ELASTIC SPLINES II: UNICITY OF OPTIMAL S-CURVES AND CURVATURE CONTINUITY

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Abstract. Given points $P_1, P_2, \dots, P_m$ in the complex plane, we are concerned with the problem of finding an interpolating curve with minimal bending energy (i.e., an optimal interpolating curve). It was shown previously that existence is assured if one requires that the pieces of the interpolating curve be s-curves. In the present article we also impose the restriction that these s-curves have chord angles not exceeding $\pi/2$ in magnitude. With this setup, we have identified a sufficient condition for the curvature continuity of optimal interpolating curves. This sufficient condition relates to the stencil angles $\{\psi_j\}$, where ψ_j is defined as the angular change in direction from segment $[P_{j-1}, P_j]$ to segment $[P_j, P_{j+1}]$. An angle Ψ ($\approx 37°$) is identified, and we show that if the stencil angles satisfy $|\psi_j| < \Psi$, then optimal interpolating curves are curvature continuous. We also prove that the angle Ψ is sharp.

As with the previous article, much of our effort is concerned with the geometric Hermite interpolation problem of finding an optimal s-curve $c_1(\alpha, \beta)$ that connects $0 + i0$ to $1 + i0$ with prescribed chord angles (α, β). Whereas existence was previously shown, and sometimes uniqueness, the present article begins by establishing uniqueness when $|\alpha|, |\beta| \le \pi/2$ and $|\alpha - \beta| < \pi$. We also prove two fundamental identities involving the initial and terminal signed curvatures of $c_1(\alpha, \beta)$ and partial derivatives, with respect to α or β, of the bending energy of $c_1(\alpha, \beta)$.

1. Introduction

Given points $P_1, P_2, \dots, P_m$ in the complex plane $\mathbb{C}$ with $P_j \neq P_{j+1}$, we are concerned with the problem of finding a *fair* curve that interpolates the given points. The present contribution is a continuation of [2] and so we adopt much of the notation used there. In particular, an **interpolating curve** is an absolutely-continuously differentiable function $F : [a, b] \to \mathbb{C}$, with F' non-vanishing, for which there exist times $a = t_1 < t_2 < \cdots < t_m = b$ such that $F(t_j) = P_j$. We treat F as a curve consisting of $m - 1$ pieces; the j-th piece of F, denoted $F_{[t_j, t_{j+1}]}$, runs from P_j to P_{j+1}. It is known (see [1]) that there does not exist an interpolating curve with minimal bending energy, except in the trivial case when the interpolation points lie sequentially along a line. In [2], it was shown that existence is assured if one imposes the additional condition that each piece of the interpolating curve be an s-curve. Here, an **s-curve** is a curve that first turns monotonically at most 180° in one direction (either counter-clockwise or clockwise) and then turns monotonically at most 180° in the opposite direction. Incidentally, a **c-curve** is an s-curve that turns in

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only one direction, and a **u-turn** is a c-curve that turns a full 180°. Associated with an s-curve $f : [a, b] \to \mathbb{C}$ (see Fig. 1) are its **breadth** $L := |f(b) - f(a)|$ and **chord angles** (α, β), defined by

$$\alpha := \arg \frac{f'(a)}{f(b) - f(a)}, \quad \beta := \arg \frac{f'(b)}{f(b) - f(a)},$$

where arg is defined with the usual range $(-\pi, \pi]$.

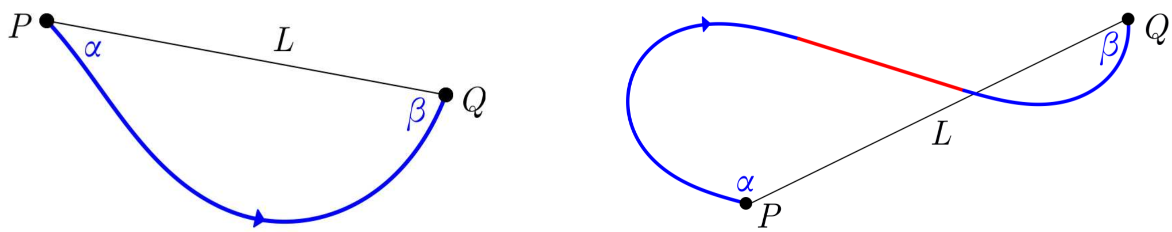


Fig. 1 **(a)** optimal s-curve of *first form* **(b)** optimal s-curve of *second form*

Note that although the chord angles are signed, our figures only indicate their magnitudes. The chord angles (α, β) of an s-curve necessarily satisfy

$$|\alpha|, |\beta| < \pi \quad \text{and} \quad |\alpha - \beta| \le \pi. \tag{1.1}$$

Defining

$$\mathcal{A}(P_1, P_2, \ldots, P_m)$$

to be the set of all interpolating curves whose pieces are s-curves, the main result of [2] is that $\mathcal{A}(P_1, P_2, \ldots, P_m)$ contains a curve (called an **elastic spline**) with minimal bending energy. Most of the effort in [2] is devoted to proving the existence of optimal s-curves. Specifically, it is shown that given distinct points P, Q and angles (α, β) satisfying (1.1), the set of all s-curves from P to Q with chord angles (α, β) contains a curve with minimal bending energy. Denoting the bending energy of such an optimal s-curve by $\frac{1}{L}E(\alpha, \beta)$, it is also shown that $E(\alpha, \beta)$ depends continuously on (α, β). In the constructive proof of existence, all optimal s-curves are described, but uniqueness is only proved in the case when the optimal curve is a c-curve, but not a u-turn. An optimal s-curve is of *first form* (resp. *second form*) if it does not (resp. does) contain a u-turn. Optimal s-curves of first form are either line segments or segments of rectangular elastica (see Fig. 1 (a)) while those of second form (see Fig. 1 (b)) contain a u-turn of rectangular elastica along with, possibly, line segments and a c-curve of rectangular elastica (see Definition 5.6 for the precise definitions). Here, 'rectangular elastica' refers to a planar curve whose signed curvature κ satisfies the differential equation $2\frac{\mathrm{d}^2\kappa}{\mathrm{d}s^2} + \kappa^3 = 0$ (see [2, pp. 190,193,205] for more details).

Elastic splines were computed in a computer program *Curve Ensemble*, written in conjunction with [7], and it was there observed that the fairness of elastic splines can be significantly degraded when pieces of second form arise. As a remedy, it was suggested in [7] that elastic splines be further restricted by requiring that chord angles of pieces satisfy

$$|\alpha|, |\beta| \le \frac{\pi}{2}. \tag{1.2}$$

The reader is referred to [8] for a detailed description of the experiments that motivate this suggestion. This additional restriction, which is stronger than (1.1), also greatly simplifies the numerical computation and theoretical development, and for these reasons, we adopt this restriction and so define

$$\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m)$$

to be the set of curves in $\mathcal{A}(P_1, P_2, \ldots, P_m)$ whose pieces have chord angles satisfying (1.2). Curves in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m)$ with minimal bending energy are called **restricted elastic splines**.

In Section 5, we show that if (1.2) holds and $(\alpha, \beta) \notin \{(\pi/2, -\pi/2), (-\pi/2, \pi/2)\}$, then the optimal s-curve from P to Q, with chord angles (α, β), is unique and of first form. The omitted cases correspond to u-turns (see Fig. 2 (a)) which fail to be unique only because one can always extend a u-turn with line segments without affecting optimality. Nevertheless, the u-turn of rectangular elastica (see Fig. 2 (b)) is the unique C^∞ optimal s-curve when $(\alpha, \beta) \in \{(\pi/2, -\pi/2), (-\pi/2, \pi/2)\}$. We mention, belatedly, that the optimality of the u-turn of rectangular elastica was first proved by Linnér and Jerome [10].

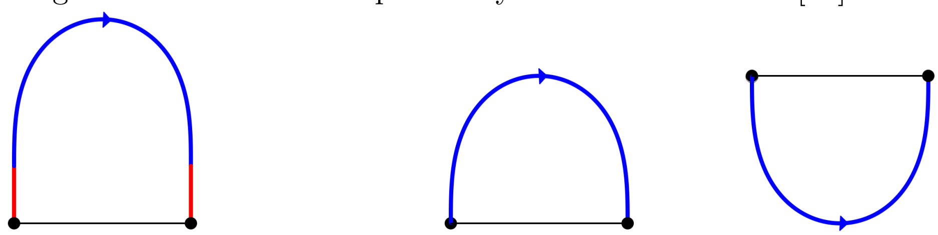


Fig. 2 **(a)** optimal u-turn **(b)** u-turns of rectangular elastica.

With unicity of optimal s-curves in hand, we can then appeal to the framework developed in [7] for assistance in proving existence and curvature continuity of restricted elastic splines. When discussing geometric curves, the notions of geometric regularity, G^1 and G^2, are preferred over the more familiar notions of parametric regularity, C^1 and C^2. A curve F has G^1 regularity if its unit tangent direction changes continuously with respect to arclength and it has G^2 regularity if, additionally, its signed curvature changes continuously with arclength. By our definition of interpolating curve (given at the outset), all interpolating curves are G^1, but not necessarily G^2.

The following will be proved in Section 6.

Proposition 1.1. *The set $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m)$ contains a curve F_{opt} with minimal bending energy. Moreover, if $F \in \mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m)$ has minimal bending energy, then each piece of F (connecting one interpolation point to the next) is G^2.*

The main concern of the present contribution is to identify conditions under which a restricted elastic spline F_{opt} will be globally G^2. This direction of inquiry is motivated by a result of Lee & Forsyth [9] (see also Brunnett [3]) which says that if an interpolating curve F has bending energy which is locally minimal (i.e., minimal among all 'nearby' interpolating curves), then F is globally G^2. The proofs in [9] and [3] employ variational calculus, but we prefer the constructive approach of [7] for its clarity and generality. We now explain our results on G^2 regularity assuming that F_{opt} is a curve in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m)$ having minimal bending energy. Note that it does not follow from Proposition 1.1 that F_{opt} is globally G^2 because it is possible for the signed curvature to have jump discontinuities across the interior nodes $P_2, P_3, \ldots, P_{m-1}$. The following is a consequence of Theorem 7.5.

Corollary 1.2. *If the chord angles at interior nodes are all (strictly) less than $\frac{\pi}{2}$ in magnitude, then F_{opt} is globally G^2.*

Proposition 1.1 and Corollary 1.2 are analogous to results of Jerome and Fisher [5, 6, 4] in that first additional constraints are imposed in order to ensure existence of an optimal curve, and then it is shown that if these additional constraints are inactive, the optimal curve is globally G^2 and its pieces are segments of rectangular elastica. These results are a good start, but they are not entirely satisfying because they shed no light on whether one can expect the added constraints to be inactive.

Our experience using the program *Curve Ensemble* is that the hypothesis of Corollary 1.2 holds when the interpolation points $\{P_j\}$ impose only mild changes in direction. This vague idea can be quantified in terms of the **stencil angles** $\{\psi_j\}$ (see Fig. 3), defined by

$$\psi_j := \arg \frac{P_{j+1} - P_j}{P_j - P_{j-1}}, \quad j = 2, 3, \dots, m-1.$$

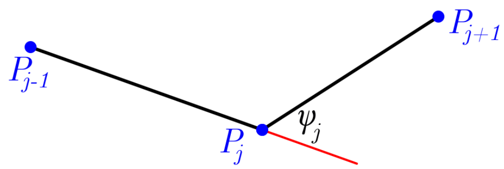


Fig. 3 the stencil angle ψ_j

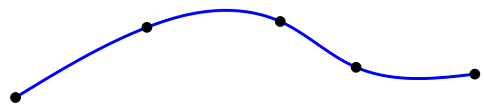


Fig. 4 a globally G^2 restricted elastic spline

The following is a consequence of Corollary 8.2.

Corollary 1.3. *Let Ψ ($\approx 37°$) be the positive angle defined in (8.1). If the stencil angles satisfy $|\psi_j| < \Psi$ for $j = 2, 3, \dots, m-1$, then the hypothesis of Corollary 1.2 holds and consequently F_{opt} is globally G^2.*

For example, the stencil angles in Fig. 4 are all less than Ψ and therefore it follows from Corollary 1.3 that the shown restricted elastic spline is globally G^2. In Section 9, we prove the following theorem which shows that the angle Ψ is sharp.

Theorem 1.4. *Let Ψ be the positive angle defined in (8.1). For all $\varepsilon > 0$, there exist points $P_1, P_2, \dots, P_m$, with stencil angles satisfying $|\psi_{m-1}| \le \Psi + \varepsilon$ and $|\psi_j| < \Psi$ for $j = 2, 3, \dots, m-2$, such that F_{opt} is not globally G^2.*

An outline of the remainder of the paper is as follows. In Section 2, we summarize some notation from [2] that is needed here, and then in sections 3 and 4 we study the relation between parameters (t_1, t_2) and the chord angles (α, β) of the segment $R_{[t_1,t_2]}$ of rectangular elastica, defined in Section 2. In Section 5 we combine Theorem 4.1 (unicity of (t_1, t_2)) with results from [2, Section 5] to prove the unicity of optimal s-curves mentioned above. With unicity of optimal s-curves in hand, we explain in Section 6 how these optimal s-curves constitute a *basic curve method* that fits into the framework of [7], and this yields Proposition 1.1. Motivated by the framework of [7], in Section 7 we prove the first fundamental identity (Theorem 7.3) and this leads to Theorem 7.5, mentioned above. The second fundamental identity (Theorem 8.6) is proved in Section 8, and this yields Corollary 8.2, mentioned above. Finally, in Section 9 we prove that the angle Ψ is sharp, as explained above.

2. Summary of Notation

The present contribution uses the same notation as in [2]; we summarize it here. A curve is a function $f : [a, b] \to \mathbb{C}$ whose derivative f' is absolutely continuous and non-vanishing. The **bending energy** of f is defined by

$$\|f\|^2 := \frac{1}{4} \int_0^S \kappa^2 \, \mathrm{d}s,$$

where S denotes the arclength of f and κ its signed curvature (the unusual factor $\frac{1}{4}$ is used to simplify some formulae related to rectangular elastica). Let $g : [c, d] \to \mathbb{C}$

be another curve. We say that f and g are **equivalent** if they have the same arclength parametrizations. They are **directly similar** if there exists a linear transformation $T(z) = c_1z+c_2$ ($c_1, c_2 \in \mathbb{C}$) such that f and $T\circ g$ are equivalent; if $|c_1| = 1$, they are called **directly congruent**. The notions of **similar** and **congruent** are the same except that T is allowed to have the form $T(z) = c_1\overline{z} + c_2$, where $\overline{z}$ denotes the complex conjugate of z.

As mentioned earlier, we call f an **s-curve** if it first turns monotonically at most 180° in one direction and then turns monotonically at most 180° in the opposite direction. An s-curve that turns in only one direction is called a **c-curve**, and a c-curve that turns a full 180° is called a **u-turn**. A non-degenerate s-curve is called a **right-left s-curve** if it first turns clockwise and then turns counter-clockwise; otherwise it is called a **left-right s-curve**. S-curves are often associated with a geometric Hermite interpolation problem, and so to facilitate this we employ the unit tangent vectors $u = (f(a), f'(a)/|f'(a)|)$ and $v = (f(b), f'(b)/|f'(b)|)$ to say that f **connects** u to v. If $g : [c, d] \to \mathbb{C}$ is a curve satisfying $(g(c), g'(c)/|g'(c)|) = (f(b), f'(b)/|f'(b)|)$, then $f \sqcup g$ denotes the concatenated curve which, for the sake of clarity, is assumed to have the arclength parametrization. Most of the s-curves that we will encounter are segments of rectangular elastica; our preferred parametrization (see Fig. 6) is $R(t) = \sin t + i\,\xi(t)$, where ξ is defined by $\xi'(t) = \dfrac{\sin^2 t}{\sqrt{1+\sin^2 t}}$, $\xi(0) = 0$. One easily verifies that ξ is odd and satisfies $\xi(t + \pi) = d + \xi(t)$, where $d := \xi(\pi)$. Since the sine function is odd and 2π-periodic, we conclude that R is odd and satisfies $R(t + 2\pi) = i\,2d + R(t)$. For later reference, we mention the following.

$$|R'(t)| = \frac{1}{\sqrt{1+\sin^2 t}}, \qquad \frac{R'(t)}{|R'(t)|} = \cos t\sqrt{1+\sin^2 t} + i\,\sin^2 t, \qquad \kappa(t) = 2\sin t,$$

$$\|R_{[a,b]}\|^2 = \frac{1}{4}\int_a^b \kappa(t)^2 |R'(t)|\,\mathrm{d}t = \xi(b) - \xi(a),$$

where $R_{[a,b]}$ denotes the restriction of R to the interval $[a, b]$.

3. The chord angles of $R_{[t_1,t_2]}$

In this section and the next, we establish relations between the **parameters** (t_1, t_2), with $t_1 < t_2$, and the chord angles (α, β) of the segment $R_{[t_1,t_2]}$ of rectangular elastica (defined in Section 2). Our primary purpose in this section is to prove Theorem 3.3 and Corollary 3.4.

Recall from Section 2 that the chord angles are given by $\alpha = \alpha(t_1, t_2) := \arg \frac{R'(t_1)}{R(t_2)-R(t_1)}$ and $\beta = \beta(t_1, t_2) := \arg \frac{R'(t_2)}{R(t_2)-R(t_1)}$. We mention that since $\xi(t)$ is increasing, it follows that the chord angles $\alpha(t_1, t_2)$ and $\beta(t_1, t_2)$ never equal π (i.e., the branch cut in the definition of arg is never crossed).

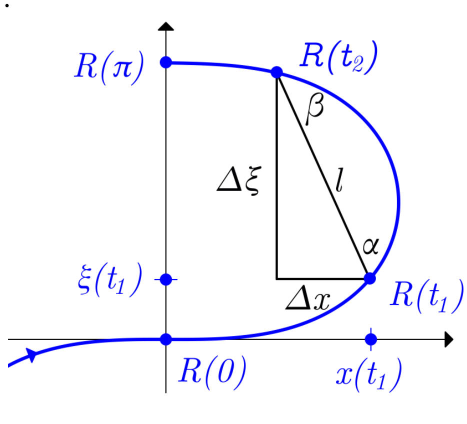


Fig. 5 notation for $R_{[t_1,t_2]}$

Assuming $t_1 < t_2$, we introduce the following notation (see Fig. 5):

$$\Delta x := \sin(t_2) - \sin(t_1), \quad \Delta\xi := \xi(t_2) - \xi(t_1), \quad l := |R(t_2) - R(t_1)|,$$

whereby $l^2 = (\Delta x)^2 + (\Delta\xi)^2$ and $\|R_{[t_1,t_2]}\|^2 = \Delta\xi$. We refer to the quantity $l\|R_{[t_1,t_2]}\|^2$ as the **normalized bending energy** of $R_{[t_1,t_2]}$ because this is the bending energy of the curve $\frac{1}{l}R_{[t_1,t_2]}$, which has been normalized to have breadth 1.

Let Q denote the mapping $(t_1,t_2) \mapsto (\alpha,\beta)$, i.e.,

$$(\alpha,\beta) =: Q(t_1,t_2).$$

We leave it to the reader to verify the following formulae for partial derivatives (these are valid for any sufficiently smooth curve):

$$\begin{aligned} \frac{\partial\alpha}{\partial t_1} &= |R'(t_1)|\left(\frac{\sin\alpha}{l} + \kappa(t_1)\right) & \frac{\partial\alpha}{\partial t_2} &= -|R'(t_2)|\frac{\sin\beta}{l} \\ \frac{\partial\beta}{\partial t_1} &= |R'(t_1)|\frac{\sin\alpha}{l} & \frac{\partial\beta}{\partial t_2} &= |R'(t_2)|\left(\frac{-\sin\beta}{l} + \kappa(t_2)\right) \end{aligned} \tag{3.1}$$

The determinant of $DQ := \begin{bmatrix} \frac{\partial\alpha}{\partial t_1} & \frac{\partial\alpha}{\partial t_2} \\ \frac{\partial\beta}{\partial t_1} & \frac{\partial\beta}{\partial t_2} \end{bmatrix}$ is therefore given by

$$\det(DQ) = |R'(t_1)||R'(t_2)|\left(\kappa(t_1)\kappa(t_2) + \kappa(t_2)\frac{\sin\alpha}{l} - \kappa(t_1)\frac{\sin\beta}{l}\right). \tag{3.2}$$

Let the *cross product* in $\mathbb{C}$ be defined by $(u_1+iv_1)\times(u_2+iv_2) := u_1v_2 - v_1u_2$. Noting that $l|R'(t_1)|\sin\alpha = (R(t_2)-R(t_1))\times R'(t_1) = -\cos t_1\Delta\xi + \xi'(t_1)\Delta x$ and $l|R'(t_2)|\sin\beta = (R(t_2)-R(t_1))\times R'(t_2) = -\cos t_2\Delta\xi + \xi'(t_2)\Delta x$, the generic formulation in (3.2) can be written specifically as:

$$\begin{aligned} \det(DQ) = {} & \frac{4\sin t_1 \sin t_2}{\sqrt{1+\sin^2 t_1}\sqrt{1+\sin^2 t_2}} + \frac{2\sin t_2}{l^2\sqrt{1+\sin^2 t_2}}\big(-\cos t_1\,\Delta\xi + \xi'(t_1)\Delta x\big) \\ & - \frac{2\sin t_1}{l^2\sqrt{1+\sin^2 t_1}}\big(-\cos t_2\,\Delta\xi + \xi'(t_2)\Delta x\big). \end{aligned} \tag{3.3}$$

Note that if both $\sin t_1 = 0$ and $\sin t_2 = 0$, then $\det(DQ) = 0$.

Lemma 3.1. *Suppose* (t_1,t_2) *belongs to the first or third set defined in Theorem 3.3. If* $\sin t_1 \sin t_2 = 0$, *then* $\det(DQ) < 0$.

Proof. We prove the lemma assuming $t_1 = 0 < t_2 < \pi$ since the proof in the other three cases is similar. Since $\xi'(0) = 0$ and $\Delta\xi > 0$, it follows from (3.3) that $\det(DQ) = \frac{2\sin t_2}{l^2\sqrt{1+\sin^2 t_2}}(-\Delta\xi) < 0$. □

If $\sin t_1 \sin t_2 \neq 0$, then (3.3) can be factored as

$$\begin{aligned} \det(DQ) &= \frac{2\Delta\xi}{l^2\sqrt{1+\sin^2 t_1}\sqrt{1+\sin^2 t_2}}\sin t_1 \sin t_2\, W(t_1,t_2), \qquad \text{where} \\ W(t_1,t_2) &:= 2\Delta\xi + \frac{(\Delta x)^2}{\Delta\xi} + \frac{\cos t_2\sqrt{1+\sin^2 t_2}}{\sin t_2} - \frac{\cos t_1\sqrt{1+\sin^2 t_1}}{\sin t_1}. \end{aligned} \tag{3.4}$$

Note that the sign of $\det(DQ)$ is the same as that of $\sin t_1 \sin t_2\, W(t_1,t_2)$.

Lemma 3.2. *If* $\sin t_1 \sin t_2 \neq 0$*, then*

$$\frac{\partial W}{\partial t_1} = \frac{\sqrt{1+\sin^2 t_1}}{(\Delta\xi)^2}\left[\frac{\cos t_1 \Delta\xi}{\sin t_1} - \frac{\sin t_1 \Delta x}{\sqrt{1+\sin^2 t_1}}\right]^2 \geq 0, \quad \textit{and}$$

$$\frac{\partial W}{\partial t_2} = -\frac{\sqrt{1+\sin^2 t_2}}{(\Delta\xi)^2}\left[\frac{\cos t_2 \Delta\xi}{\sin t_2} - \frac{\sin t_2 \Delta x}{\sqrt{1+\sin^2 t_2}}\right]^2 \leq 0.$$

Proof. We only prove the result pertaining to $\frac{\partial W}{\partial t_1}$ since the proof of the other is the same, *mutatis mutandis.* Direct differentiation yields

$$\frac{\partial W}{\partial t_1} = -2\xi'(t_1) + \frac{-2\Delta x\Delta\xi\cos t_1 + (\Delta x)^2\xi'(t_1)}{(\Delta\xi)^2} - \frac{-\sin^2 t_1\sqrt{1+\sin^2 t_1} + \frac{\cos^2 t_1 \sin^2 t_1}{\sqrt{1+\sin^2 t_1}} - \cos^2 t_1\sqrt{1+\sin^2 t_1}}{\sin^2 t_1},$$

which then simplifies to

$$\begin{aligned}\frac{\partial W}{\partial t_1} &= \frac{-2\cos t_1 \Delta x\Delta\xi + \frac{\sin^2 t_1}{\sqrt{1+\sin^2 t_1}}(\Delta x)^2}{(\Delta\xi)^2} + \left(\frac{\sqrt{1+\sin^2 t_1}}{\sin^2 t_1} - \frac{1+\sin^2 t_1}{\sqrt{1+\sin^2 t_1}}\right)\\ &= \frac{-2\cos t_1 \Delta x\Delta\xi + \frac{\sin^2 t_1}{\sqrt{1+\sin^2 t_1}}(\Delta x)^2}{(\Delta\xi)^2} + \frac{\cos^2 t_1\sqrt{1+\sin^2 t_1}}{\sin^2 t_1}.\end{aligned}$$

A simple computation then shows that this last expression can be factored as stated in the lemma. ☐

Theorem 3.3. *There exists a unique* $t^* \in (0, \pi)$ *such that* $W(-t^*, t^*) = 0$*. Moreover* $\det(DQ) < 0$ *on the following sets:*
(i) $\{(t_1, t_2) : -\pi \leq t_1 < t_2 \leq 0,\ (t_1, t_2) \neq (-\pi, 0)\}$,
(ii) $\{(t_1, t_2) : -t^* < t_1 < 0 < t_2 < t^*\}$
(iii) $\{(t_1, t_2) : 0 \leq t_1 < t_2 \leq \pi,\ (t_1, t_2) \neq (0, \pi)\}$,
(iv) $\{(t_1, t_2) : \pi - t^* < t_1 < \pi < t_2 < \pi + t^*\}$

Proof. For $-\pi < t_1 < 0 < t_2 < \pi$, the function $W(t_1, t_2)$ is analytic in both t_1 and t_2, and consequently, it follows from Lemma 3.2 that $W(t_1, t_2)$ is increasing in t_1 and decreasing in t_2. Furthermore, the function $W(-t, t)$ is analytic and decreasing for $t \in (0, \pi)$. Note that if $-\frac{\pi}{2} \leq t_1 < 0 < t_2 \leq \frac{\pi}{2}$, then $\sin t_1 < 0$ and it is clear (from (3.4)) that $W(t_1, t_2) > 0$. In particular, $W(-t, t) > 0$ for all $t \in (0, \frac{\pi}{2}]$. It is easy to verify (by inspection of (3.4)) that $\lim_{t\to\pi^-} W(-t, t) = -\infty$, and so it follows that there exists a unique $t^* \in (0, \pi)$ such that $W(-t^*, t^*) = 0$.

If (t_1, t_2) belongs to set (ii), then $W(t_1, t_2) > W(-t^*, t_2) > W(-t^*, t^*) = 0$ and since $\sin t_1 \sin t_2 < 0$, it follows that $\det(DQ) < 0$. This proves that $\det(DQ) < 0$ for all (t_1, t_2) in set (ii).

We will show that $\det(DQ) < 0$ for all (t_1, t_2) in set (i). This has already been proved in Lemma 3.1 if $0 = t_1 < t_2 < \pi$ or $0 < t_1 < t_2 = \pi$, so assume $0 < t_1 < t_2 < \pi$. As above, the function $W(t, t_2)$ is analytic and increasing for $t \in (0, t_2)$. It is easy to see (by inspection of (3.4)) that $\lim_{t\to t_2^-} W(t,t_2) = 0$, and therefore $W(t,t_2) < 0$ for all $t \in (0,t_2)$; in particular, $W(t_1,t_2) < 0$. Since $\sin t_1 \sin t_2 > 0$, we have $\det(DQ) < 0$. This completes the proof that $\det(DQ) < 0$ for all (t_1,t_2) in set (i).

Finally, if (t_1,t_2) belongs to set (iii) or set (iv), then $(t_1-\pi, t_2-\pi)$ belongs to set (i) or set (ii) and $\det(DQ(t_1,t_2)) = \det(DQ(t_1-\pi,t_2-\pi)) < 0$. □

Corollary 3.4. *Let $t^* \in (0,\pi)$ be as defined in Theorem 3.3. Then $t^* > \frac{\pi}{2}$ and $\beta(0,t^*) > \frac{\pi}{2}$. Moreover, $\beta(0,t)$ is increasing for $t \in (0,t^*]$ and decreasing for $t \in [t^*,\pi]$.*

Proof. Since $W(-t,t) > 0$ for $t \in (0,\frac{\pi}{2}]$, it is clear that $t^* > \frac{\pi}{2}$. Since $W(-t^*,t^*) = 0$, it follows from (3.4) that $\det(DQ(-t^*,t^*)) = 0$, and therefore, by (3.1), we must have

$$\kappa(-t^*)\kappa(t^*) + \kappa(t^*)\frac{\sin(\alpha(-t^*,t^*))}{l(-t^*,t^*)} - \kappa(-t^*)\frac{\sin(\beta(-t^*,t^*))}{l(-t^*,t^*)} = 0.$$

From the definition of α and β it is clear that $\alpha(-t^*,t^*) = \beta(-t^*,t^*) > 0$ and $\kappa(t^*) = -\kappa(-t^*) > 0$, so the above equality reduces to $\kappa(t^*) - \dfrac{2\sin(\beta(-t^*,t^*))}{l(-t^*,t^*)} = 0$. From the symmetry of the curve R one has $\sin(\beta(-t^*,t^*)) = \sin(\beta(0,t^*))$ and $l(-t^*,t^*) = 2l(0,t^*)$ which yields $\kappa(t^*) - \frac{\sin(\beta(0,t^*))}{l(0,t^*)} = 0$. It now follows from (3.1) that $\dfrac{\partial\beta}{\partial t_2}(0,t^*) = 0$. Moreover, the uniqueness of $t^* \in (0,\pi)$ shows (running the above argument backwards) that $t = t^*$ is the unique $t \in (0,\pi)$ where $\dfrac{\partial\beta}{\partial t_2}(0,t) = 0$. This implies that the function $\beta(0,t)$ is increasing on $(0,t^*]$ and decreasing on $[t^*,\pi]$. Consequently, $\beta(0,t^*) > \beta(0,\pi) = \frac{\pi}{2}$. □

Corollary 3.5. *There exists a unique $\bar{t} \in (0,t^*)$ such that $\beta(0,\bar{t}) = \frac{\pi}{2}$. Moreover, we have $\beta(0,t) < \frac{\pi}{2}$ for all $0 < t < \bar{t}$ and $\beta(0,t) > \frac{\pi}{2}$ for all $\bar{t} < t < \pi$.*

Proof. Since $\lim_{t\to 0^+} \beta(0,t) = 0$, $\beta(0,t^*) > \frac{\pi}{2}$, and $\beta(0,\pi) = \frac{\pi}{2}$, the result follows immediately from Corollary 3.4. □

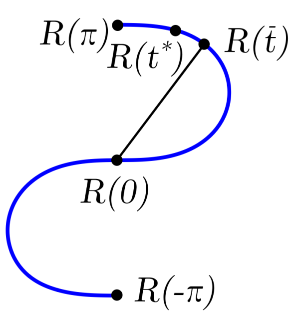


Fig. 6 $R(t)$ shown with $\bar{t}$ and t^*.

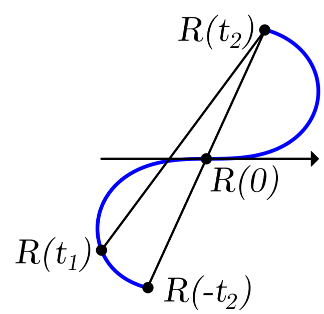


Fig. 7 $[R(t_1), R(t_2)]$ intersects negative real axis.

4. Unicity of Parameters

Recall from the previous section that the chord angles of $R_{[t_1,t_2]}$ are written as

$$(\alpha(t_1,t_2), \beta(t_1,t_2)) = Q(t_1,t_2).$$

The rectangular elastic curve R is periodic in the sense that $R(t+2\pi) = i2d + R(t)$, and it follows that $R_{[t_1+2k\pi, t_2+2k\pi]}$ is directly congruent to $R_{[t_1,t_2]}$ for all integers k. With the identification $(t_1+2k\pi, t_2+2k\pi) \equiv (t_1,t_2)$, the half-plane $Y := \{(t_1,t_2) : t_1 < t_2\}$ becomes a cylinder and we adopt the view that the chord angles $\alpha(t_1,t_2)$ and $\beta(t_1,t_2)$ are C^∞ functions defined on Y. It will be convenient, notationally, to define the subset

$$Y_{2\pi} := \{(t_1,t_2) \in Y : t_1 < t_2 < t_1 + 2\pi\}. \tag{4.1}$$

Note that the restriction $t_2 - t_1 < 2\pi$ removes pairs like $(t_1, t_2) = (-\pi, \pi)$, where both $R_{[-\pi,\pi]}$ and $R_{[-\bar{t},\bar{t}]}$ have chord angles $(\frac{\pi}{2}, \frac{\pi}{2})$.

Our purpose in this section is to prove the following.

Theorem 4.1. *For all $(\alpha, \beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \backslash \{(0,0)\}$, there exists a unique (t_1, t_2) in the cylinder $Y_{2\pi}$ such that $R_{[t_1,t_2]}$ is an s-curve with chord angles (α, β).*

Remark 4.2. Since, for $t_1 < t_2$,
(a) $R_{[t_1+\pi,t_2+\pi]}$ is directly congruent to reflections of $R_{[t_1,t_2]}$ (and hence $Q(t_1+\pi, t_2+\pi) = (-\alpha, -\beta)$ if and only if $Q(t_1, t_2) = (\alpha, \beta)$), and
(b) $R_{[-t_2,-t_1]}$ is directly congruent to the reversal of $R_{[t_1,t_2]}$ (and hence $Q(-t_2, -t_1) = (\beta, \alpha)$ if and only if $Q(t_1, t_2) = (\alpha, \beta)$), when proving Theorem 4.1, we can assume, without loss of generality, that $\alpha \geq |\beta|$.

Before proving Theorem 4.1, we first address two special cases (Propositions 4.7 and 4.10 below), which themselves require several lemmata. To avoid confusion with the functions $\alpha(t_1, t_2)$ and $\beta(t_1, t_2)$, we will write the chord angles in Theorem 4.1 as $(\widehat{\alpha}, \widehat{\beta})$, and we will assume that

$$\frac{\pi}{2} \geq \widehat{\alpha} \geq |\widehat{\beta}|, \qquad \widehat{\alpha} > 0. \tag{4.2}$$

The first proposition will be stated assuming that the parameters (t_1, t_2) satisfy

$$-\pi \leq t_1 < t_2 \leq \pi \quad \text{and} \quad t_2 - t_1 < 2\pi. \tag{4.3}$$

The following is proved in [2, Lemma 6.3].

Lemma 4.3. *For $0 < t < \pi$, $|\alpha(0, t)| < \beta(0, t)$.*

In the following lemma, the marker $\bar{t}$ is as defined in Corollary 3.5.

Lemma 4.4. *Assume (4.2), (4.3) and that $Q(t_1, t_2) = (\widehat{\alpha}, \widehat{\beta})$. The following hold.*
(i) $t_1 < 0$.
(ii) If $t_2 > 0$, then $-\bar{t} \leq t_1 < 0 < t_2 \leq \bar{t}$.

Proof. If $t_1 \geq 0$, then $R_{[t_1,t_2]}$ is a left c-curve, which contradicts $\widehat{\alpha} > 0$; hence (i). Assume $t_2 > 0$. We can assume that $t_2 \geq -t_1$, since the proof in the remaining case $t_2 < -t_1$ is similar. Assume, by way of contradiction, that $t_2 > \bar{t}$. We will show that $\beta(t_1, t_2) > \frac{\pi}{2}$ (which contradicts (4.2)). If $t_2 = -t_1$, then we must have $\bar{t} < t_2 < \pi$ (since $t_2 - t_1 < 2\pi$) and, by symmetry, $\beta(t_1, t_2) = \beta(0, t_2) > \frac{\pi}{2}$ by Corollary 3.5. So assume $t_2 > -t_1$, whereby $-\pi \leq -t_2 < t_1 < 0$ and $\bar{t} < t_2 \leq \pi$. The chord $[R(t_1), R(t_2)]$ must intersect the negative real axis (see Fig. 7), and therefore $\beta(t_1, t_2) > \beta(0, t_2) \geq \frac{\pi}{2}$. □

Let $\theta : \mathbb{R} \to [0, \pi]$ be defined by

$$\theta(t) := \arg \frac{R'(t)}{|R'(t)|} = \arg[\cos t \sqrt{1 + \sin^2 t} + i \sin^2 t],$$

where $0 \leq \sin^2 t \leq 1$ ensures that $\theta(t) \in [0, \pi]$. Several salient properties of θ are
1. $\theta(-t) = \theta(t) = \theta(2\pi + t) = \theta(2\pi - t)$ for all $t \in \mathbb{R}$;
2. $\theta(0) = 0$, $\theta(\frac{\pi}{2}) = \frac{\pi}{2}$, $\theta(\pi) = \pi$;

We can write θ explicitly as $\theta(t) = \cos^{-1}(\cos t \sqrt{1 + \sin^2 t})$, and also as

$$\theta(t) = \int_0^t \kappa(\tau) |R'(t)| \, \mathrm{d}\tau = \int_0^t \frac{2 \sin \tau}{\sqrt{1 + \sin^2 \tau}} \, \mathrm{d}\tau. \tag{4.4}$$

It follows from (4.4) that θ is C^∞ on $\mathbb{R}$ and $\theta' > 0$ on $(0, \pi)$. Let $\theta^{-1} : [0, \pi] \to [0, \pi]$ be the inverse of the restriction of θ to $[0, \pi]$. Then θ^{-1} is increasing and continuous on $[0, \pi]$ and C^∞ on $(0, \pi)$.

Lemma 4.5. *If $t_1 < t_2$, then $\theta(t_1) - \theta(t_2) = \alpha(t_1, t_2) - \beta(t_1, t_2)$.*

Proof. Since $\xi(t_1) < \xi(t_2)$, it follows that $\gamma := \arg(R(t_2) - R(t_1)) \in (0, \pi)$ and hence $\alpha(t_1, t_2) - \beta(t_1, t_2) = (\theta(t_1) - \gamma) - (\theta(t_2) - \gamma) = \theta(t_1) - \theta(t_2)$. $\square$

Lemma 4.6. *Assume (4.2) and (4.3), and define $b_0 := \theta^{-1}(\pi + \widehat{\beta} - \widehat{\alpha})$. If $Q(t_1, t_2) = (\widehat{\alpha}, \widehat{\beta})$, then $|t_2| \leq b_0$.*

Proof. By Lemma 4.5, $\theta(t_1) - \theta(t_2) = \widehat{\alpha} - \widehat{\beta}$. If $|t_2| > b_0$, then $\theta(t_1) - \theta(t_2) = \theta(t_1) - \theta(|t_2|) < \pi - \theta(b_0) = \pi - (\pi + \widehat{\beta} - \widehat{\alpha}) = \widehat{\alpha} - \widehat{\beta}$, which is a contradiction. $\square$

Proposition 4.7. *Assume (4.2). Then there exists a unique (t_1, t_2), satisfying (4.3), such that $Q(t_1, t_2) = (\widehat{\alpha}, \widehat{\beta})$.*

Note that if (4.2) holds, then $0 \leq \widehat{\alpha} - \widehat{\beta} \leq \pi$.

Proof of Proposition 4.7 in case $\widehat{\alpha} - \widehat{\beta} = \pi$. It follows from (4.2) that $\widehat{\alpha} = \frac{\pi}{2}$ and $\widehat{\beta} = -\frac{\pi}{2}$, and hence $(t_1, t_2) = (-\pi, 0)$ satisfies $Q(t_1, t_2) = (\widehat{\alpha}, \widehat{\beta})$. Conversely, if (t_1, t_2) satisfies (4.3) and $Q(t_1, t_2) = (\widehat{\alpha}, \widehat{\beta})$, then it follows from Lemma 4.5 that $\theta(t_1) = \pi$ and $\theta(t_2) = 0$; hence $t_1 = -\pi$ and $t_2 = 0$. $\square$

Proof of Proposition 4.7 in case $0 = \widehat{\alpha} - \widehat{\beta}$. Since $0 = \widehat{\alpha} - \widehat{\beta}$, we have $0 < \widehat{\alpha} = \widehat{\beta} \leq \frac{\pi}{2}$. It follows from Corollary 3.5 that there exists a unique $t_2 \in (0, \bar{t}\,]$ such that $\beta(0, t_2) = \widehat{\alpha}$. By symmetry, we have $Q(-t_2, t_2) = (\widehat{\alpha}, \widehat{\alpha})$, which establishes existence. To see uniqueness, assume (t_1', t_2') satisfies (4.3) and $Q(t_1', t_2') = (\widehat{\alpha}, \widehat{\alpha})$. It follows from Lemma 4.5 that $\theta(t_1') = \theta(t_2')$, and therefore we have $t_1' = -t_2'$. This forces $t_2' > 0$ and hence $0 < t_2' \leq \bar{t}$, by Lemma 4.4. Since $\widehat{\alpha} = \beta(-t_2', t_2') = \beta(0, t_2')$, it follows that $t_2' = t_2$, which implies $(t_1', t_2') = (-t_2, t_2)$. $\square$

Lemma 4.8. *Assume (4.2) holds with $0 < \widehat{\alpha} - \widehat{\beta} < \pi$, and define $T_1 : [-b_0, b_0] \to \mathbb{R}$ by $T_1(t_2) := -\theta^{-1}[\theta(t_2) + \widehat{\alpha} - \widehat{\beta}]$, where b_0 is as defined in Lemma 4.6. The following hold.*
(i) T_1 is continuous on $[-b_0, b_0]$ and C^∞ on $(-b_0, b_0)$.
(ii) $-\pi \leq T_1(t_2) < -|t_2|$ for all $t_2 \in [-b_0, b_0]$.
(iii) $\alpha(T_1(t_2), t_2) - \beta(T_1(t_2), t_2) = \widehat{\alpha} - \widehat{\beta}$ for all $t_2 \in [-b_0, b_0]$.
(iv) T_1 is decreasing on $[0, b_0]$.
(v) If (t_1, t_2) satisfies (4.3) and $Q(t_1, t_2) = (\widehat{\alpha}, \widehat{\beta})$, then $|t_2| \leq b_0$ and $t_1 = T_1(t_2)$.

Proof. Since $\theta(t_2) + \widehat{\alpha} - \widehat{\beta} \in [\widehat{\alpha} - \widehat{\beta}, \pi)$ for all $t_2 \in (-b_0, b_0)$, (i) is a consequence of θ^{-1} being continuous on $[0, \pi]$ and C^∞ on $(0, \pi)$.
The first inequality, $-\pi \leq T_1(t_2)$, in (ii) is clear since the range of θ^{-1} equals $[0, \pi]$. For the latter inequality, we note that $\theta(t_2) + \widehat{\alpha} - \widehat{\beta} = \theta(|t_2|) + \widehat{\alpha} - \widehat{\beta} > \theta(|t_2|)$. Since θ^{-1} is increasing, we have $-T_1(t_2) = \theta^{-1}[\theta(t_2) + \widehat{\alpha} - \widehat{\beta}] > |t_2|$, which completes (ii).
We readily obtain (iii) as a consequence of (ii) and Lemma 4.5.
Now assume $0 \leq \tau < t \leq b_0$. Since θ is increasing on $[0, \pi]$, we have $\theta(-T_1(\tau)) = \theta(\tau) + \widehat{\alpha} - \widehat{\beta} < \theta(t) + \widehat{\alpha} - \widehat{\beta} = \theta(-T_1(t))$. Since $-T_1(\tau), -T_1(t) \in (0, \pi]$ (by (ii)), we have $-T_1(\tau) < -T_1(t)$, and therefore $T_1(\tau) > T_1(t)$, which proves (iv).
Finally, assume (4.3) holds and $Q(t_1, t_2) = (\widehat{\alpha}, \widehat{\beta})$. Then $|t_2| \leq b_0$, by Lemma 4.6. Furthermore, $\theta(-T_1(t_2)) = \theta(t_2) + \widehat{\alpha} - \widehat{\beta}$, while $\theta(-t_1) = \theta(t_1) = \theta(t_2) + \widehat{\alpha} - \widehat{\beta}$, by Lemma 4.5. Since θ is increasing on $[0, \pi]$ and $-t_1, -T_1(t_2) \in (0, \pi]$, we have $t_1 = T_1(t_2)$. $\square$

Proof of Proposition 4.7 in case $0 < \widehat{\alpha} - \widehat{\beta} < \pi$. Since $\alpha(t_1, t_2)$ is C^∞ on Y, it follows from Lemma 4.8 (i) and (ii) that the function $a(t) := \alpha(T_1(t), t)$ is continuous on $[-b_0, b_0]$ and C^∞ on $(-b_0, b_0)$. We will show that $a(-b_0) < \widehat{\alpha}$ and $a(b_0) > \widehat{\alpha}$. Note that $T_1(-b_0) =$

$T_1(b_0) = -\pi$. By Lemma 4.3, and symmetry, we have $\alpha(-\pi, -b_0) < -\beta(-\pi, -b_0)$, and hence $\alpha(-\pi, -b_0) + \beta(-\pi, -b_0) < 0$. Since $\alpha(-\pi, -b_0) - \beta(-\pi, -b_0) = \widehat{\alpha} - \widehat{\beta}$, by Lemma 4.8 (iii), it follows that
$2\alpha(-\pi, -b_0) = [\alpha(-\pi, -b_0) - \beta(-\pi, -b_0)] + [\alpha(-\pi, -b_0) + \beta(-\pi, -b_0)] < \widehat{\alpha} - \widehat{\beta} \le 2\widehat{\alpha}$, by (4.2). Therefore $a(-b_0) = \alpha(-\pi, -b_0) < \widehat{\alpha}$. Since $b_0 \in (0, \pi)$, it is clear that $\gamma := \arg(R(b_0) - R(-\pi)) \in (0, \frac{\pi}{2})$, and hence $a(b_0) = \alpha(-\pi, b_0) = \theta(-\pi) - \gamma = \pi - \gamma > \frac{\pi}{2}$. Since $\widehat{\alpha} \le \frac{\pi}{2}$, this proves that $a(b_0) > \widehat{\alpha}$. Having established $a(-b_0) < \widehat{\alpha} < a(b_0)$, it now follows, by the Intermediate Value Theorem, that there exists $t_2 \in (-b_0, b_0)$ such that $a(t_2) = \widehat{\alpha}$; that is, $\alpha(T_1(t_2), t_2) = \widehat{\alpha}$. By Lemma 4.8 (iii), we also have $\beta(T_1(t_2), t_2) = \widehat{\beta}$ and therefore, with $t_1 := T_1(t_2)$, we have $Q(t_1, t_2) = (\widehat{\alpha}, \widehat{\beta})$. Note that (t_1, t_2) satisfies (4.3), by Lemma 4.8 (ii). This establishes existence.
In order to prove uniqueness, assume, by way of contradiction, that $Q(t_1', t_2') = (\widehat{\alpha}, \widehat{\beta})$ for some (t_1', t_2') satisfying (4.3), with $(t_1', t_2') \ne (t_1, t_2)$. By Lemma 4.8 (v), we have $|t_2'| \le b_0$ and $t_1' = T_1(t_2')$, and therefore $a(t_2') = \widehat{\alpha}$. We must have $t_2' \ne t_2$, since otherwise it would follow that $(t_1', t_2') = (t_1, t_2)$. We can assume, without loss of generality, that $t_2 > t_2'$. It follows by Rolle's Theorem that there exists $\tau \in (t_2', t_2)$ such that $a'(\tau) = 0$; that is, $\frac{\mathrm{d}}{\mathrm{d}t}\alpha(T_1(t), t) = 0$ at $t = \tau$. It follows from Lemma 4.8 (iii) that we also have $\frac{\mathrm{d}}{\mathrm{d}t}\beta(T_1(t), t) = 0$ at $t = \tau$. Therefore $DQ(T_1(\tau), \tau) \begin{bmatrix} T_1'(\tau) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where DQ is the 2×2 matrix defined just after (3.1), and we conclude that $DQ(T_1(\tau), \tau)$ is singular.
If $\tau \le 0$, then we have $-\pi \le T_1(\tau) < \tau \le 0$, with $(T_1(\tau), \tau) \ne (-\pi, 0)$ (since $\widehat{\alpha} - \widehat{\beta} < \pi$). Hence, $DQ(T_1(\tau), \tau)$ being singular contradicts Theorem 3.3 (i).
On the other hand, if $\tau > 0$, then $t_2 > 0$ and it follows from Lemma 4.4 that $-\bar{t} \le T_1(t_2) < 0 < t_2 \le \bar{t}$. By Lemma 4.8 (iv), $T_1(\tau) > T_1(t_2)$, and now $DQ(T_1(\tau), \tau)$ being singular contradicts Theorem 3.3 (ii), since $-\bar{t} \le T_1(t_2) < T_1(\tau) < 0 < \tau < t_2 \le \bar{t} < t^*$. □

Remark 4.9. When (4.2) holds with $\widehat{\alpha} = -\widehat{\beta}$, then the unique (t_1, t_2) mentioned in Proposition 4.7 can be found explicitly. With $\tau := \theta^{-1}(\frac{\pi}{2} - \widehat{\alpha})$, it is easy to verify that $Q(-\pi + \tau, -\tau) = (\widehat{\alpha}, -\widehat{\alpha})$.

Proposition 4.10. *Assume (4.2) holds with* $\widehat{\beta} < 0$. *If* $0 \le t_1 < \pi < t_2 \le 2\pi$ *and* $t_2 - t_1 < 2\pi$, *then* $Q(t_1, t_2) \ne (\widehat{\alpha}, \widehat{\beta})$.

Proof. Assume, by way of contradiction, that $0 \le t_1 < \pi < t_2 \le 2\pi$, $t_2 - t_1 < 2\pi$, and $Q(t_1, t_2) = (\widehat{\alpha}, \widehat{\beta})$.
Case 1: $\widehat{\alpha} = -\widehat{\beta}$.
Define $t_1' := \pi - t_2$ and $t_2' := \pi - t_1$. Then $-\pi \le t_1' < 0 < t_2' \le \pi$ and $t_2' - t_1' < 2\pi$. Applying the transformation rules mentioned in Remark 4.2, we find that $Q(t_1', t_2') = (-\widehat{\beta}, -\widehat{\alpha})$. Note that $(\alpha', \beta') := (-\widehat{\beta}, -\widehat{\alpha})$ satisfies (4.2) with $\alpha' = -\beta'$. By Proposition 4.7 and Remark 4.9, it follows that $(t_1', t_2') = (-\pi + \tau, -\tau)$, where $\tau := \theta^{-1}(\frac{\pi}{2} - \widehat{\alpha}) \in [0, \frac{\pi}{2})$. Hence $-\pi \le t_1' < t_2' \le 0$, which is a contradiction.
Case 2: $\widehat{\alpha} > -\widehat{\beta}$.
By Lemma 4.5, $\theta(t_1) - \theta(t_2) = \widehat{\alpha} - \widehat{\beta}$. Set $a := \theta^{-1}(\widehat{\alpha} - \widehat{\beta})$. If $t_1 < a$, then $\theta(t_1) - \theta(t_2) < \widehat{\alpha} - \widehat{\beta} - \theta(t_2) \le \widehat{\alpha} - \widehat{\beta}$; hence $a \le t_1 < \pi$. Furthermore, since $\pi < t_2 \le 2\pi$ and $\theta(t_2) = \theta(t_1) - \widehat{\alpha} + \widehat{\beta}$, it follows that $t_2 = T_2(t_1)$, where $T_2 : [a, \pi] \to [\pi, 2\pi]$ is defined by $T_2(t) := 2\pi - \theta^{-1}(\theta(t) - \widehat{\alpha} + \widehat{\beta})$. Note that T_2 is continuous on $[a, \pi]$. Since $\theta(2\pi - t) = \theta(t)$ for all t, it follows from Lemma 4.5 that $\alpha(t, T_2(t)) - \beta(t, T_2(t)) = \widehat{\alpha} - \widehat{\beta}$ for all $t \in [a, \pi]$. By Lemma 4.3, and symmetry, we have $\alpha(\pi, T_2(\pi)) < -\beta(\pi, T_2(\pi))$, while $\alpha(t_1, T_2(t_1)) = \widehat{\alpha} > -\widehat{\beta} = -\beta(t_1, T_2(t_1))$. By the Intermediate Value Theorem, there exists $\widetilde{t}_1 \in (t_1, \pi)$ such

that $\widetilde{\alpha} := \alpha(\widetilde{t}_1, T_2(\widetilde{t}_1)) = -\beta(\widetilde{t}_1, T_2(\widetilde{t}_1))$. Since $2\widetilde{\alpha} = \alpha(\widetilde{t}_1, T_2(\widetilde{t}_1)) - \beta(\widetilde{t}_1, T_2(\widetilde{t}_1)) = \widehat{\alpha} - \widehat{\beta} > 0$, we have $\widetilde{\alpha} > 0$. With $\widetilde{t}_2 := T_2(\widetilde{t}_1)$, we conclude that $a < \widetilde{t}_1 < \pi < \widetilde{t}_2 < 2\pi$ satisfies $Q(\widetilde{t}_1, \widetilde{t}_2) = (\widetilde{\alpha}, -\widetilde{\alpha})$, and now following down Case 1 we arrive at a contradiction. □

Proof of Theorem 4.1. As explained in Remark 4.2, we can assume, without loss of generality, that (4.2) holds. Existence is established in Proposition 4.7. In order to see that (t_1, t_2) is unique, assume that $(t_1, t_2), (t_1', t_2') \in Y_{2\pi}$ satisfy $Q(t_1, t_2) = Q(t_1', t_2') = (\widehat{\alpha}, \widehat{\beta})$. Since $\widehat{\alpha} > 0$, $R_{[t_1,t_2]}$ cannot be a left c-curve. We claim that it cannot be a left-right s-curve either. If $R_{[t_1,t_2]}$ is a left-right s-curve, then we can assume, without loss of generality, that $0 \le t_1 < \pi < t_2 \le 2\pi$. It then follows from Proposition 4.10 that $Q(t_1, t_2) \ne (\widehat{\alpha}, \widehat{\beta})$, which is a contradiction. Hence $R_{[t_1,t_2]}$ cannot be a left-right s-curve, as claimed. By elimination, $R_{[t_1,t_2]}$ is either a right c-curve or a right-left s-curve, and the same holds for $R_{[t_1',t_2']}$. We can therefore assume, without loss of generality, that (4.3) holds for both (t_1, t_2) and (t_1', t_2'), and it then follows, by Proposition 4.7, that $(t_1, t_2) = (t_1', t_2')$. □

5. Unicity of optimal s-curves

Let $\alpha, \beta \in (-\pi, \pi]$ and set $u = (0, e^{i\alpha})$ and $v = (1, e^{i\beta})$. The set $S(\alpha, \beta)$, defined to be the set of all s-curves connecting u to v, was intensely studied in [2], and it is easy to verify that $S(\alpha, \beta)$ is non-empty if and only if $(\alpha, \beta) \in \mathcal{F}$, where

$$\mathcal{F} := \{(\alpha, \beta) : |\alpha|, |\beta| < \pi \text{ and } |\alpha - \beta| \le \pi\}.$$

It is proved in [2] that $S(\alpha, \beta)$ contains a curve f_{opt} with **minimal bending energy**, i.e., satisfying $\|f_{opt}\|^2 \le \|f\|^2$ for all $f \in S(\alpha, \beta)$. The bending energy of f_{opt} is denoted

$$E(\alpha, \beta) := \|f_{opt}\|^2, \quad (\alpha, \beta) \in \mathcal{F}. \tag{5.1}$$

Curves in $S(\alpha, \beta)$ with minimal bending energy are called **optimal** curves.

Definition 5.1. For $(\alpha, \beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \backslash \{(0, 0)\}$, let $(t_1, t_2) \in Y_{2\pi}$ be as described in Theorem 4.1, and define the curve $c_1(\alpha, \beta)$ by

$$c_1(\alpha, \beta; t) := \frac{R(t_1 + t) - R(t_1)}{R(t_2) - R(t_1)}, \quad 0 \le t \le t_2 - t_1.$$

For $(\alpha, \beta) = (0, 0)$, we define $c_1(0, 0)$ to be the line segment $[0, 1]$ (i.e., $c_1(0, 0, t) := t + i0$, $0 \le t \le 1$).

Note that every curve in $S(0, 0)$ is equivalent to $c_1(0, 0)$, since the line segment is the only s-curve with chord angles $(0, 0)$. For $(\alpha, \beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \backslash \{(0, 0)\}$, since $c_1(\alpha, \beta)$ is directly similar to $R_{[t_1,t_2]}$ and $c_1(\alpha, \beta; 0) = 0$, $c_1(\alpha, \beta; t_2 - t_1) = 1$, it follows that $c_1(\alpha, \beta)$ belongs to $S(\alpha, \beta)$. An immediate consequence of Definition 5.1 and Theorem 4.1 is the following.

Proposition 5.2. *Let $(\alpha, \beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \backslash \{(0, 0)\}$. If $(t_1, t_2) \in Y_{2\pi}$ is such that $R_{[t_1,t_2]}$ is an s-curve with chord angles (α, β), then $c_1(\alpha, \beta)$ is directly similar to $R_{[t_1,t_2]}$.*

Remark 5.3. When an s-curve f contains a u-turn, the u-turn can be elongated, as described in [2, Remark 5.5], without affecting the bending energy. We will use the notational device $\lfloor f \rfloor$ to refer to the curve obtained from f by removing any elongation(s) in the u-turn(s). If f does not contain any u-turns then $\lfloor f \rfloor$ is simply f.

Our main purpose in this section is to prove the following.

Theorem 5.4. *For* $(\alpha,\beta) \in [-\frac{\pi}{2},\frac{\pi}{2}]^2\backslash\{(0,0)\}$, *the following hold.*
(i) $c_1(\alpha,\beta)$ *is an optimal curve in* $S(\alpha,\beta)$.
(ii) If $|\alpha-\beta| < \pi$, *then* $c_1(\alpha,\beta)$ *does not contain a u-turn and every optimal curve in* $S(\alpha,\beta)$ *is equivalent to* $c_1(\alpha,\beta)$.
(iii) If $|\alpha-\beta| = \pi$, *then* $c_1(\alpha,\beta)$ *is a u-turn and for all optimal curves* $f \in S(\alpha,\beta)$, $\lfloor f \rfloor$ *is equivalent to* $c_1(\alpha,\beta)$.

Remark 5.5. Since
(a) the bending energy of a curve is invariant under translations, rotations, reflections and reversals (of orientation),
(b) $c_1(\beta,\alpha)$ is directly congruent to the reversal of $c_1(\alpha,\beta)$,
(c) $c_1(-\alpha,-\beta)$ equals the reflection of $c_1(\alpha,\beta)$ about the real axis,
when proving Theorem 5.4 we can assume, without loss of generality, that $\alpha \geq |\beta|$.

Note that this reduction to 'canonical form' was first employed in [2]. The following definition is taken from [2, Def. 5.2].

Definition 5.6. A curve f is of
(i) **first form** if there exist $-\pi < t_1 < t_2 < \pi$ such that f is directly similar to $R_{[t_1,t_2]}$,
(ii) **second form** if there exists $a \geq 0$ and $t_2 \in [0,\pi]$ such that f is directly similar to

$$R_{[-\pi,0]} \sqcup [0,a] \sqcup (a + R_{[0,t_2]}).$$

The following is proved in [2, Sections 5,6].

Theorem 5.7. *Assume* $(\alpha,\beta) \in \mathcal{F}$ *satisfies* $\alpha > 0$ *and* $\alpha \geq |\beta|$. *The following hold.*
(i) There exists a curve $f_{opt} \in S(\alpha,\beta)$ *with minimal bending energy.*
(ii) If $f_{opt} \in S(\alpha,\beta)$ *has minimal bending energy, then either* f_{opt} *is of first form or* $\lfloor f_{opt} \rfloor$ *is of second form.*

In the course of proving Theorem 5.7 in [2], every optimal curve in $S(\alpha,\beta)$ is 'described', but uniqueness is only established when the optimal curve is a c-curve of first form.

Lemma 5.8. *Assume* $\frac{\pi}{2} \geq \alpha \geq |\beta|$, *and let* f *belong to* $S(\alpha,\beta)$. *If* f *is of second form, then exactly one of the following hold.*
(i) $(\alpha,\beta) = (\frac{\pi}{2},-\frac{\pi}{2})$ *and* f *is directly similar to* $R_{[-\pi,0]}$.
(ii) $(\alpha,\beta) = (\frac{\pi}{2},\frac{\pi}{2})$ *and* f *is directly similar to* $R_{[-\pi,\pi]}$.

Proof. Assume f is of second form. Then there exist $a \geq 0$ and $t_2 \in [0,\pi]$ such that f is directly similar to $g := R_{[-\pi,0]} \sqcup [0,a] \sqcup (a + R_{[0,t_2]})$. Recall that $R(-\pi) = 0 - id$ lies on the negative imaginary axis. It is clear that if $a > 0$ or $0 < t_2 < \pi$, then $\arg(a + R(t_2) - R(-\pi)) \in (0,\frac{\pi}{2})$ and hence $\alpha = \pi - \arg(a + R(t_2) - R(-\pi)) > \frac{\pi}{2}$, a contradiction. Therefore $a = 0$ and $t_2 \in \{0,\pi\}$. If $t_2 = 0$, we have (i), and if $t_2 = \pi$, we have (ii). □

Lemma 5.9. *Let* $f,g \in S(\frac{\pi}{2},\frac{\pi}{2})$ *and suppose that* f *is directly similar to* $R_{[-\pi,\pi]}$ *and* g *is directly similar to* $R_{[-\bar{t},\bar{t}]}$. *Then the bending energy of* f *is greater than that of* g.

Proof. Numerically we have $\|f\|^2 = |R(\pi) - R(-\pi)|(\xi(\pi) - \xi(-\pi)) \approx 5.74216008836904$ and $\|g\|^2 = |R(\bar{t}) - R(-\bar{t})|(\xi(\bar{t}) - \xi(-\bar{t})) \approx 5.29016586074592$; nevertheless, a mathematical proof can be extracted from [2, Section 5] as follows. Let Γ, σ and G be as defined in [2, Def. 5.3]. In [2, Summary 5.4], the chord angles $(\alpha,\beta) = (\frac{\pi}{2},\frac{\pi}{2})$ fall into Case B and the curve f corresponds to γ being the left endpoint of Γ, namely $\gamma = -\frac{\pi}{2}$; moreover, $\|f\|^2 = G(-\frac{\pi}{2})$. It follows from Theorem 4.1, that there exists a unique γ_0 in the interior of Γ such that $\sigma(\gamma_0) = 0$ and g corresponds to γ_0 with $\|g\|^2 = G(\gamma_0)$. As explained in [2, Summary 5.4], the function G has a minimum value G_{min} and, in the present situation, $G_{min} = \min\{G(-\frac{\pi}{2}), G(\gamma_0)\}$. So, in order to prove that $\|f\|^2 > \|g\|^2$, it suffices to show that $G(-\frac{\pi}{2}) > G_{min}$. Although $G'(-\frac{\pi}{2}) = 0$, using [2, Lemma 5.11], one can show that there exists $\varepsilon > 0$ such that $G' < 0$ on the interval $(-\frac{\pi}{2}, -\frac{\pi}{2} + \varepsilon]$, and it follows that $G(-\frac{\pi}{2}) > G_{min}$. □

Proof of Theorem 5.4. By Theorem 5.7 (i), there exists an optimal curve f_{opt} in $S(\alpha, \beta)$.
Case I: $(\alpha, \beta) \neq (\frac{\pi}{2}, -\frac{\pi}{2})$
We will first show that f_{opt} is of first form. Assume, by way of contradiction, that f_{opt} is not of first form. Then, by Theorem 5.7 (ii), $\lfloor f_{opt} \rfloor$ is of second form and it follows from Lemma 5.8 that $(\alpha, \beta) = (\frac{\pi}{2}, \frac{\pi}{2})$ and $\lfloor f_{opt} \rfloor$ is directly similar to $R_{[-\pi,\pi]}$. By Proposition 5.2, $c_1(\frac{\pi}{2}, \frac{\pi}{2})$ is directly similar to $R_{[-\bar{t},\bar{t}]}$. By Lemma 5.9, $\|f_{opt}\|^2 > \|c_1(\frac{\pi}{2}, \frac{\pi}{2})\|^2$, which contradicts the optimality of f_{opt}. Therefore, f_{opt} is of first form. By Definition 5.6, there exist $-\pi < t_1 < t_2 < \pi$ such that f_{opt} is directly similar to $R_{[t_1,t_2]}$. By Proposition 5.2, $c_1(\alpha, \beta)$ is directly similar to $R_{[t_1,t_2]}$, and it follows that f_{opt} is equivalent to $c_1(\alpha, \beta)$ and $c_1(\alpha, \beta)$ is optimal in $S(\alpha, \beta)$. Hence (i) and (ii).
Case II: $(\alpha, \beta) = (\frac{\pi}{2}, -\frac{\pi}{2})$
It follows from the definition of s-curve that all curves in $S(\frac{\pi}{2}, -\frac{\pi}{2})$ are right u-turns, and hence f_{opt} cannot be of first form. By Theorem 5.7 (ii), $\lfloor f_{opt} \rfloor$ is of second form and hence, by Lemma 5.8, $\lfloor f_{opt} \rfloor$ is directly similar to $R_{[-\pi,0]}$. By Proposition 5.2, $c_1(\frac{\pi}{2}, -\frac{\pi}{2})$ is directly similar to $R_{[-\pi,0]}$, and it follows that $\lfloor f_{opt} \rfloor$ is directly similar to $c_1(\frac{\pi}{2}, -\frac{\pi}{2})$ and $c_1(\frac{\pi}{2}, -\frac{\pi}{2})$ is optimal in $S(\frac{\pi}{2}, -\frac{\pi}{2})$. Hence (i) and (iii). □

6. Restricted Elastic Splines and Proof of Existence

Although written specifically for s-curves that connect $u = (0, e^{i\alpha})$ to $v = (1, e^{i\beta})$, Theorem 5.4 easily extends to general configurations (u, v). To see this, let $u := (P_1, d_1)$ and $v := (P_2, d_2)$ be two unit tangent vectors with distinct base points $P_1 \neq P_2$. The chord angles (α, β) determined by (u, v) are $\alpha = \arg \frac{d_1}{P_2 - P_1}$ and $\beta = \arg \frac{d_2}{P_2 - P_1}$. With $S(u, v)$ denoting the set of s-curves that connect u to v, and defining $T(z) := (P_2 - P_1)z + P_1$, we see that $S(\alpha, \beta)$ is in one-to-one correspondence with $S(u, v)$:

$$f \in S(\alpha, \beta) \text{ if and only if } T \circ f \in S(u, v).$$

Moreover, with $L := |P_2 - P_1|$, we have $\|f\|^2 = \frac{1}{L}\|T \circ f\|^2$. Now, assume $\alpha, \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2$ and let $c_1(\alpha, \beta)$ be the optimal curve defined in Definition 5.1. Then

$$c(u, v) := T \circ c_1(\alpha, \beta)$$

is an optimal curve in $S(u, v)$, and Theorem 5.4 translates immediately into the following.

Corollary 6.1. *Let (u, v) be a configuration with chord angles $(\alpha, \beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \backslash \{(0, 0)\}$. The following hold.*
(i) $c(u, v)$ is an optimal curve in $S(u, v)$.
(ii) If $|\alpha - \beta| < \pi$, then $c(u, v)$ does not contain a u-turn and every optimal curve in $S(u, v)$ is equivalent to $c(u, v)$.
(iii) If $|\alpha - \beta| = \pi$, then $c(u, v)$ is a u-turn and for all optimal curves $f \in S(u, v)$, $\lfloor f \rfloor$ is equivalent to $c(u, v)$ (see Remark 5.3 for the definition of $\lfloor f \rfloor$).

In the framework of [7], the curves $\{c(u, v)\}$ are called **basic curves** and the mapping $(u, v) \mapsto c(u, v)$ is called a **basic curve method**. We define the **energy** of basic curves to be the bending energy. In [7], it is assumed that the basic curve method and energy are translation and rotation invariant, and this allows one's attention to be focused on the (canonical) case where $u = (0, e^{i\alpha})$ and $v = (L, e^{i\beta})$, $L > 0$. The resulting basic curve and energy functional are denoted $c_L(\alpha,\beta)$ and $E_L(\alpha,\beta)$. In our setup, we have the two additional properties that $c_L(\alpha,\beta)$ is equivalent to $Lc_1(\alpha,\beta)$ and $E_L(\alpha,\beta) = \frac{1}{L}E_1(\alpha,\beta)$, where the latter holds because

$$E_L(\alpha,\beta) := \|c_L(\alpha,\beta)\|^2 = \|Lc_1(\alpha,\beta)\|^2 = \frac{1}{L}\|c_1(\alpha,\beta)\|^2 = \frac{1}{L}E_1(\alpha,\beta).$$

In the language of [7], we would say that the basic curve method is *scale invariant* and the energy functional is *inversely proportional to scale.* This special case is addressed in detail in [7, Section 3], and it allows us to focus our attention on the case $L = 1$ where we have, for $(\alpha,\beta) \in [-\frac{\pi}{2},\frac{\pi}{2}]^2$, the optimal curve $c_1(\alpha,\beta)$ and its energy $E_1(\alpha,\beta) := \|c_1(\alpha,\beta)\|^2$. Note that $E_1(\alpha,\beta) = E(\alpha,\beta)$ for $(\alpha,\beta) \in [-\frac{\pi}{2},\frac{\pi}{2}]^2$, where $E(\alpha,\beta)$ is defined in (5.1). The distinction between E_1 and E is that the domain of E_1 is $[-\frac{\pi}{2},\frac{\pi}{2}]^2$, while the domain of E is the larger set $\mathcal{F}$ (defined just above (5.1)). In [2, Section 7], it is shown that E is continuous on $\mathcal{F}$ and it therefore follows that E_1 is continuous on $[-\frac{\pi}{2},\frac{\pi}{2}]^2$.

The framework of [7] is concerned with the set $\widehat{\mathcal{A}}_{\pi/2}(P_1,P_2,\ldots,P_m)$ consisting of all interpolating curves whose pieces are basic curves, and the energy of such an interpolating curve $\widehat{F} = c(u_1,u_2) \sqcup c(u_2,u_3) \sqcup \cdots \sqcup c(u_{m-1},u_m)$ is defined to be the sum of the energies of its constituent basic curves:

$$\text{Energy}(\widehat{F}) := \textstyle\sum_{j=1}^{m-1} \|c(u_j,u_{j+1})\|^2 = \|\widehat{F}\|^2.$$

Note that $\widehat{\mathcal{A}}_{\pi/2}(P_1,P_2,\ldots,P_m)$ is a subset of $\mathcal{A}_{\pi/2}(P_1,P_2,\ldots,P_m)$ and energy in both sets is defined to be bending energy. Since E_1 is continuous on $[-\frac{\pi}{2},\frac{\pi}{2}]^2$, it follows from [7, Theorem 2.3] that there exists a curve in $\widehat{\mathcal{A}}_{\pi/2}(P_1,P_2,\ldots,P_m)$ with minimal bending energy. The following lemma will be needed in our proof of Proposition 1.1.

Lemma 6.2. *Given $F \in \mathcal{A}_{\pi/2}(P_1,P_2,\ldots,P_m)$, let $u_1,u_2,\ldots,u_m$ be the unit tangent vectors, with base-points $P_1,P_2,\ldots,P_m$, determined by F, and define*

$$\widehat{F} := c(u_1,u_2) \sqcup c(u_2,u_3) \sqcup \cdots \sqcup c(u_{m-1},u_m) \in \widehat{\mathcal{A}}_{\pi/2}(P_1,P_2,\ldots,P_m).$$

Then $\|F\|^2 \geq \|\widehat{F}\|^2$.

The proof of the lemma is simply that f_j, the j-th piece of F, has bending energy at least $\|c(u_j,u_{j+1})\|^2$ because it belongs to $S(u_j,u_{j+1})$.

Proof of Proposition 1.1. Since $\widehat{\mathcal{A}}_{\pi/2}(P_1,P_2,\ldots,P_m)$ is a subset of $\mathcal{A}_{\pi/2}(P_1,P_2,\ldots,P_m)$ and the former contains a curve with minimal bending energy, it follows immediately from Lemma 6.2 that the latter contains a curve with minimal bending energy. Now, assume $F \in \mathcal{A}_{\pi/2}(P_1,P_2,\ldots,P_m)$ has minimal bending energy, and let $\widehat{F}$ be as in Lemma 6.2. Then $\|F\|^2 = \|\widehat{F}\|^2$ and we must have $\|f_j\|^2 = \|c(u_j,u_{j+1})\|^2$ for $j = 1,2,\ldots,m-1$, where f_j denotes the j-th piece of F. Hence f_j is an optimal curve in $S(u_j,u_{j+1})$, and we can appeal to Corollary 6.1. Since $c(u_j,u_{j+1})$ is directly similar to a line segment or a segment of R, it is clear that $c(u_j,u_{j+1})$ is G^2. If $c(u_j,u_{j+1})$ does not contain a u-turn, then f_j is equivalent to $c(u_j,u_{j+1})$, and hence f_j is G^2. On the other hand, if $c(u_j,u_{j+1})$ equals a u-turn, then $\lfloor f_j \rfloor$ is equivalent to $c(u_j,u_{j+1})$, and now it is not obvious that f_j is G^2. However, in this case, $c(u_j,u_{j+1})$ is directly similar to $R_{[-\pi,0]}$ or $R_{[0,\pi]}$ and so its curvature is 0 at the endpoints. Since f_j is equivalent to an elongation of the u-turn $c(u_j,u_{j+1})$ (see Fig. 2 (a)), it follows that f_j is G^2. □

Remark 6.3. We see in the above proof that the distinction between minimal energy curves in $\mathcal{A}_{\pi/2}(P_1,P_2,\ldots,P_m)$ and those in $\widehat{\mathcal{A}}_{\pi/2}(P_1,P_2,\ldots,P_m)$ is fairly minor:

If $F = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{m-1}$ has minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, \dots, P_m)$, then $\widehat{F} := \lfloor f_1 \rfloor \sqcup \lfloor f_2 \rfloor \sqcup \cdots \sqcup \lfloor f_{m-1} \rfloor$ has minimal bending energy in $\widehat{\mathcal{A}}_{\pi/2}(P_1, P_2, \dots, P_m)$. Moreover, $\lfloor f_j \rfloor$ differs from f_j only when they are u-turns (i.e., when $(\alpha_j, \beta_{j+1}) = \pm(\frac{\pi}{2}, -\frac{\pi}{2})$).

7. Conditional G^2 Regularity

The following definition is taken from [7, Section 3].

Definition 7.1. Let $F \in \widehat{\mathcal{A}}_{\pi/2}(P_1, P_2, \dots, P_m)$ have minimal bending energy and let (α_j, β_{j+1}) be the chord angles of the the j-th piece of F. We say that F is **conditionally** G^2 if F is G^2 across P_j whenever the two chord angles associated with node P_j satisfy $|\beta_j|, |\alpha_j| < \pi/2$.

Let $\kappa_a(f)$ and $\kappa_b(f)$ denote, respectively, the initial and terminal signed curvatures of a curve f. The following result is an amalgam of [7, Theorems 3.3 and 3.5].

Theorem 7.2. *If there exists a constant $\mu \in \mathbb{R}$ such that*

$$-\kappa_a(c_1(\alpha,\beta)) = \mu \frac{\partial}{\partial \alpha} E_1(\alpha,\beta) \quad \text{and} \quad \kappa_b(c_1(\alpha,\beta)) = \mu \frac{\partial}{\partial \beta} E_1(\alpha,\beta) \tag{7.1}$$

for all $(\alpha,\beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2$ with $|\alpha - \beta| < \pi$, then minimal energy curves in $\widehat{\mathcal{A}}_{\pi/2}(P_1, P_2, \dots, P_m)$ are conditionally G^2.

Although [7, Theorem 3.3] is stated assuming that (7.1) holds for all $(\alpha,\beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2$, the given proof remains valid under the weaker assumption that (7.1) holds for all $(\alpha,\beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2$ with $|\alpha - \beta| < \pi$. The following result shows that condition (7.1) holds with $\mu = 2$.

Theorem 7.3. *For all $(\alpha,\beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \backslash \{(-\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, -\frac{\pi}{2})\}$,*

$$-\kappa_a(c_1(\alpha,\beta)) = 2\frac{\partial E_1}{\partial \alpha}(\alpha,\beta) \quad \text{and} \quad \kappa_b(c_1(\alpha,\beta)) = 2\frac{\partial E_1}{\partial \beta}(\alpha,\beta) \tag{7.2}$$

Proof. Fix $(\widehat{\alpha}, \widehat{\beta}) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \backslash \{(-\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, -\frac{\pi}{2})\}$. We will show that (7.2) holds at $(\widehat{\alpha}, \widehat{\beta})$. We first address the easy case $(\widehat{\alpha}, \widehat{\beta}) = (0,0)$, where $c_1(0,0)$ is a line segment. In the proof of [2, Prop. 7.6], it is shown that there exists a constant C such that $E_1(\alpha,\beta) = E(\alpha,\beta) \le C(\tan^2\alpha + \tan\alpha\tan\beta + \tan^2\beta)$ for all $(\alpha,\beta) \in [-\pi/3, \pi/3]^2$. From this it easily follows that $\nabla E_1(0,0) = [0,0]$, and since the line segment $c_1(0,0)$ has 0 curvature, we obtain (7.2) for the case $(\widehat{\alpha}, \widehat{\beta}) = (0,0)$.

We proceed assuming $(\widehat{\alpha}, \widehat{\beta}) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \backslash \{(-\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, -\frac{\pi}{2}), (0,0)\}$. Let $(\widehat{t_1}, \widehat{t_2}) \in Y_{2\pi}$ be as described in Definition 5.1, whereby $c_1(\widehat{\alpha}, \widehat{\beta})$ is directly similar to $R_{[\widehat{t_1}, \widehat{t_2}]}$ and $Q(\widehat{t_1}, \widehat{t_2}) = (\widehat{\alpha}, \widehat{\beta})$. Keeping in mind that $Y_{2\pi}$ (defined in (4.1)) is an open cylinder, the restriction $(\widehat{\alpha}, \widehat{\beta}) \notin \{(-\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, -\frac{\pi}{2})\}$ ensures that $(\widehat{t_1}, \widehat{t_2})$ does not identify with either $(-\pi, 0)$ or $(0, \pi)$. It thus follows from Lemma 4.4, and symmetry, that $(\widehat{t_1}, \widehat{t_2})$ identifies with a pair (τ_1, τ_2) that belongs to one of the four sets listed in Theorem 3.3, and therefore $\det(DQ(\widehat{t_1}, \widehat{t_2})) < 0$. Since Q is C^∞ on $Y_{2\pi}$, it follows that there exists an open neighborhood $N \subset Y_{2\pi}$ of $(\widehat{t_1}, \widehat{t_2})$ such that Q is injective on N, $\det(DQ) < 0$ on N, $Q(N)$ is an open neighborhood of $(\widehat{\alpha}, \widehat{\beta})$, and Q^{-1} is C^∞ on $Q(N)$.

We mention that it is possible that N contains pairs (t_1, t_2) for which $R_{[t_1,t_2]}$ is not an

s-curve or $Q(t_1,t_2) \notin [-\frac{\pi}{2},\frac{\pi}{2}]^2$. This is a minor difficulty.
We define $E^* : Q(N) \to [0,\infty)$ as follows. For $(\alpha,\beta) \in Q(N)$,

$$E^*(\alpha,\beta) := l\|R_{[t_1,t_2]}\|^2, \text{ where } (t_1,t_2) := Q^{-1}(\alpha,\beta) \text{ and } l := |R(t_1) - R(t_2)|.$$

Claim. If $(\alpha,\beta) \in Q(N) \cap [-\frac{\pi}{2},\frac{\pi}{2}]^2$, then $E^*(\alpha,\beta) = E_1(\alpha,\beta)$ and $c_1(\alpha,\beta)$ is directly congruent to $\frac{1}{l}R_{[t_1,t_2]}$.

proof. Assume $(\alpha,\beta) \in Q(N) \cap [-\frac{\pi}{2},\frac{\pi}{2}]^2$. Since $(t_1,t_2) \in Y_{2\pi}$ and $Q(t_1,t_2) = (\alpha,\beta)$, it follows from Theorem 4.1 and Definition 5.1 that $c_1(\alpha,\beta)$ is directly similar to $R_{[t_1,t_2]}$. Consequently, $c_1(\alpha,\beta)$ is directly congruent to $\frac{1}{l}R_{[t_1,t_2]}$ and $E_1(\alpha,\beta) := \|c_1(\alpha,\beta)\|^2 = E^*(\alpha,\beta)$, as claimed.

We recall, from Section 2, that the curvature of R is given by $\kappa(t) = 2\sin t$, and hence $\kappa_a(c_1(\alpha,\beta)) = 2l\sin t_1$ and $\kappa_b(c_1(\alpha,\beta)) = 2l\sin t_2$. So with the claim in view, in order to establish (7.2) at $(\widehat{\alpha},\widehat{\beta})$, it suffices to show that

$$[-l\sin t_1, l\sin t_2] = \nabla E^*(\alpha,\beta), \text{ for all } (\alpha,\beta) \in Q(N). \tag{7.3}$$

The bending energy of $R_{[t_1,t_2]}$ (see Section 2) is given by $\|R_{[t_1,t_2]})\|^2 = \xi(t_2) - \xi(t_1) =: \Delta\xi$, and hence $E^*(\alpha,\beta) = l\Delta\xi$. Defining $\widetilde{E} : N \to [0,\infty)$ by $\widetilde{E}(t_1,t_2) := l\Delta\xi$, we have $\widetilde{E} = E^* \circ Q$, and therefore, since DQ is nonsingular on N, (7.3) is equivalent to

$$[-l\sin t_1, l\sin t_2]DQ = \nabla\widetilde{E}(t_1,t_2), \text{ for all } (t_1,t_2) \in N.$$

This can be written explicitly as

$$\begin{aligned} -l\sin t_1\frac{\partial\alpha}{\partial t_1} + l\sin t_2\frac{\partial\beta}{\partial t_1} &= \frac{\partial}{\partial t_1}(l\Delta\xi) \\ -l\sin t_1\frac{\partial\alpha}{\partial t_2} + l\sin t_2\frac{\partial\beta}{\partial t_2} &= \frac{\partial}{\partial t_2}(l\Delta\xi) \end{aligned}$$

Using (3.1) and the formulae above (3.3) the first equality is proved as follows.

$$\begin{aligned} -l\sin t_1\frac{\partial\alpha}{\partial t_1} + l\sin t_2\frac{\partial\beta}{\partial t_1} &= |R'(t_1)|\sin\alpha\Delta x - l\sin t_1|R'(t_1)|\kappa(t_1) \\ &= (-\cos t_1\Delta\xi + \xi'(t_1)\Delta x)\frac{\Delta x}{l} - 2l\xi'(t_1) \\ &= (-\cos t_1\Delta\xi + \xi'(t_1)\Delta x)\frac{\Delta x}{l} - \frac{\Delta x^2 + \Delta\xi^2}{l}\xi'(t_1) - l\xi'(t_1) \\ &= -\frac{-\cos t_1\Delta x - \xi'(t_1)\Delta\xi}{l}\Delta\xi - l\xi'(t_1) = \frac{\partial}{\partial t_1}(l\Delta\xi). \end{aligned}$$

We omit the proof of the second equality since it is very similar. □

Corollary 7.4. *E_1 is C^∞ on $[-\frac{\pi}{2},\frac{\pi}{2}]^2\backslash\{(-\frac{\pi}{2},\frac{\pi}{2}),(\frac{\pi}{2},-\frac{\pi}{2}),(0,0)\}$*

Proof. Fix $(\widehat{\alpha},\widehat{\beta}) \in [-\frac{\pi}{2},\frac{\pi}{2}]^2\backslash\{(-\frac{\pi}{2},\frac{\pi}{2}),(\frac{\pi}{2},-\frac{\pi}{2}),(0,0)\}$ and let N and E^* be as in the proof above. Then E^* is C^∞ on $Q(N)$, an open neighborhood of $(\widehat{\alpha},\widehat{\beta})$. The desired conclusion is now a consequence of the Claim in the above proof. □

Together, Theorem 7.2 and Theorem 7.2 imply that minimal energy curves in $\widehat{\mathcal{A}}_{\pi/2}(P_1,P_2,\ldots,P_m)$ are conditionally G^2; we can now prove that this also holds for the larger set $\mathcal{A}_{\pi/2}(P_1,P_2,\ldots,P_m)$.

Theorem 7.5. *Let $F = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{m-1}$ have minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m)$. Then F is G^2 across P_j (i.e., $\kappa_b(f_{j-1}) = \kappa_a(f_j)$) whenever the two chord angles associated with node P_j satisfy $|\beta_j|, |\alpha_j| < \pi/2$.*

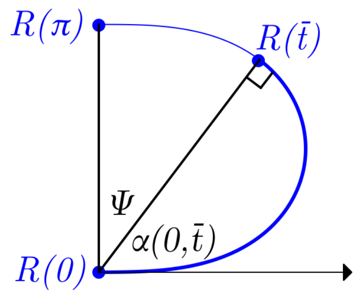
Proof. Let $\widehat{F} \in \widehat{\mathcal{A}}_{\pi/2}(P_1, P_2, \ldots, P_m)$ be as in Lemma 6.2. As explained in Remark 6.3, $\widehat{F} = \lfloor f_1 \rfloor \sqcup \lfloor f_2 \rfloor \sqcup \cdots \sqcup \lfloor f_{m-1} \rfloor$. Let $j \in \{2, 3, \ldots, m-1\}$ be such that $|\beta_j|, |\alpha_j| < \pi/2$. By Theorem 7.2 and Theorem 7.2, $\widehat{F}$ is G^2 across P_j, and hence $\kappa_b(\lfloor f_{j-1} \rfloor) = \kappa_a(\lfloor f_j \rfloor)$. Since $|\alpha_j| < \pi/2$, the piece f_j cannot be a u-turn and it follows that $\lfloor f_j \rfloor = f_j$. Similarly, since $|\beta_j| < \pi/2$, we have $\lfloor f_{j-1} \rfloor = f_{j-1}$, and therefore $\kappa_b(f_{j-1}) = \kappa_a(f_j)$. ☐

8. G^2 Regularity

With $\bar{t}$ as described in Corollary 3.5, let Ψ (see Fig. 8) denote the positive angle defined by

(8.1) $$\Psi := \frac{\pi}{2} - |\alpha(0, \bar{t})|.$$

Fig. 8



Our purpose in this section is to prove the following theorem and corollary.

Theorem 8.1. *Let $\widehat{F} \in \widehat{\mathcal{A}}_{\pi/2}(P_1, P_2, \ldots, P_m)$ have minimal bending energy. If the stencil angle at node P_j satisfies $|\psi_j| < \Psi$, then the two chord angles at P_j satisfy $|\beta_j|, |\alpha_j| < \frac{\pi}{2}$.*

Corollary 8.2. *Let $F \in \mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m)$ have minimal bending energy. If the stencil angle at node P_j satisfies $|\psi_j| < \Psi$, then the two chord angles at P_j satisfy $|\beta_j|, |\alpha_j| < \frac{\pi}{2}$ and consequently F is G^2 across node P_j.*

Our proof of Theorem 8.1 employs the following result which is essentially [7, Theorem 5.1] but specialized to the present context.

Theorem 8.3. *Suppose that for every $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ there exists $\beta^*(\alpha)$, with $|\beta^*(\alpha)| \leq \frac{\pi}{2} - \Psi$, such that*

(8.2) $$\operatorname{sign}\left(\frac{\partial E_1}{\partial \beta}(\alpha, \beta)\right) = \operatorname{sign}(\beta - \beta^*(\alpha)) \text{ for all } \beta \text{ satisfying } |\beta| \leq \frac{\pi}{2} \text{ and } |\beta - \alpha| < \pi.$$

Let $\widehat{F} \in \widehat{\mathcal{A}}_{\pi/2}(P_1, P_2, \ldots, P_m)$ be a curve with minimal bending energy. If P_j is a point where the stencil angle ψ_j satisfies $|\psi_j| < \Psi$, then the two chord angles associated with node P_j satisfy $|\beta_j|, |\alpha_j| < \frac{\pi}{2}$.

Proof. Employing the symmetry $E_1(\alpha, \beta) = E_1(\beta, \alpha)$, conditions (i) and (ii) in the hypothesis of [7, Theorem 5.1] reduce simply to the single condition

(8.3) $$\operatorname{sign}\left(\frac{\partial E_1}{\partial \beta}(\alpha, \beta)\right) = \operatorname{sign}(\beta - \beta^*(\alpha)) \text{ for all } |\beta| \leq \frac{\pi}{2}.$$

Therefore Theorem 8.3, with (8.2) replaced by (8.3), is an immediate consequence of [7, Theorem 5.1]. Note that the only distinction between (8.2) and (8.3) is that (8.2) is moot when (α,β) equals $(\pi/2,-\pi/2)$ or $(-\pi/2,\pi/2)$. With a slight modification (specifically: rather than showing that $f'(\Omega)>0$ and $f'(\psi_2-\Omega)<0$, one instead shows that there exists $\varepsilon>0$ such that $f'(\beta)>0$ for $\Omega-\varepsilon<\beta<\Omega$ and $f'(\beta)<0$ for $\psi_2-\Omega<\beta<\psi_2-\Omega+\varepsilon$), the proof of [7, Theorem 5.1] also proves Theorem 8.3. □

The appearance of (8.2), rather than (8.3), in Theorem 8.3 is simply a consequence of the (unproven) fact that $\frac{\partial E_1}{\partial\beta}(\alpha,\beta)=0$ when (α,β) equals $(\pi/2,-\pi/2)$ or $(-\pi/2,\pi/2)$. This distinction is not without consequence. In [7, Theorem 5.1], the conclusion is obtained when $\psi_i\le\Psi$, while in Theorem 8.3 we require $\psi_i<\Psi$.

Definition 8.4. One easily verifies that $\beta(0,0):=\lim_{t\to0^+}\beta(0,t)=0$ and $\alpha(0,0):=\lim_{t\to0^+}\alpha(0,t)=0$. Let $\phi:[0,\pi]\to[0,\infty)$ be defined by $\phi(t):=\beta(0,t)$. Then ϕ is continuous and, by Corollary 3.4, ϕ is increasing on $[0,t^*]$ and decreasing on $[t^*,\pi]$. Let ϕ^{-1} denote the inverse of ϕ restricted to $[0,t^*]$. We define $\beta^*:[-\phi(t^*),\phi(t^*)]\to\mathbb{R}$ by

$$\beta^*(\alpha):=\operatorname{sign}(\alpha)\,\alpha(0,\phi^{-1}(|\alpha|))\qquad\text{(see Fig. 9 below where } \alpha>0 \text{ and } t_\alpha:=\phi^{-1}(\alpha)).$$

Lemma 8.5. *The function β^* is continuous, odd, and decreasing. Moreover, the following hold.*
(i) $|\beta^*(\alpha)|\le\frac{\pi}{2}-\Psi$ for all $\alpha\in[-\frac{\pi}{2},\frac{\pi}{2}]$.
(ii) On $[0,\frac{\pi}{2}]$, the function $\gamma\mapsto\Psi-\beta^*(\gamma)$ increases continuously from Ψ to $\frac{\pi}{2}$.
(iii) On $[0,\frac{\pi}{2}]$, the function $\gamma\mapsto\gamma+\beta^*(\gamma)$ increases continuously from 0 to Ψ.

Proof. The function $t\mapsto\alpha(0,t)$ is continuous and, by (3.1), decreasing on $[0,\pi]$. Since $\alpha(0,0)=0$ and ϕ^{-1} is continuous and increasing, it follows that β^* is continuous, odd, and decreasing. Consequently, for $\alpha\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $|\beta^*(\alpha)|\le|\beta^*(\frac{\pi}{2})|=|\alpha(0,\bar t)|=\frac{\pi}{2}-\Psi$, which proves (i). Noting that $\beta^*(0)=0$ and $\beta^*(\frac{\pi}{2})=\Psi-\frac{\pi}{2}$, (ii) follows from β^* being continuous and decreasing.
In pursuit of (iii), define $g(t):=\phi(t)+\alpha(0,t)$, $0\le t\le\pi$. Then g is continuous on $[0,\pi]$ and C^∞ on $(0,\pi]$ and, by (3.1), we have $g'(t)=2|R'(t)|(\sin t-\frac{1}{\ell}\sin\phi(t))$, where $\ell:=\ell(t):=|R(t)|$. Using the identity $\ell|R'(t)|\sin\phi(t)=-\xi(t)\cos t+\xi'(t)\sin t$ (see the discussion just below (3.2)), we find that

$$g'(t)=\tfrac{2}{\ell^2}|R'(t)|\xi(t)\left(\cos t\sqrt{1+\sin^2 t}+\xi(t)\sin t\right),\ \ 0<t\le\pi.$$

Using the 'dot product' in $\mathbb{C}$ (defined by $(x+iy)\cdot(u+iv):=xu+yv$), we have

$$\begin{aligned}\ell\cos\phi(t)&=R(t)\cdot\frac{R'(t)}{|R'(t)|}\\&=(\sin t+i\xi(t))\cdot\left(\cos t\sqrt{1+\sin^2 t}+i\sin^2 t\right)=\sin t\left(\cos t\sqrt{1+\sin^2 t}+\xi(t)\sin t\right),\end{aligned}$$

and therefore $g'(t)=\dfrac{2}{\ell\sin t}|R'(t)|\xi(t)\cos\phi(t)$, $0<t\le\pi$. It follows from Corollary 3.5 that $0<\phi(t)<\frac{\pi}{2}$ for $t\in(0,\bar t)$ and hence $g'>0$ on $(0,\bar t)$. Therefore, g is increasing on $[0,\bar t\,]$. Recall that on $[0,\frac{\pi}{2}]$, ϕ^{-1} increases continuously from 0 to $\bar t$. Since the function $\gamma\mapsto\gamma+\beta^*(\gamma)$ equals $g\circ\phi^{-1}$, it follows that it increases continuously on $[0,\frac{\pi}{2}]$ from $g(0)=0$ to $g(\bar t)=\frac{\pi}{2}+(\Psi-\frac{\pi}{2})=\Psi$, which proves (iii). □

Theorem 8.6. *With β^* as defined in Definition 8.4, (8.2) holds for all $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.*

Proof. Fix $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
Case 1: $0 < \alpha \le \frac{\pi}{2}$.

Set $B = [-\frac{\pi}{2}, \frac{\pi}{2}]\backslash\{\alpha - \pi\}$. It follows from Corollary 7.4 that the function $\beta \mapsto E_1(\alpha, \beta)$ is C^∞ on B, and, from Theorem 7.3, we have that $\frac{\partial E_1}{\partial \beta}(\alpha,\beta) = \frac{1}{2}\kappa_b(c_1(\alpha,\beta))$ for $\beta \in B$. Note that if $(t_1,t_2) \in Y_{2\pi}$ is such that $c_1(\alpha,\beta)$ is directly similar to $R_{[t_1,t_2]}$, then $\operatorname{sign}\left(\frac{\partial E_1}{\partial \beta}(\alpha,\beta)\right) = \operatorname{sign}(\sin t_2)$ since the signed curvature of $R(t)$ is $\kappa(t) = 2\sin t$.

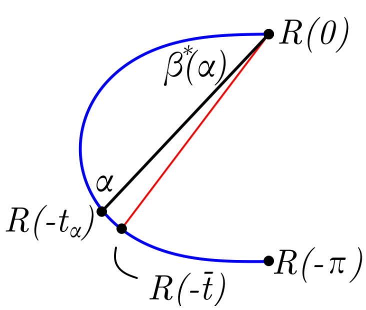


Fig. 9 the parameter $-t_\alpha$

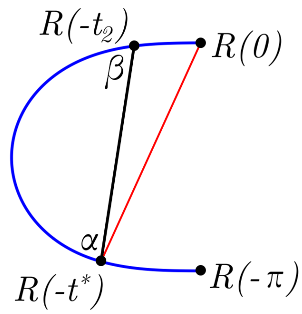


Fig. 10 the parameter $-t_2$

Set $t_\alpha := \phi^{-1}(\alpha)$. Then $R_{[-t_\alpha,0]}$ (see Fig. 9) has chord angles $(\alpha, \beta^*(\alpha))$ and hence $\operatorname{sign}\left(\frac{\partial E_1}{\partial \beta}(\alpha,\beta^*(\alpha))\right) = \operatorname{sign}(\sin 0) = 0$.

Claim: If $\beta \in B$ is such that $\frac{\partial E_1}{\partial \beta}(\alpha,\beta) = 0$, then $\beta = \beta^*(\alpha)$.

Proof. Assume $\beta \in B$ is such that $\frac{\partial E_1}{\partial \beta}(\alpha,\beta) = 0$. Let $(t_1,t_2) \in Y_{2\pi}$ be as described in Definition 5.1, whereby $c_1(\alpha,\beta)$ is directly similar to $R_{[t_1,t_2]}$. Then $\sin(t_2) = 0$. Since $R_{[t_1,t_2]}$ is an s-curve, we can assume, without loss of generality, that $-\pi \le t_1 < t_2 \le \pi$ or $0 \le t_1 < \pi < t_2 \le 2\pi$, and it follows that $t_2 \in \{0, \pi, 2\pi\}$. We will show, by elimination, that $t_2 = 0$. Recall that $t_2 - t_1 < 2\pi$ by definition of $Y_{2\pi}$. If $t_2 = 2\pi$, then $0 < t_1 < \pi$ and it follows that $R_{[t_1,2\pi]}$ contains a u-turn, contradicting Theorem 5.4 (ii). If $t_2 = \pi$ and $0 \le t_1 < \pi$, then $\alpha < 0$, a contradiction. Finally, if $t_2 = \pi$ and $-\pi < t_1 < 0$, then $R_{[t_1,\pi]}$ contains a u-turn, contradicting Theorem 5.4 (ii). Therefore, we must have $t_2 = 0$. The definition of B ensures that $(\alpha,\beta) \ne (\frac{\pi}{2}, -\frac{\pi}{2})$, and therefore $-\pi < t_1 < 0$. By symmetry, we have $\beta(0,-t_1) = \alpha$. By Corollaries 3.4 and 3.5, we must have $-t_1 = t_\alpha$, and it follows that $\beta = \beta^*(\alpha)$; hence the claim.

Note that $R_{[-t_\alpha,t_\alpha]}$ has chord angles (α,α) and hence $\operatorname{sign}\left(\frac{\partial E_1}{\partial \beta}(\alpha,\alpha)\right) = \operatorname{sign}(\sin t_\alpha) > 0$. Since $\alpha > 0 > \beta^*(\alpha)$, it follows from continuity that $\operatorname{sign}\left(\frac{\partial E_1}{\partial \beta}(\alpha,\beta)\right) > 0$ for $\beta \in B$ with $\beta > \beta^*(\alpha)$.

Now, in order to complete the proof (of Case I), it suffices to show that there exists $\beta \in B$ such that $\operatorname{sign}\left(\frac{\partial E_1}{\partial \beta}(\alpha,\beta)\right) < 0$. Since $\alpha(-t^*,0) = \beta(0,t^*) > \frac{\pi}{2} \ge \alpha$, it follows that there exists $-t_2 \in (-t^*,0)$ such that $\alpha(-t^*,-t_2) = \alpha$. Set $\beta := \beta(-t^*,-t_2) < 0$ (see Fig. 10). It is easy to verify that $|\beta| < \frac{\pi}{2}$ and therefore $\beta \in B$. Note that $\operatorname{sign}\left(\frac{\partial E_1}{\partial \beta}(\alpha,\beta)\right) = \operatorname{sign}(\sin(-t_2)) < 0$. This completes the proof for Case I.

Case II: $-\frac{\pi}{2} \le \alpha < 0$.

This case follows from Case I and the symmetries

$$E_1(-\alpha,-\beta) = E_1(\alpha,\beta) \text{ and } \beta^*(-\alpha) = -\beta^*(\alpha).$$

Case III: $\alpha = 0$.

Note that $\beta^*(0) = 0$ and, by Theorem 7.3, $\frac{\partial E_1}{\partial \beta}(0,0) = 0$.

Claim: If $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is such that $\frac{\partial E_1}{\partial \beta}(0,\beta) = 0$, then $\beta = 0$.

Proof. By way of contradiction, assume $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\backslash\{0\}$ is such that $\frac{\partial E_1}{\partial \beta}(0,\beta) = 0$. Let $(t_1, t_2) \in Y_{2\pi}$ be such that $c_1(0,\beta)$ is directly similar to $R_{[t_1,t_2]}$. As before, we have $\sin(t_2) = 0$ and we can assume, without loss of generality, that $-\pi \le t_1 < t_2 \le \pi$ or $0 \le t_1 < \pi < t_2 \le 2\pi$; hence $t_2 \in \{0, \pi, 2\pi\}$. But now, since $\alpha = 0$, $R_{[t_1,t_2]}$ must be a non-degenerate s-curve, and it follows that $R_{[t_1,t_2]}$ contains a u-turn, contradicting Theorem 5.4 (ii); hence the claim.

The symmetry $E_1(0,\beta) = E_1(0,-\beta)$ ensures that $\frac{\partial E_1}{\partial \beta}(0,-\beta) = -\frac{\partial E_1}{\partial \beta}(0,\beta)$ and therefore it suffices to show that $\frac{\partial E_1}{\partial \beta}(0,\beta) > 0$ for all $\beta \in (0, \frac{\pi}{2}]$. Define $g(\beta) := E_1(0,\beta)$, $\beta \in [0, \frac{\pi}{2}]$ so that $g'(\beta) = \frac{\partial E_1}{\partial \beta}(0,\beta)$. Then g is continuous on $[0, \frac{\pi}{2}]$ and is C^∞ on $(0, \frac{\pi}{2}]$. It follows from the claim that $\operatorname{sign}(g')$ is nonzero and constant on $(0, \frac{\pi}{2}]$. If $\operatorname{sign}(g') = -1$ on $(0, \frac{\pi}{2}]$, then we would have $E_1(0, \frac{\pi}{2}) < E_1(0,0) = 0$, which is a contradiction; therefore $\operatorname{sign}(g') = 1$ on $(0, \frac{\pi}{2}]$ and this completes the proof of the final case. □

We note that Theorem 8.1 is now a consequence of Lemma 8.5 (i), Theorem 8.3, and Theorem 8.6.

Proof of Corollary 8.2. Let $\widehat{F} \in \widehat{\mathcal{A}}_{\pi/2}(P_1, P_2, \ldots, P_m)$ be as in Lemma 6.2, and recall (see Remark 6.3) that $\widehat{F}$ has minimal bending energy in $\widehat{\mathcal{A}}_{\pi/2}(P_1, P_2, \ldots, P_m)$. Let $j \in \{2, 3, \ldots, m-1\}$ be such that $|\psi_j| < \Psi$. It follows from Theorem 8.1 that the two chord angles at node P_j satisfy $|\beta_j|, |\alpha_j| < \frac{\pi}{2}$, and therefore, by Theorem 7.5, F is G^2 across node P_j. □

9. Proof of Theorem 1.4

As in previous sections, let interpolation points $P_1, P_2, \ldots, P_m$, with $P_j \neq P_{j+1}$, be given and we continue with the notation $\psi_j := \arg \frac{P_{j+1} - P_j}{P_j - P_{j-1}}$ for the stencil angle at node P_j, $j = 2, 3, \ldots, n-1$. Furthermore, for an interpolating curve F, let the chord angles of the j-th piece (connecting P_j to P_{j+1}) be denoted by (α_j, β_{j+1}) and its breadth by $L_j := |P_j - P_{j+1}|$. Our main results, Theorem 7.5, Theorem 8.1, and Corollary 8.2, are actually valid in more general situations than stated, specifically, the five possible scenarios described in [7, Remark 2.1]. We have been addressing the first scenario where the interpolating curve is free at both ends P_1 and P_n. Scenarios 2,3,4 involve possible clamps at the end points, while the last scenario only considers interpolating curves that are closed (i.e., periodic). Our main results are easily adapted to these other scenarios by simply assuming that when interpolating curves are clamped at nodes P_1 or P_m, then the resulting chord angles, α_1 or β_m, belong to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. These generalizations are valid because the proofs are obtained by holding all directions (of an optimal interpolating curve) fixed except the direction at the interior node P_j being examined. In this section, we will employ the second and third scenarios of [7, Remark 2.1], where the first direction or the last, but not both, is clamped. For this let $\mathcal{A}_{\pi/2}(\widehat{\alpha}; P_1, P_2, \ldots, P_m)$ (resp. $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m; \widehat{\beta})$) denote the subset of curves $F \in \mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m)$ satisfying $\alpha_1 = \widehat{\alpha}$ (resp. $\beta_m = \widehat{\beta}$). With this notation, Corollary 8.2 becomes the following.

Corollary 9.1. *Let $\widehat{\alpha}, \widehat{\beta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and let F be an interpolating curve that has minimal bending energy in $\mathcal{A}_{\pi/2}(\widehat{\alpha}; P_1, P_2, \ldots, P_m)$ or in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m; \widehat{\beta})$. If the stencil angle at node P_j satisfies $|\psi_j| < \Psi$, then the two chord angles at P_j satisfy $|\beta_j|, |\alpha_j| < \frac{\pi}{2}$ and consequently F is G^2 across node P_j.*

Our main purpose in this section is to prove Theorem 1.4 and thereby demonstrate that the angle Ψ is sharp in Corollary 8.2. We will begin, however, with easier prey, namely showing that Ψ is sharp in Corollary 9.1.

Proposition 9.2. *Set $P_1 := -1 + i0$ and $P_2 := 0 + i0$. For all $\varepsilon \in (0, \frac{\pi}{180})$, there exist $\delta > 0$ and a point $P_3 \neq P_2$, with $\psi_2 = \Psi + \varepsilon$, such that for all $\widehat{\alpha} \in [\frac{\pi}{2} - \delta, \frac{\pi}{2}]$ and for all minimal energy curves $F \in \mathcal{A}_{\pi/2}(\widehat{\alpha}; P_1, P_2, P_3)$, F is not G^2 across P_2.*

Lemma 9.3. *In the notation of Section 6, let $u := (P_1, d_1)$ and $v := (P_2, d_2)$ be two unit tangent vectors with $L := |P_1 - P_2| > 0$, and let $S(u, v)$ be the set of s-curves that connect u to v. Assume that the chord angles determined by u, v satisfy $(\alpha, \beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2$. If $f : [a, b] \to \mathbb{C}$ is an optimal curve in $S(u, v)$, then κ, the signed curvature of f, satisfies $|\kappa(t)| < \frac{4\pi}{L}$, $t \in [a, b]$.*

Proof. If $(\alpha, \beta) = (0, 0)$, then f is a line segment and the claim is clear; so assume $(\alpha, \beta) \neq (0, 0)$. We first address the case when $f = \lfloor f \rfloor$ (see Remark 5.3). Then f is directly congruent to $Lc_1(\alpha, \beta)$ and, by Definition 5.1, there exists $(t_1, t_2) \in Y_{2\pi}$ such that $c_1(\alpha, \beta)$ is directly congruent to $\frac{1}{\ell} R_{[t_1,t_2]}$, where $\ell := |R(t_2) - R(t_1)|$. Since $t_2 - t_1 < 2\pi$ and the speed of R is at most 1, it follows that $\ell < 2\pi$. Since the signed curvature of R ranges from -2 to 2, it follows that the signed curvature of f ranges from $-2\frac{\ell}{L}$ to $2\frac{\ell}{L}$; hence $|\kappa(t)| < \frac{4\pi}{L}$, $t \in [a, b]$.
The remaining case, $f \neq \lfloor f \rfloor$, arises when $|\alpha - \beta| = \pi$ and f is obtained by elongating the u-turn $\lfloor f \rfloor$. The desired inequality, $|\kappa(t)| < \frac{4\pi}{L}$, applies to $\lfloor f \rfloor$ (by the first case) and so it also applies to f since the extending line segments have curvature 0. □

Lemma 9.4. *Let $0 < a_1 \leq a_2 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} < b_1 \leq b_2 \leq \frac{\pi}{2}$ and define*
$$\nu := \inf\{|\kappa_b(c_1(\alpha, \beta))| : a_1 \leq \alpha \leq a_2 \text{ and } b_1 \leq \beta \leq b_2\}.$$
If $\beta^*(a_1) < b_1$, then $\nu > 0$.

Proof. Assume $\beta^*(a_1) < b_1$ and set $X := \{(\alpha, \beta) : a_1 \leq \alpha \leq a_2 \text{ and } b_1 \leq \beta \leq b_2\}$. By Theorem 7.3, $\kappa_b(c_1(\alpha, \beta)) = 2\frac{\partial E_1}{\partial \beta}(\alpha, \beta)$ for all $(\alpha, \beta) \in X$, and so by Corollary 7.4, $\kappa_b(c_1(\alpha, \beta))$ is a continuous function of $(\alpha, \beta) \in X$. We will show that if $\nu = 0$, then $\beta^*(a_1) \geq b_1$. Assume that $\nu = 0$. Then, since X is compact, there exists $(\alpha, \beta) \in X$ such that $\kappa_b(c_1(\alpha, \beta)) = 0$, and it follows by Theorem 8.6 that $\beta = \beta^*(\alpha)$. Since β^* is decreasing, we have $b_1 \leq \beta = \beta^*(\alpha) \leq \beta^*(a_1)$. □

Proof of Proposition 9.2. Fix $\varepsilon \in (0, \frac{\pi}{180})$ and set $P_3 := L\exp(i\psi_2)$, where $\psi_2 := \Psi + \varepsilon$ and $L > 0$ is yet to be determined. Recall that $\beta^*(\frac{\pi}{2}) = -(\frac{\pi}{2} - \Psi)$, and let $\delta \in (0, \frac{\pi}{2})$ be determined by $\beta^*(\frac{\pi}{2} - \delta) = -(\frac{\pi}{2} - \Psi - \frac{\varepsilon}{2})$. Let ν be as defined in Lemma 9.4 with $[a_1, a_2] := [\frac{\pi}{2} - \delta, \frac{\pi}{2}]$ and $[b_1, b_2] := [\psi_2 - \frac{\pi}{2}, \frac{\pi}{2}]$. Since $\beta^*(\frac{\pi}{2} - \delta) = \Psi + \frac{\varepsilon}{2} - \frac{\pi}{2} < \Psi + \varepsilon - \frac{\pi}{2} = \psi_2 - \frac{\pi}{2}$, it follows by Lemma 9.4 that $\nu > 0$.
Now, assume $L > \frac{4\pi}{\nu}$ and fix $\widehat{\alpha} \in [\frac{\pi}{2} - \delta, \frac{\pi}{2}]$. Let $F = f_1 \sqcup f_2$ have minimal bending energy in $\mathcal{A}_{\pi/2}(\widehat{\alpha}; P_1, P_2, P_3)$. Note that the feasible range of β_2 is $[\psi_2 - \frac{\pi}{2}, \frac{\pi}{2}]$, and it follows, by Theorem 5.4, that f_1 is directly congruent to $c_1(\widehat{\alpha}, \beta_2)$. On one side we have $|\kappa_b(f_1)| = |\kappa_b(c_1(\widehat{\alpha}, \beta_2))| \geq \nu$, while on the other side we have, by Lemma 9.3, $\nu > |\kappa_a(f_2)|$. Hence F is not G^2 across P_2. □

Our proof of Theorem 1.4 mimics that of Proposition 9.2, except that we are not allowed to employ clamps. In order to get around this, we introduce the notion of a *soft clamp.*

Definition 9.5. Let $I \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$ be an interval. A sequence of points $P_1, P_2, \ldots, P_m$ imposes a **soft clamp on α_{m-1} with range** I if the following hold:
(i) For $j = 2, 3, \ldots, m - 1$, $|\psi_j| < \Psi$.

(ii) For all $\widehat{\beta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, if F has minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m; \widehat{\beta})$, then $\alpha_{m-1} \in I$.

We leave it to the reader to verify that the property of imposing a soft clamp on α_{m-1} with range I is invariant under translations, rotations and scalings of the points $P_1, P_2, \ldots, P_m$, while reflecting the points about a line changes the range to $-I$. An example of a soft clamp is given in the following.

Lemma 9.6. *Let $\varepsilon \in (0, \frac{\pi}{180})$ and set $P_1 := -1 + i0$, $P_2 := 0 + i0$, and $P_3 := L\exp(i\psi_2)$, where $\psi_2 := \Psi - \frac{\varepsilon}{2}$. If L is sufficiently large, then P_1, P_2, P_3 imposes a soft clamp on α_2 with range $[-\frac{\pi}{2}, \varepsilon - \Psi]$.*

Proof. Set $\nu := \inf\{|\kappa_b(c_1(\beta^*(\beta), \beta))| : \beta \in [\frac{\varepsilon}{2}, \frac{\pi}{2}]\}$. We claim that $\nu > 0$. Suppose, to the contrary, that $\nu = 0$. By Theorem 7.3 and Corollary 7.4, the function $\beta \mapsto \kappa_b(c_1(\beta^*(\beta), \beta))$ is continuous on $[\frac{\varepsilon}{2}, \frac{\pi}{2}]$, and it follows that there exists $\beta_0 \in [\frac{\varepsilon}{2}, \frac{\pi}{2}]$ such that $\kappa_b(c_1(\beta^*(\beta_0), \beta_0)) = 0$. It follows from Theorem 8.6 that $\kappa_b(c_1(\beta_0, \beta^*(\beta_0))) = 0$ and therefore, by symmetry, $\kappa_a(c_1(\beta^*(\beta_0), \beta_0)) = 0$. Hence $c_1(\beta^*(\beta_0), \beta_0)$ begins and ends with 0 curvature. From this it follows that $c_1(\beta^*(\beta_0), \beta_0)$ is either a line segment or a u-turn, but since $\beta_0 \notin \{-\frac{\pi}{2}, 0\}$, we must have $(\beta^*(\beta_0), \beta_0) = (-\frac{\pi}{2}, \frac{\pi}{2})$, which contradicts Lemma 8.5 (iii). Therefore, $\nu > 0$ as claimed.
Fix $L > \frac{4\pi}{\nu}$, and let $\widehat{\beta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be arbitrary. Assume $F = f_1 \sqcup f_2$ has minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, P_3; \widehat{\beta})$. By Corollary 9.1, F is G^2 across P_2, so $\kappa_b(f_1) = \kappa_a(f_2)$. By Lemma 9.3, we have $\nu > |\kappa_a(f_2)|$. Since F has minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, P_3; \widehat{\beta})$, it follows that f_1 has minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2; \beta_2)$ and therefore, by Theorem 8.6 (and symmetry), $\alpha_1 = \beta^*(\beta_2)$; hence, f_1 is equivalent to $c_1(\beta^*(\beta_2), \beta_2)$. The feasible range of β_2 is $\beta_2 \in [\psi_2 - \frac{\pi}{2}, \frac{\pi}{2}]$. If $\beta_2 \in [\frac{\varepsilon}{2}, \frac{\pi}{2}]$, then $|\kappa_b(f_1)| = |\kappa_b(c_1(\beta^*(\beta_2), \beta_2))| \geq \nu$, which contradicts $\kappa_b(f_1) = \kappa_a(f_2)$. Therefore, $\beta_2 \in [\psi_2 - \frac{\pi}{2}, \frac{\varepsilon}{2})$ and it follows that $\alpha_2 = \beta_2 - \psi_2 < \varepsilon - \Psi$; hence $\alpha_2 \in [-\frac{\pi}{2}, \varepsilon - \Psi]$. $\square$

Note that if we reflect the three points in Lemma 9.6 about the real axis, we obtain points $Q_1 := 1 + i0$, $Q_2 := 0 + i0$, $Q_3 := L\exp(-\psi_2)$ which impose a soft clamp on α_2 with range $[\Psi - \varepsilon, \frac{\pi}{2}]$. Starting with this initial soft clamp, we can build up inductively to construct soft clamps with ranges closer to $\{\frac{\pi}{2}\}$.

Proposition 9.7. *Let $\gamma \in (0, \frac{\pi}{2})$ and suppose that $P_1, P_2, \ldots, P_m$ imposes a soft clamp on α_{m-1} with range $[\gamma, \frac{\pi}{2}]$. Then for all $\varepsilon > 0$, there exists $P_{m+1} \neq P_m$ such that $P_1, P_2, \ldots, P_m, P_{m+1}$ imposes a soft clamp on α_m with range $[-\frac{\pi}{2}, -(\Psi - \beta^*(\gamma) - \varepsilon)]$.*

Proof. We can assume, without loss of generality, that $P_{m-1} = -1 + i0$ and $P_m = 0 + i0$. Fix $\varepsilon \in (0, \frac{\pi}{180})$ and set $P_{m+1} := L\exp(i\psi_m)$, where $\psi_m := \Psi - \frac{\varepsilon}{2}$ and $L > 0$ is yet to be determined. Let ν be as defined in Lemma 9.4 with $[a_1, a_2] := [\gamma, \frac{\pi}{2}]$ and $[b_1, b_2] := [\beta^*(\gamma) + \frac{\varepsilon}{2}, \frac{\pi}{2}]$. Since $\beta^*(a_1) < b_1$, it follows from Lemma 9.4 that $\nu > 0$.
Fix $L > \frac{4\pi}{\nu}$, and let $\widehat{\beta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Assume $F = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{m-1} \sqcup f_m$ has minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m, P_{m+1}; \widehat{\beta})$. Note that f_{m-1} is directly congruent to $c_1(\alpha_{m-1}, \beta_m)$. Since $f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{m-1}$ has minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m; \beta_m)$, it follows from the soft clamp property that $\alpha_{m-1} \in [\gamma, \frac{\pi}{2}]$. The feasible range of β_m is $[\psi_m - \frac{\pi}{2}, \frac{\pi}{2}]$, but we will show that $\beta_m < \beta^*(\gamma) + \frac{\varepsilon}{2}$. Assume, to the contrary, that $\beta_m \in [\beta^*(\gamma) + \frac{\varepsilon}{2}, \frac{\pi}{2}]$. By Lemma 9.3, $|\kappa_a(f_m)| < \nu$, while $|\kappa_b(f_{m-1})| = |\kappa_b(c_1(\alpha_{m-1}, \beta_m))| \geq \nu$; hence F is not G^2 across P_m, contradicting Corollary 9.1. Therefore, $\beta_m < \beta^*(\gamma) + \frac{\varepsilon}{2}$. It now follows that $-\frac{\pi}{2} \leq \alpha_m = \beta_m - \psi_m < \beta^*(\gamma) + \frac{\varepsilon}{2} - \psi_m = \beta^*(\gamma) - \Psi + \varepsilon$. $\square$

We again note that reflecting the points $P_1, P_2, \ldots, P_{m+1}$ about the real axis yields points that impose a soft clamp on α_m with range $[\Psi - \beta^*(\gamma) - \varepsilon, \frac{\pi}{2}]$. With Lemma 9.6

and Proposition 9.7 in hand, the question which remains is whether we can construct soft clamps with range $[\frac{\pi}{2}-\delta, \frac{\pi}{2}]$ for arbitrarily small $\delta > 0$. Let the sequence $\{\gamma_j\}$ be defined by $\gamma_1 := \Psi$ and $\gamma_{j+1} := \Psi - \beta^*(\gamma_j)$, $j = 1, 2, 3, \ldots$. It follows from Lemma 8.5 (ii) that $\{\gamma_j\}$ is well-defined, increasing, and bounded from above by $\frac{\pi}{2}$. Set $\gamma := \lim_{j\to\infty} \gamma_j$, and note that $\gamma \in (\Psi, \frac{\pi}{2}]$ must satisfy $\gamma = \Psi - \beta^*(\gamma)$; that is, $\gamma + \beta^*(\gamma) = \Psi$. By Lemma 8.5 (iii), $\gamma = \frac{\pi}{2}$.

Theorem 9.8. *For all $\delta > 0$, there exists a sequence $P_1, P_2, \ldots, P_m$ that imposes a soft clamp on α_{m-1} with range $[\frac{\pi}{2}-\delta, \frac{\pi}{2}]$.*

Proof. Let k be the first index such that $\gamma_k > \frac{\pi}{2} - \delta$. Define $g(\gamma) := \Psi - \beta^*(\gamma)$, $\gamma \in [0, \frac{\pi}{2}]$, and note that $\gamma_{j+1} := g(\gamma_j)$. Since g is continuous and $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_k < \frac{\pi}{2}$, it follows that there exists $\varepsilon > 0$ sufficiently small such that the finite sequence $\{\widehat{\gamma}_j\}$, defined by $\widehat{\gamma}_1 := \Psi - \varepsilon$ and $\widehat{\gamma}_{j+1} := g(\widehat{\gamma}_j) - \varepsilon$, $j = 1, 2, \ldots, k-1$, is well-defined and satisfies $\widehat{\gamma}_k > \frac{\pi}{2} - \delta$. By Lemma 9.6, there exist points P_1, P_2, P_3 that impose a soft clamp on α_2 with range $[\widehat{\gamma}_1, \frac{\pi}{2}]$, and then repeated applications of Proposition 9.7 yield a sequence $P_1, P_2, \ldots, P_{k+2}$ that imposes a soft clamp on α_{k+1} with range $[\widehat{\gamma}_k, \frac{\pi}{2}]$. $\square$

Proof of Theorem 1.4. Fix $\varepsilon \in (0, \frac{\pi}{180})$ and let $\delta > 0$ and P_1, P_2, P_3 be as in Proposition 9.2, but let us name them Q_1, Q_2, Q_3 instead. By Theorem 9.8, there exists a sequence $P_1, P_2, \ldots, P_m$ that imposes a soft clamp on α_{m-1} with range $[\frac{\pi}{2}-\delta, \frac{\pi}{2}]$. We can assume, without loss of generality, that $P_{m-1} = Q_1$ and $P_m = Q_2$. Set $P_{m+1} := Q_3$, and note that $|\psi_j| < \Psi$ for $j = 2, 3, \ldots, m-1$, while $\psi_m = \Psi + \varepsilon$. Let $F_{opt} \in \mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m, P_{m+1})$ have minimal bending energy, say $F_{opt} = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{m-1} \sqcup f_m$. We will show that F_{opt} is not G^2 across P_m. The chord angles of f_{m-1} are (α_{m-1}, β_m). Since $f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{m-1}$ has minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_m; \beta_m)$, it follows from the soft clamp property that $\alpha_{m-1} \in [\frac{\pi}{2}-\delta, \frac{\pi}{2}]$. Since $f_{m-1} \sqcup f_m$ has minimal bending energy in $\mathcal{A}_{\pi/2}(\alpha_{m-1}; Q_1, Q_2, Q_3)$, it follows from Proposition 9.2 that $f_{m-1} \sqcup f_m$ is not G^2 across Q_2; therefore F_{opt} is not G^2 across P_m. $\square$

Remark 9.9. The quantity $\rho := \max_{1\le j<m} \max\{\frac{L_{j+1}}{L_j}, \frac{L_j}{L_{j+1}}\}$ serves as a mesh ratio, and our main results, Corollary 1.3 and Theorem 1.4, are obtained without any constraints on ρ. Indeed, it is clear in the proof of Proposition 9.7 that $\rho \to \infty$ as $\varepsilon \to 0$. We expect that the critical angle Ψ will increase if one constrains the mesh ratio.

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