,2}$ and $()_2 b_{1,5} = 42()^{1,6} + 6()^{3,2}$ have their 'support' in that of $b_{1,6} = 7()^{1,6} + 3()^{3,2}$, but neither is a scalar multiple of $b_{1,6}$. Hence, one is in b_Z and the other is not. Which is which depends on the ordering of the columns of $[b_\Xi, ()_1 b_\Xi, ()_2 b_\Xi]$. Assume the ordering such that $()_2 b_{1,5} \in b_{\mathrm{Z}}$. Then, since we already know that $()_1 b_{1,5} \in b_{\mathrm{Z}}$, (5) holds trivially for $\xi = (1,5)$. Further, $()_1 b_{0,6} = 4b_{1,6} - (1/2)()_2 b_{1,5}$, hence $M_1 b_{0,6} = 4b_{1,6}$, while we already know that $()_2 b_{1,6} \in b_{\mathrm{Z}}$ therefore, $M_2 M_1 b_{0,6} = 0$. On the other hand, $()_2 b_{0,6} = 7()^{0,7} + 3()^{3,3}$ has its 'support' in that of $b_{0,7} = ()^{0,7} + 3()^{3,3} + 5()^{4,0}$ but is not a scalar multiple of it, hence is in b_{Z}, and therefore already $M_2 b_{0,6} = 0$. Thus, (5) also holds for $\xi = (0,6)$.

$\xi = (0,5), (1,4)$: Both $()_1 b_{0,5} = 42()^{1,5} + 18()^{3,1}$ and $()_2 b_{1,4} = 210()^{1,5} + 6()^{3,1}$ have their 'support' in that of $b_{1,5} = 42()^{1,5} + 6()^{3,1}$ but $()^{3,1} = b_{3,1}$ was already identified as an element of b_{Z}, hence neither $()_1 b_{0,5}$ nor $()_2 b_{1,4}$ is in b_{Z}. But, since $()^{3,1} \in b_{\mathrm{Z}}$, and so $b_{1,5} = N b_{1,5} = N(42()^{1,5})$, we have $M_1 b_{0,5} = b_{1,5}$ and $M_2 b_{1,4} = 5b_{1,5}$. Since we already know that $()_1 b_{1,5} \in b_{\mathrm{Z}}$, it follows that $M_1 M_2 b_{1,4} = 0$ while we already know that $()_1 b_{1,4} \in b_{\mathrm{Z}}$, hence already $M_1 b_{1,4} = 0$. Therefore, (5) holds for $\xi = (1,4)$. Further, we already know that $()_2 b_{1,5} \in b_{\mathrm{Z}}$, hence $M_2 M_1 b_{0,5} = 0$, while $()_2 b_{0,5} = 42()^{0,6} + 18()^{2,2}$ has the same 'support' as $b_{0,6} = 7()^{0,6} + 9()^{2,2}$ but is not a scalar multiple of it, hence is in b_{Z} and, therefore, already $M_2 b_{0,5} = 0$, showing that (5) holds for $\xi = (0,5)$.

$\xi = (0,4)$: $()_2 b_{0,4} = 210()^{0,5} + 18()^{2,1} = 5b_{0,5} - 72b_{2,1}$, with $b_{2,1} \in b_{\mathrm{Z}}$, hence $()_2 b_{0,4}$ is not in b_{Z} and $M_2 b_{0,4} = 5b_{0,5}$, therefore $M_1 M_2 b_{0,4} = 5M_1 b_{0,5} = 5b_{1,5}$, the last equation from the preceding paragraph. On the other hand, $()_1 b_{0,4} = 210()^{1,4} + 18()^{3,0} = b_{1,4} + 12b_{3,0}$, with both $b_{1,4}$ and $b_{3,0}$ in F, hence $()_1 b_{0,4}$ is not in b_{Z}, and $M_1 b_{0,4} = b_{1,4} + 12b_{3,0}$, therefore, since $()_2 b_{3,0} = b_{3,1} \in b_{\mathrm{Z}}$, $M_2 M_1 b_{0,4} = M_2 b_{1,4} = 5b_{1,5}$, the last equation from the preceding paragraph. Thus, (5) holds for $\xi = (0,4)$.

$\xi = (1,3)$: We already know that $()_1 b_{1,3} = b_{2,3} \in b_{\mathrm{Z}}$ and therefore already $M_1 b_{1,3} = 0$, while $()_2 b_{1,3} = ()^{1,4} = (b_{1,4} - 6b_{3,0})/210 \in F$, therefore $210 M_1 M_2 b_{1,3} = M_1 b_{1,4} = 0$, thus (5) holds for $\xi = (1,3)$.

For the remaining $\xi \in \Xi$, each b_ξ is a monomial, hence $()_j b_\xi$ is again a monomial, and either in F or not and, if not, then its exponent is in

$$\partial\Xi_0 \;:=\; \{(2,3), (2,2), (2,1), (3,1), (4,0)\}.$$

Moreover, $()_1()_2 b_\xi$ is in F iff $()_2()_1 b_\xi$ is. Hence, (5) also holds for the remaining $\xi \in \Xi$. This finishes the proof that, for this F and N, (b) of Theorem 1 holds.

It remains to show that, nevertheless, (a) of Theorem 1 does not hold. For this, observe that $()^{2,1}$ and $()^{4,0}$ are in $\ker N$, as is, e.g., $()_2 b_{1,6} = 7()^{1,7} + 3()^{3,3}$, hence $p = ()^{1,7} + ()^{3,3} + ()^{5,0}$ is in the ideal generated by $\ker N$, making it impossible for N to be the restriction to $\Pi_1(F)$ of an ideal projector P with range F since this would place the nontrivial p in both $\ker P$ and $\operatorname{ran} P$. $\square$

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