STABILITY OF INTERPOLATING ELASTICA

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ABSTRACT. Interpolating elastica are the extremals for the functional $\int_0^s k^2(s)ds$, which is the integral of the square of the curvature with respect to arc length, in the family of plane curves that interpolate at (not prescribed) arc lengths $s_0 < s_1 < \cdots < s_n$ a prescribed configuration of points $p_0, p_1, \ldots, p_n$. If at one or both terminals the slope is prescribed, the extremal is said to be angle-constrained, otherwise free. The curvature functional represents the elastic strain energy of a thin elastic beam of indefinite length with sleeve supports anchored at $p_0, p_1, \ldots, p_n$, which allow the beam to slide through without friction and to rotate freely (except at the end supports if angle-constrained). The interpolating elastica are also known as nonlinear spline curves. It is known that the infimum of the strain energy is 0 in all cases, hence cannot be attained if the points $p_0, p_1, \ldots, p_n$ do not lie on a ray. On the other hand, interpolating elastica are known to exist for a variety of configurations, and this report investigates whether these extremals make the strain energy a local minimum or not (i.e., whether they are "stable" or "unstable"). Several general stability criteria are established and they are used to decide the stability of some specific elastica.

SIGNIFICANCE AND EXPLANATION

It is an old technique of draftsmen to use a mechanical spline to pass a smooth curve through a prescribed set of points in a plane. Curves which are obtained in this way (interpolating elastica, also called non-linear spline curves) may be considered as the equilibrium positions of thin elastic beams which are constrained to pass through short, frictionless, freely rotating sleeve supports, anchored at the interpolation points. The strain energy of such a beam is given by the integral of the square of the curvature with respect to arc length, and equilibrium requires that the position be such that the energy be minimal for the given interpolation conditions. However, a global minimum cannot be attained (except in the trivial case of the unbent beam) since the energy can be made arbitrarily small by using sufficiently large loops between the supports. Instead one looks for local minima which guarantee stability against small perturbations. In this report some general stability criteria are established and some specific interpolating elastica are investigated for stability. Except for a few previous isolated observations these seem to be the first proven results on the stability of interpolating elastica.

[The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.]

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1. Introduction

Elastica are the plane curves with "normal representation" $s \mapsto \theta(s)$ (s denotes arc length and $\theta(s)$ the angle of inclination at s) which are solutions of the differential equation

$$\frac{1}{2}(\theta'(s))^2 = \lambda[\sin(\theta - \theta_1) - \alpha] \tag{1.1}$$

where λ, θ_1 and α are real constants (see, e.g. [1, Article 263]). (1.1) is the Euler equation for the variational problem

$$\delta \int_0^{\overline{s}} \theta'^2 = 0, \int_0^{\overline{s}} \cos\theta = b, \int_0^{\overline{s}} \sin\theta = d \tag{1.2}$$

where $\overline{s}, b, d$ are prescribed (see above reference or [2, Prop. 3.2]). The integral $\int_0^{\overline{s}} \theta'^2$ represents (with the proper choice of unit) the strain energy of a thin elastic beam of uniform cross section of length $\overline{s}$, and the side conditions in (1.2) specify the relative position of the ends of the bent beam.

The elastica described by (1.1), when considered for all values of s, have infinitely many inflection points, $\theta'(s) = 0$ when $\sin(\theta(s) - \theta_1) = \alpha$, and are therefore called inflectional elastica (see [1, loc cit.]). Below we will consider only elastica for which $\alpha = 0$; geometrically speaking, these are curves for which the variation of θ between consecutive inflection points is π. We refer to them as **simple** elastica. All the simple elastica are obtained from a particular one by similarity transformations.

The **interpolating elastica** (so named by M. A. Malcolm in [3]) consist of finitely many subarcs of the simple elastica, fitted together so that a smooth curve with continuous curvature results which has jump discontinuities of the derivative of the curvature only at the "knots" $p_1, \ldots, p_{n-1}$. Such an interpolating elastica E with normal representation θ is the solution of the variational problem

$$\delta \int_{s_0}^{s_n} \theta'^2 = 0, \int_{s_{i-1}}^{s_i} \cos\theta = b_i, \int_{s_{i-1}}^{s_i} \sin\theta = d_i \quad (i = 1, \ldots, n) \tag{1.3}$$

where the b_i, d_i are prescribed (they are the coordinates of the vector $\overline{p_i - p_{i-1}}$, but the arc lengths $s_0 < s_1 < \cdots < s_n$ of the terminals p_0, p_n and of the knots $p_1, \ldots, p_{n-1}$ are varied (see [2, loc. cit.] or [3, Sec. 21]). If the ends p_0, p_1 are "free" then the natural boundary conditions

$$\theta'(0) = 0, \quad \theta'(s_n) = 0 \tag{1.3a}$$

are appended to (1.3). Frequently we shall be concerned with "angle-constrained" interpolating elastica; in this case we are given

$$\theta(0) = \alpha, \quad \theta(s_n) = \beta. \tag{1.3b}$$

The solutions of (1.3), (1.3a) represent possible equilibrium positions (stable or not) of a thin elastic beam of indefinite length which is constrained to pass through frictionless freely rotating small sleeves anchored at the positions $p_0, p_1, \ldots, p_n$. If the sleeves at the terminals p_0, p_n are pinned then (1.3b) replaces (1.3a). The interpolating elastica are a reasonable mathematical model for the mechanical spline used by draftsmen to pass a smooth curve through the given points $p_0, p_1, \ldots, p_n$. They are also called nonlinear (interpolating) splines (see, e.g., [4]) and were referred to as **extremal interpolants** for the configuration $\{p_0, p_1, \ldots, p_n\}$ in [2]. We still will refer to them by this name in the sequel.

The solutions of (1.3) are definitely not absolute minima, except in the trivial case where $p_0, p_1, \ldots, p_n$ lie (in this order) along a ray (and moreover, $\alpha = \beta = 0$ in case of end conditions (1.3b)). This was first pointed out by the authors of [5]. $\displaystyle\int_0^{s_n} \theta'^2$ can be made arbitrarily small by using large interpolating circular loops. The solutions are often referred to as local minima, although no proofs are given that they are indeed extrema of this kind. Only in [2, Theorem 6.1] was it proved that the nontrivial simple elastica interpolating 2 points are nonstable, i.e., they do not represent local minima of $\displaystyle\int_{s_0}^{s_1} \theta'^2$. It is the objective of this paper to establish, for several known extremal interpolants, whether they are local minima or not (stable or unstable).

The fact that the extremal interpolants do not represent minima nor, in general, local minima of the functional $\displaystyle\int \theta'^2$ is, probably, the major reason for the lack of general existence results and of good computational procedures. For an existence proof limited to length-restricted extremals, see [6]. In [7, Theorem 3] it is proved that if there is a length-restricted extremal of "unstable length", there is also an interpolating local minimum, but no nontrivial length-restricted extremal of unstable length is presented. Existence of length-prescribed extremals and of unrestricted extremal interpolants close to a ray interpolant is proved in [2, Appendix and Theorem 7.4], where also many examples of specific interpolants are given, which were not known before. For a survey of old and new computational procedures, see [3]. In the discussion of stability (that is, whether the extremals are local minima or not) we naturally restrict ourselves to cases where existence of extremal interpolants has been proved or is postulated.

In Section 2 the variational equations for interpolating splines in normal representation are derived, without recourse to Lagrange multiplier theory, and as a preparation for the computation of the second variation. In Section 3 the second variation is used for stability criteria (Jacobi's condition): an explicitly given quadratic functional must be positive-definite, or equivalently, a nonconventional linear second-order boundary value problem must have only positive eigenvalues. In Section 4 it is proved that interpolating splines close (in a precise sense) to stable ones are stable and those close to strongly unstable ones are unstable. This result is then used to prove that splines that interpolate configurations close to a ray configuration (whose existence was proved in [2]) are stable (even in the case of free terminals). This is probably the first general existence proof for locally minimizing interpolants which are not length-restricted. In Section 5 it is proved that the extremal 2-point interpolant consisting of $n \geq 1$ complete loops of the simple elastica is unstable even if angle-constrained (in [2] the instability was proved for the free elastica). If the angle-constrained 2-point interpolants is a proper subarc of one loop of the simple elastica (hence has no inflection point) then it is stable, and any angle-constrained 2-point interpolant that contains one complete loop of the simple elastica is unstable. The proof for these last results is contained in Section 6; it is built mainly on the discovery of the eigenfunction belonging to the eigenvalue 0 for the second variational equation that goes with the one-loop angle-constrained simple elastica. By an extension of this method it is proved in Section 7 that if an angle-constrained interpolant contains an interior inflection point then it is stable if it contains neither the left nor the right "stability focus". These are points on the simple elastica which are situated symmetrically with respect to the inflection point, not far from the neighboring inflection points. If the angle-constrained 2-point interpolant with one inflection point contains both stability foci it is unstable. The general result on the stability of such 2-point interpolants is stated with the use of what we call “conjugate points”. If p is a point on a simple elastica arc containing one inflection point there is a conjugate point $p_\star$ defined by a transcendental equation, and it is also given a geometric interpretation (p and $p_\star$ are on opposite sides of the inflection point; if p is a stability focus then $p_\star$ is the other stability focus). The angle-constrained elastica is stable if and only if it contains no pair of conjugate points. If the 2-point extremal interpolant is free at one end and angle-constrained at the other end, then it is stable if and only if it contains no stability focus. Section 8 contains the most important stability results. It is first proved that a necessary condition for the stability of extremal N-point interpolants is that each arc between consecutive nodes be “proper”, i.e., internal arcs do not contain a pair of conjugate points, and the terminal arcs do not contain a stability focus. Then a computable “stability function” of $(N-2)$ real variables is defined for the extremal N-point interpolant under investigation, which has a critical value at the point that corresponds to the extremal. It is proved that the extremal is stable if and only if the critical value is a local minimum. These results are applied to decide the stability of some 3-point and 4-point extremal interpolants. In this connection it is also shown that the often repeated claim (first appearing in [5]) that a certain 4 -point configuration has no interpolating elastica is false. In the last section we show that the closed extremals which interpolate the vertices of a regular n-gon ($n \neq 3$) (their existence is proved in [2, Sec. 8]) are stable.

2. The Euler-Lagrange conditions for the interpolating spline in normal representation

Let $s \mapsto \theta(s)$, $0 \le s \le \overline{s}$ be the normal representation of an admissible interpolant C for the configuration $\{p_0, p_1, \dots, p_n\}$. Here s denotes the arc length along the curve C and $\theta(s)$ the angle that C makes at arc length s with a reference line. The interpolation conditions are

$$\int_{s_{i-1}}^{s_i} \cos\theta(s)ds = b_i, \qquad \int_{s_{i-1}}^{s_i} \sin\theta(s)ds = d_i, \qquad i = 1, \dots, n \tag{2.1}$$

where b_i, d_i are given numbers, and the nodes $0 = s_0 < s_1 < \cdots < s_n = \overline{s}$ are the arc lengths at which C passes through the interpolation points $p_0, p_1, \dots, p_n$ ($s_1, \dots, s_n$ vary with C). We assume $b_i^2 + d_i^2 > 0$, hence $p_{i-1} \neq p_i$ $(i = 1, \dots, n)$.

Much of the paper deals with **angle-constrained** interpolants, in which case the angles

$$\theta(0) = \alpha, \qquad \theta(\overline{s}) = \beta \tag{2.2}$$

are prescribed. If $\theta(0)$ and/or $\theta(\overline{s})$ is not prescribed the corresponding terminal of C is said to be **free**, and the corresponding natural end conditions for an extremal interpolant turn out to be

$$\theta'(0) = 0, \qquad \theta'(\overline{s}) = 0 \tag{2.3}$$

The functional which is made stationary by an extremal interpolant E is the potential energy (or curvature functional)

$$\int_0^{\overline{s}} [\theta'(s)]^2. \tag{2.4}$$

The comparison functions are taken from the Sobolev space $W_{1,2} = W_{1,2}[0,S]$ of functions $\theta : [0,S] \to \mathbb{R}$, which are absolutely continuous and have derivatives θ' in $L_2[0,S]$ with norm $\left(\int_0^S (\theta^2 + \theta'^2)\right)^{1/2}$. S is a prescribed positive number large enough so that the functions in $W_{1,2}$ satisfying conditions (2.1) and (2.2) (if imposed) form a subset with nonempty interior. In this paper we do not deal with the existence of extremal interpolants, but we start with a known extremal E_0 and investigate whether it is stable or not. In this case, we may take $S = \overline{s} + \delta$ where $\overline{s}$ is the length of E_0 and δ is an arbitrary positive number.

Let $s \mapsto \theta_0(s)$ be the normal representation of E_0 and $0 = s_0 < s_1 < \cdots < s_n = \overline{s}$ the interpolation nodes. For fixed real numbers $\tau_1, \ldots, \tau_n$ and fixed functions η, ξ in $W_{1,2}$, which we assume to have piecewise continuous derivatives with jumps only at $s_1, \ldots, s_{n-1}$, consider the family of comparison curves C_ε, given parametrically by

$$
\begin{aligned}
&\theta_\varepsilon(t) = \theta_0(t) + \varepsilon\eta(t) + \varepsilon^2\xi(t), \qquad 0 \le t \le \overline{s} \\
&s_\varepsilon(t) = \sum_{j=1}^{i-1}(1+\varepsilon\tau_j)(s_j - s_{j-1}) + (1+\varepsilon\tau_i)(t - s_{i-1}), \qquad s_{i-1} \le t \le s_i
\end{aligned}
\tag{2.5}
$$

where $\theta_\varepsilon(t)$, $s_\varepsilon(t)$ denote the angle of inclination and the arc length of C_ε at t. If $\varepsilon \in \mathbb{R}$ is sufficiently small then C_ε is in a prescribed neighborhood of E_0. The interpolation conditions (2.1) require

$$
(1+\varepsilon\tau_i)\int_{s_{i-1}}^{s_i} \cos\theta_\varepsilon(t)dt = b_i, \qquad (1+\varepsilon\tau_i)\int_{s_{i-1}}^{s_i} \sin\theta_\varepsilon(t)dt = d_i, \qquad i = 1, \ldots, n. \tag{2.6}
$$

For definiteness, we assume $d_i \neq 0$ $(i = 1, \ldots, n)$. Then equating terms in ε^1 in (2.6) gives

$$
\tau_i = -\frac{1}{d_i}\int_{s_{i-1}}^{s_i} (\cos\theta_0)\eta \tag{2.7a}
$$

$$
\int_{s_{i-1}}^{s_i} (b_i\cos\theta_0 + d_i\sin\theta_0)\eta = 0, \tag{2.7b}
$$

and equating terms in ε^2 gives

$$
\begin{aligned}
&2b_i\tau_i^2 + \int_{s_{i-1}}^{s_i} (\eta^2\cos\theta_0 + 2\xi\sin\theta_0) = 0 \\
&2d_i\tau_i^2 + \int_{s_{i-1}}^{s_i} (\eta^2\sin\theta_0 - 2\xi\cos\theta_0) = 0 \\
&i = 1, \ldots, n.
\end{aligned}
\tag{2.7c}
$$

The value of the potential energy for the curve C_ε is

$$
U(\varepsilon) = \int_0^{\overline{s}} \left[\frac{\theta'_\varepsilon(t)}{s'_\varepsilon(t)}\right]^2 s'_\varepsilon(t)dt = \sum_{i=1}^n (1+\varepsilon\tau_i)^{-1}\int_{s_{i-1}}^{s_i} {\theta'_\varepsilon}^2. \tag{2.8}
$$

Set

$$u_i = \int_{s_{i-1}}^{s_i} {\theta_0'}^2, \quad U_0 = \sum_{i=1}^n u_i. \tag{2.9}$$

Expand (2.8) in powers of ε, using (2.5):

$$\begin{aligned} U(\varepsilon) = U_0 + \varepsilon\left[2\int_0^{\overline{s}} \theta_0'\eta' + \sum_{i=1}^n \tau_i u_i\right] \\ + \varepsilon^2\left[2\int_0^{\overline{s}}\theta_0'\xi' - 2\sum_{i=1}^n \tau_i \int_{s_{i-1}}^{s_i}\theta_0'\eta' + \sum_{i=1}^n \tau_i^2 u_i + \int_0^{\overline{s}} {\eta'}^2\right] + O(\varepsilon^3). \end{aligned} \tag{2.10}$$

Since U_0 is a stationary value of the potential energy, we must have, using (2.7a),

$$\sum_{i=1}^n \int_{s_{i-1}}^{s_i} (2\theta_0'\eta' + \frac{u_i}{d_i}\cos\theta_0\eta) = 0$$

and this must be true for every η for which (2.7b) holds and for which

$$\eta(0) = 0 \quad \text{and/or} \quad \eta(\overline{s}) = 0 \tag{2.7d}$$

if E_0 is angle-constrained. From this one infers, by usual arguments of the calculus of variations (carried out in detail in [2]), that θ_0' is continuous, θ_0'' is continuous between consecutive interpolation nodes, and there exist constants $\lambda_i \in \mathbb{R}$ such that

$$\begin{gathered} 2\theta_0''(s) - \lambda_i d_i \sin\theta_0(s) - \left(\frac{u_i}{d_i} + \lambda_i b_i\right)\cos\theta_0(s) = 0, \\ s_{i-1} < s < s_i, \quad i = 1, \dots, n. \end{gathered} \tag{2.12}$$

Moreover, conditions (2.3) must hold for θ_0 if the terminals are free.

Integration of (2.12) gives

$${\theta_0'}^2(s) + \lambda_i d_i \cos\theta_0(s) - \left(\frac{u_i}{d_i} + \lambda_i b_i\right)\sin\theta_0(s) = \delta_i$$

and another integration from s_{i-1} to s_i shows that $\delta_i = 0$. Thus,

$${\theta_0'}^2(s) + \lambda_i^1\cos\theta_0(s) + \lambda_i^2 \sin\theta_0(s) = 0, \quad s_{i-1} \le s \le s_i, \quad i = 1, \dots, n \tag{2.14}$$

where we have set

$$\lambda_i^1 = \lambda_i d_i, \quad \lambda_i^2 = -\lambda_i b_i - \frac{u_i}{d_i}. \tag{2.14}$$

To determine the multipliers λ_i^1, λ_i^2 we use the fact that θ_0 and θ_0' are continuous, hence

$$\left(\lambda_{i+1}^1 - \lambda_i^1\right)\cos\theta_0(s_i) + \left(\lambda_{i+1}^2 - \lambda_i^2\right)\sin\theta_0(s_i) = 0, \quad i = 1, \dots, n-1. \tag{2.15}$$

Conditions (2.15) together with the interpolation conditions (2.1) and end conditions $\theta_0(0) = \alpha$ (or $\theta_0'(0) = 0$), $\theta_0(\overline{s}) = \beta$ (or $\theta_0'(\overline{s}) = 0$), are $3n+1$ independent conditions for the $3n+1$ unknowns $\lambda_i^1, \lambda_i^2, s_i$ $(i = 1, \dots, n)$ and $\theta_0(0)$, which together with the differential equation (2.13) determine the interpolating elastica θ_0. There may be many solutions of these equations, as shown in [2], but the distinct solutions are isolated.

The assumption $d_i \neq 0$ $(i = 1, \dots, n)$ was made only to avoid case splitting. The obtained result remains true as long as $b_i^2 + d_i^2 > 0$ for $i = 1, \dots, n$.

3. Stability Criteria

We now look at the quadratic terms in the expansion (2.10) for the potential energy:

$$\begin{aligned}
&\int_0^{\bar{s}} \eta'^2 - 2\sum_{i=1}^{n} \tau_i \int_{s_{i-1}}^{s_i} \theta_0'\eta' + \sum_{i=1}^{n} \tau_i^2 u_i + 2\int_0^{\bar{s}} \theta_0'\xi', \\
&\tau_i = -\frac{1}{d_i}\int_{s_{i-1}}^{s_i} (\cos\theta_0)\eta = \frac{1}{b_i}\int_{s_{i-1}}^{s_i} (\sin\theta_0)\eta, \quad u_i = \int_{s_{i-1}}^{s_i} {\theta_0'}^2.
\end{aligned} \tag{3.1}$$

Using(2.7c), (2.12), (2.13) and (2.14), we can eliminate ξ in (3.1):

$$\begin{aligned}
2\int_{s_{i-1}}^{s_i} \theta_0'\xi'' - 2\theta_0'\Big|_{s_{i-1}}^{s_i} &= -2\int_{s_{i-1}}^{s_1} \theta_0''\xi \\
&= -\lambda_i d_i \int_{s_{i-1}}^{s_i} (\sin\theta_0)\xi - \left(\frac{u_i}{d_i} + \lambda_i b_i\right)\int_{s_{i-1}}^{s_i} (\cos\theta_0)\xi \\
&= \lambda_i d_i \left[\frac{1}{2}\int_{s_{i-1}}^{s_i} (\cos\theta_0)\eta^2 + b_i\tau_i^2\right] - \left(\frac{u_i}{d_i} + \lambda_i b_i\right)\left[\frac{1}{2}\int_{s_{i-1}}^{s_i} (\sin\theta_0)\eta^2 + d_i\tau_i^2\right] \\
&= -\frac{1}{2}\int_{s_{i-1}}^{s_i} {\theta_0'}^2\eta^2 - \tau_i^2 u_i.
\end{aligned} \tag{3.2}$$

Thus, since $(\theta_0'\xi)(s_i - 0) = (\theta_0'\xi)(s_i + 0)$ and $(\theta_0'\xi)(0) = (\theta_0'\xi)(\bar{s}) = 0$:

$$2\int_0^{\bar{s}} \theta_0'\xi' = -\frac{1}{2}\int_0^{\bar{s}} {\theta_0'}^2\eta^2 - \sum_{i=1}^{n} \tau_i^2 u_i$$

and (3.1) becomes

$$\int_0^{\bar{s}} (\eta'^2 - 2\sum_{i=1}^{n} \tau_i \int_{s_{i-1}}^{s_i} \theta_0'\eta') - \frac{1}{2}\int_0^{\bar{s}} {\theta_0'}^2\eta^2. \tag{3.3}$$

We introduce the subspace $V(\theta_0)$ of $W_{1,2}[0,\bar{s}]$:

$$\begin{aligned}
V_0(\theta_0) = \{\eta \in W_{1,2}[0,\bar{s}] : &\int_{s_{i-1}}^{s_i} (b_i\cos\theta_0 + d_i\sin\theta_0)\eta = 0 \text{ for } i = 1, \dots, n; \\
&\eta(0) = 0 \text{ and/or } \eta(\bar{s}) = 0 \text{ if } E_0 \text{ is angle-constrained at the corresponding terminal}\}.
\end{aligned} \tag{3.4}$$

and the quadratic form $Q(\theta_0, \cdot)$ with domain V_0:

$$Q(\theta_0, \eta) = \int_0^{\bar{s}} (\eta'^2 - \frac{1}{2}{\theta_0'}^2\eta^2) + 2\sum_{i=1}^{n} d_i^{-1}\int_{s_{i-1}}^{s_i} (\cos\theta_0)\eta \int_{s_{i-1}}^{s_i} \theta_0'\eta'. \tag{3.5}$$

It is understood that the factor $d_i^{-1}\int_{s_{i-1}}^{s_i} (\cos\theta_0)\eta$ is replaced by $-b_i^{-1}\int_{s_{i-1}}^{s_i} (\sin\theta_0)\eta$ if $d_i = 0$. If $Q(\theta_0, \eta) \le 0$ for some $\eta \ne 0$ then the stationary value U_0 is not a strict local minimum of the potential energy, i.e., the extremal interpolant E_0 is not stable. If $Q(\theta_0, \eta) > 0$ for each $\eta \ne 0$ then the potential energy is larger than U_0 for every admissible interpolant $\theta \ne \theta_0$ in some $W_{1,2}[0, S]$ neighborhood of θ_0 (not only for those of the form (2.5)), hence U_0 is a strict local minimum and E_0 is a stable extremal. We have obtained

Proposition 3.1. *The possibly angle-constrained extremal interpolant E_0 with the normal representation $s \mapsto \theta_0(s)$, interpolation nodes $0 = s_0 < \cdots < s_n = \overline{s}$, and interpolation data $\int_{s_{i-1}}^{s_i} \cos\theta_0 = b_i$, $\int_{s_{i-1}}^{s_i} \sin\theta_0 = d_i$, is stable if and only if the quadratic form (3.5) with domain (3.4) is positive definite, i.e., $Q(\theta_0, \eta) > 0$ for every $\eta \neq 0$.*

Set now

$$Q_\star = \inf\{Q(\theta_0, \eta) : \eta \in V_0 \text{ and } \int_0^{\overline{s}} \eta^2 = 1\}. \tag{3.6}$$

Clearly $Q_\star > -\infty$. Also $\int_0^{\overline{s}} \eta'^2$ is bounded for $\eta \in V(\theta_0)$, $\int_0^{\overline{s}} \eta^2 = 1$, $Q(\theta_0, \eta) \leq Q_\star + 1$. By familiar arguments it follows that the continuous form Q attains the value $Q_\star$ for some $\eta_\star \in V_0(\theta_0)$, $\int_0^{\overline{s}} \eta_\star^2 = 1$. The Euler equation for $\eta_\star$ is:

$$\begin{aligned}
&\eta_\star''(s) + \frac{1}{2}\theta_0'^2(s)\eta_\star(s) - d_i^{-1}\Big(\int_{s_{i-1}}^{s_i} \theta_0'\eta_\star'\Big)\cos\theta_0(s) + d_i^{-1}\Big(\int_{s_{i-1}}^{s_i} \cos\theta_0\eta_\star\Big)\theta_0''(s) \\
&\quad + \rho_i[b_i\cos\theta_0(s) + d_i\sin\theta_0(s)] + \mu_\star\eta_\star(s) = 0, \\
&\quad s_{i-1} \leq s \leq s_i, \quad i = 1, \ldots, n; \quad \eta_\star' \text{ continuous.}
\end{aligned}$$

The multipliers $\rho_i \in \mathbb{R}$ result from the side conditions $\int_{s_{i-1}}^{s_i} (b_i\cos\theta_0 + d_i\sin\theta_0)\eta = 0$, and $\mu_\star \in \mathbb{R}$ from the condition $\int_0^{\overline{s}} \eta^2 = 1$. It should be understood that $d_i^{-1}\cos\theta_0$ in the two integral terms of (3.7a) is replaced by $-b_i^{-1}\sin\theta_0$ if $d_i = 0$. (3.7a) is supplemented by the conditions of (3.4)

$$\int_{s_{i-1}}^{s_i} (b_i\cos\theta_0 + d_i\sin\theta_0)\eta_\star = 0, \quad i = 1, \ldots, n, \tag{3.7b}$$

and

$$\eta(0) = 0 \text{ or } \eta_\star'(0) = 0 \quad \text{and} \quad \eta_\star(\overline{s}) = 0 \quad \text{or} \quad \eta_\star'(\overline{s}) = 0 \tag{3.7c}$$

depending on whether θ_0 is angle-constrained or free. Besides we have the conditions

$$\theta_0'(s_i)\left[d_i^{-1}\int_{s_{i-1}}^{s_i} \cos\theta_\star\eta_\star - d_{i+1}^{-1}\int_{s_i}^{s_{i+1}} \cos\theta_0\eta_\star\right] = 0, \quad i = 1, \ldots, n-1 \tag{3.7d}$$

resulting from the fact that the coefficient of η' in (3.5) is discontinuous. If $\theta_0'(s_i) \neq 0$ for the internal nodes $s_1, \ldots, s_{n-1}$ then (3.7d) combined with (3.7c) requires:

$$d_i^{-1}\int_{s_{i-1}}^{s_i} \cos\theta_0\eta_\star = -b_i^{-1}\int_{s_{i-1}}^{s_i} \sin\theta_0\eta_\star = \text{ constant for } i = 1, \ldots, n. \tag{3.8}$$

The multipliers ρ_i can be eliminated from (3.7a). We integrate (3.7a) over the interval (s_{i-1}, s_i) and obtain:

$$\begin{aligned}
\rho_i(b_i^2 + d_i^2) &= \eta_\star'(s_i - 1) - \eta_\star'(s_i) - \frac{1}{2}\int_{s_{i-1}}^{s_i} \theta_0'^2\eta_\star + b_i d_i^{-1}\int_{s_{i-1}}^{s_i} \theta_0'\eta_\star' \\
&\quad + d_i^{-1}(\theta_0'(s_{i-1}) - \theta_0'(s_i))\int_{s_{i-1}}^{s_i} \cos\theta_0\eta_\star - \mu_\star\int_{s_{i-1}}^{s_i} \eta_\star.
\end{aligned} \tag{3.9}$$

With (3.9) substituted in (3.7a), we obtain the equation

$$\eta_\star''(s) + \frac{1}{2}{\theta_0'}^2\eta_\star(s) + \beta_i(\eta_\star)\cos\theta_0(s) + \delta_i(\eta_\star)\sin\theta_0(s) + \mu_\star\eta_\star(s) = 0, \quad s_{i-1} \le s \le s_i, \quad i = 1, \dots, n \tag{3.10}$$

where the β_i and δ_i are well-defined linear functionals, depending only on θ_0. (3.10) together with (3.7b, c, d) is a nonconventional linear boundary-value problem for $\eta_\star$, $\mu_\star$ being the eigenvalue. Introduce the linear operator R with domain $D(R)$ of functions $\eta : [0, \overline{s}] \to \mathbb{R}$, with η' continuous on $[0, \overline{s}]$, η'' continuous on each $[s_{i-1}, s_i]$ and η satisfying conditions (3.7 b, c, d), defined by:

$$(R\eta)(s) = -\eta''(s) - \frac{1}{2}{\theta_0'}^2(s)\eta(s) - \beta_i(\eta)\cos\theta_0(s) - \delta_i(\eta)\sin\theta_0(s), \quad s_{i-1} \le s \le s_i, \quad i = 1, \dots, n \tag{3.11}$$

Then the above eigenvalue problem may be stated as:

$$R\eta = \mu\eta. \tag{3.12}$$

A simple calculation shows that if $\int_0^{\overline{s}} \eta^2 = 1$ then

$$\mu = \int_0^{\overline{s}} \eta R\eta = Q(\theta_0, \eta). \tag{3.13}$$

Therefore, $\mu_\star = Q(\theta_0, \eta_\star)$ is the smallest eigenvalue of R.

We conclude that the form Q is positive definite if and only if $\mu_\star > 0$, or equivalently, all the eigenvalues of R are positive. We have obtained

Proposition 3.2. *The extremal interpolant E_0 is stable if and only if the operator R defined above has only positive eigenvalues.*

The following proposition provides a useful sufficient condition for instability of interpolating elastica.

Proposition 3.3. *Suppose E_0 with normal representation $s \mapsto \theta_0(s)$ $(s_1 \le s \le s_n)$ is an angle-constrained extremal interpolant for some configuration $\{p_1, p_2, \dots, p_n\}$. Suppose E is another extremal interpolant (angle-constrained or free), with normal representation $s \mapsto \theta(s)$ $(s_0 \le s \le s_{n+1}, s_0 \le s_1, s_{n+1} \ge s_n)$, where θ is an extension of θ_0 with no additional knot; thus θ'' is continuous at $s_1(s_n)$ if $s_0 < s_1$ $(s_{n+1} > s_n)$. Then E is also unstable.*

Proof. The extremal E, which interpolates the configuration $\{p_0, p_2, \dots, p_{n-1}, p_{n+1}\}$ can also be considered as an extremal $\overline{E}$ which interpolates the configuration with $p_1(p_n)$ inserted between p_0 and p_2 (p_{n-1} and p_{n+1}) if $s_0 < s_1$ $(s_{n+1} > s_n)$. Let $\overline{\theta}$ denote θ in this identification. Since E_0 is unstable there exists, by Proposition 3.1, $\eta_0 \in V_0(\theta_0)$ such that $Q(\theta_0, \eta_0) \le 0$. In particular, $\eta_0(s_1) = \eta_0(s_n) = 0$. Let $\overline{\eta}$ be defined as an extension of η_0:

$$\begin{aligned} \overline{\eta}(s) &= \eta_0(s), \quad s_1 \le s \le s_n \\ &= 0, \quad s_0 \le s \le s_1 \text{ and } s_n \le s \le s_{n+1}. \end{aligned} \tag{3.14}$$

It is easily checked that $\overline{\eta} \in V_0(\overline{\theta})$ and $Q(\overline{\theta}, \overline{\eta}) = Q(\theta_0, \eta_0) \le 0$. It follows, again by Proposition 3.1, that $\overline{E}$ is unstable. Since E is obtained from $\overline{E}$ by the removal of constraints, E is unstable. □

Let E_0 of Proposition 3.3 be angle-constrained at one terminal only, say at p_n. For this case we have the

Corollary. *If the unstable extremal of Proposition 3.3 is angle-constrained only at p_n and θ is an extension of θ_0 to $s_0 \le s \le s_{n+1}$ with θ'' continuous at s_n then E is unstable.*

The proof of this is an obvious modification of that for Proposition 3.3.

Another useful sufficient condition is expressed in the following

Proposition 3.4. *Suppose E is an interpolating elastica angle-constrained at none, one, or both terminals and E_i is a subarc between consecutive nodes of E. If E_i, considered as a 2-point extremal interpolant which is angle-constrained at the terminals which are internal nodes of E, is unstable then E is.*

Proof. Suppose $s \mapsto \theta(s) \quad (0 \le s \le \bar{s})$ is the normal representation of E and $s \mapsto \theta_i(s) \quad (s_{i-1} \le s \le s_i)$ is the restriction of θ which represents E_i. Since E_i is unstable there exists $\eta_i \in V_0(\theta_i), \quad \eta_i \ne 0$, such that $Q(\theta_i, \eta_i) \le 0$. In particular, $\eta_i(s_{i-1}) = 0$ and/or $\eta_i(s_i) = 0$ if $i \ge 2$ and/or $i \le n-1$. The extension η_i with value 0 on $[0, s_{i-1})$ (if $i \ge 2$) and on $(s_i, \bar{s}]$ (if $i \le n-1$) is continuous, and clearly $\eta \in V_0(\theta), \quad Q(\theta, \eta) = Q(\theta_i, \eta_i) \le 0$. Hence E is unstable. $\square$

4. Extremals close to stable ones

If E_0 is an extremal interpolant of some configuration $\{p_0, p_1, \dots, p_n\}$, $s \mapsto \theta_0(s)$, $0 \le s \le \bar{s}$, is its normal representation, and $0 = s_0 < s_1 < \cdots < s_n = \bar{s}$ are its interpolation nodes, then $t \mapsto \theta_0(\bar{s}t) = \widetilde{\theta}_0(t)$, $0 \le t \le 1$, is the normal representation of an extremal interpolant $\widetilde{E}_0$ for the configuration $\{p_0/\bar{s}, p_1/\bar{s}, \dots, p_n/\bar{s}\}$, with interpolation nodes $0 = t_0 < t_1 < \cdots < t_n = 1$, $t_i = s_i|\bar{s}$. If the terminals of $\widetilde{E}_0$ are free or angle-constrained, so are those of E_0. Clearly, $\widetilde{E}_0$ is stable if and only if E_0 is. In the following we will often use the standardized normal representation of elastica.

Let F_1 denote the metric space of functions $\theta : [0,1] \to \mathbb{R}$, for which there are real numbers $\alpha = \alpha(\theta)$, $\beta = \beta(\theta)$ such that the equations

$$\begin{aligned} &\frac{1}{2}\theta'^2(t) = \alpha \sin\theta(t) - \beta\cos\theta(t) \\ &\theta'(t) = \alpha\cos\theta(t) + \beta\sin\theta(t), \quad 0 \le t \le 1, \end{aligned} \tag{4.1}$$

hold with the distance functional $d_1(\theta_1, \theta_2) = \max_{0 \le t \le 1} |\theta_1(t) - \theta_2(t)|$. For the proof of the proposition below we will use the following

Lemma. *The functionals α, β from F_1 to $\mathbb{R}$ are continuous.*

Proof. First, $\alpha(\theta), \beta(\theta)$ are uniquely defined, for if (4.1) and also $\frac{1}{2}\theta'^2 = \alpha_1 \sin\theta - \beta_1 \cos\theta$, $\theta'' = \alpha_1\cos\theta + \beta_1 \sin\theta$ hold, then $0 = (\alpha - \alpha_1)\sin\theta(t) - (\beta - \beta_1)\cos\theta(t) = (\alpha - \alpha_1)\cos\theta(t) + (\beta - \beta_1)\sin\theta(t)$, hence $\alpha = \alpha_1$ and $\beta = \beta_1$. Clearly, (4.1) is equivalent to

$$\theta(t) - (1-t)\theta(0) - t\theta(1) = \int_0^1 g(t,\tau)[\alpha\cos\theta(\tau) + \beta\sin\theta(\tau)]\, d\tau, \tag{4.2}$$

where

$$g(t,\tau) = \begin{cases} (t-1)\tau & , \quad 0 \le \tau \le t, \\ (\tau - 1)t & , \quad t \le \tau \le 1. \end{cases}$$

The uniqueness of α, β in (4.2) implies that the continuous functions $x = \int_0^1 g(\cdot,\tau)\cos\theta(\tau)d\tau$, $y = \int_0^1 g(\cdot,\tau)\sin\theta(\tau)d\tau$ are linearly independent. Therefore, the Gramian $\int x^2 \int y^2 - (\int xy)^2$ is $\neq 0$. Thus, if (4.2) is dot-multiplied by x and y respectively, two independent linear scalar equations for α, β are obtained, whose solution demonstrates the assertion of the Lemma. $\square$

We also observe that the functionals $\alpha(\theta), \beta(\theta)$ are uniquely determined by the restriction of θ to any subinterval of [0,1].

Now let F_n denote the class of interpolating elastica E with normal representation $t \mapsto \theta(t)$, $0 \le t \le 1$, all satisfying the same type of end conditions (free or angle contraints) and having $(n+1)$ interpolation nodes $0 = t_0 < t_1 < \cdots < t_n = 1$, where $t_i = t_i(\theta)$ for $i = 1, \dots, n-1$. We also assume that no two consecutive interpolation points of E coincide. F_n is made into a metric space by use of the distance functional

$$d(\theta_1,\theta_2) = \max_{i=1,\dots,n-1} |t_i(\theta_1) - t_i(\theta_2)| + \max_{0\le t\le 1} |\theta_1(t) - \theta_2(t)|. \tag{4.4}$$

We now prove

Proposition 4.1. *Suppose E_0 is an interpolating elastica with normal representation $\theta_0 \in F_n$, which is stable. Then there exists $\delta > 0$ such that every interpolating elastica E with normal representation $\theta \in F_n$ for which $d(\theta_0,\theta) < \delta$ is also stable.*

Proof. For every $\theta \in F_n$ we have by (2.12)

$$\begin{aligned} &\frac{1}{2}\theta'^2(t) = \alpha_i(\theta)\sin\theta(t) - \beta_i(\theta)\cos\theta(t), \\ &\theta''(t) = \alpha_i(\theta)\cos\theta(t) + \beta_i(\theta)\sin\theta(t), \\ &t_{i-1}(\theta) \le t \le t_i(\theta), \quad i = 1,\dots,n. \end{aligned} \tag{4.5}$$

We first take $\delta_1 > 0$ so that $d(\theta,\theta_0) < \delta_1$ implies

$$(t_{i-1}(\theta_0), t_i(\theta_0)) \cap (t_{i-1}(\theta), t_i(\theta)) \neq \phi \text{ for } i = 1,\dots,n. \tag{4.6}$$

It then follows from the above Lemma that we can find $\delta_2 > 0, \delta_2 < \delta_1$ so that for $d(\theta,\theta_0) < \delta_2$:

$$|\alpha_i(\theta) - \alpha_i(\theta_0)| < 1, \quad |\beta_i(\theta) - \beta_i(\theta_0)| < 1, \quad i = 1,\dots,n,$$

hence by (4.5)

$$\max_{0\le t\le 1} |\theta''(t)| \le M$$

for some M. Thus, the family $\{\theta' : \theta \in F_n, d(\theta,\theta_0) < \delta_2\}$ is equicontinuous:

$$|\theta'(t') - \theta'(t'')| \le M|t' - t''|. \tag{4.7}$$

Suppose $\varepsilon > 0$ is given. Using the Lemma again and Equations (4.5), we can choose $\delta_3 > 0, \delta_3 \le \delta_2$ so that $d(\theta,\theta_0) < \delta_3$ implies $|\alpha_i(\theta) - \alpha_i(\theta_0)| + |\beta_i(\theta) - \beta_i(\theta_0)|$ is so small for $i = 1,\dots,n$ that (4.5) yields

$$\begin{aligned} |\theta'(t) - \theta_0'(t)| < \frac{\varepsilon}{3} \quad \text{for} \quad & t \in [t_{i-1}(\theta_0), t_i(\theta_0)] \cap [t_{i-1}(\theta), t_i(\theta)], \\ & i = 1,\dots,n. \end{aligned} \tag{4.8}$$

Let the overlapping of the two intervals in (4.8) occur so that $t_{i-1}(\theta_0) \leq t_{i-1}(\theta) < t_i(\theta_0) \leq t_i(\theta)$. If $\delta_4 > 0, \delta_4 \leq \delta_3$ is such that $M|t_{i-1}(\theta) - t_{i-1}(\theta_0)| < \varepsilon/3$ for $d(\theta, \theta_0) < \delta_4$ then by (4.7) and (4.8), for $t_{i-1}(\theta_0) \leq t \leq t_{i-1}(\theta)$:

$$\begin{aligned}|\theta'(t) - \theta_0'(t)| \leq & |\theta_0'(t) - \theta_0'(t_{i-1}(\theta_0))| + |\theta_0'(t_{i-1}(\theta_0)) - \theta'(t_{i-1}(\theta_0))| \\ & + |\theta'(t_{i-1}(\theta_0)) - \theta'(t)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\end{aligned}$$

and the same result is obtained for $t_i(\theta_0) \leq t \leq t_i(\theta)$. Altogether one finds that for $d(\theta, \theta_0) < \delta_4$:

$$|\theta'(t) - \theta_0'(t)| < \varepsilon \quad 0 \leq t \leq 1. \tag{4.9}$$

For $\theta \in F_n, \eta \in W_{1,2}[0,1]$ defined (compare (3.5)):

$$Q(\theta, \eta) = \int_0^1 (\eta'^2 \frac{1}{2} \theta'^2 \eta^2) + 2 \sum_{i=1}^{n} \left[\int_{t_{i-1}}^{t_i} (\cos\theta)\eta \Big/ \int_{t_{i-1}}^{t_i} \sin\theta \right] \int_{t_{i-1}}^{t_i} \theta\eta', \tag{4.10}$$

where t_i stands for $t_i(\theta)$. In (4.10) it is assumed that $\int_{t_{i-1}}^{t_i} \sin\theta \neq 0$; if $\int_{t_{i-1}}^{t_i} \sin\theta = 0$ then the term in brackets is to be replaced by $-\left[\int_{t_{i-1}}^{t_i} (\sin\theta)\eta \Big/ \int_{t_{i-1}}^{t_i} \cos\theta \right]$. If follows from (4.9) that one can find, for a given bounded set $B \subset W_{1,2}[0,1]$, $\delta_5 > 0$, $\delta_5 \leq \delta_4$, such that

$$|Q(\theta, \eta) - Q(\theta_0, \eta)| < 2\varepsilon \tag{4.11}$$

for all $\eta \in B$ and $\theta \in F_n$, $d(\theta, \theta_0) < \delta_5$.

For $\theta \in F_n$ we also define the subspace $V_0(\theta)$ of $W_{1,2}[0,1]$ (see (3.4)):

$$\begin{aligned}V_0(\theta) = \{\eta \in W_{1,2}[0,1] : & \int_{t_{i-1}}^{t_i} dt \int_{t_{i-1}}^{t_i} d\tau\, \eta(t) \cos[\theta(t) - \theta(\tau)] = 0, \\ & i = 1, \dots, n; \text{ and } \eta(0) = 0 \text{ and/or } \eta(1) = 0 \quad \text{if the} \\ & \text{elements of } F_n \text{ are angle-constrained at the corresponding terminal}\}.\end{aligned} \tag{4.12}$$

For the given bounded set $B \subset W_{1,2}[0,1]$ (B is then totally bounded in $L_2[0,1]$) one can choose $\delta_6 > 0, \delta_6 \leq \delta_5$, so that the L_2-distance of the sets $V_0(\theta) \cap B, V_0(\theta_0) \cap B$ is arbitrary small if $d(\theta, \theta_0) < \delta_6$. From this, together with (4.11), one concludes that $\delta > 0$ can be found such that $d(\theta, \theta_0) < \delta$ implies

$$\inf_{\eta \in V_0(\theta_0), \int_0^1 \eta^2 = 1} Q(\theta_0, \eta) \quad - \quad \inf_{\eta \in V_0(\theta), \int_0^1 \eta^2 = 1} Q(\theta, \eta) \quad < \quad \inf_{\eta \in V_0(\theta_0), \quad \eta^2 = 1,} Q(\theta_0, \eta) \tag{4.13}$$

hence that, by Proposition 3.1, E is stable. $\square$

Proposition 4.2. *If θ_0 in Proposition 3.1 is strongly unstable (i.e., $\inf_{\eta \in V(\theta_0), \int \eta^2 = 1} Q(\theta_0, \eta) < 0$), then there is $\delta > 0$ such that the elastica E for which $d(\theta_0, \theta) < \delta$ are also unstable.*

We apply Proposition 4.1 to extremals which interpolate configurations close to the ray configuration. Suppose E_0 is the extremal interpolant with normal representation

$\theta_0(t) = 0, 0 \le t \le 1$, which interpolates the ray configuration $\{p_0^0, p_1^0, \dots, p_n^0\}$, where $p_i^0 = (t_i^0, 0)$, $0 = t_0^0 < t_1^0 < \cdots < t_n^0 = 1$, and has free terminals. It was proved in [2, Theorem 7.4] that, given $\varepsilon > 0$, there exists $\delta > 0$ such that for every configuration $\{p_0, p_1, \dots, p_n\}$ with $|p_i - p_i^0| < \delta$ there is a unique extremal interpolant E_ε with free ends and normal representation θ_ε for which $d(\theta_0, \theta_\varepsilon) < \varepsilon$. Now θ_0 is stable. In fact, $Q(\theta_0, \eta) = \int_0^1 {\eta'}^2$ and $V_0(\theta_0) = \{\eta : \int_{t_{i-1}}^{t_i} \eta = 0, \quad i = 1, \dots, n\}$. In particular, we must have $\int_0^1 \eta = 0$ for $\eta \in V_0(\theta_0)$, and it follows that $Q(\theta_0, \eta) \ge 4\pi^2 \int_0^1 \eta^2$. Thus we have obtained

Proposition 4.3. *For every configuration sufficiently close to the ray configuration there exists a unique stable extremal interpolant with free terminals that is close to the trivial interpolant.*

Of course, this proposition holds, a fortiori, for extremal interpolants with angle constraints.

5. Instability of the 2-point interpolants $E_n, E_n^\star$

If the configuration to be interpolated consists of two points p_0, p_1 then the elastica E_0 has normal representation $\theta_0 \in C_2[0,1]$, satisfying the equations (see (2.12), (2.13)):

$$\frac{1}{2}{\theta_0'}^2 - \lambda^1 \sin\theta_0 + \lambda^2 \cos\theta_0 = 0, \quad \theta_0'' - \lambda^1 \cos\theta_0 - \lambda^2 \sin\theta_0 = 0.$$

In the sequel we will arrange it so that $\theta_0' = 0$ when $\theta_0 = 0$ or π; then these equations become

$$\frac{1}{2}{\theta_0'}^2 = \lambda_0 \sin\theta_0, \quad \theta_0'' = \lambda_0 \cos\theta_0 \tag{5.1}$$

for some $\lambda_0 \in \mathbb{R}$. If $p_0 = (0,0)$, $p_1 = (0,d)$, $d > 0$, then the interpolation conditions are

$$\int_0^1 \cos\theta_0 = 0, \quad \int_0^1 \sin\theta_0 = d.$$

If $\theta_0(0) = \alpha$, $\theta_0(1) = \beta$, $0 \le \alpha \le \pi$, $0 \le \beta \le \pi$, then by (5.1):

$$(2\lambda_0)^{1/2} = |\int_\alpha^\beta \sin^{-1/2} u du|, \quad (2\lambda_0)^{1/2} d = |\int_\alpha^\beta \sin^{1/2} u du|. \tag{5.2}$$

However, these formulas for λ_0 and d are correct only if $\theta_0'(t) \ne 0$ for $0 < t < 1$ (i.e., E_0 has no internal inflection point); otherwise they must be modified, as will be done below.

The quadratic form (3.5) becomes

$$Q(\theta_0, \eta) = \int_0^1 ({\eta'}^2 - \lambda_0 \sin\theta_0 \eta^2) - (2\lambda_0/d)(\int_0^1 \cos\theta_0 \eta)^2 \tag{5.3}$$

and it is to be minimized on the space (see (3.4)):

$$V_0(\theta_0) = \{\eta \in W_{1,2}[0,1] : \int_0^1 \eta \sin\theta_0 = 0; \\ \eta(0) = 0 \text{ and/or } \eta(1) = 0 \text{ if } E_0 \text{ is angle-constrained}\}. \tag{5.4}$$

a. We first investigate the stability of the extremal $E_n (n \geq 1)$ with free terminals which has $(n-1)$ internal and 2 terminal inflection points. E_n consists of n arcs, congruent to E_1, which is the basic nontrivial 2-point extremal interpolant (see [2, Sec. 5]). If θ_n is the normal representation of E_n and we choose $\theta_n(0) = 0$ then $\theta_n(t)$ varies from 0 to π to 0 to $\pi \cdots$ to $\frac{1}{2}[1-(-1)^n]\pi$ as t varies from 0 to $1/n$ to $2/n$ to $\cdots$ to n/n. The points k/n $(k = 1, \dots, n-1)$ are the internal inflection points. The total variation of θ_n is $Va(\theta_n) = n\pi$. We have

$$
\begin{aligned}
\frac{1}{2}{\theta'_n}^2(t) &= \lambda_n \sin\theta_n(t) \\
\theta_n(1/n+t) &= \pi - \theta_n(t), \quad \theta_n(2/n+t) = \theta_n(t) \\
\theta_n(0) &= 0.
\end{aligned}
\tag{5.5}
$$

Formulas (5.2) are now replaced by

$$
(2\lambda_n)^{1/2} = n\int_0^\pi \sin^{-1/2} u du, \quad (2\lambda_n)^{1/2} d = n\int_0^\pi \sin^{1/2} u du. \tag{5.6}
$$

We choose

$$
\eta = \theta_n - d^{-1}\int_0^1 \theta_n \sin\theta_n.
$$

Then $\int_0^1 \eta\sin\theta_n = 0$, hence $\eta \in V_0(\theta_n)$. Since $\int_0^1 \cos\theta_n = 0$, we have $\int_0^1 \eta\cos\theta_n = \int_0^1 \theta_n\cos\theta_n$, and (5.3) becomes

$$
Q(\theta_n,\eta) = 2\lambda_n d - \lambda_n\int_0^1 \theta_n^2\sin\theta_n + (\lambda_n/d)(\int_0^1 \theta_n\sin\theta_n)^2 - (2\lambda_n/d)(\int_0^1 \theta_n\cos\theta_n)^2. \tag{5.8}
$$

We use

$$
(\int_0^1 \theta_n\sin\theta_n)^2 \leq \int_0^1 \theta_n^2\sin\theta_n \int_0^1 \sin\theta_n = d\int_0^1 \theta_n^2\sin\theta_n
$$

and find

$$
Q(\theta_n,\eta) \leq (2\lambda_n/d)[d^2 - (\int_0^1 \theta_n\cos\theta_n)^2] \tag{5.9}
$$

To evaluate the integral term in (5.9) we first assume n even. Then

$$
\begin{aligned}
\int_0^1 \theta_n\cos\theta_n &= \frac{n}{2}\left[\int_0^{1/n} \theta_n\cos\theta_n - \int_0^{1/n} (\pi-\theta_n)\cos\theta_n\right] \\
&= -(n/2\lambda_n)2\int_0^{1/n} {\theta'_n}^2 = -2d.
\end{aligned}
\tag{5.10}
$$

We find the same result for n odd. (5.9), (5.10) show $Q(\theta_n,\eta) < 0$. Thus, we have proved that E_n is unstable. This was also proved in [2], but by a different method.

b. We now show that the above extremal is, for $n \geq 2$, also unstable if angle-constrained at both ends. Let this extremal be denoted as $E_n^\star$. If $\theta_n^\star$ is its normal representation then $\theta_n^\star$ minimizes $\int_0^1 \theta'^2$ among the functions that satisfy the interpolation conditions $\int_0^1 \cos\theta = 0$, $\int_0^1 \sin\theta = d$ and the end conditions $\theta(0) = 0$, $\theta(1) = \frac{1}{2}[1-(-1)^n]\pi$. $E_n^\star$ coincides with E_n of paragraph **a.**, hence $\theta_n^\star = \theta_n$. $\eta \in V(\theta_n^\star)$ now requires $\eta(0) = \eta(1) = 0$ besides $\int_0^1 \eta \sin\theta_n = 0$. We choose

$$\begin{aligned} \eta_\star(t) &= \theta_n'(t), \quad 0 \le t \le 2/n \\ &= 0, \quad 2/n \le t \le 1. \end{aligned} \tag{5.11}$$

Then, clearly, $\eta_\star \in V_0(\theta_n^\star)$, and also $\int_0^1 \eta_\star \cos\theta_n = 0$. Thus (5.3) becomes

$$\begin{aligned} Q(\theta_n^\star, \eta_\star) &= \int_0^{2/n} ({\eta_\star'}^2 - \lambda_n \sin\theta_n \eta_\star^2) \\ &= \int_0^{2/n} (\lambda_n^2 \cos^2\theta_n - 2\lambda_n^2 \sin^2\theta_n) = \lambda_n^2[2/n - 3\int_0^{2/n} \sin^2\theta_n]. \end{aligned} \tag{5.12}$$

But, using (5.5), (5.6) and integration by parts, we find

$$\int_0^{2/n} \sin^2\theta_n = (2/\sqrt{2\lambda_n}) \int_0^\pi \sin^{3/2} u du = (2/3\sqrt{2\lambda_n}) \int_0^\pi \sin^{-1/2} u du = 2/3n. \tag{5.13}$$

Thus, $Q(\eta_\star) = 0$, and this proves instability of the extremal $E_n^\star$, $\quad n \ge 2$.

In the next section it will be proved that $E_1^\star$ is also unstable.

6. Two-point angle-constrained interpolants with no inflection point.

In this section we prove that 2-point angle-constrained interpolants are stable if they have no inflection point, and are unstable if they have at least 2 inflection points.

Proposition 6.1. *A 2-point angle-constrained extremal interpolation E with no inflection point is stable.*

Proof. If E has no inflection point then E is a proper subarc of the basic 2-point extremal E_1 (see Sec.5). Clearly E is contained in another proper subarc $\widetilde{E}$ of E_1 which has an axis of symmetry. By Proposition 3.3 it suffices to prove that the angle-constrained extremal $\check{E}$ is stable. Let $t \mapsto \check{\theta}(t)$ $\quad (0 \le t \le 1)$ be the normal representation of $\check{E}$ and $(0,0), (0,d), (d > 0)$ the coordinates of the terminals, with $\theta = 0$ along the positive $x-$axis. Then we have the following equations for $\check{\theta}$:

$$\begin{aligned} \frac{1}{2}\check{\theta}'^2(t) &= \lambda \sin\check{\theta}(t), \quad 0 \le t \le 1 \\ \check{\theta}(t) &= \pi - \check{\theta}(1-t) \\ \check{\theta}(0) &= \alpha, \quad 0 < \alpha < \pi/2 \\ d &= \int_0^1 \sin\check{\theta} = 2(2\lambda)^{-1/2} \int_\alpha^{\pi/2} \sin^{1/2} u du \\ (2\lambda)^{1/2} &= 2\int_\alpha^{\pi/2} \sin^{-1/2} u du. \end{aligned} \tag{6.1}$$

It follows from Proposition 4.1 that $\check{E}$ is stable for all α sufficiently close to $\pi/2$. Hence, if $\check{E}$ is unstable for some $\alpha > 0$, there exists a smallest $\alpha = \alpha_0$, $\quad 0 < \alpha_0 < \pi/2$, for which $\check{E} = E_0$ (correspondingly, θ_0, λ_0) is unstable. It then follows, by Proposition 4.2, that $\inf\{Q(\theta_0, \eta) : \eta \in V_0(\theta_0), \int \eta^2 = 1\} = 0$, hence there exists $\eta_0 \in V_0(\theta_0)$, $\eta_0 \neq 0$, such that $Q(\theta_0, \eta_0) = 0$. We will show that this is not the case.

By (3.4) and (3.5) we have

$$\begin{aligned} V_0(\theta_0) &= \{\eta \in W_{1,2}[0,1] : \int_0^1 \eta \sin\theta_0 = 0\} \\ Q(\theta_0, \eta) &= \int_0^1 (\eta'^2 - \lambda_0 \sin\theta_0 \eta^2) - (2\lambda_0/d)(\int_0^1 \cos\theta_0 \eta)^2 \end{aligned} \tag{6.2}$$

$\inf\{Q(\theta_0, \eta) : \eta \in V(\theta_0), \int \eta^2 = 1\} = Q(\theta_0, \eta_0) = 0$ implies (see Proposition 3.2 and Equations (3.7a,b)) that η_0 satisfies the following system for some $\rho_0 \in \mathbb{R}$:

$$\begin{aligned} &\eta_0''(t) + \lambda_0 \sin\theta_0(t)\eta_0(t) + \sigma_0 \cos\theta_0(t) + \rho_0 \sin\theta_0(t) = 0 \\ &\eta_0(0) = \eta_0(1) = 0, \quad \int_0^1 \eta_0 \sin\theta_0 = 0, \quad \eta_0 \neq 0 \\ &\sigma_0 = (2\lambda_0/d) \int_0^1 \eta_0 \cos\theta_0. \end{aligned} \tag{6.3}$$

The equation $\eta'' + \lambda_0 \sin\theta_0 \eta = 0$ has the general solution

$$\eta = c_0\theta_0' + c_1\theta_0'\gamma_0, \quad \gamma_0(t) = \int_0^t (1/\sin\theta_0(\tau))d\tau. \tag{6.4}$$

By using the method of variation of parameters one finds for the general solution of the differential equation in (6.3):

$$\eta_0(t) = -(\sigma_0/2\lambda_0)t\theta_0'(t) - (\rho_0/\lambda_0) + c_0\theta_0'(t) + c_1\theta_0'(t)\gamma_0(t). \tag{6.5}$$

$\eta_0(0) = \eta_0(1) = 0$ give, since $\theta_0'(0) = \theta_0'(1) := \kappa_0$

$$c_0 = \rho_0/\lambda_0\kappa_0, \quad c_1 = \sigma_0/2\lambda_0\gamma_0(1). \tag{6.6}$$

By the use of integration by parts one finds

$$\int_0^1 \eta_0 \cos\theta_0 = -(\sigma_0/2\lambda_0)(\sin\alpha_0 - d) + c_1(\gamma_0(1)\sin\alpha_0 - 1) \tag{6.7}$$

and since this must equal $(d\sigma_0/2\lambda_0)$ by (6.3), one obtains

$$-(\sigma_0/2\lambda_0)\sin\alpha_0 + c_1(\gamma_0(1)\sin\alpha_0 - 1) = 0. \tag{6.8}$$

(6.8) together with (6.6) gives $\sigma_0 = 0, \quad c_1 = 0$. Thus, we are left with

$$\eta_0 = (\rho_0/\lambda_0\kappa_0)(\theta_0' - \kappa_0). \tag{6.9}$$

The final condition $\int_0^1 \eta_0 \sin\theta_0 = 0$ yields

$$(\rho_0/\lambda_0\kappa_0)(2\cos\alpha_0 - \kappa_0 d) = 0. \tag{6.10}$$

Since $\rho_0 = 0$ implies $\eta_0 = 0$, we must have

$$0 = G(\alpha_0) := \cos\alpha_0 - \kappa_0 d/2.$$

By (6.1) we have $\kappa_0 = \theta_0'(0) = (2\lambda_0 \sin\alpha_0)^{1/2}$, $\quad d/2 = (2\lambda_0)^{-1/2}\int_{\alpha_0}^{\pi/2} \sin^{1/2} u du$, hence $G(0) = 1$, $\quad G(\pi/2) = 0$, $\quad G'(\alpha) = -(1/2)\sin^{-1/2}\alpha\cos\alpha \int_{\alpha}^{\pi/2} \sin^{1/2} u < 0$ for $0 < \alpha < \pi/2$. Therefore no η_0, ρ_0 satisfying (6.3) exist, and the proof of Proposition 6.1 is complete. $\square$

We prove next:

Proposition 6.2. *A 2-point extremal interpolant E (angle-constrained or free) with 2 or more inflection points is unstable.*

Proof. If E has at least 2 inflection points then E contains the basic 2-point extremal E_1 (see Sec. 5). By Proposition 3.3. it suffices to prove that $E_1^\star$, which is E_1 with angle-constraint, is unstable. We do this by exhibiting $\eta_1 \in V_0(\theta_1)$, $\quad \eta_1 \neq 0$, for which $Q(\theta_1, \eta_1) = 0$. As in the preceding proof, this will be the case if for some $\rho_1 \in \mathbb{R}$:

$$\begin{aligned}
&\eta_1'' + \lambda_1\eta_1\sin\theta_1 + \sigma_1\cos\theta_1 + \rho_1\sin\theta_1 = 0\\
&\eta_1(0) = \eta_1(1) = 0, \quad \int_0^1 \eta_1\sin\theta_1 = 0, \quad \eta_1 \neq 0\\
&\sigma_1 = (2\lambda_1/d)\int_0^1 \eta_1\cos\theta_1.
\end{aligned} \tag{6.12}$$

This system is satisfied by

$$\rho_1 = 0, \quad \eta_1(t) = (1-2t)\theta_1'(t).$$

Indeed, one computes

$$\begin{aligned}
&\eta_1'' + \lambda_1\eta_1\sin\theta_1' = -\sigma_1\cos\theta_1', \quad \sigma_1 = 4\lambda_1\\
&\int_0^1 \eta_1\cos\theta_1 = 2d = d\sigma_1/2\lambda_1\\
&\eta_1(0) = \eta_1(1) = \int_0^1 \eta_1\sin\theta_1 = 0.
\end{aligned}$$

Here we have used $\theta_1(0) = \theta_1(1) = \theta_1'(0) = \theta_1'(1) = 0$. $\square$

7. Two-point angle-constrained interpolants with one inflection point.

If the 2-point angle-constrained extremal E contains one inflection point (either at one end or internally) then the problem of stability is more complex. If one proceeds from the inflection point 0 along E in one or the other direction to a terminal one traverses a proper subarc of the basic extremal E_1 (see Sec. 5). There is a point on E_1, close to the far terminal, - its precise location is given below - on which the stability of E depends. We call this point a **stability focus**. E may contain the right, the left or neither stability focus. We prove

Proposition 7.1. *A 2-point angle-constrained extremal interpolant E with one inflection point is stable if E contains no stability focus.*

Proof. E contains neither stability focus as one proceeds from the inflection point to one or the other terminal, hence is a subarc of another extremal $\widehat{E}$, which is symmetric with respect to the inflection point and also contains no stability focus. By Proposition 3.3 it suffices to prove that the angle-constrained extremal $\widehat{E}$ is stable. Let $t \mapsto \widehat{\theta}(t)(0 \le t \le 1))$ be the normal representation of $\widehat{E}$ and $(-b/2, -d/2), \quad (b/2, d/2), \quad (b > 0, d > 0)$ the coordinates of the terminals, with $\theta = 0$ along the positive $x-$axis. We then have:

$$
\begin{aligned}
\frac{1}{2}\widehat{\theta'}^2(t) &= \widehat{\lambda} \sin \widehat{\theta}(t, \quad 0 \le t \le 1) \\
\widehat{\theta}(t) &= \widehat{\theta}(1-t) \\
\widehat{\theta}(0) &= \alpha, \quad 0 < \alpha < \pi, \quad \widehat{\theta}(1/2) = \widehat{\theta}'(1/2) = 0 \\
b &= \int_0^1 \cos \widehat{\theta} = 2(2\widehat{\lambda})^{-1/2} \int_0^\alpha \cos u \cdot \sin^{-1/2} u du \\
d &= \int_0^1 \sin \widehat{\theta} = 2(2\widehat{\lambda})^{-1/2} \int_0^\alpha \sin^{1/2} u du \\
(2\widehat{\lambda})^{1/2} &= 2 \int_0^\alpha \sin^{-1/2} u du.
\end{aligned}
\tag{7.1}
$$

It follows from Proposition 4.2 that $\widehat{E}$ is stable for all α sufficiently small. Further if $\alpha = \pi$, E contains 2 inflection points, hence is unstable. Thus there is a smallest $\alpha = \alpha_\star, \quad 0 < \alpha_\star < \pi$, for which $\widehat{E} = \widehat{E}_\star$ (correspondingly, $\theta_\star, \lambda_\star$) is unstable. As one proceeds along this $\widehat{E}_\star$ from the inflection point to one of the terminals one reaches the (left or right) stability focus, mentioned in the statement of the proposition.

By Proposition 4.1, we are left to find $\alpha_\star$ and $\theta_\star$, so that

$$
\inf\{Q(\theta_\star, \eta) : \eta \in V_0(\theta_\star), \int \eta^2 = 1\}, \quad Q(\theta_\star, \eta) = 0
$$

where $\widehat{\theta} = \theta_\star$ satisfies (7.1), with α replaced by $\alpha_\star$. By (3.4) and (3.5) we have

$$
\begin{aligned}
V_0(\theta_\star) &= \{\eta \in W_{1,2}[0,1] : \int_0^1 \eta(b \cos \theta_\star + d \sin \theta_\star) = 0\} \\
Q(\theta_\star, \eta) &= \int_0^1 (\eta'^2 - \lambda_\star \sin \theta_\star \eta^2) - (2\lambda_\star/d)(\int_0^1 \eta \cos \theta_\star)^2.
\end{aligned}
\tag{7.3}
$$

The infimum 0 of $Q(\theta_\star, \eta)$ is attained for $\eta = \eta_\star \in V(\theta_\star)$ if (see Equations (3.7a,b)) $\eta_\star$ satisfies the following system for some $\rho_\star \in \mathbb{R}$:

$$
\begin{aligned}
&\eta_\star'' + \lambda_\star \eta_\star \sin \theta_\star + \sigma_\star \cos \theta_\star + \rho_\star (b \cos \theta_\star + d \sin \theta_\star) = 0, \\
&\eta_\star(0) = \eta_\star(1) = 0, \quad \int_0^1 \eta_\star (b \cos \theta_\star + d \sin \theta_\star) = 0, \quad \eta_\star \ne 0, \\
&\sigma_\star (2\lambda_\star/d) \int_0^1 \eta_\star \cos \theta_\star.
\end{aligned}
\tag{7.4}
$$

Using the general solution

$$\eta(t) = -[(\sigma_\star + \rho_\star b)/2\lambda_\star]t\theta'_\star(t) - \rho_\star d/\lambda_\star + c_0\theta'_\star(t) + c_1\gamma_\star(t)$$

(7.5)
$$\gamma_\star(t) = \begin{cases} -\theta'_\star(t)\int_0^t (1/\sin\theta_\star(\tau))d\tau & \text{for } 0 \le t < 1/2 \\ 2 & \text{for } t = 1/2 \\ \theta'_\star(t)\int_t^1 (1/\sin\theta_\star(\tau))d\tau & \text{for } 1/2 < t \le 1 \end{cases}$$

of the differential equation in (7.4), one finds after lengthy calculations,

(7.6)
$$\eta_\star(t) = (1-2t)\theta'_\star(t) - \theta'_\star(0), \quad \rho_\star = \lambda_\star\theta'_\star(0)/d.$$

Using integration by parts and the relations, following from (7.1):

(7.7)
$$2\sin\alpha_\star = \kappa_\star^2/\lambda_\star, \quad b = -2\kappa_\star/\lambda_\star, \quad \text{where} \quad \kappa_\star = \theta'_\star(0)$$

one obtains

(7.8)
$$\sigma_\star = (2\lambda_\star/d)\int_0^1 \eta_\star\cos\theta_\star = 4\lambda_\star - \lambda_\star\kappa_\star b/d.$$

Then one verifies readily that (7.6) solves the differential equation in (7.4); also $\eta_\star(0) = \eta_\star(1) = 0$ and $\int_0^1 \eta_\star\sin\theta_\star = 2\cos\alpha_\star - 2b - \kappa_\star d$, hence

(7.9)
$$\int_0^1 \eta_\star(b\cos\theta_\star + d\sin\theta_\star) = -2\kappa_\star^3/\lambda^2 + 2d\cos\alpha_\star - \kappa_\star d^2.$$

Thus, all the conditions of (7.4) are satisfied if the quantity (7.9) is 0, or using (7.1) and (7.7) and the abbreviation

(7.10)
$$S(\alpha) = \int_0^\alpha \sin^{1/2} u du, \quad 0 \le \alpha \le \pi,$$
$$F(\alpha_\star) := \sin^{1/2}\alpha_\star S^2(\alpha_\star) + \cos\alpha_\star S(\alpha_\star) + 2\sin^{3/2}\alpha_\star = 0.$$

$\alpha_\star$ is the unique root between $\pi/2$ and π of (7.10). Since $F(\pi/2) > 0$ and $F(\pi) < 0$, there is a root between $\pi/2$ and π, and since $F'(\alpha) < 0$, the root is unique (a rough estimate shows $\alpha \approx 171°$).

We have shown that $\widehat{\theta}$, given by (7.1), with $\alpha < \alpha_\star$ is stable and this completes the proof of Proposition 7.1. □

The result in Proposition 7.1 is sharp because we have

Proposition 7.2. *A 2-point angle-constrained extremal interpolant E with one inflection point is unstable if E contains the two stability foci.*

Proof. In the proof of Proposition 7.1 it was seen that the extremal $\widehat{E}_\star$ whose terminals are the stability foci is unstable. By Proposition 3.3 E, which contains $\widehat{E}_\star$, is unstable.

There remains the case where the 2-point angle-constrained interpolant E contains one inflection point and one stability focus. We may assume that the normal representation θ of E is a solution of

(7.11)
$$\frac{1}{2}[\theta'(t)]^2 = \lambda\sin\theta(t), \quad 0 \le t \le 1$$
$$\theta(0) = \alpha, \quad \theta(1) = \beta$$

for some $\lambda \in \mathbb{R}$, where

$$\begin{aligned} &0 < \alpha < \alpha_\star \le \beta < \pi \\ &\theta(t_0) = \theta'(t_0) = 0 \quad \text{for a unique} \quad t_0. \end{aligned} \tag{7.12}$$

The numbers λ and t_0 are determined from the relations

$$\sqrt{2\lambda} = \int_0^\alpha \sin^{-1/2} u\, du + \int_0^\beta \sin^{-1/2} u\, du, \quad \sqrt{2\lambda} t_0 = \int_0^\alpha \sin^{-1/2} u\, du. \tag{7.13}$$

It is seen that $t_0 \le 1/2$ and

$$\theta(t_0 - \tau) = \theta(t_0 + \tau), \quad 0 \le \tau \le t_0. \tag{7.14}$$

We now show that for each α, $0 < \alpha < \alpha_\star$, there exists a unique $\beta = \beta_\star(\alpha)$ such that the extremal E is stable if $\beta < \beta_\star(\alpha)$ and is unstable if $\beta \ge \beta_\star(\alpha)$. We say, the point on E for which θ has the value $\beta_\star(\alpha)$ is **conjugate** to the point for which θ has the value α. As α approaches $\alpha_\star$ (from below) $\beta_\star(\alpha)$ approaches $\alpha_\star$ from above, hence the stability foci of Proposition 7.1 are the special case of conjugate points where $\beta_\star(\alpha) = \alpha$. □

We now prove

Proposition 7.3. *Suppose E is an angle-constrained extremal 2-point interpolant with normal representation $t \mapsto \theta(t)$, $0 \le t \le 1$, which contains one inflection point and for which $\theta(0) = \alpha$, $0 \le \alpha \le \alpha_\star$ (see (7.10)), $\theta(1) = \beta \ge \alpha_\star$. E is stable if and only if $\beta < \beta_\star(\alpha)$, where $\beta_\star(\alpha)$ is the unique root between $\alpha_\star$ and π of Equation (7.17) below. As α increases from 0 to $\alpha_\star$, $\beta_\star(\alpha)$ strictly decreases from π to $\alpha_\star$.*

Proof. By Propositions 6.2 and 7.1 E is stable if $\beta < \alpha_\star$ and unstable if $\beta = \pi$. Let $\beta_\star(\alpha)$ denote the smallest value of β for which E is unstable and let $\theta_\star$ be the normal representation of the extremal $E_\star$ for which $\theta_\star(0) = \alpha$, $\theta_\star(1) = \beta_\star(\alpha)$. There must then exist $\eta_\star \in V_0(\theta_\star)$, $\int \eta^2 = 1$, such that

$$\inf\{Q(\theta_\star, \eta) : \eta \in V_0(\theta_\star), \int \eta^2 = 1\} = Q(\theta_\star, \eta_\star) = 0. \tag{7.15}$$

As in the proof of Proposition 7.1, we have for $\eta_\star$ the system (7.4). The general solution of the differential equation in (7.4) is given by (7.5). One computes, using integration by parts,

$$\begin{aligned} &\int_0^1 \gamma_\star \cos\theta_\star = 1, \quad \int_0^1 \gamma_\star \sin\theta_\star = 2/\theta'_\star(1) - 2/\theta'_\star(0) \\ &\int_0^1 t\cos\theta_\star(t) \cdot \theta'_\star(t) dt = \sin\beta - d, \quad \int_0^1 t\sin\theta_\star(t) \cdot \theta'_\star(t) dt = b - \cos\beta. \end{aligned} \tag{7.16}$$

The four conditions $\eta_\star(0) = \eta_\star(1) = 0$, $\int_0^1 \eta_\star \cos\theta_\star = d\sigma_\star/2\lambda$, $\int_0^1 (b\cos\theta_\star + d\sin\theta_\star)\eta_\star = 0$ for $\eta_\star \in V_0(\theta_\star)$, and the condition $\eta_\star \ne 0$, then lead to the equation

$$\begin{aligned} H(\alpha,\beta) := {} & (\sin\alpha\sin\beta)^{1/2}(S(\alpha) + S(\beta))^2 \\ & + (\sin^{1/2}\alpha\cos\beta + \sin^{1/2}\beta\cos\alpha)(S(\alpha) + S(\beta)) \\ & + 2(\sin\alpha\sin\beta)^{1/2}(\sin^{1/2}\alpha + \sin^{1/2}\beta)^2 = 0 \end{aligned} \tag{7.17}$$

for $\beta = \beta_\star(\alpha)$. One finds that the function $\beta \mapsto H(\alpha,\beta)$ is strictly decreasing for $\alpha_\star \le \beta \le \pi$. Also, if $0 < \alpha < \alpha_\star$,

$$H(\alpha,\pi) < 0, \quad H(\alpha,\alpha) = 4F(\alpha)\sin^{-1/2}\alpha > 0, \tag{7.18}$$

where F is the function in (7.10) and $F(\alpha) > 0$ since $\alpha < \alpha_\star$. It follows that $\beta_\star(\alpha)$ is uniquely defined by (7.17). Then $\theta = \theta_\star$, with $\theta_\star(0) = \alpha$, $\theta_\star(1) = \beta_\star(\alpha)$, is the normal representation of an extremal $E_\star$, for which there exists $\eta_\star \in V_0(\theta_\star)$, with $\eta_\star = 0$, such that (7.15) holds. Therefore, $E_\star$ is unstable, and by Proposition 3.3, E is unstable if E contains $E_\star$, i.e., if $\beta \ge \beta_\star(\alpha)$.

The function $\alpha \mapsto \beta_\star(\alpha)$ is nonincreasing. For if $\beta_\star(\alpha_1) < \beta_\star(\alpha_2)$ for $\alpha_2 > \alpha_1$, then the angle-constrained unstable extremal E_1 with $\theta_1(0) = \alpha_1$, $\theta_1(1) = \beta_\star(\alpha_1)$, is contained in the extremal E_2 with $\theta_2(0) = \alpha_2$, $\theta_2(1) = \beta_\star(\alpha_1)$, which is stable since $\beta_\star(\alpha_1) < \beta_\star(\alpha_2)$. This is a contradiction to Proposition 3.3. Actually, $\beta_\star$ is strictly decreasing, for if $\beta_\star(\alpha_1) = \beta_\star(\alpha_2)$ for $\alpha_2 > \alpha_1$ then $\beta_\star(\alpha)$ is constant for $\alpha_1 \le \alpha \le \alpha_2$, which is impossible since the function $\beta_\star$ is analytic. Clearly, $\beta_\star(0) = \pi$ and $\beta_\star(\alpha_\star) = \alpha_\star$, thus the proposition is completely proved. $\square$

We proceed to give a geometric interpretation of conjugate points on a simple elastica curve. At the same time we obtain the precise range of angles that an arc of the elastica, which contains one inflection point, can make with the chord connecting the endpoints.

Let E be the simple elastica of Proposition 7.3, $t \mapsto \theta(t)$ $(0 \le t \le 1)$ its normal representation, $\theta(0) = \alpha$, $\theta(1) = \beta$, with $0 \le \alpha \le \beta \le \pi$, and let p_0, p_1 be the local vectors of the terminals of E. In the original interpolation problem the length of the vector $p_1 - p_0$ and the angles A, B that E makes with $p_1 - p_0$ at the endpoint are prescribed. More precisely, let A, B denote the angle in $(-\pi,\pi)$ from the vector $p_1 - p_0$ to the oriented curve E at p_0, p_1, respectively. We investigate the relationship between α, β and A, B. Clearly,

$$\alpha - \beta = A - B. \tag{7.19a}$$

If the inflection point is taken as the origin of a Cartesian coordinate system xy, with the positive x-axis along $\theta = 0$, then the point $(t, \theta(t))$ on E has coordinates

$$x = \int_{t_0}^{t} \cos\theta(\tau)d\tau = (2/\sqrt{2\lambda})\sin^{1/2}\theta(t)\,\mathrm{sgn}(t - t_0)$$
$$y = \int_{t_0}^{t} \sin\theta(\tau)d\tau = (1/\sqrt{2\lambda})S(\theta(t))\,\mathrm{sgn}(t - t_0).$$

Expressing the slope of the vector $p_1 - p_0$, we obtain

$$[S(\alpha) + S(\beta)]/[2\sin^{1/2}\alpha + 2\sin^{1/2}\beta] = \tan(\alpha - A). \tag{7.19b}$$

Since p_1 is above and to the right of p_0 it follows that

$$0 < \alpha - A \le \pi/2 \tag{7.19c}$$

For each pair (α,β), with $0 \le \alpha \le \beta \le \pi$, $\alpha + \beta > 0$, there is a unique pair (A, B) with $-\pi < A < B < \pi$ determined by Equations (7.19a,b,c) (actually $A \ge -\pi/2$).

Let $B(A;\alpha,\beta)$ be the angle B for fixed A, α, β, and set

$$B^\star(A) = \sup_{0\le\alpha\le\beta\le\pi} B(A;\alpha,\beta) = B(A;\alpha_A,\beta_A). \tag{7.20}$$

It is readily found that if $\alpha_A = 0$ then $A = -\pi/2$ and $\beta_A = \pi$, and if $\beta_A = \pi$ then $A = -\pi/2$ and $\alpha_A = 0$; also if $\alpha_A = \beta_A$ then $\alpha_A = \alpha_\star$ (solution of (7.10)) and $A = B^\star(A)$. We write

$$A^\star = B(A^\star;\alpha_\star,\alpha_\star) = \sup_{0\le\alpha\le\beta\le\pi} B(A^\star;\alpha,\beta) \tag{7.21}$$

($A^\star \approx 99.5°$). It follows that if A is neither $-\pi/2$ nor $A^\star$ then the supremum in (7.20) is attained in the interior of the region $0 \le \alpha \le \beta \le \pi$. Thus, (α_A, β_A) make $B = A - \alpha + \beta$ a maximum under the side condition (7.19b). It follows that $\alpha = \alpha_A, \beta = \beta_A$ satisfy the equations

$$\begin{aligned} 1 + \mu(\partial/\partial\alpha)[S(\alpha) + S(\beta) - 2\tan(\alpha - A)(\sin^{1/2}\alpha + \sin^{1/2}\beta)] = 0 \\ -1 + \mu(\partial/\partial\beta)[S(\alpha) + S(\beta) - 2\tan(\alpha - A)(\sin^{1/2}\alpha + \sin^{1/2}\beta)] = 0. \end{aligned} \tag{7.22}$$

Elimination of the multiplier μ, and use of (7.19b) yield

$$H(\alpha_A, \beta_A) = 0 \tag{7.23}$$

where H is the function (7.13). Thus $\sup B(A;\alpha,\beta)$ is attained for conjugate values α_A, β_A. This is also true in the excluded cases $A = -\pi/2, \quad A = A^\star$ since $(0,\pi), \quad (\alpha_\star,\alpha_\star)$ are conjugate pairs. Each conjugate pair (α,β) occurs in this characterization; for if α,β are used in (7.19b,c) a unique A is obtained for which $\alpha = \alpha_A, \beta = \beta_A$. We have proved

Proposition 7.4. *Suppose $A, \quad -\pi < A < \pi$, is such that there exists a simple elastica E with terminals p_0, p_1, which contains an inflection point and which makes the angle A with the vector $p_1 - p_0$ at p_0. Then the largest angle B that E can make with $p_1 - p_0$ at p_1 is obtained if p_0, p_1 are conjugate points of E. Conversely, each pair of conjugate points is characterized in this way.*

We proceed to determine the range of angles A, B that a simple elastica with one inflection point can make with the chord joining the endpoints. Because of symmetry it suffices to determine the half where $A \le B$, which we denote as $R_{A\le B}$. If $0 \le A \le A^\star$ (see (7.21)) then the interval $\{A : A \le B \le B^\star(A)\}$ is in $R_{A\le B}$ $\quad$ ($B^\star(A)$ as in (7.20)). If $A > A^\star$ then there is no $B \ge A$ such that $(A,B) \in R_{A\le B}$; this follows from the above discussion. Let us assume now $A < 0$. By (7.19c), we have $A \ge -\pi/2$; so fix $A, \quad 0 < A \le -\pi/2$. Substitute $B - A + \alpha$ for β in (7.19b), which then defines B as a function of α. It is easily found that $\partial B/\partial\alpha$ at $\alpha = 0$ is $+\infty$. B takes on its minimum $B_\star(A)$ for $\alpha = 0$, hence by (7.19a,b)

$$S(B_\star(A) - A) = 2\sin^{1/2}(B_\star(A) - A)\tan(-A). \tag{7.24}$$

It is easy to see that the interval $\{A : B_\star(A) \le A \le B^\star(A)\}$ is in $R_{A\le B}$. In Summary, we have

$$R_{A\le B} = \{-\pi/2 \le A < 0 : B_\star(A) \le B \le B^\star(A)\} \cup \{0 \le A \le A^\star : A \le B \le B^\star(A)\}.$$

Remark. *The general $(A,B) \in R_{A\le B} \cup R_{B\le A}$ is the image of two pairs (α,β), hence arises from two distinct simple elastica $E_{(A,B)}$. If angle-constrained, no more than one of these is stable. There may be no stable elastica at all for $(A,B) \in R_{A\le B} \cup R_{B\le A}$. Thus, if $0 \le A \le A^\star, \quad B = B^\star(A)$, then $B = B(A;\alpha_A,\beta_A)$ and there is a unique $E_{(A,B)}$, whose terminals are at the conjugate α_A, β_A. By Proposition 7.3, the angle-constrained $E_{(A,B)}$ is not stable. It seems probable that this happens only on the boundary of $R_{A\le B} \cup R_{B\le A}$.*

The last proposition of this section deals with 2-point interpolants with angle constraint at only one end.

Proposition 7.5. *A 2-point extremal interpolant E which is angle-constrained at one terminal and free at the other is stable if and only if E contains no stability focus.*

Proof. For the normal representation $t \mapsto \theta(t) \quad (0 \leq t \leq 1)$ of E we may assume

$$\begin{aligned}
&\frac{1}{2}\theta'^2(t) = \lambda \sin\theta(t), \quad 0 \leq t \leq 1 \\
&\theta(0) = \theta'(0) = 0; \quad \theta(1) = \beta > 0 \quad \text{prescribed} \\
&\int_0^1 \cos\theta = b \quad \text{and} \quad \int_0^1 \sin\theta = d \quad \text{prescribed.}
\end{aligned} \tag{7.25}$$

If β is sufficiently small then E is clearly stable. If E is stable for some $\beta_1 > 0$ then, by the Corollary to Proposition 3.4, E is stable for each $\beta < \beta_1$. On the other hand, E is not stable if $\beta = \pi$ since in this case E is unstable even if angle-constrained. It follows that there exists $\beta_\star, \quad 0 < \beta_\star < \pi$, such that E is stable for $\beta < \beta_\star$, but unstable for $\beta > \beta_\star$. By the same arguments as in the earlier part of this section we conclude that we have

$$\inf\{Q(\theta,\eta) : \eta \in V_0(\theta), \int \eta^2 = 1\} = Q(\theta, \eta_\star) = 0 \tag{7.26}$$

where

$$V_0(\theta) = \{\eta \in W_{1,2}[0,1] : \eta(1) = 0, \quad \int_0^1 \eta(b\cos\theta + d\sin\theta) = 0\}. \tag{7.27}$$

For $\eta_\star$ we have the conditions (compare (7.4)):

$$\begin{aligned}
&\eta_\star'' + \lambda\eta_\star \sin\theta + \sigma\cos\theta + \rho(b\cos\theta + d\sin\theta) = 0 \\
&\sigma = (2\lambda/d)\int_0^1 \eta_\star \cos\theta, \quad \eta_\star'(0) = 0, \quad \eta_\star(1) = 0, \quad \eta_\star \neq 0.
\end{aligned} \tag{7.28}$$

The condition $\eta_\star'(0) = 0$ results from the fact that if $\eta = \eta_\star$ minimizes $Q(\theta, \eta)$ then $\eta_\star$ must satisfy the free boundary condition $\eta_\star'(0) = 0$. Proceeding as in the proof of Proposition 7.1, one finds

$$\eta_\star(t) = t\theta'(t) - \theta'(1) \tag{7.29}$$

is a solution provided β (which enters (7.28) through λ, $(2\lambda)^{1/2} = \int_0^\beta \sin^{-1/2} u du$) is a zero of F, cf. (7.10). Thus, $\beta_\star = \alpha_\star$, the previously found stability focus. $\square$

8. The stability function

In the two preceding sections the stability problem was settled for all extremal 2-point interpolants. Let $E^\star$ now be an extremal, interpolating a general $(n+1)$-points configuration $\{p_0, p_1, \ldots, p_n\}$, and free at the terminals p_o, p_n. For ease of formulation we introduce the

Definition. *A subarc $E_i^\star$ of $E^\star$ between two consecutive interior nodes $p_{i-1}, p_i \quad (2 \leq i \leq n-1)$ is said to be* **proper** *if $E_i^\star$ contains no pair of conjugate points. The terminal arcs $E_1^\star$ and $E_n^\star$ are proper if they contain no stability focus.*

By Proposition 3.4, 7.3 and 7.5, $E^\star$ is unstable if any of the subarcs $E_i^\star$ is not proper. We state this important result as

Proposition 8.1. *A necessary condition for stability of an extremal interpolant with free terminals is that each arc between consecutive interpolation nodes be proper.*

It should be observed that by assuming all arcs are proper we do not exclude the presence of inflection points. However we will exclude, with little loss of generality, inflection points at the knots. We say $E^\star$ is **decomposable** if p_m for some m between 1 and $n-1$ is an inflection point, otherwise $E^\star$ is indecomposable. If $E^\star$ is decomposable then the subarcs E_a from p_0 to p_m and E_b from p_m to p_n are (free) extremal interpolants, and it is readily seen that $E^\star$ is stable or unstable if both E_a and E_b are stable or unstable, respectively (the case where one of the E_a, E_b is stable, the other unstable, is omitted).

For indecomposable extremal interpolants $E^\star$ which satisfy the necessary condition of Proposition 8.1 we find a computable function $U^\star$ of $n-1$ variables ($n+1$ is the number of interpolation nodes) with the property that $E^\star$ is stable if and only if $U^\star$ has a local minimum at the critical point corresponding to $E^\star$.

Let $s \mapsto \theta^\star(s) \quad (0 \le s \le s_n^\star)$ be the normal representation of $E^\star$, with interpolation nodes $0 = s_0^\star < s_1^\star < \cdots < s_n^\star$. Then for uniquely defined $\lambda_1^\star, \dots, \lambda_n^\star, \quad \mu_1^\star, \dots, \mu_n^\star$ we have

$$\begin{aligned}
&\theta^{\star''}(s) + \lambda_i^\star \sin\theta^\star(s) - \mu_i^\star \cos\theta^\star(s) = 0, \\
&\frac{1}{2}\theta^{\star'2}(s) - \lambda_i^\star \cos\theta^\star(s) - \mu_i^\star \sin\theta^\star(s) = 0, \quad s_{i-1}^\star \le s \le s_i^\star \\
&\int_{s_{i-1}^\star}^{s_i^\star} \cos\theta^\star = b_i, \quad \int_{s_{i-1}^\star}^{s_i^\star} \sin\theta^\star = d_i, \quad i = 1, \dots, n \\
&\theta^{\star'}(0) = 0, \quad \theta^\star(s_n^\star) = 0.
\end{aligned} \tag{8.1}$$

In addition to (8.1) we have the corner conditions

$$\theta_i^{\star'}(s^\star - 0) = \theta^{\star'}(s_i^\star + 0), \quad i = 1, \dots, n-1 \tag{8.2}$$

The potential energy for $E^\star$ is

$$U_0(E^\star) = \int_0^{s_n^\star} \theta^{\star'2} = \sum_{i=1}^n 2(\lambda_i^\star b_i + \mu_i^\star d_i). \tag{8.3}$$

We choose an arbitrary number $\delta > 0$, set $s_n^\star + \delta = S$, and extend $\theta^\star$ to the interval $[0, S]$ by setting $\theta^\star(s) = \theta^\star(s_n^\star)$ for $s_n^\star < s \le S$. Every function θ in this section is in the space $W_{1,2} = W_{1,2}[0, S]$ and is constant on some interval $[s_n, S]$, where $0 < s_n = s_n(\theta) < S$.

As stated above, we assume each subarc $E_i^\star (i = 1, \dots, n)$ of $E^\star$ is proper and also that $\theta^{\star'}(s_i) \ne 0$ for $i = 1, \dots, n-1$. We set $\theta^\star(s_i^\star) = \alpha_i^\star \quad (i = 0, 1, \dots, n)$. For every $(n-1)$-tuple $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ sufficiently close to $\alpha^\star = (\alpha_1^\star, \dots, \alpha_{n-1}^\star)$ the system

$$\begin{aligned}
&\theta''(s) + \lambda_i \sin\theta(s) - \mu_i \cos\theta(s) = 0, \\
&\frac{1}{2}\theta'^2(s) - \lambda_i \cos\theta(s) - \mu_i \sin\theta(s) = 0, \quad s_{i-1} \le s \le s_i \\
&\int_{s_{i-1}}^{s_i} \cos\theta = b_i, \quad \int_{s_{i-1}}^{s_i} \sin\theta = d_i, \qquad i = 1, \dots, n \\
&\theta'(0) = 0, \quad \theta'(s_n) = 0, \\
&\theta(s_j) = \alpha_j, \qquad\qquad j = 1, \dots, n-1
\end{aligned} \tag{8.4}$$

has a unique solution $\theta \in W_{1,2}$, with $\lambda_i, \mu_i \in \mathbb{R}$, $\quad 0 = s_1 < s_2 < \cdots < s_n < S$, in a sufficiently small preassigned neighborhood of $\theta^\star$. This follows readily from the fact that each of the arcs $E_i^\star$ is proper. (8.4) is system (8.1) with additional conditions $\theta(s_j) = \alpha_j$ replacing the conditions (8.2). We let θ denote the solution of (8.4), E_α the $\{p_0, p_1, \ldots, p_n\}$ interpolant represented by θ_α. The potential energy for E_α is

$$U_0(E_0) = \int_0^{S_n} {\theta'_\alpha}^2 = \sum_{i=1}^n 2(\lambda_i b_i + \mu_i d_i). \tag{8.5}$$

We now introduce the function

$$U^\star(\alpha) = U_0(E_\alpha) \tag{8.6}$$

and call it the **stability function** (associated with the extremal $E^\star$). It is defined in a neighborhood of $\alpha^\star$. We prove

Proposition 8.2. *There is a neighborhood $N(\alpha^\star) \subset \mathbb{R}^{n-1}$ of $\alpha^\star$ such that $\alpha^\star$ is the unique critical point in $N(\alpha^\star)$ of the function $U^\star$.*

Proof. Let $W_{1,2}^0$ denote the metric space of functions $\theta \in W_{1,2}^\star$ which interpolate the points p_i at nodes $s_i = s_i(\theta)$, $\quad \Delta = s_0 < s_1 < \cdots < s_n < s^\star$, with the metric

$$d^0(\theta_1, \theta_2) = \max_{i=1,\ldots,n} |s_i(\theta_1) - s_i(\theta_2)| + |\theta_1(0) - \theta_2(0)| + \left\{ \int_0^s (\theta_1' - \theta_2')^2 \right\}^{1/2}. \tag{8.7}$$

We can choose $\delta^\star > 0$ so that the following three conditions are satisfied: (i) $\theta^\star$ is the only extremal in

$$N(\theta^\star) = \{\theta \in W_{1,2}^0 : d^0(\theta, \theta^\star) \le \delta^\star\}; \tag{8.8i}$$

(ii) for each α in

$$N(\alpha^\star) = \{\alpha \in \mathbb{R}^{n-1} : |\alpha - \alpha^\star| \le \delta^\star\} \tag{8.8ii}$$

system (8.4) has a unique solution $\theta_\alpha \in N(\theta^\star)$ and each restriction $\theta_{\alpha|[s_{i-1}, s_i]}$ $\quad (i = 1, \ldots, n)$ is proper; (iii) for $j = 1, \ldots, n-1$

$$\operatorname{sgn} \theta'_\alpha(s_j - 0) = \operatorname{sgn} \theta'_\alpha(s_j + 0) = \operatorname{sgn} {\theta^\star}'(s_j^\star). \tag{8.8iii}$$

To prove the proposition it suffices to show that $\alpha \in N(\alpha^\star)$ is a critical point of $U^\star(\alpha)$ if and only if $\alpha = \alpha^\star$.

By (8.5) we have

$$U^\star(\alpha) = \sum_{i=1}^n (\lambda_i b_i + \mu_i d_i) \tag{8.9}$$

where the $\lambda_i = \lambda_i(\alpha)$, $\quad \mu_i = \mu_i(\alpha)$ are determined from the interpolation and end conditions:

$$\begin{aligned} B_i(\alpha_{i-1}, \alpha_i, \lambda_i, \mu_i) &:= \int_{s_{i-1}}^{s_i} \cos\theta_\alpha - b_i = 0 \\ D_i(\alpha_{i-1}, \alpha_i, \lambda_i, \mu_i) &:= \int_{s_{i-1}}^{s_i} \sin\theta_\alpha - d_i = 0, \quad i = 1, \ldots, n \\ E_i(\alpha_0, \lambda_1, \mu_1) &:= \lambda_1 \cos\alpha_0 + \mu_1 \sin\alpha_0 = 0 \\ E_n(\alpha_n, \lambda_n, \mu_n) &:= \lambda_n \cos\alpha_n + \mu_n \sin\alpha_n = 0. \end{aligned} \tag{8.10}$$

We now seek critical points of $U^\star$ as a function of $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ and the accessory variables $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n, \alpha_0, \alpha_n$, under the $2n+2$ side conditions (8.10). If α is a critical point then there exist multipliers $\rho_i, \sigma_i \quad (i = 1, \dots, n)$ and ω_1, ω_n such that

$$\frac{\partial}{\partial \gamma} \left\{ \sum_{k=1}^{n} (\lambda_k b_k + \mu_k d_k + \rho_k B_k + \sigma_k D_k) + \omega_1 E_1 + \omega_n E_n \right\} = 0$$

where γ stands for each of the variables $\alpha_i, \lambda_i, \mu_i$.

Let first $i \quad (2 \le i \le n-1)$ be such that $E_i^\star$ has no inflection point. Then $\operatorname{sgn} \theta^{\star'}(s_{i-1}^\star) = \operatorname{sgn} \theta^{\star'}(s_i^\star) = 1$, say, and, by (8.8iii), $\theta'_\alpha(s) > 0$ for $s_{i-1} \le s \le s_i$. Thus, using (8.4), we find

$$\begin{aligned} B_i(\alpha_{i-1}, \alpha_i, \lambda_i, \mu_i) &= \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-1}(u) \cos u du - b_i \\ D_i(\alpha_{i-1}, \alpha_i, \lambda_i, \mu_i) &= \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-1}(u) \sin u du - d_i, \end{aligned} \tag{8.12}$$

where

$$\kappa_i(u) = (2\lambda_i \cos u + 2\mu_i \sin u)^{1/2}. \tag{8.13}$$

Using $\gamma = \lambda_i$ and $\gamma = \mu_i$ in (8.11), one obtains

$$\begin{aligned} b_i - \rho_i \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \cos^2 - \sigma_i \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \sin \cdot \cos = 0 \\ d_i - \rho_i \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \cos \cdot \sin - \sigma_i \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \sin^2 = 0 \end{aligned}$$

or, since $b_i = \int_{s_{i-1}}^{s_i} \cos \theta_\alpha = \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-1} \cos = \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} (2\lambda_i \cos + 2\mu_i \sin) \cos$:

$$\begin{aligned} (2\lambda_i - \rho_i) \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \cos^2 + (2\mu_i - \sigma_i) \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \cos \cdot \sin = 0 \\ (2\lambda_i - \rho_i) \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \cos \cdot \sin + (2\mu_i - \sigma_i) \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \sin^2 = 0 \end{aligned} \tag{8.14}$$

By the Schwarz inequality

$$\left(\int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \cos \cdot \sin \right)^2 < \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \cos^2 \int_{\alpha_{i-1}}^{\alpha_i} \kappa_i^{-3} \sin^2$$

(equality cannot hold), hence (8.14) gives

$$\rho_i = 2\lambda_i, \quad \sigma_i = 2\mu_i. \tag{8.15}$$

If $i = 1$ then $\theta^{\star'}(s_1^\star) \neq 0$ (since $E_1^\star$ is proper), say $\theta^{\star'}(s_1^\star) > 0$, and also $\theta'_\alpha(s) > 0$ for $0 < s \le s_1$, hence (8.12) holds for $i = 1$ (the integrals involved are improper). To avoid the divergent integrals in (8.14), we set

$$B_1 = \lambda_1 F_1 + \mu_1 G_1, \quad D_1 = \mu_1 F_1 - \lambda_1 G_1 \tag{8.16i}$$

where

$$
\begin{aligned}
F_1 &= (\lambda_1^2+\mu_1^2)^{-1}\left[\int_{\alpha_0}^{\alpha_1}\kappa_1^{-1}(\lambda_1\cos+\mu_1\sin)-\lambda_1 b_1-\mu_1 d_1\right]\\
&= \frac{1}{2}(\lambda_1^2+\mu_1^2)^{-1}\left[\int_{\alpha_0}^{\alpha_1}\kappa_1-\lambda_1 b_1-\mu_1 d_1\right]\\
G_1 &= (\lambda_1^2+\mu_1^2)^{-1}\left[\int_{\alpha_0}^{\alpha_1}\kappa_1^{-1}(-\lambda_1\sin+\mu_1\cos)+\lambda_1 d_1-\mu_1 b_1\right]\\
&= (\lambda_1^2+\mu_1^2)^{-1}(\kappa_1(\alpha_1)+\lambda_1 d_1-\mu_1 b_1).
\end{aligned}
\tag{8.16ii}
$$

Using these expressions in (8.11), one can carry out the differentiations with respect to $\gamma=\lambda_1$ and $\gamma=\mu_1$, and one obtains (8.15) for $i=1$. The same result is obtained for $i=n$.

Finally if $j\quad(2\le j\le n-2)$ is such that $\operatorname{sgn}\theta^{\star\prime}(s_{j-1}^\star)=-\operatorname{sgn}\theta^{\star\prime}(s_j^\star)=1$, say, (hence $E_j^\star$ has an inflection point), then by (8.8iii), $\theta_\alpha'(s)$ also changes sign in (s_{j-1},s_j), and (8.12) is replaced by

$$
\begin{aligned}
B_j(\alpha_{j-1},\alpha_j,\lambda_j,\mu_j) &= \left(\int_{\alpha_{j-1}}^{\beta_j}-\int_{\beta_j}^{\alpha_j}\right)(\kappa_j^{-1}\cos)-b_j\\
D_j(\alpha_{j-1},\alpha_j,\lambda_j,\mu_j) &= \left(\int_{\alpha_{j-1}}^{\beta_j}-\int_{\beta_j}^{\alpha_j}\right)(\kappa_j^{-1}\sin)-d_j
\end{aligned}
\tag{8.17}
$$

where $\kappa_j(\beta_j)=0,\quad \alpha_{j-1}<\beta_j,\beta_j>\alpha_j$. To differentiate the improper integrals one replaces the B_j,D_j by functions F_j,G_j analogous to (8.16), then (8.11) for $\gamma=\lambda_j$ and $\gamma=\mu_j$ again yields (8.15) for $i=j$. It should be observed that β_j depends on $\alpha_{j-1},\alpha_j,\lambda_j,\mu_j$, but $\partial F_j/\partial\gamma$ and $\partial G_j/\partial\gamma$ do not contain terms $\partial\beta_j/\partial\gamma$. We have now established (8.15) for $i=1,\dots,n$.

We next choose $\alpha_i\quad(i=1,\dots,n-1)$ for γ in (8.11) and obtain

$$(\rho_i\cos\alpha_i+\sigma_i\sin\alpha_i)\kappa_i^{-1}(\alpha_i)=(\rho_{i+1}\cos\alpha_i+\sigma_{i+1}\sin\alpha_i)\kappa_{i+1}^{-1}(\alpha_i)$$

or, using (8.13) and (8.15): $\kappa_i(\alpha_i)=\kappa_{i+1}(\alpha_i)$, i.e.

$$\theta_\alpha'(s_i-0)=\theta_\alpha'(s_i+0),\quad i=1,\dots,n-1. \tag{8.18}$$

Furthermore, by (8.4), $\theta_\alpha'(0)=\theta_\alpha'(s_n)=0$. Thus we have shown that if α is a critical point of $U^\star(\alpha)$ then θ_α satisfies (8.1) and (8.2), hence $\theta_\alpha=\theta^\star,\alpha=\alpha^\star$. That conversely $U^\star(\alpha^\star)=U_0(E^\star)$ is a critical value of $U^\star$ follows immediately from the fact that $U_0(E^\star)$ is a stationary value of U_0. Proposition 8.2 is proved. □

The stability function $U^\star$ attains a minimum in the compact set $N(a^\star)$, say

$$U^\star(\alpha_{\min})=\min_{\alpha\in N(\alpha^\star)}U^\star(\alpha).$$

If $\alpha_{\min}$ is a critical point of $U^\star$ (i.e., $\alpha_{\min}$ is in the interior of $N(\alpha^\star)$) then, by the preceding proposition, $\alpha_{\min}=\alpha^\star$ and $E^\star$ minimizes the potential energy U_0 among all E_α with $\alpha\in N(\alpha^\star)$. The theorem below will show that in this case $E^\star$ minimizes U_0 among all the $\{p_0,p_1,\dots,p_n\}$-interpolants sufficiently close to $E^\star$, hence that $E^\star$ is stable. On the other hand, if $U^\star(\alpha^\star)$ is not a local minimum of $U^\star$ then there are interpolants E_α arbitrarily close to $E^\star$ for which $U_o(E_\alpha)=U^\star(\alpha)<U^\star(\alpha^\star)=U_0(E^\star)$, hence $E^\star$ is unstable. Thus, we arrive at the following effective stability criterion:

Theorem. *Suppose the indecomposable extremal interpolant $E^\star = E_{\alpha^\star}$ has only proper subarcs $E_i^\star$. Then $E^\star$ is stable if and only if the stability function $U^\star$ has a local minimum at $\alpha^\star$.*

Proof. The proof depends critically on the following result which we formulate as a lemma. □

Lemma. *There exists a neighborhood $N_0(\theta^\star) \subset N(\theta^\star)$ such that $U_0(C) \geq U_0(E_\alpha)$ for each C with normal representation $\theta \in N_0(\theta^\star)$. Here $\alpha = \{\alpha_1, \dots, \alpha_{n-1}\}, \quad \alpha_i = \theta(s_i(\theta))$.*

Proof of Lemma. Since each internal (terminal) arc of E_α between consecutive interpolation nodes, if considered as a 2-point extremal interpolant with two (one) angle constraints, is stable it is true that $U_0(C) \geq U_0(E_\alpha)$ for C sufficiently close to E_α, α fixed. The lemma asserts that this inequality holds in a neighborhood that is independent of α.

We may assume θ in the form (2.5):

$$\theta = \theta_\alpha + \varepsilon\eta + \varepsilon^2\xi(\varepsilon) \tag{8.19}$$

with $\eta \in V_0(\theta_\alpha), \quad \eta(s_i(\theta_\alpha)) = 0 \quad (i = 1, \dots, n-1), \quad d^0(0,\eta) \leq 1, \quad d^0(0,\xi) \leq 1$. We also may assume $d_i = \displaystyle\int_{s_{i-1}(\theta_\alpha)}^{s_i(\theta_\alpha)} \sin\theta_\alpha \neq 0$ (the integral is independent of α), otherwise d_i should be replaced by b_i. Then by (2.10), (3.5)

$$\begin{aligned} &\int_0^S \theta'^2 = \int_0^S {\theta'_\alpha}^2 + \varepsilon^2 Q(\theta_\alpha,\eta) + R(\varepsilon), \\ &Q(\theta_\alpha,\eta) = \int_0^S (\eta'^2 - \frac{1}{2}{\theta'_\alpha}^2\eta^2) + 2\sum_{i=1}^n d_i^{-1} \int_{s_{i-1}(\theta_\alpha)}^{s_i(\theta_\alpha)} (\cos\theta_\alpha)\eta \int_{s_{i-1}(\theta_\alpha)}^{s_i(\theta_\alpha)} \theta'_\alpha\eta. \end{aligned} \tag{8.20}$$

where $R(\varepsilon)/\varepsilon^2 \to 0$ as $\varepsilon \to 0$, uniformly for $\theta \in N(\theta^\star), \quad \alpha \in N(\alpha^\star)$. The mappings $\alpha \mapsto s_i(\theta_\alpha) \quad (i = 1, \dots, n)$, $\alpha \mapsto \theta_\alpha$, from $N(\alpha^\star)$ to $\mathbb{R}$, $N(\theta^\star)$, respectively, are continuous, and so is the mapping

$$\alpha \mapsto \inf_{\eta \in V_0(\theta_\alpha), d^0(\theta,\eta) \leq 1} Q(\theta_\alpha,\eta) := q_\alpha \tag{8.21}$$

Since $q_\alpha > 0$ for each $\alpha \in N(\alpha^\star)$ it follows that $q_\star = \inf_{\alpha \in N(\alpha^\star)} q_\alpha > 0$ and, by (8.19), $\displaystyle\int \theta'^2 \geq \int {\theta'_\alpha}^2 + \frac{1}{2}\varepsilon^2 Q(\theta_\alpha,\eta)$ for all sufficiently small ε, say $|\varepsilon| \leq \varepsilon_0$. We can now choose the neighborhood $N_0(\theta^\star) \subset N(\theta^\star)$ so that $\theta \in N_0(\theta^\star)$ may be represented in the form (8.19) with $|\varepsilon| \leq \varepsilon_0$. Then $\displaystyle\int \theta'^2 \geq \int {\theta'_\alpha}^2$, which proves the lemma. □

Proof of the Theorem. We need to prove only the sufficiency of the condition. Thus, we assume there exists $\varepsilon_1 > 0$ such that $U^\star(\alpha^\star) \leq U^\star(\alpha)$ for $|\alpha - \alpha^\star| \leq \varepsilon_1$. If the neighborhood $N_1(\theta^\star) \subset N_0(\theta^\star)$ is sufficiently small then $|\alpha - \alpha^\star| \leq \varepsilon_1$ for each $\theta \in N_1(\theta^\star), \quad \alpha = \{\theta(s_i(\theta))\}$. Using the Lemma, we have

$$\int_0^s {\theta^{\star}}'^2 = U^\star(\alpha^\star) \leq U^\star(\alpha) = \int_0^s {\theta'_\alpha}^2 \leq \int_0^s \theta'^2,$$

hence $U_0(E^\star)$ is a local minimum. □

We present two examples, which illustrate the effectiveness of the propositions in this section.

Example 1. Suppose we have the configuration $\{p_0, p_1, p_2\}$ where $p_0 = (0,0)$, $p_1 = (1,0)$, $p_2 = (1,d)$. Without loss we may assume $d \geq 1$. It is easy to see that, for each d, there is an extremal interpolant $E^\star$, which makes the angle $\alpha^\star$ with the vector $p_1 - p_0$ at p_1, where $\alpha^\star$ varies from $\pi/4$ to 0 as d varies from 1 to ∞. Here the stability function $U^\star$ is a function of a single variable α, which has been computed by Dr. D. Pence. It is found that $U^\star(\alpha^\star)$ is a local minimum for each d. By the Theorem, the above $E^\star$ is a stable extremal.

Example 2. Suppose the configuration to be interpolated is $\{p_1, p_2, p_3, p_4\}$ with $p_1 = (a, 0)$, $p_2 = (1,0)$, $p_3 = (0,1)$, $p_4 = (0,a)$, where $-\infty < a < 1$. This configuration with $a = 0.5$ was mentioned first in the note [5] as an example for which there is no interpolating elastica, and this claim was, without examination, repeated in many subsequent publications. However, there are interpolating elastica, for each a, in particular there is one which is symmetric with respect to the symmetry axis of the configuration. This can be seen as follows. Let C_β be the symmetric interpolant of $\{p_1, p_2, p_3, p_4\}$ which is uniquely defined by the following conditions. C_β has continuous slope; the arcs $C_{1\beta}, C_{2\beta}, C_{3\beta}$ between the interpolation nodes are simple elastica; $C_{1\beta}$ and $C_{3\beta}$ have curvature 0 at p_1 and p_4 respectively; the tangent vector along $C_{1\beta}$ turns through the angle β. Clearly, for $\beta = \pi$, the curvature at p_2 jumps from 0 to a negative value; and for some $\beta < \pi$ the curvature at p_2 jumps from a negative value to 0. Therefore there is some value (it is unique) $\beta_\star$, $0 < \beta_\star < \pi$, such that $C_{\beta_\star}$ has continuous curvature at p_2 (thus also at p_3, in fact everywhere), and this is an extremal interpolant of $\{p_1, p_2, p_3, p_4\}$. In this way, for each a, $-\infty < a < 1$, a unique extremal interpolant, $E^\star$, is defined. We will see that each of these extremals is unstable. The results are based on computations carried out by Dr. D. Pence.

If $a \geq a^\star$, where $a^\star \approx -.27$, then the terminal arcs of $E^\star$ are improper, hence $E^\star$ is unstable by Proposition 8.1. If $a < a^\star$ then the hypotheses of the above Theorem are satisfied. Instead of the stability function $U^\star(\alpha) = U_0(E_{\alpha^\star}), \alpha = (\alpha_1, \alpha_2)$, we use the function of one variable which is the restriction of $U^\star(\alpha)$ to $\alpha_2 = 3\pi/4 + \alpha_1$ (i.e., we consider only symmetric perturbations E_α of $E^\star$). The computed results show that $\alpha^\star$ is not a minimum point of this function, hence $E^\star$ is not stable.

The question whether there are stable extremal interpolants for the configuration $\{p_1, \dots, p_4\}$ (which would necessarily be nonsymmetric) remains open.

9. Stability of closed extremals interpolating regular polygons

An extremal interpolant E with normal representation $s \mapsto \theta(s)$ $(0 \leq s \leq \bar{s})$ is said to be closed if

$$\theta(0) = \theta(\bar{s}), \quad \theta'(0) = \theta'(\bar{s}). \tag{9.1}$$

Let $p_0, p_1, \dots, p_{n-1}$ be the vertices of a regular n-gon $(n \geq 3)$. In [2, Sec. 8] it was shown that there exist closed extremals that interpolate the configuration $\{p_0, p_1, \dots, p_{n-1}, p_n = p_0\}$. In particular, there is one, $\overset{\circ}{E}_n$, which has no inflection points. Let $s \mapsto \overset{\circ}{\theta}_n(s)$, $0 \leq s \leq n$, be its normal representation. Its total variation $Va(\overset{\circ}{\theta}_n)$ is minimal, $Va(\overset{\circ}{\theta}_n) = 2\pi$. We prove

Proposition 9.1. *The closed extremal* $\overset{\circ}{E}_n (n \geq 3)$ *is stable.*

Proof. The course $\overset{\circ}{E}_n$ consists of n congruent arcs, each of length 1, and the increment of angle along each arc is $2\pi/n$. We write $\overset{\circ}{\theta}$ for its normal representation and define $\overset{\circ}{\theta}(s+n) = \overset{\circ}{\theta}(s)$. Then

$$\overset{\circ}{\theta}(s) = \overset{\circ}{\theta}(s-1) + 2\pi/n. \tag{9.2}$$

We assume $p_0 = (0,0)$, and set $p_k - p_{k-1} = (b_k, d_k) \quad (k = 1, \dots, n)$ with $b_1 = 0, \quad d_1 = d > 0$. Then

$$b_k = \int_{k-1}^{k} \cos\overset{\circ}{\theta} = -d\sin(k-1)2\pi/n, \quad d_k = \int_{k-1}^{k} \sin\overset{\circ}{\theta} = d\cos(k-1)2\pi/n. \tag{9.3}$$

Because of symmetry we have

$$\overset{\circ}{\theta}(0) = \pi/2 - \pi/n, \quad \overset{\circ}{\theta}(1/2) = \pi/2. \tag{9.4}$$

Also,

$$\begin{aligned} &\frac{1}{2}(\overset{\circ}{\theta}{}'(s))^2 = \overset{\circ}{\lambda}\sin\overset{\circ}{\theta}(s), \quad 0 \le s \le n \\ &(2\overset{\circ}{\lambda})^{1/2} = 2\int_0^{\pi/n} \cos^{-1/2} u du, \quad (2\overset{\circ}{\lambda})^{1/2} d = 2\int_0^{\pi/n} \cos^{1/2} u du. \end{aligned} \tag{9.5}$$

The quadratic form (3.5) becomes in this case

$$\begin{aligned} Q(\overset{\circ}{\theta}, \eta) = &\sum_{k=0}^{n-1} \int_0^1 [\eta'^2(t+k)dt - \frac{1}{2}(\overset{\circ}{\theta}{}'(t+k))^2 \eta^2(t+k)]dt \\ &- 2\sum_{k=0}^{n-1} (\overset{\circ}{\lambda}/d_k)(\int_0^1 \eta(t+k)\cos\overset{\circ}{\theta}(t+k)dt)^2. \end{aligned}$$

By (9.2) and (9.5)

$$(1/d_k)\int_0^1 \eta(t+k)\cos\overset{\circ}{\theta}(t+k)dt = (1/d)\int_0^1 \eta(t+k)\cos\overset{\circ}{\theta}(t)dt,$$

thus

$$Q(\overset{\circ}{\theta}, \eta) = \sum_{k=0}^{n-1} \left\{ \int_0^1 [\eta'^2(t+k)dt - \frac{1}{2}(\overset{\circ}{\theta}{}'(t))^2 \eta^2(t+k)]dt - (2\overset{\circ}{\lambda}/d)\left(\int_0^1 \eta(t+k)\cos\overset{\circ}{\theta}(t)dt\right)^2 \right\}. \tag{9.6}$$

This form is to be minimized on the space (3.4):

$$V_0(\overset{\circ}{\theta}) = \{\eta \in \overset{\circ}{W}_{1,2} : \int_{k-1}^{k} (b_k \cos\overset{\circ}{\theta} + d_k \sin\overset{\circ}{\theta}) = 0, \quad k = 1, \dots, n\}$$

Here $\overset{\circ}{W}_{1,2}$ denotes the $W_{1,2}$-space of functions of period n. Using (9.3), we find

$$V_0(\overset{\circ}{\theta}) = \{\eta \in \overset{\circ}{W}_{1,2} : \int_0^1 \eta(t+k)\sin\overset{\circ}{\theta}(t)dt = 0, \quad k = 0, 1, \dots, n-1\} \tag{9.7}$$

Put $\eta(t+k) = \eta_k(t)(k = 0, 1, \dots, n-1)$. Clearly $Q(\overset{\circ}{\theta}, \eta_k) = Q(\overset{\circ}{\theta}, \eta_0)$ and $\eta_k \in V_0(\overset{\circ}{\theta})$ if $\eta_0 \in V_0(\overset{\circ}{\theta})$. If $Q(\overset{\circ}{\theta}, \eta)$ attains its infimum for $\eta = \eta_0$, then also for $\eta = \widetilde{\eta} = (1/n)(\eta_0 + \eta_1 + \cdots + \eta_{n-1})$, and η has period 1. For $\widetilde{\eta}$ of period 1 (9.5) becomes

$$(1/n)Q(\overset{\circ}{\theta}, \eta) = \int_0^1 (\eta'^2 - \overset{\circ}{\lambda}\eta^2 \sin \overset{\circ}{\theta}) - (2\eta\overset{\circ}{\lambda}/d) \left(\int_0^1 \eta \cos \overset{\circ}{\theta}\right)^2 \tag{9.8}$$

and (9.7) requires $\int_0^1 \eta \sin \overset{\circ}{\theta} = 0$. Thus, η must change sign in $(0, 1)$ and we conclude

$$\int_0^1 \eta'^2/\eta^2 \geq \int_0^1 (d \sin 2\pi t/dt)^2 \Big/ \int_0^1 (\sin 2\pi t)^2 = 4\pi^2. \tag{9.9}$$

From (9.5) we have the estimates

$$\begin{gathered} (2\overset{\circ}{\lambda})^{1/2} < (2\pi/n) \cos^{-1/2} \pi/n \\ d > \cos \pi/n. \end{gathered} \tag{9.10}$$

With (9.9), (9.10) substituted in (9.8), we find

$$(1/n)Q(\overset{\circ}{\theta}, \eta) \geq 4\pi^2(1 - (1/n)\tan^2 \pi/n - 1/2n^2 \cos \pi/n) \int_0^1 \eta^2, \tag{9.11}$$

thus $\eta \mapsto Q(\overset{\circ}{\theta}, \eta)$ is positive definite for $n \geq 3$. By Proposition 1, $\overset{\circ}{\theta}$ is stable.

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