**Fourier analysis of the approximation power of principal shift-invariant spaces**

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ABSTRACT

The approximation order provided by a directed set $\{s_h\}_{h>0}$ of spaces, each spanned by the $h\mathbb{Z}^d$-translates of one function, is analyzed. The "near-optimal" approximants of [R2] from each $s_h$ to the exponential functions are used to establish upper bounds on the approximation order. These approximants are also used on the Fourier transform domain to yield approximations for other smooth functions, and thereby provide lower bounds on the approximation order. As a special case, the classical Strang-Fix conditions are extended to bounded summable generating functions.

The second part of the paper consists of a detailed account of various applications of these general results to spline and radial function theory. Emphasis is given to the case when the scale $\{s_h\}$ is obtained from $s_1$ by means other than dilation. This includes the derivation of spectral approximation orders associated with smooth positive definite generating functions.

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## 1. Introduction

Spaces spanned by finitely or countably many translates of one or several functions play an important role in spline theory, radial basis function theory, sampling theory and wavelet theory. Spline theory stresses the case when the generating functions are compactly supported, while sampling theory singles out the case when the spectrum (i.e., the support of the Fourier transform) of the generating functions is compact. In the radial basis function theory, neither of these is assumed, and instead, the computational simplicity as well as the positive definiteness (i.e., the positivity of the Fourier transform) of the generating functions is preferred. Finally, wavelet theory focuses on the interrelation between the initial space and its dyadic dilates. In all these areas, the underlying space $s$ is meant for approximation or decomposition of functions, and thus, the determination of its approximation properties is of basic significance.

The present literature is mainly concerned with a space $s$ which is the algebraic or topological span of the *integer* translates of one generating function $\psi$. More precisely, we hold a collection $\{\psi_h\}_{h\in I}$ of complex-valued measurable functions defined on $\mathbb{R}^d$, where $I$ is either the open interval $(0 \,\mathrm{d} h_0)$, or a discrete subset of such an interval (e.g., $\{2^{-n} : \ n \in \mathbb{N}\}$). For each $h$, we look at all linear combinations $\sum_{\alpha\in h\mathbb{Z}^d} \psi_h(\cdot - \alpha)a(\alpha)$, for which this sum converges in a certain sense, and denote by $s_h$ the space of all limit functions obtained in this way. Roughly speaking, we call $s_h$ *the span of the* $h\mathbb{Z}^d$*-translates of* $\psi_h$, and this is an exact description of $s_h$ in case $\psi_h$ is of compact support, a case in which the sum $\sum_{\alpha\in h\mathbb{Z}^d} \psi_h(\cdot - \alpha)a(\alpha)$ is locally finite, and hence arbitrary linear combinations are allowed in this sum. Approximation properties are primarily studied via *approximation orders*: for the given scale $\{s_h\}_h$, one examines the rate of decay (as $h \to 0$) of $\mathrm{dist}\,(f, s_h)$, where $f$ varies over some space of admissible functions, which must contain all test functions in $\mathcal{D}(\mathbb{R}^d)$ (namely, all $C^\infty(\mathbb{R}^d)$ compactly supported functions), and the distance $\mathrm{dist}\,(f, s_h)$ between $f$ and $s_h$ is measured in some norm, usually a $p$-norm ($1 \le p \le \infty$), or a weighted $p$-norm. We say that the **approximation order of the scale** $\{s_h\}_h$ **is** $k$ (or $O(h^k)$), for some positive (usually integer) $k$, if, for every admissible $f$, $\mathrm{dist}\,(f, s_h) = O(h^k)$, with a constant that depends on $f$ (and clearly not on $h$), while, for some admissible $f$, $\mathrm{dist}\,(f, s_h) \ne o(h^k)$.

Although a discussion of the above model can already be found in Schoenberg's work [S] (for univariate functions), the first comprehensive analysis of approximation orders was carried out about twenty years ago primarily by people from the finite element group, the best known reference for which is [SF]. Strang and Fix considered the "compactly supported stationary case", namely, when $\psi_1$ is compactly supported and $\psi_h$ is its $h$-dilate (i.e., $\psi_h = \psi_1(\cdot/h)$), and showed (for the 2-norm) that approximation orders are characterized by the *polynomials* in $s_1$. Some modifications and improvements of these results (known these days as "The Strang-Fix Conditions") can be found in [DM2], [BJ] and [R2]. However [DR], the polynomials in spline spaces are unrelated to approximation orders if the $\{\psi_h\}_h$ are not the dilates of one function. Discussion of approximation orders for compactly supported piecewise-exponentials $\{\psi_h\}_h$ can be found in [DR], [BR] and [LJ]. We know of no study of approximation orders for general compactly supported $\{\psi_h\}_h$.

In the study of the above problem, one usually considers separately the questions of *lower bounds* and *upper bounds* on the approximation order (and hopes of course to match them). The standard approach to lower bounds is via the quasi-interpolation argument: first, a space $H \subset \cap_h s_h$ is identified, and then the local approximation properties of $H$ are converted to approximation orders of $\{s_h\}_h$ with the aid of local linear operators (=quasi-interpolants) whose restriction to $H$ is the identity. The space $H$ consists of polynomials in the stationary case, and of exponential-polynomials in the piecewise-exponential case, but need not to be so in general (cf. [BAR]). Further, the condition $H \subset s_h$, all $h$, is convenient, but not essential, as the quasi-interpolation argument of [R2] shows. For earlier constructions of quasi-interpolants see, e.g., [SF] and [BF]. An updated discussion, together with a partial bibliography, can be found in [B2] and [BR].

In contrast to lower bounds, there does not seem to exist a standard approach to the upper bound question. We already mentioned [SF] and [BJ], and we add [LC], [JL] and [HL], where weaker forms of approximation orders ("local", "controlled-local") are characterized, under the assumption that the generating functions are either compactly supported or maintain a high order 0 at $\infty$ (where "high" is defined relative to the desired approximation order, and *several* generating functions are allowed in each $h$-layer). However, all these results are confined to the stationary case, and further, the fast decay at $\infty$ that is required from the generating functions excludes various functions of interest. Sharp upper bounds on the approximation order of polynomial box spline spaces and exponential box spline spaces (integer direction case) were derived in [BH] and [LJ] respectively, based on the local structure of the spline space, which in general is a rare possibility (see the box spline section in this paper). Optimal schemes for approximating bounded exponentials in the non-scaling (still, compactly supported) case were introduced in [R2]. These results will be presented in the sequel, since they form the starting point for the upper bound analysis here.

We introduce and analyze in this paper a new approach for the determination of the approximation orders of the scale $\{s_h\}_h$. In this approach, only modest decay rates are required of the generating function $\psi_h$ (e.g., some maximal function $\psi_h^{\#}$ should be integrable), and the questions of upper bounds and lower bounds are attacked almost simultaneously, so that, for all special cases studied here, they match each other and the approximation order is determined. Using Fourier analysis methods, we further need not restrict our attention to integral approximation orders. On the other hand, for the lower bound part, we place some smoothness conditions on the generating functions, which are met in all examples we know from the radial basis function theory, but exclude splines of low smoothness, so that we have here the usual smoothness-localization trade-off. This approach makes no use of quasi-interpolation arguments; in particular, polynomial or exponential reproduction is not required. In addition, the approximation scheme is constructive enough for the determination of realistic estimates for the constant which is hidden in the $O(h^k)$ expression.

In spite of the generality of the results here, we are able to apply them directly to obtain upper and matching lower bounds for the case when the generating function is a(n exponential) box spline with rational directions. We believe that none of the methods now in the literature could provide either bounds. We show the important fact that many of the lower bounds known for radial basis (and related) functions underestimate the correct approximation order, and explain this phenomenon. Finally, we show that the use of basic mollifiers for the generating function (e.g., the Gaussian kernel) leads, if properly used, to *infinite* approximation orders.

As mentioned, lower bounds on the approximation order were derived previously with the aid of quasi-interpolants, and the difficulty we observed in the implementation of this method encouraged us to start the work reported here. While the quasi-interpolation argument is an extremely useful and powerful tool in the compactly supported stationary case, its application in other known situations is complicated. For example, for piecewise-exponentials, the space $H$ of exponential-polynomials in $\cap_h s_h$ might be hard to determine, its local approximation properties might be even harder to analyze (cf. [DR]), and the lower bounds attained in this way might underestimate the true approximation order (all these three are valid difficulties in the exponential box spline/rational direction case). But the major drawback of the quasi-interpolation argument appears in the area of radial basis functions (cf. [P] and the references therein). In almost all examples there, $\psi_h$ is the $h$-dilate of $\psi_1$, hence one expects to use polynomial reproduction in the quasi-interpolation argument. Still, if the function $\psi_1$ does not decay fast enough, standard polynomial reproduction arguments (namely, Poisson's summation formula) do not apply. Further, even if all desired polynomials are shown to be reproduced, more subtle information on the rates of decay of $\psi_1$ is required, [DJLR], [Bu1-3]. At the outset of our present study, we tried to apply to these cases the quasi-interpolation argument from [R2], which involves only bounded exponentials, but found that, although the polynomial reproduction argument can be circumvented in this way, no better approximation orders are obtained.

The crux of all the analysis here is the linkage between the Fourier transform and Fourier series via the periodization argument, and which is best expressed by Poisson's summation formula. Starting with [S], this tool has always been the chief Fourier analysis argument for polynomial/exponential reproduction. The results of this work show that the periodization argument is not only an important technical tool, but is at the center of the approximation order analysis: the rearrangement of the error into Fourier series allows us to distinguish between terms that can be reduced by an optimal selection of the approximant, and terms that can be small only because of the good approximation properties of the spaces $\{s_h\}_h$.

We have chosen in this paper to focus on the $L_\infty$ case, namely, measure the error in the $\infty$-norm, primarily since this substantially simplifies the analysis of upper bounds (by making the exponential functions admissible for approximation). On the other hand, this norm is probably one of the harder choices in the lower bound analysis (certainly when compared to the 2-norm): Since the approximation is performed entirely in the Fourier transform domain, we needed to bound the Sobolev (or potential) norm of the function to be approximated in terms of its Fourier transform, and thus our notion of "admissibility" falls short of the usual Sobolev space. Further, as we already mentioned before, our lower bound conditions exclude generating functions of low smoothness, and this is again related to the choice of the norm: the error in the approximation scheme can be written and analyzed in terms of certain Fourier multipliers, whose Fourier transform is explicitly known. However, to obtain sharp results with the aid of these multipliers requires, because of the use of the $\infty$-norm, information about the behaviour of the multiplier in the original domain, which, as a rule, is not easily accessible.

Throughout the paper, $C$ stands for the unit cube $[-1/2 \,\mathrm{d} 1/2]^n$, and $B_\eta$ for the $L_2(\mathrm{IR}^d)$-ball of radius $\eta$ centered at the origin. We use the notation $e_\theta$, $\theta \in \mathbb{R}^d$, for the *complex* exponential

$$e_\theta : x \mapsto e^{i\theta\cdot x},$$

and denote by $\phi{*'}$ the **semi-discrete convolution**

$$\phi{*'} : f \mapsto \sum_{\alpha\in\mathbb{Z}^d} \phi(\cdot - \alpha) f(\alpha),$$

where $f$ is any function defined (at least) on $\mathbb{Z}^d$. The Fourier transform of the summable function $f$ is defined by

$$\widehat{f}(\theta) := \int_{\mathbb{R}^d} e_{-\theta}(t) f(t)\, dt,$$

and is extended by duality to all distributions in $\mathcal{D}'(\mathbb{R}^d)$. We also make use of the **discrete Fourier transform** (or **symbol**) $\widetilde{f}$ of the function $f$ (of polynomial growth), defined as

$$\widetilde{f} := \sum_{\alpha\in\mathbb{Z}^d} e_{-\alpha} f(\alpha).$$

Note that

$$\widetilde{f}(w) = (f{*'}e_w)(0) = (e_{-w} f{*'}1)(0) \tag{1.1}$$

in case $f_{|_{\mathbb{Z}^d}} \in \ell_1(\mathbb{Z}^d)$. We denote by $\Pi$ the ring of all polynomials in $d$ variables, and $\Pi_n$ is the subspace of polynomials of degree at most $n$. Also, $\Pi_{<n} := \Pi_{n-1}$.

As a rule, $\alpha$, $\beta$ are generic points of $\mathbb{Z}^d$, $2\pi\mathbb{Z}^d$, respectively, and $\theta$, $w$ are generic points of the Fourier transform domain. Also, the default norm is $\|\cdot\| := \|\cdot\|_\infty$, while, for $x \in \mathbb{R}^d$, $|x|_p$ is its $p$-norm, and $|x| := |x|_2$ is its Euclidean norm.

## 2. BOUNDS ON THE APPROXIMATION ORDER

### 2.1. Principal shift-invariant spaces

We are interested in characterizing the **approximation order** of the spaces $\{s_h\}_h$. This is, by definition, the maximal nonnegative $k$ for which

$$\operatorname{dist}_\infty(f, s_h) = O(h^k), \text{ when } h \to 0,$$

for every $k$-admissible $f$.

In order for this definition to make any sense, we need to define precisely the spaces $\{s_h\}_h$, as well as explain the notion of "$k$-admissible". We start with the former.

We take $s_h$ to be an appropriate closure of the linear hull of the $h\mathbb{Z}^d$-translates of some function, its **generating** function. Specifically, we take $s_h$ to consist of functions of the form $\sum_{\alpha\in h\mathbb{Z}^d}\psi_h(\cdot-\alpha)a(\alpha)$, with, possibly, some restriction imposed on the coefficient sequence $a$. Because of the nature of the results in this paper, it is convenient to scale up $s_h$, i.e., to look at the space

$$S_h := \{f(h\cdot):\ f\in s_h\}. \tag{2.1}$$

The space $S_h$ is a **principal shift-invariant** space, which means, by definition, that it is "spanned" by the *integer* translates of one generating function $\phi_h$ (which happens to be $\psi_h(h\cdot)$). Denoting by

$$S(\phi)$$

the principal shift-invariant space generated by the integer translates of $\phi$, we can then write $S_h = S(\phi_h)$. Since the $\infty$-norm is scale-invariant, we have

$$\mathrm{dist}_\infty(f, s_h) \;=\; \mathrm{dist}_\infty(f(h\cdot), S(\phi_h)),$$

hence the change from $s_h$ to the scaled space $S_h = S(\phi_h)$ requires nothing more than switching from $f$ to the correspondingly scaled $f(h\cdot)$. As a simple example, note that in the stationary case, when $\psi_h$ is the $h$-dilate of $\psi_1$, the scale-up procedure undoes the dilation and hence $\phi_h = \phi_1 = \psi_1$ for all $h$. In other words, $S_h$ does not change with $h$.

Thus our setting is as follows: we hold in hand a collection $\{S_h\}_h$ of spaces, each of which is a principal shift-invariant space generated by some $h$-dependent function $\phi_h$. Then for a "reasonable" function $f$, we consider the quantities $\mathrm{dist}_\infty(f(h\cdot), S_h)$. Whenever these quantities decay to 0 like $h^k$, we say that $\{S_h\}_h$ provides approximation order $k$ for $f$. If $\mathrm{dist}_\infty(f(h\cdot), S_h) = O(h^k)$ for all $k$-admissible functions, then we say that $\{S_h\}_h$ **provides approximation order** $k$.

We have not yet defined the topology used in the definition of the principal shift-invariant space $S(\phi)$. While the derivation of lower bounds is largely independent of the topology used in the definition of this "spline" space (since only a small subset of the space is usually employed in the analysis), upper bounds are intimately related to the way $S(\phi)$ is defined: results on upper bounds become stronger with the weakening of the topology in which the limits $\sum_{\alpha\in\mathbb{Z}^d}\phi(\cdot-\alpha)a(\alpha)$ are calculated. In the absence of a standard definition for the space $S(\phi)$, we have chosen here the following one, which is motivated by the particular way in which we shall derive upper bounds in the next section.

**Definition.** *The* **principal shift-invariant space** $S(\phi)$ *is the space of all locally bounded functions* $\phi *' a$*, for which the double sum*

$$\phi *'(\phi *' a) = \sum_{\alpha,\beta\in\mathbb{Z}^d}\phi(x-\beta)\phi(\beta-\alpha)a(\alpha)$$

*is absolutely convergent for every* $x\in\mathbb{R}^d$*.*

If $\phi$ has compact support, then $S(\phi)$ contains $\phi *' a$ for arbitrary $a$. Furthermore, if $\phi$ has some decay at $\infty$, then $S(\phi)$ contains all $\phi *' a$ for which $a$ does not grow too fast at $\infty$. Here is a sample proposition:

**Proposition 2.2.** *Assume that, for every $p \in \Pi_n$, the series $\phi *' p$ converges pointwise absolutely to a locally bounded function, and let $A_n$ be the space of all sequences $a : \mathbb{Z}^d \to \mathbb{C}$ of (at most) polynomial growth $n$ at $\infty$. Then $\phi *' A_n \subset S(\phi)$. In particular, $\phi *' A_n \subset S(\phi)$ in case $|\phi(x)| = O(|x|^{-m})$ for some $m > n + d$, as $x \to \infty$.*

**Proof:** We will show that, for $a \in A_n$, $(\phi *' a)_{|_{\mathbb{Z}^d}} \in A_n$, from which it will follow (because of the assumption on $\phi$) that $\phi *'(\phi *' a)$ converges absolutely to a locally bounded function, and therefore, by the definition of $S(\phi)$, $\phi *' a \in S(\phi)$. Without loss, we may assume that both $\phi$ and $a$ are nonnegative (otherwise take absolute values).

By assumption, we can find a constant const such that $\|\phi *' p\|_{L_\infty(C)} \le \text{const}$ for all normalized monomials $p : x \mapsto x^\alpha/\alpha!$, $\alpha \in \mathbb{Z}^d_+$, $|\alpha|_1 \le n$. It follows that

$$\|\phi *' p\|_{L_\infty(C)} \le \text{const} \max_{|\gamma|_1 \le n} |D^\gamma p(0)|$$

for all $p \in \Pi_n$. Now, let $y \in \mathbb{R}^d$, and set $y =: t_y + \alpha_y$, with $t_y \in C$ and $\alpha_y \in \mathbb{Z}^d$. Since $(\phi *' p)(\cdot + \alpha_y) = \phi *'(p(\cdot + \alpha_y))$, we deduce that $(\phi *' p)(y)$ is the value at $t_y$ of $\phi *'(p(\cdot + \alpha_y))$, and therefore, by the argument above, $|\phi *' p(y)| \le \text{const} \max_{|\gamma|_1 \le n} |D^\gamma p(\alpha_y)|$. Thus $\phi *' p = O(|\cdot|^{\deg p})$ at $\infty$, and hence $(\phi *' p)_{|_{\mathbb{Z}^d}} \in A_n$ for any $p \in \Pi_n$.

As for $\phi *' a$, by definition of $A_n$, $a \in A_n$ can be bounded by some $p \in \Pi_n$ (in the sense that $a(\alpha) \le p(\alpha)$ for all $\alpha$), hence $\phi *' a$ is dominated by $\phi *' p$ and therefore $(\phi *' a)_{|_{\mathbb{Z}^d}} \in A_n$. □

For the approach taken in this paper, it is important that the sum $\phi *' e_\theta$ be well-defined for any exponential $e_\theta$, $\theta \in \mathbb{R}^d$. Therefore, we assume that each operator $\phi *'$ is well-defined and bounded as a map from $\ell_\infty$ to $L_\infty$, and denote the corresponding norm by $\|\phi *'\|$. Some conditions related to the boundedness of $\|\phi *'\|$ are recorded in the following proposition whose proof is standard.

**Proposition 2.3.** *The norm of the operator $\phi *'$ is $\| \sum_{\alpha \in \mathbb{Z}^d} |\phi(\cdot - \alpha)| \|$, hence, this operator is bounded if and only if the series $\sum_{\alpha \in \mathbb{Z}^d} |\phi(\cdot - \alpha)|$ is pointwise convergent to a bounded function.*

This proposition implies that $\phi \in L_1(\mathbb{R}^d)$ whenever $\phi *'$ is bounded, and, hence, that the Fourier transform $\widehat{\phi}$ of $\phi$ is a well-defined continuous function. Also, a sufficient condition for the boundedness of $\phi *'$ is the integrability of the maximal function $\phi^\#(x) := \|\phi\|_{L_\infty(x+C)}$.

### 2.2. Admissibility

Next, we turn to the definition of the space of admissible functions associated with the $\infty$-norm:

**Definition.** *A function $f$ of at most polynomial growth at $\infty$ is termed here $k$-**admissible** if $(1 + |\cdot|^k)\widehat{f}$ is a Radon measure of finite total mass. For such a function $f$, we denote by*

$$\|f\|'_k$$

*the total mass of $(1 + |\cdot|^k)\widehat{f}$.*

It follows that $f$ is $k$-admissible (for some $k \ge 0$) only if $\widehat{f}$ is a measure of finite total mass. In particular, any admissible $f$ is bounded. It is worthwhile to keep in mind two examples of admisssible functions. The first is the exponential $f = e_\theta$, $\theta \in \mathbb{R}^d$. In this case, $\widehat{f} = \delta_{-\theta}$, and since $\widehat{f}$ is compactly supported, $f$ is admissible of all orders. However, $\|f\|'_k = 1 + |\theta|^k$, and this grows with $k$ and/or $\theta$. The other example occurs when $\widehat{f}$ is a function. In this case, $f$ is $k$-admissible whenever $(1 + |\cdot|^k)\widehat{f} \in L_1(\mathbb{R}^d)$.

As usual, in case $k$ is integral, the admissibility condition can be interpreted in terms of the $k$th order derivatives of $f$:

**Proposition 2.4.** *A function $f$ is $k$-admissible for some $k \in \mathbb{Z}_+$ if and only if the Fourier transforms of $f$ and of all its $k$th order derivatives are measures of finite total mass.*

**Proof:** Let $f_\alpha$ be the $\alpha$th order (distributional) derivative of $f$, hence $\widehat{f_\alpha} : w \mapsto (iw)^\alpha \widehat{f}(w)$, and choose $c_\alpha$ so that $\sum_\alpha c_\alpha |x^\alpha| = |x|_1^k$ for $x \in \mathbb{R}^d$, i.e., $c_\alpha = \binom{k}{\alpha}$ for $|\alpha|_1 = k$ and $c_\alpha = 0$ otherwise. Then $\sum_\alpha c_\alpha |\widehat{f_\alpha}| = |\cdot|_1^k |\widehat{f}|$, therefore, if $\widehat{f_\alpha}$ is a measure of finite total mass for each $\alpha \in \mathbb{Z}_+^d$ with $|\alpha|_1 = k$, then so is $|\cdot|_1^k |\widehat{f}|$, hence so is $|\cdot|^k |\widehat{f}|$. Thus, if also $\widehat{f}$ is of finite mass, then we conclude that so is $(1 + |\cdot|^k)|\widehat{f}|$. The converse is even simpler: if $f$ is $k$-admissible, then $\widehat{f}$, as well as $w \mapsto (iw)^\alpha \widehat{f}(w)$ for $|\alpha|_1 = k$, are majorized by a measure of finite mass (viz. $(1+|\cdot|^k)\widehat{f}$), and hence the Fourier transform of $f$ and of all its derivatives of order $k$ are measures of finite mass. □

## 2.3. Upper bounds

We obtain upper bounds for the approximation order by considering approximation to exponentials $e_\theta$, $\theta \in \mathbb{R}^d$. Our starting point is the following result from [R2]:

**Result 2.5.** *Let $\theta \in \mathbb{R}^d$, and assume that the sequence $\{\phi_h\}_h$ satisfies the following conditions:*
*(a)* $\operatorname{supp} \phi_h \subset B$, *for all $h$, and for some $h$-independent compact $B$.*
*(b) The functions $\{\phi_h\}_h$ are uniformly bounded.*
*Then,*

$$\|\phi_h *' e_{h\theta} - \widetilde{\phi}_h(h\theta) e_{h\theta}\| \le c \operatorname{dist}_\infty(e_{h\theta}, S(\phi_h)). \tag{2.6}$$

The proof provided here will make use of the following condition, which is a consequence of (a)+(b), but implies only (b):
*(ab)* $\sup_h \|\phi_h *'\| < \infty$.

**Proof:** Fix $h$, and let $f \in S(\phi_h)$. Since $\phi_h *' g = g *' \phi_h$ for all $g \in S(\phi_h)$, by [B1], and also $e_\vartheta *' \phi_h = \widetilde{\phi}_h(\vartheta) e_\vartheta$ for any $\vartheta$, we have

$$\|\phi_h *' e_{h\theta} - \widetilde{\phi}_h(h\theta) e_{h\theta}\| \le \|\phi_h *' e_{h\theta} - \phi_h *' f\| + \|f *' \phi_h - e_{h\theta} *' \phi_h\| \le 2\|\phi_h *'\| \, \|e_{h\theta} - f\|. \tag{2.7}$$

Since $f \in S(\phi_h)$ was arbitrary, the result follows, with $c = 2 \sup_h \|\phi_h *'\|$. □

Since the key to the above argument is the "flip" property: $\phi *' f = f *' \phi, f \in S(\phi)$, we can extend the result to any $\phi$, compactly supported or not, with that property. *Our particular definition of the space $S(\phi)$ was chosen primarily to ensure the "flip" property.*

**Flip Lemma 2.8.** *For every $f \in S(\phi)$,*

$$\phi *' f = f *' \phi.$$

**Proof:** The argument follows the one given in [B1]. We fix $f \in S(\phi)$ and $x \in \mathbb{R}^d$, and wish to show that both $\phi *' f(x)$ and $f *' \phi(x)$ converge, and to the same limit. Since $f = \phi *' a$ for some sequence $a$, we write explicitly

$$(\phi *' f)(x) = \sum_{\alpha \in \mathbb{Z}^d} \phi(x - \alpha) \sum_{\beta \in \mathbb{Z}^d} \phi(\alpha - \beta) a(\beta).$$

By the definition of $S(\phi)$, this double sum is absolutely convergent, hence we may rearrange terms (and replace $\alpha$ by $\alpha+\beta$) to get:

$$\sum_{\alpha\in\mathbb{Z}^d}\phi(\alpha)\sum_{\beta\in\mathbb{Z}^d}\phi(x-(\alpha+\beta))a(\beta)=f*'\phi(x).$$

$\square$

**Theorem 2.9.** *Assume that the operator $\phi*'$ is bounded. Then, for any $\theta\in\mathbb{R}^d$,*

$$\|\phi*'e_{h\theta}-\widetilde{\phi}(h\theta)e_{h\theta}\|\le 2\|\phi*'\|\operatorname{dist}_\infty(e_{h\theta},S(\phi)). \tag{2.10}$$

**Proof:** Repeat the proof of Result 2.5, but replace the reference to [B1] by a reference to Lemma 2.8. $\square$

Theorem 2.9 can be interpreted in two different ways. On the one hand, it suggests that a 'near-optimal' approximant for the exponential $e_{h\theta}$ from

$$S_h:=S(\phi_h)$$

is provided by (the supposedly well-defined) $\widetilde{\phi}_h(h\theta)^{-1}\phi_h*'e_{h\theta}$. The following corollary records this fact.

**Corollary 2.11.** *Assuming that $\phi_h*'$ is bounded and that $\widetilde{\phi}_h(h\theta)\ne 0$, we get*

$$\|\widetilde{\phi}_h(h\theta)^{-1}\phi_h*'e_{h\theta}-e_{h\theta}\|\le 2\frac{\|\phi_h*'\|}{|\widetilde{\phi}_h(h\theta)|}\operatorname{dist}_\infty(e_{h\theta},S_h). \tag{2.12}$$

We note that the ratio $\|\phi_h*'\|/|\widetilde{\phi}_h(h\theta)|$ is independent of the way $\phi_h$ is normalized, and hence, the right hand side of (2.12) is independent of the particular normalization we choose for $\phi_h$. But the estimate (2.12) is useful for the derivation of bounds for the approximation order only in case the sequence $\{\|\phi_h*'\|/\widetilde{\phi}(h\theta)\}_h$ is bounded.

Fortunately, Theorem 2.9 can be used directly to derive *upper* bounds. We simply observe that, in case the operators $\{\phi_h*'\}_h$ are uniformly bounded, Theorem 2.9 shows that $\operatorname{dist}_\infty(e_{h\theta},S_h)=O(h^k)$ only if the same holds for $\|\phi_h*'e_{h\theta}-\widetilde{\phi}_h(h\theta)e_{h\theta}\|$, i.e., for $\|e_{-h\theta}(\phi_h*'e_{h\theta})-\widetilde{\phi}_h(h\theta)\|$ (since $|e_{h\theta}(x)|=1$ for every $\theta\in\mathbb{R}^d$ and every $x\in\mathbb{R}^d$). Since $e_{-h\theta}(\phi_h*'e_{h\theta})=(e_{-h\theta}\phi_h)*'1$, we obtain

$$\|(e_{-h\theta}\phi_h)*'1-\widetilde{\phi}_h(h\theta)\|\le c\operatorname{dist}_\infty(e_{h\theta},S_h). \tag{2.13}$$

Thus, if we assume that we have approximation order $k$, then we must have

$$\|(e_{-h\theta}\phi_h)*'1-\widetilde{\phi}_h(h\theta)\|=O(h^k). \tag{2.14}$$

Since the function $(e_{-h\theta}\phi_h)*'1$ is $\mathbb{Z}^d$**-periodic**, (2.14) implies that its Fourier coefficients (excluding the 0'th coefficient) must be of size $O(h^k)$. Furthermore, (2.14) implies, in particular, that

$$\|(e_{-h\theta}\phi_h)*'1-\widetilde{\phi}_h(h\theta)\|_{L_2(C)}=O(h^k), \tag{2.15}$$

which means that the 2-norm of the Fourier coefficient sequence for this periodic function is of order $O(h^k)$. Since $\widetilde{\phi}_h(h\theta)$ is part of the constant term of this function, these coefficients are $(e_{-h\theta}\phi_h)*'1)\hat{}(\beta)$ for $\beta \in 2\pi\mathbb{Z}^d\backslash 0$. We compute

$$
\begin{aligned}
((e_{-h\theta}\phi_h)*'1)\hat{}(\beta) &= \int_C \sum_{\alpha\in\mathbb{Z}^d} e_{-h\theta}(t-\alpha)\phi_h(t-\alpha)e_{-\beta}(t)\,dt \\
&= \sum_{\alpha\in\mathbb{Z}^d} \int_{C-\alpha} e_{-h\theta}(t)\phi_h(t)e_{-\beta}(t)\,dt \qquad (2.16)\\
&= \int_{\mathbb{R}^d} \phi_h(t)e_{-h\theta-\beta}(t)\,dt \qquad = \widehat{\phi}_h(h\theta+\beta),
\end{aligned}
$$

where, for the second equality, the fact that $\phi_h \in L_1$ was used. Therefore, we conclude that

$$\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} |\widehat{\phi}_h(h\theta+\beta)|^2 = O(h^{2k}).$$

As a matter of fact, nothing in the above arguments requires the approximation order to behave like a power of $h$, and we thus obtain the following.

**The Upper Bound Theorem 2.17.** *Assume that the $\phi_h*'$ are bounded, and let $\theta \in \mathbb{R}^d$. If*

$$\operatorname{dist}(e_{h\theta}, S(\phi_h)) = O(\rho_\theta(h))$$

*for some (univariate) function $\rho_\theta$, then, for every $h$,*

$$\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} |\widehat{\phi}_h(h\theta+\beta)|^2 \le \text{const}\|\phi_h*'\|\rho_\theta(h)^2.$$

*In particular, if we normalize $\{\phi_h\}_h$ to obtain a uniformly bounded $\{\phi_h*'\}_h$, then $\{S(\phi_h)\}_h$ provides approximation order $k$ to the exponential function $e_\theta$ , $\theta \in \mathbb{R}^d$, only if*

$$\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} |\widehat{\phi}_h(h\theta+\beta)|^2 \le c_\theta h^{2k}.$$

Note that the above implies that, in order to obtain $k$-approximation order, it is necessary to have

$$|\widehat{\phi}_h(h\theta+\beta)| \le c_\theta h^k, \quad \beta \in 2\pi\mathbb{Z}^d\backslash 0,\ \theta \in \mathbb{R}^d. \qquad (2.18)$$

It is this slightly weaker condition that we use throughout the paper in order to obtain upper bounds on the approximation orders.

It is remarkable that the result avoids an application of Poisson's summation formula (namely, the convergence of the Fourier series of $(e_{-h\theta}\phi_h)*'1$ was not required), and hence no smoothness conditions were imposed on $\{\phi_h\}_h$. Also, no "regularity" condition was needed in the upper bound theorem, i.e., neither $\{\widehat{\phi}_h(0)\}_h$ nor $\{\widetilde{\phi}_h(0)\}_h$ were required to stay away from 0. (However, it is plausible that, in the singular cases, this upper bound overestimates the actual approximation order by the order of the zero of $h \mapsto \widetilde{\phi}_h(h\theta)$ at $h = 0$.)

The upper bounds were derived under the assumption that the exponential function $e_\theta$ is admissible, hence should be approximated well. Under a stronger assumption on the rates of decay of each $\phi_h$ at $\infty$, we can show that the Upper Bound Theorem 2.17 remains valid even if we only wish to approximate the test functions in $\mathcal{D}$, i.e., infinitely smooth compactly supported functions.

**Theorem 2.19.** *Assume that* $\{\|\phi_h *'\|\}_h$ *is bounded and, in addition,* $\{\phi_h\}_h$ *satisfies the condition*

$$\sum_{|\alpha|\geq 1/h} |\phi_h(x-\alpha)| \leq ch^k, \quad x \in C. \tag{2.20}$$

*Then* $\{S(\phi_h)\}_h$ *provides approximation order* $k$ *to all functions in* $\mathcal{D}$ *only if, for every* $\theta \in \mathbb{R}^d$ *and every* $\beta \in 2\pi\mathbb{Z}^d\backslash 0$,

$$|\widehat{\phi}_h(h\theta+\beta)| \leq c_\theta h^k,$$

*where* $c_\theta$ *is independent of* $\beta$.

**Proof:** We fix $\theta \in \mathbb{R}^d$, and choose $f \in \mathcal{D}$ with $\|f\| = 1$ such that $f = e_\theta$ on some neighborhood of 0, e.g., on $3C$. Let $x \in C$, $h \leq 1$, and set $f_h := f(h\cdot)$. Then, since $|\alpha| \leq 1/h$ implies that $C - \alpha \subseteq 3C/h$, we have $f_h(x-\alpha) = e_{h\theta}(x-\alpha)$ for $|\alpha| \leq 1/h$, and therefore

$$|(f_h - e_{h\theta}) *' \phi_h(x)| \leq 2 \sum_{|\alpha|\geq 1/h} |\phi_h(\alpha)| = O(h^k).$$

In a similar fashion, $|\phi_h *' (f_h - e_{h\theta})(x)| = O(h^k)$, too.

As in Result 2.5, we obtain that, for each $g \in S(\phi_h)$,

$$\|\phi_h *' f_h - f_h *' \phi_h\| \leq \|\phi_h *' (f_h - g)\| + \|(f_h - g) *' \phi_h\| \leq 2\|\phi_h *'\| \, \|f_h - g\|.$$

Therefore, for $x \in C$ we obtain that

$$\begin{aligned}|(e_{-h\theta}\phi_h) *' 1(x) - \widetilde{\phi}_h(h\theta)| &= |\phi_h *' e_{h\theta}(x) - \widetilde{\phi}_h(h\theta) e_{h\theta}(x)| = |\phi_h *' e_{h\theta}(x) - e_{h\theta} *' \phi_h(x)| \\ &\leq |\phi_h *' f_h(x) - f_h *' \phi_h(x)| + O(h^k) \leq 2\|\phi_h *'\| \operatorname{dist}_\infty(f_h, S_h) + O(h^k).\end{aligned}$$

Since the function $e_{-h\theta}\phi_h *' 1$ is $\mathbb{Z}^d$-periodic, and since we assume the boundedness of $\{\phi_h *'\}_h$, we conclude that

$$\|(e_{-h\theta}\phi_h) *' 1 - \widetilde{\phi}(h\theta)\| \leq \operatorname{const} \operatorname{dist}_\infty(f_h, S_h) + O(h^k).$$

Finally, in the proof of the Upper Bound Theorem 2.17 we have observed that $\{\widehat{\phi}(h\theta+\beta)\}_{\beta\in 2\pi\mathbb{Z}^d}$ are the Fourier coefficients of the function $(e_{-h\theta}\phi_h) *' 1$, and therefore, if $\{S_h\}_h$ provides approximation order $k$ for $f$, we must have

$$|\widehat{\phi}(h\theta+\beta)| \leq ch^k, \quad \beta \in 2\pi\mathbb{Z}^d\backslash 0,$$

where $c$ depends on $f$, i.e., on $\theta$, but is independent of $\beta$. □

**Lower bounds**

The upper bound analysis provides us with near-optimal approximants $\varepsilon_{h\theta}$ from $S_h$ to the exponential $e_{h\theta}$. We use closely related approximants to establish lower bounds on the approximation orders provided by $\{S_h\}_h$. The idea of producing an approximant from $S_h$ for the function $f_h := f(h\cdot)$ is very simple: we write

$$f_h(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\theta) e_{h\theta}(x)\, d\theta,$$

hence can provide the approximant to $f_h$ in the form

$$(2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\theta) \varepsilon_{h\theta}(x)\, d\theta.$$

Of course, we still need to make sure that this approximant lies in $S_h$, and that $f$ is "reasonable" enough for the above integration to make sense. But, even in such a case, the above approximation scheme is "too global" (in the sense that all the Fourier transform information of $f$ is taken into account), and therefore, in order to simplify our error analysis, we use a suitable nonnegative continuous cut-off function $\sigma$ with support near the origin. For convenience, we assume that $\operatorname{supp}\sigma$ lies in the ball $B_\eta = \{x \in \mathbb{R}^d : |x| \le \eta\}$, that $\sigma$ is 1 on $B_{\eta/2}$, and that $\|\sigma\| = 1$. Furthermore, since the approximation scheme should be applicable to any admissible function $f$, we only know that $\widehat{f}$ is a measure, and therefore prefer the notation $d\widehat{f}(\theta)$ to the notation $\widehat{f}(\theta)\, d\theta$. Thus, our approximation scheme takes the form:

$$f_h(x) := f(hx) \;\approx\; R_h f(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} \varepsilon_{h\theta}(x)\sigma(h\theta) d\widehat{f}(\theta). \tag{2.21}$$

Then

$$\begin{aligned} (2\pi)^d |R_h f - f_h| &\le \int_{\mathbb{R}^d} |\varepsilon_{h\theta}\sigma(h\theta) - e_{h\theta}|\; |d\widehat{f}|(\theta) \\ &\le \int_{\mathbb{R}^d} |\sigma(h\theta) - 1|\; |d\widehat{f}|(\theta) + \int_{\mathbb{R}^d} \sigma(h\theta) |\varepsilon_{h\theta}/e_{h\theta} - 1|\; |d\widehat{f}|(\theta). \end{aligned} \tag{2.22}$$

If now $f$ is $k$-admissible, then

$$(2h/\eta)^{-k} \int_{\mathbb{R}^d} (1 - \sigma(h\theta))\; |d\widehat{f}|(\theta) \le \int_{\mathbb{R}^d} |\theta|^k (1 - \sigma(h\theta))\; |d\widehat{f}|(\theta) \quad \underset{h\to 0}{\longrightarrow} \quad 0$$

(since the integrand vanishes on $B_{\eta/(2h)}$), and therefore the first integral in the last line of (2.22) is $o(h^k)$. For the second integral, we need to look more carefully at the 'periodized' error

$$\varepsilon_{h\theta}/e_{h\theta} - 1. \tag{2.23}$$

## 2.4. Lower bounds: analysis

With the assumption that $\widehat{\phi}_h(h\theta) \neq 0$, we choose the 'near optimal' approximation $\varepsilon_{h\theta}$ from $S_h$ to $e_{h\theta}$ in the form

$$\varepsilon_{h\theta} := \frac{\phi_h *' e_{h\theta}}{\widehat{\phi}_h(h\theta)}. \tag{2.24}$$

With this choice, the approximation $R_h f$ takes the more explicit form

$$R_h f := \phi_h *' f_h^*, \tag{2.25}$$

with $f_h^*$ the bounded analytic function

$$f_h^*(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{ihx\cdot\theta}}{\widehat{\phi}_h(h\theta)} \sigma(h\theta)\, d\widehat{f}(\theta). \tag{2.26}$$

In particular, this makes clear that $R_h f$ is indeed an element of $S_h$.

Further, the 'periodized' error takes the form

$$\varepsilon_{h\theta}/e_{h\theta} - 1 = \frac{(e_{-h\theta}\phi_h)*'1}{\widehat{\phi}_h(h\theta)} - 1.$$

Recall from (2.16) that

$$((e_{-h\theta}\phi_h)*'1)\widehat{\ }(\beta) = \widehat{\phi}_h(h\theta + \beta), \quad \beta \in 2\pi\mathbb{Z}^d.$$

Consequently the Fourier series of the periodized error has the form

$$\sum_{\beta\in 2\pi\mathbb{Z}^d} \frac{\widehat{\phi}_h(h\theta+\beta)}{\widehat{\phi}_h(h\theta)} e_\beta - 1 = \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} \frac{\widehat{\phi}_h(h\theta+\beta)}{\widehat{\phi}_h(h\theta)} e_\beta,$$

and it always converges to the periodized error, at least in a distributional sense. While the estimates for this sum provided by summability methods seem to be hard to analyze, we can expect that, for a smooth $\phi_h$, the series will converge absolutely. In such a case, we obtain the following important estimate.

**Theorem 2.27.** *For any $h$, for which $\phi_h *'$ is bounded and $\widehat{\phi}_h(h\theta) \neq 0$,*

$$\|\varepsilon_{h\theta}/e_{h\theta} - 1\| = \|\frac{\phi_h *' e_{h\theta}}{\widehat{\phi}_h(h\theta)} - e_{h\theta}\| \le \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} \frac{|\widehat{\phi}_h(h\theta+\beta)|}{|\widehat{\phi}_h(h\theta)|}. \tag{2.28}$$

It may seem surprising that we did not use here the approximation

$$e_{h\theta} \approx \frac{\phi_h *' e_{h\theta}}{\widetilde{\phi}_h(h\theta)} \tag{2.29}$$

derived during the discussion of upper bounds. The reason is simple. Recall from (1.1) that $\widetilde{\phi}_h(h\theta) = (\phi_h *' e_{h\theta})(0) = (e_{-h\theta}\phi_h *' 1)(0)$, and this equals $\sum_{\beta\in 2\pi\mathbb{Z}^d}(e_{-h\theta}\phi_h *' 1)\widehat{\ }(\beta)$. Therefore, by (2.16), $\widetilde{\phi}_h(h\theta) = \widehat{\phi}_h(h\theta) + \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0}\widehat{\phi}_h(h\theta+\beta)$. Hence if one of the two sums

$$\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0}\frac{|\widehat{\phi}_h(h\theta+\beta)|}{|\widetilde{\phi}_h(h\theta)|} \quad\text{and}\quad \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0}\frac{|\widehat{\phi}_h(h\theta+\beta)|}{|\widehat{\phi}_h(h\theta)|}$$

goes to zero with $h$, then they go to zero at exactly the same rate, since $\lim_{h\to 0}\widetilde{\phi}_h(h\theta)/\widehat{\phi}_h(h\theta) = 1$ in such a case. This means that we lose nothing in the estimate (2.28) if we use the approximation $\varepsilon_{h\theta} = \frac{\phi_h *' e_{h\theta}}{\widehat{\phi}_h(h\theta)}$ instead of $\frac{\phi_h *' e_{h\theta}}{\widetilde{\phi}_h(h\theta)}$, but gain simplicity, since the Fourier transform $\widehat{\phi}_h$ is usually more readily accessible than the symbol $\widetilde{\phi}_h$.

In the sequel we exclusively use the right hand side of (2.28) to bound $\|\varepsilon_{h\theta}/e_{h\theta} - 1\|$, hence obtain positive approximation orders only when the right hand side of (2.28) tends to 0 with $h$.

### 2.5. Lower bounds: synthesis

With the bound (2.28) in hand, we return to the error estimate

$$(2\pi)^d|R_hf - f_h| \le \int_{\mathbb{R}^d}(1-\sigma(h\theta))\,|d\widehat{f}|(\theta) + \int_{\mathbb{R}^d}\sigma(h\theta)|\varepsilon_{h\theta}/e_{h\theta} - 1|\,|d\widehat{f}|(\theta); \tag{2.30}$$

see (2.22). Having observed earlier that the first integral is $o(h^k)$ whenever $f$ is $k$-admissible, we now want to make the second integral $O(h^k)$. By Theorem 2.27, we have in hand the estimate

$$\int_{\mathbb{R}^d}\sigma(h\theta)|\varepsilon_{h\theta}/e_{h\theta} - 1|\,|d\widehat{f}|(\theta) \le \int_{B_{\eta/h}}\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0}\frac{|\widehat{\phi}_h(h\theta+\beta)|}{|\widehat{\phi}_h(h\theta)|}\,|d\widehat{f}|(\theta). \tag{2.31}$$

We know from the upper bound discussion that approximation order $k$ requires that $\frac{|\widehat{\phi}_h(h\theta+\beta)|}{|\widehat{\phi}_h(h\theta)|} = O(h^k)$ for fixed $\theta$ uniformly in $\beta$. This suggests the assumption that

$$\|\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0}\frac{|\widehat{\phi}_h(\cdot+\beta)|}{|\widehat{\phi}_h|}\|_{L_\infty(B_\eta)} = O(h^k).$$

Unfortunately, except for the case of spectral approximation orders (see Section 3.3), not many examples of interest satisfy this assumption. We choose instead the following more subtle condition, in which we employ functions $\nu_h(x)$ to describe the behavior of $\frac{|\widehat{\phi}_h(x+\beta)|}{|\widehat{\phi}_h(x)|}$ for $x$ near 0. In fact, for any indexed set $\{\nu_h\}_h$ of positive functions on $B_\eta$, we have

$$\int_{B_{\eta/h}}\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0}\frac{|\widehat{\phi}_h(h\theta+\beta)|}{|\widehat{\phi}_h(h\theta)|}\,|d\widehat{f}|(\theta) \le \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0}\|\frac{\widehat{\phi}_h(\cdot+\beta)}{\nu_h\widehat{\phi}_h}\|_{L_\infty(B_\eta)}\int_{B_{\eta/h}}\nu_h(h\theta)\,|d\widehat{f}|(\theta). \tag{2.32}$$

Consequently,

$$\int_{\mathbb{R}^d}\sigma(h\theta)|\varepsilon_{h\theta}/e_{h\theta} - 1|\,|d\widehat{f}|(\theta) \le A(\nu,\eta)\int_{B_{\eta/h}}\nu_h(h\theta)\,|d\widehat{f}|(\theta), \tag{2.33}$$

with

$$A(\nu,\eta) := \sup_h \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} \|\frac{\widehat{\phi}_h(\cdot+\beta)}{\nu_h\widehat{\phi}_h}\|_{L_\infty(B_\eta)}.$$

**The Synthesis Condition.** *We say that* $\{\phi_h\}_h$ *satisfies the* $\nu$**-synthesis condition** *if, for some indexed set* $\{\nu_h\}_h$ *of functions defined in some neighborhood of 0, and some* $\eta>0$, $A(\nu,\eta)$ *is finite. This means that the number*

$$A(\nu) := \inf_\eta A(\nu,\eta) \tag{2.34}$$

*is finite.*

Now notice that $\int_{\mathbb{R}^d}(|h\theta|^k+h^k)\,|d\widehat{f}|(\theta) = h^k\|f\|'_k$. This suggests the choice

$$\nu_h(x) := |x|^k + h^k. \tag{2.35}$$

We call the $\nu$-synthesis condition with respect to this $\nu$ **the synthesis condition of order** $k$.

With this, we infer from (2.30) and (2.33) the following:

**The Lower Bound Theorem 2.36.** *Assume that each* $\phi_h*'$ *is bounded, and that* $\{\phi_h\}_h$ *satisfies the synthesis condition of order* $k$*. Then, for every* $k$*-admissible* $f$, $\|R_hf - f(h\cdot)\| = O(h^k)$*, with* $R_hf$ *defined as in (2.21). In particular,* $\{S(\phi_h)\}_h$ *provides approximation order* $k$*.*

As a matter of fact, the above discussion provides also the following significant information on the constant in the $O(h^k)$ expression.

**Corollary 2.37.** *Assuming the synthesis condition of order* $k$*, we have, for every* $k$*-admissible* $f$,

$$\|\phi_h*'f_h^* - f_h\| \le (2\pi)^{-d}h^k\|f\|'_kA(\nu) + o(h^k),$$

*with* $A(\nu)$ *as defined in (2.34) (for* $\nu_h(x) := |x|^k + h^k$*).*

Stronger results can be obtained under more restrictive assumptions on the functions $\nu$ in the synthesis condition. For example, if $\nu_h(x)$ is independent of $x$ (but, of course, depends on $h$), then (2.30) and (2.33) allow us to conclude the following improved version of the last corollary:

**Corollary 2.38.** *If* $\{\phi_h\}_h$ *satisfies the* $\nu$*-synthesis condition, with each* $\nu_h$ *being a constant function, then, for every* $k$*-admissible function* $f$,

$$\|\phi_h*'f_h^* - f_h\| \le (2\pi)^{-d}\nu_h(0)\|f\|'_0A(\nu) + o(h^k).$$

*In particular, if* $\nu_h(0) = o(h^k)$*, then so is* $\mathrm{dist}_\infty(f_h,S_h)$*.*

## 3. Applications

We apply here the general results of the previous section to the three main families of generating functions: generating functions obtained by differencing another generating function, $h$-independent generating functions, and generating functions which are compactly supported uniformly in $h$. These families overlap, and many specific examples fall into two or three of these categories (e.g., polynomial box splines satisfy all of the above conditions).

### 3.1. The stationary case

We use the terminology "the stationary case" if the functions $\{\phi_h\}_h$ are actually independent of $h$. This means that all the spaces $S_h = S(\phi_h)$ are the same, or, in other words, that the original spaces $\{s_h\}_h$ are obtained by dilating $S(\phi)$ (with $\phi := \phi_1$, say). In this case, it is convenient to speak of the approximation order **provided by** $\phi$ and mean by this the approximation order provided by $\{S_h = S(\phi)\}_h$. The upper bound theorem specializes in this case to the following.

**Theorem 3.1.** *Let $\phi$ be a measurable function whose associated operator $\phi*'$ is bounded, and whose Fourier transform is $k$ times differentiable at every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ for some $k \in \mathbb{Z}_+$. Then $\phi$ provides approximation order $k$ only if $\widehat{\phi}$ has a zero of order $k$ at every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$.*

**Proof:** Since $\phi_h = \phi$ for every $h$, we must have, by the Upper Bound Theorem 2.17,

$$\widehat{\phi}(h\theta + \beta) = O(h^k), \quad \text{all } \theta \in \mathbb{R}^d.$$

By assumption, $\widehat{\phi}(x + \beta) = p(x) + o(|x|^k)$ for some $p \in \Pi_k$. Therefore, by taking sufficiently (but finitely) many $\theta$'s, we conclude that $p = 0$, i.e., that $\widehat{\phi}$ has a zero of order $k$ at each $\beta \in 2\pi\mathbb{Z}^d\backslash 0$. $\square$

The stationary case was analyzed in great detail in the literature, especially for a compactly supported $\phi$, and the various results for this case are usually in terms of "Strang-Fix Conditions", [SF]. We record below the following version of these conditions, whose sufficiency follows from [DM1] and [B1], while their necessity was proved in [R2] (and previously, in a weaker form, in [BJ]).

**Result 3.2.** *Assume that $\phi$ is a bounded measurable compactly supported function that satisfies $\phi*'1(0) \neq 0$, and let $k \in \mathbb{N}$. Then, $\phi$ provides approximation order $k$ for every $f$ in the Sobolev space $W^k_\infty$ if and only if $\Pi_{<k} \subset S(\phi)$.*

Exploiting the Upper Bound Theorem 2.17, we are able to extend the "only if" statement in Result 3.2 as follows:

**Theorem 3.3.** *Let $\phi$ be a bounded measurable compactly supported function. Then the approximation order provided by $\phi$ to functions in $\mathcal{D}$ cannot exceed the degree of the least degree polynomial $p$ that satisfies $\phi*'p \notin \Pi$.*

**Proof:** By Theorem 2.19, the Upper Bound Theorem 2.17 holds here (although only approximation to functions in $\mathcal{D}$ is required). Thus, by Theorem 3.1 (and the fact that $\widehat{\phi}$ is entire), $\widehat{\phi}$ must have a zero of order $k$ at each $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, in case the approximation order is $k$. Now, it is known (cf. e.g., [BR]) that, for a compactly supported distribution $\phi$, $\phi{*}'\Pi_{<k} \subset \Pi$ if and only if $\widehat{\phi}$ has a zero of order $k$ at each $\beta \in 2\pi\mathbb{Z}^d\backslash 0$. Thus the degree of the least degree polynomial $p$ for which $\phi{*}'p \notin \Pi$ is an upper bound for the appoximation order. $\square$

In case $\widetilde{\phi}(0) \neq 0$, the above result reproduces the "only if" statement of Result 3.2, since it is well-known (cf. [B1]) that in such a case $\phi{*}'p \in \Pi$ if and only if $p \in S(\phi)$.

The above result does not admit a direct extension to the case of global support, since in general the operator $\phi{*}'$ need not to be defined on polynomials in that case. However, we always have the following:

**Theorem 3.4.** *Assume that $\phi{*}'$ is bounded. Then the approximation order provided by $\phi$ is positive only if $\phi{*}'1 = \mathrm{const}$.*

**Proof:** By Theorem 3.1, $\widehat{\phi}$ vanishes on $2\pi\mathbb{Z}^d\backslash 0$. Let $\{\eta_n\}$ be an approximate identity, and define $\tau_n := \eta_n * \phi$. Then, for every $n$, $\widehat{\tau}_n$ vanishes on $2\pi\mathbb{Z}^d\backslash 0$, and hence, by Poisson's summation formula,

$$\tau_n{*}'1 = \sum_{\beta\in 2\pi\mathbb{Z}^d} \widehat{\tau}_n(\beta)e_\beta = \widehat{\phi}(0).$$

Thus, $\eta_n * (\phi{*}'1) = \widehat{\phi}(0)$ for every $n$, and by letting $n \to \infty$, we obtain $\phi{*}'1 = \widehat{\phi}(0)$. $\square$

Similar results can be obtained with respect to higher degree polynomials, if we assume that $\phi{*}'$ is well-defined on such polynomials. We omit these details here.

We now turn to the synthesis condition, and examine the quotient $\frac{\widehat{\phi}(x+\beta)}{\widehat{\phi}(x)}$. In many of the examples, this ratio can be factored into two terms

$$\frac{\widehat{\phi}(x+\beta)}{\widehat{\phi}(x)} = K(x+\beta)H(x),$$

where $K(x)$ decays fast enough at $\infty$, while $H(x)$ has a zero at the origin of high enough order. In such a situation, we obtain the following result.

**Theorem 3.5.** *Suppose that $\phi{*}'$ is bounded and $\widehat{\phi}(0) \neq 0$. Suppose further that, for some continuous $K$ and smooth $H$,*

$$\frac{\widehat{\phi}(x+\beta)}{\widehat{\phi}(x)} = K(x+\beta)H(x),$$

*and that, for some $\eta > 0$,*

$$\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} \|K(\cdot+\beta)\|_{L_\infty(B_\eta)} < \infty.$$

*If $K(\beta) \neq 0$ for some $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, then the approximation order provided by $\phi$ equals the order of the zero of $H$ at the origin.*

**Proof:** Let $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ be such that $K(\beta) \neq 0$. Let $k$ be the exact order of the zero of $H$ at the origin. Since $H$ is smooth at 0, there exists $\theta \in \mathbb{R}^d\backslash 0$ such that $H(h\theta) \neq o(h^k)$, and consequently,

$$\frac{\widehat{\phi}(h\theta+\beta)}{\widehat{\phi}(h\theta)} = K(h\theta+\beta)H(h\theta) \neq o(h^k).$$

Since $\phi *'$ is bounded, we can appeal to the Upper Bound Theorem 2.17 to conclude that the relevant approximation order is at most $k$.

Next, we show that the synthesis condition of order $k$ holds: by the assumption on $H$, $|\cdot|^{-k}H$ is bounded in a neighborhood of the origin. Thus, for a small enough $\eta$ and any $x \in B_\eta$,

$$\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} \frac{|K(x+\beta)H(x)|}{|x|^k} \leq \frac{|H(x)|}{|x|^k} \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} \|K(\cdot+\beta)\|_{L_\infty(B_\eta)} < \infty.$$

This shows that the synthesis condition of order $k$ holds here, and hence, by the Lower Bound Theorem 2.36, the approximation order is at least $k$. □

### 3.2. The differencing case

Almost all the generating functions now in the literature belong to this family. Here, one starts with a function $\phi$ which has polynomial or even exponential growth at $\infty$, but for which $p(D)\phi$ decays at $\infty$ for some linear differential operator $p(D)$ with constant coefficients (e.g., $\phi$ is a fundamental solution of $p(D)$). This means that, away from the zero set of $p(i\cdot)$, $\widehat{\phi}$ coincides with $K/p(i\cdot)$, for some smooth function $K$. To obtain from $\phi$ a function in (a suitable closure of) $S(\phi)$ which decays at $\infty$ (namely, to localize $\phi$), or, equivalently, to remove the singularities of $\widehat{\phi}$, one approximates the differential operator $p(D)$ by a (finite- or infinite-) difference operator T which is supported on $\mathbb{Z}^d$, i.e., approximates the polynomial $p(i\cdot)$ by some periodic function $u$. In order to make this process feasible and as simple as possible, the real variety of the polynomial $p(i\cdot)$ should be extremely simple. For example, in the box spline case $p$ is chosen as a product of (more or less arbitrary) linear polynomials, while for typical radial basis (and related) functions, the operator $p(D)$ is elliptic (for this reason we will refer to the latter class of generating functions as belonging to “the elliptic case”), hence one has only to resolve the singularity of $\widehat{\phi}$ at the origin. For $h > 0$, one replaces $p(D)$ by the operator $p_h(D) := h^{\deg p}p(h^{-1}D)$, associates $p_h(D)$ with a function $\phi_h$ in a way analogous to the case $h = 1$ (so that now $p_h(i\cdot)\widehat{\phi}_h = K$, for the same $K$ as before), and repeats the differencing process. In case $p$ is homogeneous, $p_h$ does not change with $h$, hence $\phi_h = \phi$, all $h$.

For box splines, one easily obtains in this way a compactly supported function $\mathrm{T}\phi$ ([BH], [R1]). In contrast, in the (homogeneous) elliptic case, the major effort was devoted to the localization process, aiming at constructing T in such a way that $(\mathrm{T}\phi)\widehat{\ }$ be as smooth as possible and $(\mathrm{T}\phi)\widehat{\ } - 1$ have a high order zero at the origin. The simple approximation scheme then suggested (cf. e.g., [J], [Bu1-3], [DJLR], [P]) is $f(h\cdot) \approx \mathrm{T}\phi *' f(h\cdot)$, and the convergence rate is $O(h^k|\log h|)$ where $k$ depends on the rate of decay of $\mathrm{T}\phi$ at $\infty$ and on the order of the zero of $(\mathrm{T}\phi)\widehat{\ } - 1$ at the origin, but never exceeds $\deg p$. Under further assumptions on the decay rates of $\mathrm{T}\phi$ at $\infty$, the approximation rates were improved to $O(h^k)$, but in some cases it was proved that $O(h^k|\log h|)$ is the exact order of this approximation scheme. None of the above references provides upper bounds on the approximation order. Details for specific cases are given in the next section.

We show now that *the exact approximation order in all elliptic cases in the literature is the degree of the underlying differential operator* $p(D)$, *regardless of the decay rates of* $\mathrm{T}\phi$ *at* $\infty$, *or the smoothness of* $(\mathrm{T}\phi)\widehat{}$ *at the origin.*

**Theorem 3.6.** *Let* $p(D)$ *be a* homogeneous *elliptic operator of order* $k > d$, $d$ *being the spatial dimension. Assume that* $\phi$ *satisfies the equation*

$$p(i\cdot)\widehat{\phi} = K,$$

*where* $K$ *is a continuous bounded function,* $K(0) \neq 0$, *and* $K(\beta) \neq 0$ *for some* $\beta \in 2\pi\mathbb{Z}^d\backslash 0$. *Assume that, for some sequence* $c$ *of polynomial growth, the sum* $\phi *' c$ *converges distributionally to a function* $\psi$ *whose corresponding* $\psi *'$ *is bounded and whose Fourier transform does not vanish at 0. Then the approximation order provided by* $\psi$ *is (exactly)* $k$.

**Proof:** By the definition of $\psi$,

$$p(i\cdot)\widehat{\psi} = uK, \tag{3.7}$$

where $u$ (the Fourier transform of $c$) is a $2\pi\mathbb{Z}^d$-periodic tempered distribution. Since $\psi *'$ is bounded by assumption, $\widehat{\psi}$ is continuous (recall the remarks following Proposition 2.3). Since $K$ is continuous and $K(0) \neq 0$ (and the left side of (3.7) is continuous), we conclude that $u$ coincides, on some neighborhood $\Omega$ of the origin, with some continuous function. Therefore

$$\widehat{\psi} = \frac{uK}{p(i\cdot)} \quad \text{on } 2\pi\mathbb{Z}^d + \Omega\backslash 0.$$

Since $p(D)$ is elliptic, $|p| \geq \mathrm{const}|\cdot|^k$ for some const $> 0$, while $\mathrm{const}'|\cdot|^k \geq |p|$ for some const$'$, by the homogeneity of $p$. Thus, for small enough $x$ and for $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, by the periodicity of $u$,

$$\left|\frac{\widehat{\psi}(x+\beta)}{\widehat{\psi}(x)}\right| = \left|\frac{K(x+\beta)p(ix)}{p(i(x+\beta))K(x)}\right| \leq \mathrm{const}|x|^k|\beta|^{-k}.$$

Since $k > d$, we see that $\psi$ satisfies the synthesis condition of order $k$, hence (by the Lower Bound Theorem 2.36) the approximation order is *at least* $k$.

Now, choose $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ such that $K(\beta) \neq 0$. Since $p(D)$ is homogeneous and elliptic of order $k$, $p(ih\theta)/h^k = i^k p(\theta) \neq 0$. Hence, for any $\theta \in \mathbb{R}^d\backslash 0$,

$$h^{-k}\frac{\widehat{\psi}(h\theta+\beta)}{\widehat{\psi}(h\theta)} = h^{-k}\frac{K(h\theta+\beta)p(ih\theta)}{p(i(h\theta+\beta))K(h\theta)} \underset{h\to 0}{\longrightarrow} i^k\frac{K(\beta)p(\theta)}{p(i\beta)K(0)} \neq 0.$$

Since $\widehat{\psi}$ does not vanish on some neighborhood of the origin, the Upper Bound Theorem 2.17 applies to yield that the approximation order is *at most* $k$. □

The extension of the above theorem to non-homogeneous elliptic operators is straightforward, and is omitted only because of lack of examples of this type in the present literature. A discussion of non-elliptic cases appears in the box spline section.

As mentioned before, in case $\psi := \mathrm{T}\phi$ decays only slowly at $\infty$, the lower bounds on the approximation order now in the literature usually underestimate the actual approximation order. To explain this, it is instructive to compare the approximation scheme suggested here in the lower bound analysis with the simpler scheme

$$f(h\cdot) \approx \psi *' f(h\cdot). \tag{3.8}$$

Assume, for simplicity, that $\widehat{f} \in \mathcal{D}$, hence $f(hx) = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(w) e_{hx}(w)\, dw$. The approximation scheme here is of the form $f(h\cdot) \approx (2\pi)^{-d} \psi *' \int_{\mathbb{R}^d} \frac{\widehat{f}(w)}{\widehat{\psi}(hw)} e_{hx}(w)\, dw$. Taking into account the fact that the scheme here is proved to be optimal (in terms of approximation orders), the optimality of (3.8) depends on the behaviour of the difference $\widehat{\psi} - 1$ around the origin, or more precisely, on the order of the zero which $\widehat{\psi} - 1$ has at the origin. Indeed, the difference operator T is meant to produce a high order zero, but, since $\widehat{\phi}$ is smooth away from the origin, a high order zero of $\widehat{\psi} - 1$ at the origin implies that $\widehat{\psi}$ is *globally* smooth, hence $\psi$ decays fast at infinity, which is contrary to the assumption here.

Furthermore, resolving the singularity of $\widehat{\phi}$ at the origin with the aid of a trigonometric polynomial or another smooth periodic function might be hard in case this singularity is not of integral order (e.g., some log singularity or fractional singularity). This explains why in some cases it was impossible to remove the log factor in the approximation order by further differencing ([Bu3], [DJLR]).

### 3.3. Spectral approximation orders

The analysis of the stationary case shows that, for this case, the vanishing of $\widehat{\phi}$ on $2\pi\mathbb{Z}^d\backslash 0$ is necessary for obtaining positive approximation orders. However, high approximation orders can at times be obtained from spaces spanned by generating functions whose Fourier transform vanishes nowhere on $\mathbb{R}^d$, even though the function scale $\{\phi_h\}_h$ involves the dilates of a single function.

Suppose that, for some function $\phi$ and some neighborhood $\Omega$ of the origin,

$$\|\frac{\widehat{\phi}(\cdot + \beta/\lambda)}{\widehat{\phi}}\|_{L_\infty(\Omega/\lambda)}$$

decays fast to 0 not only as $\beta \to \infty$, but also as $\lambda \to 0$. In this case, we may choose

$$\phi_h := \phi(\lambda(h)\cdot)$$

for appropriately selected decreasing $\{\lambda(h)\}_h$, with the only limit on the approximation order being the rate of decay of $\|\frac{\widehat{\phi}(\cdot+\beta/\lambda)}{\widehat{\phi}}\|_{L_\infty(\Omega/\lambda)}$ as $\lambda \to 0$.

To simplify the analysis, we assume throughout the discussion that $\phi$ satisfies the following condition

$$\rho(\lambda, \beta) := \|\frac{\widehat{\phi}(\cdot + \beta/\lambda)}{\widehat{\phi}}\|_{L_\infty(\Omega/\lambda)} \le c(s, \varepsilon, \Omega) e^{-(|\beta|-\varepsilon)^s/\lambda^s}, \quad \beta \in 2\pi\mathbb{Z}^d\backslash 0,\ \lambda > 0 \tag{3.9}$$

for some positive $s$, some sufficiently small $\varepsilon$, and some 0-neighborhood $\Omega$. We have chosen this particular condition since it is satisfied by functions whose Fourier transform decays exponentially at infinity, as the following proposition shows. Other decay rates can be treated along the same lines.

**Proposition 3.10.** *Assume that $\widehat{\phi}$ satisfies the condition*

$$0 < c_1 \le |\widehat{\phi}(x)| e^{|x|^s} \le c_2 < \infty, \quad x \in \mathbb{R}^d, \tag{3.11}$$

*for some positive $s$. Then $\phi$ satisfies (3.9) with the same $s$ and for any $\varepsilon < 2\pi$, provided $\Omega = B_{\varepsilon/2}$.*

**Proof:** It follows from (3.11) that, for any $x \in B_\pi$, $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, and $\lambda > 0$,

$$\left|\frac{\widehat{\phi}((x+\beta)/\lambda)}{\widehat{\phi}(x/\lambda)}\right| \le \text{const} e^{-(|x+\beta|^s - |x|^s)/\lambda^s} \le \text{const} e^{-(|\beta|-2|x|)^s/\lambda^s},$$

since $\min_{|y|=|x|}(|y+\beta|^s - |y|^s) = (|\beta|-|x|)^s - |x|^s \ge (|\beta|-2|x|)^s$, using the fact that $|\beta| \ge 2\pi > 2|x|$. Thus, for any $\varepsilon < 2\pi$, (3.9) holds with $\Omega = B_{\varepsilon/2}$. □

Assuming (3.9), we define $\phi_h := \phi(\lambda(h)\cdot)$ for some positive sequence $\{\lambda(h)\}_h$. This implies that, for $\beta \in 2\pi\mathbb{Z}^d\backslash 0$,

$$\rho(\lambda(h), \beta) = \left\|\frac{\widehat{\phi}_h(\cdot+\beta)}{\widehat{\phi}_h}\right\|_{L_\infty(\Omega)} \le c e^{-(|\beta|-\varepsilon)^s/\lambda(h)^s},$$

and we are led to the following result:

**Theorem 3.12.** *Assume that $\phi(t\cdot)*'$ is bounded for every $t > 0$, that $\widehat{\phi}$ vanishes nowhere, and that (3.9) holds for some neighborhood $\Omega$ of the origin, some $\varepsilon < \pi/2$, and some $s > 0$. Define $\phi_h := \phi(\lambda(h)\cdot)$. If*

$$e^{-a/\lambda(h)^s} \le ch^m, \quad \textit{with } a := (2\pi - 3\varepsilon)^s, \tag{3.13}$$

*then the approximation order provided by $\{S_h = S(\phi_h)\}_h$ is at least $m$, and $\operatorname{dist}_\infty(f(h\cdot), S_h) \le c' h^m \|f\|'_0 + o(h^k)$ for every $k$-admissible $f$ in this case. In particular, if $\lambda(h) = O(h^r)$ for some positive $r$, then $\operatorname{dist}_\infty(f(h\cdot), S(\phi_h)) = o(h^k)$ for every $k$-admissible $f$ and* every *$k$. Moreover, if $\lambda(h) = h$, then $\operatorname{dist}_\infty(f(h\cdot), S_h) = O(e^{-a/h^s})$ for very smooth functions $f$ (e.g., functions whose Fourier transform is a compactly supported measure).*

**Proof:** Since the argument here will be used also in the sequel, we prefer to provide the main part of the proof in the form of a separate lemma:

**Lemma 3.14.** *Let $\{\phi_h\}_h$ be a sequence of functions, with corresponding positive bounded sequence $\{\lambda(h)\}_h$, which satisfy the following three conditions:*
*(a) Each $\phi_h*'$ is bounded;*
*(b) each $\widehat{\phi}_h$ vanishes nowhere in some $h$-independent neighborhood $\Omega$ of the origin;*

*(c) with $\varepsilon$ and $s$ as in Theorem 3.12,*

$$\|\frac{\widehat{\phi}_h(\cdot+\beta)}{\widehat{\phi}_h}\|_{L_\infty(\Omega)} \le c(s,\varepsilon,\Omega)e^{-(|\beta|-\varepsilon)^s/\lambda(h)^s}, \quad \beta\in 2\pi\mathbb{Z}^d\backslash 0. \tag{3.15}$$

*Then $\{\phi_h\}_h$ satisfies the conclusions of Theorem 3.12.*

**Proof of the Lemma:** For $\beta\in 2\pi\mathbb{Z}^d\backslash 0$, let $K(\beta)$ be the open ball of radius $\varepsilon<\pi/2$ centered at $\beta-2\varepsilon\frac{\beta}{|\beta|}$. Then the balls $K(\beta)$, $\beta\in 2\pi\mathbb{Z}^d\backslash 0$, while sharing the same volume $b$, are pairwise disjoint, and their union is disjoint from $B_{2\pi-3\varepsilon}$. Furthermore, with $F(x):=e^{-|x|^s/\lambda}$ for some $\lambda>0$, we have

$$\inf_{x\in K(\beta)} F(x) = e^{-(|\beta|-\varepsilon)^s/\lambda}.$$

Therefore, for $\lambda\le\lambda_0$,

$$\begin{aligned}\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} e^{-(|\beta|-\varepsilon)^s/\lambda} &\le 1/b\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0}\int_{K(\beta)}F(t)\,dt\\ &\le 1/b\int_{\mathbb{R}^d\backslash B_{2\pi-3\varepsilon}}F(t)\,dt\\ &= c(d)\int_{2\pi-3\varepsilon}^{\infty} r^{d-1}e^{-r^s/\lambda}\,dr\\ &= c(\varepsilon,d,s,\lambda_0)e^{-(2\pi-3\varepsilon)^s/\lambda}.\end{aligned} \tag{3.16}$$

Using (3.16), we can now derive conditions on $\{\lambda(h)\}_h$ which ensure that the $\nu$-synthesis condition holds with $\nu_h(x)=h^m$. Since $\widehat{\phi}_h$ vanishes nowhere on $\Omega$ and (3.15) holds, then, in view of (3.16), in order to check that the $\nu$-synthesis condition holds, we need only to verify that, for $a=(2\pi-3\varepsilon)^s$ and some constant $c>0$ and for all small enough $h$,

$$e^{-a/\lambda(h)^s}\le ch^m,$$

which is (3.13), and the first claim of Theorem 3.12 then follows from Corollary 2.38. The second claim follows from the first, since it is clear that (3.13) holds for any positive $m$ (and sufficiently small $h$) in case $\lambda(h)=h^r$ for some $r>0$.

Finally, in the case $\lambda(h)=h$, the $\nu$-synthesis condition holds even for $\nu_h(x)=e^{-a/h^s}$. Substituting this into Corollary 2.38, and recalling that the $o(h^k)$ term there is the first term of (2.30), we obtain the desired result since, for a band limited function, the first term of (2.30) is 0 for small enough $h$. □

To complete the proof of the theorem, we need to show that the assumptions on $\{\phi_h\}_h$ made in the theorem are stronger than those assumed in the lemma. Since any $\phi_h$ in the theorem is $\phi(\lambda\cdot)$ for some $\lambda$, conditions (a) and (b) in the lemma follow respectively from the assumptions in the theorem that $\phi(t\cdot)*'$ is bounded for any $t>0$, and that $\widehat{\phi}$ vanishes nowhere. As to condition (c) in the Lemma, it is implied by (3.9), since now $\widehat{\phi}_h=c_h\widehat{\phi}(\cdot/\lambda(h))$. □

Proposition 3.10 describes a simple condition which implies (3.9). A simple condition which guarantees the boundedness of $\phi(t\cdot)*'$ for every $t > 0$ is that $|\phi(x)| = O(|x|^{-(d+\delta)})$ for some $\delta > 0$, as $x \to \infty$. Thus we conclude the following.

**Corollary 3.17.** *Assume that $|\phi(x)| = O(|x|^{-(d+\delta)})$ at $\infty$ for some $\delta > 0$, and that (3.11) holds for some $s > 0$. Then the requirements of Theorem 3.12, whence its conclusions, hold.*

So far, we have considered the case when $\phi*'$ is bounded. However, the analysis here applies also to cases when $\phi*'$ is *not* bounded, but boundedness can be obtained by localization, i.e., by differencing the original $\phi$. In this process a certain precaution is required: if $\phi$ is the non-localized function and $\psi := \mathrm{T}\phi$ is its localization (with T involving only integer translates), then, in general, it is not desirable to define our sequence $\{\phi_h\}_h$ by scaling $\psi$, i.e., a definition of the form $\phi_h := \psi(\lambda(h)\cdot)$ should be avoided. The reason for this is that in this way we scale also the difference operator, hence obtain a difference operator that involves non-integer translates (see the box spline discussion, in which the deteriorating effect of non-integer translates on approximation orders is detailed; furthermore, the zeros of the Fourier transform of T then prevent us from finding a domain $\Omega$ where none of the $\phi_h$ vanish). What can be done is to generate first a sequence $\phi_h := \phi(\lambda(h)\cdot)$, and then to localize each $\phi_h$ separately with the aid of a difference operator $\mathrm{T}_h$ (each of which employs only integer translates). Here is a typical result in this direction.

**Corollary 3.18.** *Assume that, for a given $\phi$, $\widehat{\phi}$ coincides with some non-vanishing function on $\mathbb{R}^d\backslash 0$, and that $1/\widehat{\phi}$ extends to a continuous function on all of $\mathbb{R}^d$. Assume further that $\phi$ satisfies (3.9) for some $s$, $\varepsilon$ and $\Omega$. Let $\lambda(h)$ be decreasing in $h$, and let $\{\mathrm{T}_h\}_h$ be a set of difference operators (each using only integer translates) such that, with $\phi_h := \mathrm{T}_h\phi(\lambda(h)\cdot)$, the operators $\phi_h*'$ are bounded, and such that the functions $\widehat{\phi_h}$ vanish nowhere on $\Omega$. Then, all the results stated in Theorem 3.12 hold with respect to this $\{\phi_h\}_h$.*

**Proof:** The result follows from Lemma 3.14 as soon as we verify all the conditions specified there. The boundedness of each $\phi_h*'$ as well as the nonvanishing of $\widehat{\phi_h}$ on $\Omega$ are assumed here. It remains therefore to consider the ratios

$$\frac{\widehat{\phi_h}(\cdot + \beta)}{\widehat{\phi_h}}$$

in order to verify (3.15). Let $u_h$ be the Fourier transform of $\mathrm{T}_h$. Then $u_h$ is $2\pi\mathbb{Z}^d$-periodic, and therefore $\frac{\widehat{\phi_h}(\cdot+\beta)}{\widehat{\phi_h}} = \frac{\widehat{\phi}((\cdot+\beta)/\lambda(h))}{\widehat{\phi}(\cdot/\lambda(h))}$. Thus, (3.15) is implied by (3.9), which is assumed here. □

Specific generating functions, for which one is able to obtain *infinite* approximation orders are discussed in [M], [MN2] and [BuD1,2], and our interest in this topic was stimulated by a discussion with N. Dyn. We remark that Madych and Nelson derived their results in the more general context of *scattered* translates.

### 3.4. Box splines

Let $\Xi \in \mathbb{R}^{d\times m}$ be of full rank $d$ and with no 0 column. We will also consider $\Xi$ as the multiset $\{\xi : \ \xi \in \Xi\}$ of its columns and therefore mean by $Y \subset \Xi$ that $Y$ is a matrix obtained from $\Xi$ by omitting some columns. Let $\lambda := \{\lambda_\xi\}_{\xi\in\Xi}$ be an arbitrary set of complex scalars. The (exponential) box spline $M := M_{\Xi,\lambda}$ is defined via its Fourier transform as follows:

$$\widehat{M}(w) := \prod_{\xi\in\Xi} \int_0^1 e^{(\lambda_\xi - i\xi\cdot w)t}\, dt, \tag{3.19}$$

i.e., it is the convolution product of the functionals

$$M_\xi : f \mapsto \int_0^1 e^{\lambda_\xi t} f(t\xi)\, dt, \quad \xi \in \Xi.$$

In general, $M$ is a measure supported in $\Xi[0\,\mathrm{d}1]^m$. Since we assumed that $\operatorname{rank}\Xi = d$, the box spline $M$ is a bounded function, of global smoothness $k(\Xi) - 2$, with

$$k := k(\Xi) := \min\{\#Y : \ Y \subset \Xi, \ \operatorname{rank}(\Xi\backslash Y) < d\}. \tag{3.20}$$

For a generic choice of $\lambda \in \mathbb{C}^m$, $M$ is a piecewise-exponential function (called an exponential box spline, or simple exponential box spline); otherwise, it is a piecewise-exponential-polynomial function. Piecewise-polynomials are obtained by the choice $\lambda = 0$ (polynomial box splines, or box splines). As might be anticipated from their definition, box splines are obtained by differencing a specific fundamental solution of the equation $p(D)f = \delta$, with

$$p(x) := \prod_{\xi\in\Xi} (\lambda_\xi - \xi\cdot x),$$

and thus (cf. the discussion in the section on the differencing case), polynomial box splines are refined by scaling, i.e., $M_h := M$, while, for general box splines, $M_h$ is defined by

$$\widehat{M_h}(w) := \prod_{\xi\in\Xi} \int_0^1 e^{(h\lambda_\xi - i\xi\cdot w)t}\, dt. \tag{3.21}$$

The point of this refinement is that in this way the local structure is preserved, i.e., the pieces of $\{M_h(\cdot/h)\}_h$ (for fixed $\Xi$ and $\lambda$) all belong to the same finite-dimensional exponential-polynomial space. It is important to note that box splines fall into the differencing case (as defined in §3.2) only when $\Xi \in \mathbb{Z}^{d\times m}$, since otherwise the difference operator used in the localization of the fundamental solution employs translations in non-integer directions.

Results on approximation orders for box splines can be found in [BD], [BH], [DM1], ($\Xi \in \mathbb{Z}^{d\times m}$, $\lambda = 0$, see also [BHR]), [R1], [DR], [LJ] (same $\Xi$, general $\lambda$), [RS] (general $\Xi$, $\lambda = 0$). Neither upper bounds nor lower bounds on the approximation order are known for general $\Xi$ and $\lambda$. In what follows, we will derive upper bounds for the approximation order of any box spline (i.e., general $\Xi$ and $\lambda$), and, in case the spline is smooth enough, will provide also matching lower bounds. Since the integral case (i.e., $\Xi \in \mathbb{Z}^{d\times m}$) is the one mostly explored in the literature, and since our results apply to this case almost directly, we found it instructive to begin with this special case. Before doing that, we remark that the operator sequence $\{M_h *'\}_h$ ($\Xi$ and $\lambda$ fixed) is always uniformly bounded. This follows from the fact that $\operatorname{supp} M_h \subset \Xi[0\,\mathrm{d}1]^m$ and that the functions $\{M_h\}_{h\le 1}$ are uniformly bounded. The latter claim can be verified as follows ([DR]): let $N$ be the box spline associated with the same direction set $\Xi$, but with $\lambda = 0$. From the definition of the box spline $M_h$, it follows that, as a linear functional,

$$M_h : f \mapsto \int_{[0\,\mathrm{d}1]^m} e^{h\lambda\cdot t} f(\Xi t)\, dt, \quad f \in C(\mathbb{R}^d),$$

and thus, for $h \le 1$,

$$|M_h(f)| \le c_\lambda N(|f|) \le c_\lambda \|N\|\, \|f\|_{L_1(\Xi[0\,\mathrm{d}1]^m)}.$$

Consequently, $\|M_h\| \le c_\lambda \|N\|$, all $h \le 1$. For later reference, we record this fact below.

**Proposition 3.22.** *For a box spline $M$, $\{M_h *'\}_{h\le 1}$ is uniformly bounded.*

### 3.5. Box splines: integral case

We assume here $\Xi \in \mathbb{Z}^{d\times m}$ (and $\operatorname{rank}(\Xi) = d$).

To start with, we note that (3.21) implies that $\{1/\widehat{M_h}\}_h$ is uniformly bounded in a neighborhood $\Omega$ of the origin. This means that, for the analysis of upper bounds on the approximation order, we may replace the quantities $\{\widehat{M_h}(h\theta + \beta)\}_h$ in the Upper Bound Theorem 2.17 by the ratios

$$\frac{\widehat{M_h}(h\theta+\beta)}{\widehat{M_h}(h\theta)} = \prod_{\xi\in\Xi} \frac{h(\lambda_\xi - i\xi\cdot\theta)}{h(\lambda_\xi - i\xi\cdot\theta) - i\xi\cdot\beta}, \quad \beta \in 2\pi\mathbb{Z}^d\backslash 0. \tag{3.23}$$

Here we have used the fact that $\xi \in \mathbb{Z}^d$ and $\beta \in 2\pi\mathbb{Z}^d$ implies that $\xi\cdot\beta \in 2\pi\mathbb{Z}$, i.e., that $e^{-i\xi\cdot\beta} = 1$. We now fix $\theta \in \mathbb{R}^d$ such that $\lambda_\xi - i\xi\cdot\theta \ne 0$, all $\xi \in \Xi$. By the definition (3.20) of $k(\Xi)$, there exists $Y \subset \Xi$ such that $\operatorname{rank}(Y) < d$, $\#(\Xi\backslash Y) = k(\Xi)$ and $\operatorname{rank}(Y \cup \{\xi\}) = d$, all $\xi \in (\Xi\backslash Y)$. Since $\operatorname{rank}(Y) < d$ and $Y \subset \Xi \in \mathbb{Z}^{d\times m}$, we can find $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ such that $Y^T\beta = 0$. Then, since $\operatorname{rank}(Y\cup\{\xi\}) = d$ for $\xi \in (\Xi\backslash Y)$, we must have $\xi\cdot\beta \ne 0$ such $\xi$. Therefore,

$$h^{-k(\Xi)} \frac{\widehat{M_h}(h\theta+\beta)}{\widehat{M_h}(h\theta)} = \prod_{\xi\in(\Xi\backslash Y)} \frac{\lambda_\xi - i\xi\cdot\theta}{h(\lambda_\xi - i\xi\cdot\theta) - i\xi\cdot\beta} \underset{h\to 0}{\longrightarrow} \prod_{\xi\in(\Xi\backslash Y)} \frac{\lambda_\xi - i\xi\cdot\theta}{-i\xi\cdot\beta} \ne 0.$$

Invoking the Upper Bound Theorem 2.17 (which is applicable due to Proposition 3.22), we thus obtain the following result, which was first established in [LJ] by different means:

**Theorem 3.24.** *Let $\{S_h\}_h$ be the spline spaces spanned by the integer translates of the box splines $\{M_h\}_h$ (resp.) as defined in (3.21). Assume that $\Xi$ is integral. Then the approximation order provided by $\{S_h\}_h$ does not exceed the number $k(\Xi)$ defined in (3.20).*

The fact that the approximation order from the box spline spaces $\{S_h\}_h$ is *at least* $k(\Xi)$ was proved in [DR]. The following theorem reproduces this result, but only for sufficiently smooth box splines.

**Theorem 3.25.** *Let $\{S_h\}$ be box spline spaces as in Theorem 3.24. For every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, define*

$$L_\beta := \{\xi \in \Xi : \ |\xi \cdot \beta| \neq 0\}. \tag{3.26}$$

*Assume that*

$$\sum_{\beta \in 2\pi\mathbb{Z}^d\backslash 0} \prod_{\xi \in L_\beta} \frac{1}{|\xi \cdot \beta|} < \infty. \tag{3.27}$$

*Then the approximation order provided by the spaces $\{S_h\}_h$ is $k(\Xi)$.*

**Proof:** By Theorem 3.24, we only need to show that the approximation order is *at least* $k(\Xi)$. Firstly, note that $\#L_\beta \geq k(\Xi)$ for every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$. Secondly, all factors in (3.23) with $\xi \cdot \beta = 0$ equal 1, hence

$$\frac{\widehat{M}_h(w+\beta)}{\widehat{M}_h(w)} = \prod_{\xi \in L_\beta} \frac{(h\lambda_\xi - i\xi \cdot w)}{(h\lambda_\xi - i\xi \cdot w) - i\xi \cdot \beta}.$$

Since $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ and $L_\beta \subset \mathbb{Z}^d\backslash 0$, it follows that $|\xi \cdot \beta| \geq 2\pi$ for every $\xi \in L_\beta$. Thus, choosing $\eta := \min\{\frac{1}{|\xi|} : \ \xi \in \Xi\}$, we conclude that for small enough $h$ (e.g., $h \leq \min_\xi(\pi - 1)/\lambda_\xi$) and all $w \in B_\eta$,

$$\begin{aligned} \left|\frac{\widehat{M}_h(w+\beta)}{\widehat{M}_h(w)}\right| \leq \prod_{\xi \in L_\beta} \frac{h|\lambda_\xi| + |\xi||w|}{|\xi \cdot \beta|/2} &\leq \text{const}(h + |w|)^{\#L_\beta} \prod_{\xi \in L_\beta} \frac{1}{|\xi \cdot \beta|} \\ &\leq \text{const}(h + |w|)^{k(\Xi)} \prod_{\xi \in L_\beta} \frac{1}{|\xi \cdot \beta|}. \end{aligned}$$

Since we assume (3.27), it follows that $\{M_h\}_h$ satisfies the synthesis condition of order $k(\Xi)$, and the desired result thus follows from the Lower Bound Theorem 2.36. □

**Remark**. It should be noted that the condition (3.27) in the theorem above is active, namely, it does exclude box splines of low smoothness, and in this regard the theorem is weaker than the original result in [DR]. For example, if $d = 1$, then $L_\beta = \Xi$ for every $\beta \in 2\pi\mathbb{Z}\backslash 0$, and therefore condition (3.27) holds if and only if $m := \#\Xi > 1$. For a general $d$, it is easy to see that the condition $k(\Xi) \geq d + 1$ implies (3.27) but not vice versa. It would be nice to know whether the arguments in the theorem can be extended to box splines of low smoothness, particularly since a similar gap appears below in the non-integer extension of Theorem 3.25.

## 3.6. Box splines: non-integral directions

In case $\Xi \in \mathbb{Z}^{d\times m}$, the condition $\xi \cdot \beta \neq 0$ (used in the definition (3.26) of $L_\beta$) is equivalent to $\xi \cdot \beta \in 2\pi\mathbb{Z}\backslash 0$, thus leading to the simple formula (3.23) for the ratio $\widehat{M}_h(h\theta+\beta)/\widehat{M}_h(h\theta)$. For general $\Xi$, we obtain such a simple expression only for the factors corresponding to $\xi$ in

$$K_\beta := \{\xi \in \Xi : \ \xi \cdot \beta \in 2\pi\mathbb{Z}\backslash 0\}.$$

As it turns out, the factors corresponding to $\xi \in L_\beta\backslash K_\beta$ are of no help for the approximation order. In particular, the approximation order is now given by

$$k'(\Xi) := \min\{\#K_\beta : \ \beta \in 2\pi\mathbb{Z}^d\backslash 0\} \tag{3.28}$$

rather than by the possibly larger $k(\Xi) = \min\{\#L_\beta : \ \beta \in 2\pi\mathbb{Z}^d\backslash 0\}$ (cf., (3.20) and (3.26)). The following result has been proved in [RS]:

**Result 3.29.** *The approximation order provided by any* polynomial *box spline $M$ (i.e., $\lambda = 0$) equals the integer $k'(\Xi)$ defined in (3.28).*

Result 3.29 generalizes the original result from [BH] in which polynomial box splines with integral direction set were considered. Both proofs are based on the identification of the polynomials in $S(M)$ (cf. Result 3.2). Furthermore, the extensions of the result of [BH] to arbitrary $\lambda$'s made use of the exponential-polynomial space in $S$ (lower bounds, [DR]), and the local structure of the box spline $M$ (upper bounds, [LJ]). However, for a non-integral $\Xi$ and nonzero $\lambda$, the study of the exponential-polynomials in $S$ or the local structure of $M$ seems to fall short: the approximation properties of the relevant exponential-polynomial space provide lower bounds on the approximation order which, in some cases, *underestimate* the correct order, and the local structure of $M$ provides upper bounds on the approximation order which, usually, overestimate the correct order.

In contrast, the analysis here of the approximation orders for general box splines follows the outline of the analysis, given in the preceding section, for a box spline with integer directions, with some necessary modifications. In this way, the results of the preceding section are shown to hold for a general $\Xi$, with $k'(\Xi)$ replacing $k(\Xi)$ (which is a true generalization since we have $k'(\Xi) = k(\Xi)$ for $\Xi \in \mathbb{Z}^{d\times m}$).

Let $\beta \in 2\pi\mathbb{Z}^d\backslash 0$. A straightforward computation then shows that, for any $\theta \in \mathbb{R}^d$ and $\xi \in \Xi\backslash K_\beta$,

$$\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(\beta+h\theta))t}\,dt \ \underset{h\to 0}{\longrightarrow} \ \int_0^1 e^{-i\xi\cdot\beta t}\,dt \neq 0.$$

On the other hand, if $\xi \in K_\beta$ and $\theta \in \mathbb{R}^d$ satisfies $\lambda_\xi - i\xi\cdot\theta \neq 0$, then we get that, for this $\xi$,

$$h^{-1}\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(\beta+h\theta))t}\,dt \ \underset{h\to 0}{\longrightarrow} \ \frac{\lambda_\xi - i\xi\cdot\theta}{-i\xi\cdot\beta} \neq 0.$$

If we now choose $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ for which $\#K_\beta = k'(\Xi)$, and choose $\theta \in \mathbb{R}^d$ such that $\lambda_\xi - i\xi\cdot\theta \neq 0$ for every $\xi \in K_\beta$, we conclude from the above arguments that

$$|\widehat{M}_h(h\theta+\beta)| \neq o(h^{k'(\Xi)}).$$

Furthermore, Proposition 3.22 ensures that $\{M_h *'\}_h$ is uniformly bounded, and therefore, via the Upper Bound Theorem 2.17, we finally arrive at the following generalization of Theorem 3.24.

**Theorem 3.30.** *For each $h > 0$, let $S_h = S(M_h)$ be the spline space spanned by the integer translates of the box spline $M_h$ defined in (3.21). Then the approximation order provided by $\{S_h\}_h$ does not exceed $k'(\Xi)$ (as given in (3.28)).*

In an analogous way, Theorem 3.25 can be generalized to non-integral $\Xi$ as follows:

**Theorem 3.31.** *Let $\{S_h\}$ be box spline spaces as in Theorem 3.30. For every $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, define*

$$L_\beta := \{\xi \in \Xi : \ |\xi \cdot \beta| \neq 0\}. \tag{3.32}$$

*Assume further that $\Xi \in \mathbb{Q}^{d\times m}$ and that*

$$\sum_{\beta \in 2\pi\mathbb{Z}^d\backslash 0} \prod_{\xi \in L_\beta} \frac{1}{|\xi \cdot \beta|} < \infty. \tag{3.33}$$

*Then the approximation order provided by the spaces $\{S_h\}_h$ is $k'(\Xi)$.*

**Proof:** Since we know, by Theorem 3.30, that the approximation order is at most $k'(\Xi)$, we need only to prove that it is at least $k'(\Xi)$. Since $\{|\widehat{M}_h|^{-1}\}$ is uniformly bounded in some neighborhood of the origin (for small enough $h$), it suffices, for an application of the Lower Bound Theorem 2.36, to consider the quantities $\widehat{M}_h(\cdot + \beta)$, $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ and $h$ small (rather than the ratio $\widehat{M}_h(\cdot + \beta)/\widehat{M}_h$). We now fix $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ and consider three different cases of $\xi \in \Xi$:

**Case $\xi \cdot \beta = 0$ (i.e., $\xi \in \Xi\backslash L_\beta$):** In this case, for sufficiently small $h$ and for every $w \in \mathbb{R}^d$, we have

$$|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot w)t}\, dt| < 2.$$

**Case $\xi \cdot \beta \notin 2\pi\mathbb{Z}$ (i.e, $\xi \in L_\beta\backslash K_\beta$):** Here we use the estimate

$$|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(w+\beta))t}\, dt| < \frac{3}{|h\lambda_\xi - i\xi\cdot(w+\beta)|}, \tag{3.34}$$

valid for all $w$ and sufficiently small $h$. Now, since $\Xi \in \mathbb{Q}^{d\times m}$, there exists $n \in \mathbb{Z}$ such that $n\Xi \in \mathbb{Z}^{d\times m}$, and thus, since $\xi \cdot \beta \neq 0$ (and $\beta \in 2\pi\mathbb{Z}^d\backslash 0$), we have $|\xi \cdot \beta| \geq 2\pi/n$. Thus, (3.34) shows that for sufficiently small $h$ and $w$,

$$|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(w+\beta))t}\, dt| < \frac{4}{|\xi \cdot \beta|}.$$

**Case $\xi \cdot \beta \in 2\pi\mathbb{Z}\backslash 0$ (i.e, $\xi \in K_\beta$):** In this final case,

$$|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot(w+\beta)t}\, dt| = \frac{|e^{h\lambda_\xi - i\xi\cdot w} - 1|}{|h\lambda_\xi - i\xi\cdot(w+\beta)|}. \tag{3.35}$$

The denominator in the right hand side of (3.35) can be estimated as in the previous case, while the numerator, for sufficiently small $h$ and $|w|$, can be bounded by $c\,(h + |w|)$, hence we finally obtain in this case the estimate

$$|\int_0^1 e^{(h\lambda_\xi - i\xi\cdot w)t}\, dt| \leq c\,\frac{h + |w|}{|\xi \cdot \beta|}.$$

Combining all these estimates, we obtain that, for some $\beta$-independent neighborhood $\Omega$ of the origin and for some $\beta$-independent constant $c$ we have, for small enough $h$ (where the "smallness" of $h$ is again $\beta$-independent) and all $w \in \Omega$,

$$|\widehat{M}_h(w+\beta)| \le c\,(h+|w|)^{\# K_\beta} \prod_{\xi \in L_\beta} \frac{1}{|\xi \cdot \beta|} \le c\,(h+|w|)^{k'(\Xi)} \prod_{\xi \in L_\beta} \frac{1}{|\xi \cdot \beta|}.$$

Application of the Lower Bound Theorem 2.36, in view of the assumption (3.33), then completes the proof. □

## 4. Examples

**Example 4.1**. Assume that $\phi$ satisfies the following conditions: (a) $\widehat{\phi} \in C(\mathbb{R}^d \backslash 0)$; (b) for some $\delta > 0$, $|\widehat{\phi}(x)| = O(|x|^{-d-\delta})$ as $x \to \infty$; (c) for some $\mu \ge 0$, $|\widehat{\phi}(x)| \sim |x|^{-\mu}$ as $x \to 0$. Under various additional conditions on $\phi$, ($\phi$ is radially symmetric, $\widehat{\phi}$ vanishes nowhere, and others; cf. [Bu3;pp. 72-74]), it is shown in [Bu3] that there exists $c : \mathbb{Z}^d \to \mathbb{C}$ such that $\phi *' c$ converges absolutely and uniformly on compact sets to a function $\chi$ which decays like $|x|^{-d-\mu}$ at $\infty$, and which is a *cardinal function*, i.e., satisfies $\chi(\alpha) = \delta_{\alpha,0}$, $\alpha \in \mathbb{Z}^d$. [Bu3] then proceeds to show that the error in the approximation scheme

$$f(h\cdot) \approx \chi *' f(h\cdot)$$

behaves like $O(h^{-\mu})$ or $O(h^{-\mu}|\log h|)$, depending on the decay rate of $\chi$ at $\infty$ and the type of $\mu$ (non-integer, integer, even integer), and that in some cases these rates are sharp (for the above approximation scheme); cf. [Bu3;Cor. 5-12]. Our result in this regard is the following:

**Theorem 4.2.** *Assume that $\chi$ is a function for which $\chi *'$ is bounded, and $\widehat{\chi}(0) \ne 0$. Assume further that $\widehat{\chi}$ can be factored,*

$$\widehat{\chi} = u\widehat{\phi},$$

*with $u$ a $2\pi\mathbb{Z}^d$-periodic distribution which coincides with some bounded continuous function on a neighborhood $\Omega$ of the origin, and $\widehat{\phi}$ a distribution which coincides with a continuous function on $(2\pi\mathbb{Z}^d + \Omega)\backslash 0$. Assume, finally, that $\widehat{\phi}$ satisfies the following conditions:*

$$\sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} \|\widehat{\phi}(\cdot + \beta)\|_{L_\infty(\Omega)} < \infty; \tag{4.3}$$

*and*

$$c|\widehat{\phi}(x)| \ge |x|^{-\mu}, \quad x \in \Omega.$$

*Then the approximation order provided by $\chi$ is at least $\mu$. Moreover, if for some $\theta \in \mathbb{R}^d \backslash 0$, $h^\mu \widehat{\phi}(h\theta)$ is bounded for small enough $h$, and $\widehat{\phi}(\beta) \ne 0$ for some $\beta \in 2\pi\mathbb{Z}^d \backslash 0$, then the approximation order provided is exactly $\mu$.*

**Proof:** The proof is very similar to that of Theorem 3.6. By assumption, $\widehat{\phi}$ is a well-defined function on $(2\pi\mathbb{Z}^d + \Omega)\backslash 0$, and

$$\|\frac{\widehat{\chi}(\cdot + \beta)}{|\cdot|^\mu\, \widehat{\chi}}\|_{L_\infty(\Omega)} = \|\frac{\widehat{\phi}(\cdot + \beta)}{|\cdot|^\mu\, \widehat{\phi}}\|_{L_\infty(\Omega)} \le c\|\widehat{\phi}(\cdot + \beta)\|_{L_\infty(\Omega)}.$$

In view of assumption (4.3), we therefore conclude that the synthesis condition of order $\mu$ holds here, and consequently, by the Lower Bound Theorem 2.36, the approximation order is at least $\mu$.

For the sake of upper bounds, we adopt the additional assumptions in the theorem, and note that then, for the same $\beta$ as in the theorem, we have $|\frac{h^{-\mu}\widehat{\phi}(h\theta+\beta)}{\widehat{\phi}(h\theta)}| \ge c > 0$, for small enough $h$. On the other hand, $|\frac{h^{-\mu}\widehat{\phi}(h\theta+\beta)}{\widehat{\phi}(h\theta)}| = |\frac{h^{-\mu}\widehat{\chi}(h\theta+\beta)}{\widehat{\chi}(h\theta)}|$, and since $\widehat{\chi}$ is continuous and does not vanish at 0, we conclude that $|h^{-\mu}\widehat{\chi}(h\theta+\beta)| \ge c_1 > 0$. Application of the Upper Bound Theorem 2.17 implies that the approximation order is at most $\mu$. □

As in the differencing case, the above result shows that approximation orders provided by $S(\chi)$ are essentially independent of the localization process (i.e., the type of periodic function $u$ which connects $\widehat{\phi}$ and $\widehat{\chi}$). The only requirements on $\chi$ are that $\widehat{\chi}(0) \neq 0$ and that $\chi *'$ be bounded. For this reason, we will refer in what follows to such approximation orders as "the approximation orders provided by $\phi$". Here are some specific examples which are covered by the last theorem. These examples were treated by several authors under various restrictions on the underlying parameter $\gamma$ (see below) and the parity $d$ of the spatial dimension. A partial list of references includes [J], [M], [MN1,2], [DJLR], [Bu1-4] and [P]. These references provide localization processes as well as lower bounds on the approximation order, the latter being of the form $d+k$ or $O(h^{d+k}|\log h|)$ for some $k \le \gamma$ whose value depends on the quality of the localization process. Here are the details.

(1): $\phi = |\cdot|^\gamma$, $\gamma \in \mathbb{R}_+ \backslash 2\mathbb{Z}_+$. Since $\widehat{\phi} = c|\cdot|^{-d-\gamma}$ in the complement of the origin, $\phi$ satisfies the assumptions of Theorem 4.2 with $\mu := d+\gamma$, hence, indeed, this theorem shows that, whatever localization process is employed, the approximation order is exactly $d+\gamma$.

(2): $\phi = |\cdot|^\gamma \log|\cdot|$, $\gamma \in 2\mathbb{N}$. In this case, $\widehat{\phi}$ coincides on $\mathbb{R}^d \backslash 0$ with the function $c|\cdot|^{-(d+\gamma)}$ (and so, this case complements the previous one). Since $\phi$ satisfies the assumptions of Theorem 4.2, with $\mu := d+\gamma$, this theorem shows that $d+\gamma$ is the exact approximation order, regardless of the localization process and the parity of $d$.

(3): $\phi := (|\cdot|^2+\lambda^2)^{\gamma/2}$, $\gamma > -d$, $\gamma \notin 2\mathbb{Z}_+$. Here, $\widehat{\phi} = c|\cdot|^{-(d+\gamma)}K$, with $c \neq 0$, $K$ continuous, vanishing nowhere, and decaying exponentially to 0 at $\infty$. Thus, $\phi$ satisfies the assumptions of Theorem 4.2 with $\mu := d+\gamma$. Further consideration of this $\phi$, in which $\lambda$ depends on $h$ to obtain spectral orders, will be given later.

(4): $\phi = (|\cdot|^2+\lambda^2)^{\gamma/2}\log(|\cdot|^2+\lambda^2)$, $\gamma \in 2\mathbb{Z}_+$, $d$ even. In this case, $\widehat{\phi}$ admits a similar expression to that of the previous case, and again we obtain approximation order $d+\gamma$.

**Example 4.4**. We discuss here several examples in which spectral approximation orders are obtained.

(1) The Gaussian kernel: $\phi = e^{-|\cdot|^2/4}$. It was proved in [Bu3] that cardinal interpolation using this function does not allow reproduction of any polynomials. Theorem 3.1 here shows that further, since $\widehat{\phi} = ce^{-|\cdot|^2}$ vanishes nowhere, the approximation order provided by $\phi$ is 0. However, employing Corollary 3.17, we see that Theorem 3.12 applies here (with $s = 2$). In particular, we obtain the following result, which corresponds to the choice $\lambda(h) = \sqrt{h}$ in Theorem 3.12.

**Corollary 4.5.** *For the choice* $\phi_h := e^{-h|\cdot|^2}$*, we have* $\mathrm{dist}_\infty(f(h\cdot), S_h) \le c_k h^k \|f\|_0' + o(h^k)$ *for every* $k$*-admissible* $f$ *and for every* $k$.

Note that, in terms of the original generating functions $\{\psi_h\}_h$ (i.e., prior to the scale-up), the above corollary suggests the choice $\psi_h = e^{-|\cdot|^2/h}$.

An identical analysis can be made with respect to other smooth functions. E.g., we can take $\phi = (|\cdot|^2 + 1)^{-(d+1)/2}$. In this case, $\widehat{\phi} = ce^{-|\cdot|}$, and Corollary 3.17, hence Theorem 3.12, apply, with $s = 1$, so that we get, for example, the following result.

**Corollary 4.6.** *The conclusion of Corollary 4.5 holds also for* $\phi_h := (|\cdot|^2 + h^r)^{-(d+1)/2}$*, for any negative* $r$.

**Proof:** We only need to observe, in the application of Theorem 3.12, that, for this $\phi$, $\phi(\lambda\cdot) = c(\lambda)(|\cdot|^2 + \lambda^{-2})^{-(d+1)/2}$. □

**Example 4.7**. We continue with spectral orders: In the two examples above, the operator $\phi *'$ was bounded. However, as Corollary 3.18 suggests, our analysis also applies to cases when $\phi *'$ is not bounded. For instance, we can take either $\psi_\lambda := (|\cdot|^2 + \lambda^{-2})^{\gamma/2}$, $\gamma > -d$, $\gamma \notin 2\mathbb{Z}_+$, or $\psi_\lambda = (|\cdot|^2 + \lambda^{-2})^{\gamma/2} \log(|\cdot|^2 + \lambda^{-2})$, $\gamma \in 2\mathbb{Z}_+$, $d$ even; (note that these functions have been considered before, but in a different context). In both cases $\widehat{\psi}_\lambda = c_\lambda K(\cdot/\lambda)|\cdot|^{-d-\gamma}$ on $\mathbb{R}^d\backslash 0$, where $K \in C^\infty(\mathbb{R}^d\backslash 0)$, $K(x) \sim e^{-|x|}(1+|x|^{(d+\gamma-1)/2})$ on all of $\mathbb{R}^d$, all derivatives of $K$ of orders $< 2d+\gamma$ are in $L_1(\mathbb{R}^d)$, and derivatives of $K$ of any order (regarded as functions on $\mathbb{R}^d\backslash 0$) are rapidly decaying at $\infty$ (all these properties can be derived from the known properties of the modified Bessel functions [AS], since $K = |\cdot|^{(d+\gamma)/2} K_{(d+\gamma)/2}$, with $K_\nu$ being the modified Bessel function of third kind and order $\nu$). Now, assume that $\phi_h$ is a localization of $\psi_{\lambda(h)}$, namely, $\widehat{\phi}_h = u_h \widehat{\psi}_{\lambda(h)}$ for some $2\pi\mathbb{Z}^d$-periodic $u_h$, and $\phi_h *'$ is bounded. Since the only singularity of $\widehat{\psi}_{\lambda(h)}$ is at the origin, we can assume that $u_h$ does not vanish on some punctured $h$-independent neighborhood $\Omega\backslash 0$ of the origin (this, in turn, forces the use of an infinite-difference operator, except in some special cirumstances). This ensures that $\widehat{\phi}_h$ does not vanish on $\Omega\backslash 0$, and we further assume that $u_h$ is chosen such that $\widehat{\phi}_h(0) \neq 0$. If, at this point, we prefer to fix $\lambda(h)$, i.e., if we do not change $\phi_h$ with $h$, then we obtain a special example of the stationary case, and in such a case the factor in $\widehat{\phi}_h$ which determines the approximation order is $|\cdot|^{-(d+\gamma)}$, as can be observed from Theorem 4.2. However, if we change $\lambda(h)$ with $h$, as we do in the context of spectral orders, the dominant factor in $\widehat{\phi}_h$ becomes $K(\cdot/\lambda(h))$. In such a case, Corollary 3.18 implies the following:

**Corollary 4.8.** *Let* $\{\psi_{\lambda(h)}\}_h$ *be as above, and let* $\phi_h$*,* $h > 0$*, be a localization of* $\psi_{\lambda(h)}$*, namely,* $\widehat{\phi}_h = u_h \widehat{\psi}_{\lambda(h)}$ *for some* $2\pi\mathbb{Z}^d$*-periodic function* $u_h$*, and* $\phi_h *'$ *is bounded. Assume further that* $u_h$ *vanishes nowhere on some* $h$*-independent punctured neighborhood* $\Omega\backslash 0$ *of the origin and that* $\widehat{\phi}_h(0) \neq 0$*. Then all results stated in Theorem 3.12 hold with respect to this* $\{\phi_h\}_h$ *and with* $s = 1$.

**Proof:** We wish to apply Corollary 3.18, hence need to verify that all the conditions required there hold in our case. By assumption, the operators $\{\phi_h *'\}_h$ are bounded. Also, the various assumptions on the zeros of $u_h$ and $\widehat{\phi}_h$, together with the fact that $\widehat{\phi}$ vanishes nowhere (on $\mathbb{R}^d\backslash 0$), imply that $\widehat{\phi}_h$ vanishes nowhere on $\Omega$. Thus, in order to apply Corollary 3.18, it remains to verify that $\phi := (|\cdot|^2+1)^{\gamma/2}$ (or $\phi := (|\cdot|^2+1)^{\gamma/2}\log(|\cdot|^2+1)$) satisfies (3.9). For this, we note that the fact that $K \sim e^{-|\cdot|}(1+|\cdot|^{(d+\gamma-1)/2})$ implies that

$$\frac{\widehat{\phi}((x+\beta)/\lambda)}{\widehat{\phi}(x/\lambda)} \le \text{const}\frac{e^{-|(x+\beta)/\lambda|}}{e^{-|x/\lambda|}}$$

for sufficiently small $x$ and $\lambda$ and for all $\beta \in 2\pi\mathbb{Z}^d\backslash 0$. This implies, as in Proposition 3.10, that $\widehat{\phi}$ satisfies (3.9) with $s=1$. $\square$

Some improvements of the above result are available. First, we have neglected the "positive" role of the factor $|\cdot|^{-(d+\gamma)}$. In fact, let $\phi_h := \psi_{\lambda(h)}$ as before. Then, for $x \in B_\varepsilon$ and $\beta \in 2\pi\mathbb{Z}^d\backslash 0$ (and, say, with $\lambda(h) \le 1$), the estimate

$$\frac{1+(\frac{|x+\beta|}{\lambda(h)})^{(d+\gamma-1)/2}}{|x+\beta|^{d+\gamma}\,(1+|x/\lambda(h)|)^{(d+\gamma-1)/2})} \le \text{const}\,\lambda(h)^{-(d+\gamma-1)/2}\,|\beta|^{-(d+\gamma+1)/2}$$

holds (for the case $d+\gamma \ge 1$; otherwise, the factor $\lambda(h)^{-(d+\gamma-1)/2}$ can be removed from the above bound and the subsequent analysis becomes simpler). Thus, for the case $d+\gamma \ge 1$, we get that

$$\begin{aligned}\frac{\widehat{\phi}_h(x+\beta)}{\widehat{\phi}_h(x)} &= \frac{K(\frac{x+\beta}{\lambda(h)})\,|x|^{d+\gamma}}{K(x/\lambda(h))\,|x+\beta|^{d+\gamma}} \\ &\le \text{const}\,|x|^{d+\gamma}\frac{e^{-|(x+\beta)/\lambda(h)|}\,(1+(\frac{|x+\beta|}{\lambda(h)})^{(d+\gamma-1)/2})}{e^{-|x/\lambda(h)|}\,|x+\beta|^{d+\gamma}\,(1+(|x/\lambda(h)|)^{(d+\gamma-1)/2})} \\ &\le \text{const}\,|x|^{d+\gamma}\,e^{-(|\beta|-2\varepsilon)/\lambda(h)}/\lambda(h)^{(d+\gamma-1)/2}.\end{aligned}$$

We can estimate the sum $\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} e^{-(|\beta|-2\varepsilon)/\lambda(h)}$ as in the proof of Lemma 3.14 to obtain that, for $a := 2\pi - 6\varepsilon$,

$$\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} \|\frac{\widehat{\phi}_h(\cdot+\beta)}{|\cdot|^{d+\gamma}\,\widehat{\phi}}\|_{L_\infty(B_\varepsilon)} \le \text{const}\; e^{-a/\lambda(h)}/\lambda(h)^{(d+\gamma-1)/2}.$$

By changing $a$ if needed, and assuming that $\lambda(h)$ is small enough, we can replace $e^{-a/\lambda(h)}/\lambda(h)^{(d+\gamma-1)/2}$ by $e^{-a/\lambda(h)}$, which means the $\nu$-synthesis condition holds here with

$$\nu_h(x) = e^{-a/\lambda(h)}|x|^{d+\gamma}.$$

Substituting these $\{\nu_h\}_h$ into (2.33) and combining (2.33) with (2.30), we get the following:

**Theorem 4.9.** *Let $\{\phi_h\}_h$ be as in Corollary 4.8. If the sequence $\{\lambda(h)\}_h$ satisfies $e^{-a/\lambda(h)} \le ch^m$ for some non-negative $m$ and some $a < 2\pi$, then for every $k$-admissible function $f$, and every $j \le \min\{d+\gamma, k\}$,*

$$\text{dist}_\infty(f(h\cdot), S(\phi_h)) \le \text{const}\; h^{m+j}\|f\|'_j + o(h^k). \tag{4.10}$$

*In particular, if $k \geq d+\gamma$, then*

$$\mathrm{dist}_\infty(f(h\cdot), S(\phi_h)) \leq \text{const } h^{m+d+\gamma}\|f\|'_{d+\gamma} + o(h^k).$$

*Here the constant is independent of $f$ and $h$.*

The last theorem implicitly suggests the optimal choice of $\{\lambda(h)\}_h$: if $f$ is $k$-admissible, then one should choose $m$ to be any number larger than $k-\min\{d+\gamma, k\}$, for that would make the first term in (4.10) go to 0 faster than the second one.

A second remark here concerns the type of difference operators that can be used in the localization process leading to the sequence $\{\phi_h\}_h$. As previously explained, we cannot scale $\mathrm{T}_1$ to obtain $\mathrm{T}_h$, namely (assuming without loss that $\lambda(1)=1$), we cannot take $u_h := u_1(\cdot/\lambda(h))$. On the other hand, in the present context, the singularity of $\widehat{\phi}$ at the origin is *homogeneous*, i.e., the product of $\widehat{\phi}$ by a homogeneous function $G$ (viz. $|\cdot|^{d+\gamma}$) is a continuous function which does not vanish at 0 (nor anywhere else, for that matter). This suggests that the same difference operator that is used to localize $\phi$, can also be used to localize any scale of $\phi$. Here is a sample statement in this direction:

**Proposition 4.11.** *Let $\phi$ be as in Corollary 4.8, and assume that $d+\gamma \geq 1$. Let $u$ be a smooth $2\pi\mathbb{Z}^d$-periodic function such that $|\cdot|^{-(d+\gamma)}u$ has bounded derivatives in a neighborhood of the origin up to order $d+1$ inclusive. Assume that $\psi$ is a function which satisfies $\widehat{\psi} = u\widehat{\phi}(\cdot/\lambda)$ for some positive $\lambda$. Then $\psi *'$ is bounded.*

**Proof:** Recall that $\widehat{\phi}(\cdot/\lambda) = cK(\cdot/\lambda)|\cdot|^{-(d+\gamma)}$, where all derivatives of $K$, hence of $K(\cdot/\lambda)$, up to any order $< 2d+\gamma$, are in $L_1$. Combining this with the present assumption on $u$ (and using the fact that $2d+\gamma > d+1$ here), we conclude that the derivatives of $\widehat{\psi} = cuK(\cdot/\lambda)|\cdot|^{-(d+\gamma)}$ of orders up to $d+1$ are integrable around the origin. Further, away from the origin, all derivatives of $K$, hence of $K(\cdot/\lambda)$, are rapidly decaying, while the derivatives of $u$ are bounded (by virtue of the smoothness and periodicity of this function) and the derivatives of $|\cdot|^{-(d+\gamma)}$ are of (at most) polynomial growth. This proves that all derivatives of $\widehat{\psi}$ up to order $d+1$ are in $L_1(\mathbb{R}^d)$. Consequently, $\psi = o(|\cdot|^{-(d+1)})$ at $\infty$, which implies that $\psi *'$ is bounded, as claimed. $\square$

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