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BICUBIC SPLINE INTERPOLATION

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1. Introduction. Let values $u_{ij} = u(x_i, y_j)$ be given at the mesh-points (x_i, y_j) of a rectangular mesh, $(i = 0, \cdots, I; j = 0, \cdots, J)$; let the normal derivatives be given at the boundary points of the mesh, i.e., $p_{ij} = u_x(x_i, y_j)$ for $i = 0, I$ and $j = 0, \cdots, J$, and $q_{ij} = u_y(x_i, y_j)$ for $i = 0, \cdots, I$ and $j = 0, J$; finally, let $s_{ij} = u_{xy}(x_i, y_j)$ be given at the four corners of the mesh. The problem is to fit a "smooth" function $u(x, y) \in C^2$ through these given values.

The bicubic spline interpolation method to be described yields a piecewise bicubic polynomial function $u(x, y)$. This is defined in each rectangular cell $R_{ij} : x_{i-1} \leqq x \leqq x_i ; y_{j-1} \leqq y \leqq y_j$ of the grid as a bicubic polynomial, i.e.,

$$u(x, y) = c_{ij}(x, y) = \sum_{m,n=0}^{3} \alpha_{mn}^{ij}(x - x_{i-1})^m(y - y_{j-1})^n, \ (x, y) \in R_{ij} .$$

It is shown in §3 that there exists exactly one such piecewise bicubic polynomial function which assumes the given values and is of class C^2. In §§4–5, an efficient procedure for computing the coefficients α_{mn}^{ij} is described.

2. Linearized Spline Interpolation. Bicubic spline interpolation is a two-dimensional analog of "linearized spline interpolation" (cf. [1, p. 258]) for functions of one variable. Some apparently unpublished properties of this interpolation procedure are needed to establish this analogy. A short résumé of linearized spline interpolation is, therefore, given here.

For functions of one variable, linearized spline interpolation defines a function $u(x)$ of class C^2 which assumes given values $u_i = u(x_i)$ at given points x_i, $i = 0, \cdots, I$, $x_0 < x_1 < \cdots < x_I$, and given slopes $p_i = u'(x_i)$ at the two endpoints x_0 and x_I. The interpolating function is a cubic polynomial in each of the intervals $[x_{i-1}, x_i]$, $i = 1, \cdots, I$. The points x_i, $i = 0, \cdots, I$, are called the joints of $u(x)$. Let $S(z; z_1, z_2, \cdots, z_n)$, $z_1 < z_2 < \cdots < z_n$, denote the linear space of all functions $u(z)$ of class C^2 on the interval $[z_1, z_n]$, which are equal to a cubic polynomial in each of the intervals $[z_{i-1}, z_i]$, $i = 2, \cdots, n$, i.e., piecewise cubic.

Theorem 1. For each set $\{u_0, u_1, \cdots, u_I, p_0, p_I\}$ of values there exists exactly one $u(x) \in S(x; x_0, \cdots, x_I)$ such that

$$(1) \qquad u(x_i) = u_i, \qquad i = 0, \cdots, I, \qquad u'(x_0) = p_0, \qquad u'(x_I) = p_I .$$

Proof. We first recall a well-known result.

Lemma 1. There is exactly one cubic polynomial $c(x) = \sum_{m=0}^{3} \alpha_m(x - a)^m$ which assumes given values for $c(x)$ and $c'(x)$ at the endpoints of any interval $[a, b]$, $a \neq b$. This polynomial is

$$(2) \qquad \begin{aligned} c(x) = c(a) + c'(a)(x - a) &+ \left[3\frac{c(b) - c(a)}{(b - a)^2} - \frac{c'(b) + 2c'(a)}{b - a}\right](x - a)^2 \\ &+ \left[-2\frac{c(b) - c(a)}{(b - a)^3} + \frac{c'(b) + c'(a)}{(b - a)^2}\right](x - a)^3. \end{aligned}$$

The first statement follows from the fact that the determinant of the matrix connecting $c(a)$, $c'(a)$, $c(b)$, $c'(b)$ and the four coefficients α_m is $(b - a)^{-4} \neq 0$ for $b \neq a$. Equation 2 then follows by inspection.

Corollary. If u_i and p_i are given for $i = 0, \cdots, I$, then there exists exactly one piecewise cubic polynomial $u(x) \in C^1$ with joints $x_0, \cdots, x_I$, which satisfies $u(x_i) = u_i$ and $u'(x_i) = p_i$, $i = 0, \cdots, I$.

Lemma 2. Let x_0, x_1, x_2 be such that $\Delta x_0 = x_1 - x_0 \neq 0$ and $\Delta x_1 = x_2 - x_1 \neq 0$, but not necessarily $x_0 \neq x_2$. Let $v(x)$ and $w(x)$ be cubic polynomials satisfying $v(x_1) = w(x_1) = u_1$ and $v'(x_1) = w'(x_1) = p_1$. Then $v''(x_1) = w''(x_1)$ if and only if

$$(3) \qquad \begin{aligned} &\Delta x_1 v'(x_0) + 2(\Delta x_1 + \Delta x_0)p_1 + \Delta x_0 w'(x_2) \\ &\qquad = 3\left[\frac{\Delta x_0}{\Delta x_1}(w(x_2) - u_1) + \frac{\Delta x_1}{\Delta x_0}(u_1 - v(x_0))\right]. \end{aligned}$$

Proof. Set $a = x_1$, $b = x_0$, $c(x) \equiv v(x)$ in (2); then

$$v''(x_1) = \frac{2}{-\Delta x_0}\left[3\frac{v(x_0) - u_1}{-\Delta x_0} - v'(x_0) - 2p_1\right].$$

Similarly, set $a = x_1$, $b = x_2$, $c(x) \equiv w(x)$ in (2); then

$$w''(x_1) = \frac{2}{\Delta x_1}\left[3\frac{w(x_2) - u_1}{\Delta x_1} - w'(x_2) - 2p_1\right].$$

Thus $w''(x_1) = v''(x_1)$ if and only if (3) holds.

Corollary. Let $u(x)$ be a piecewise cubic polynomial of class C^1 with joints $x_0, \cdots, x_I$. For given $u_i = u(x_i)$, $i = 0, \cdots, I$, and $p_0 = u'(x_0)$, $p_I = u'(x_I)$, there exists exactly one set of values $p_i = u'(x_i)$, $i = 1, \cdots, I - 1$, such that $u(x) \in C^2$.

Proof. By Lemma 2, the continuity of $u''(x)$ for $u(x) \in C^1$ is equivalent to a set of $I - 1$ linear equations

$$(4) \qquad \begin{aligned} &\Delta x_i p_{i-1} + 2(\Delta x_i + \Delta x_{i-1})p_i + \Delta x_{i-1} p_{i+1} \\ &\qquad = 3\left[\Delta x_{i-1}\frac{\Delta u_i}{\Delta x_i} + \Delta x_i \frac{\Delta u_{i-1}}{\Delta x_{i-1}}\right], \qquad i = 1, \cdots, I - 1, \end{aligned}$$

for the $I - 1$ unknowns, p_i, $i = 1, \cdots, I - 1$. The tridiagonal matrix of this linear system is strictly diagonally dominant, hence* has only non-zero eigenvalues and is thus non-singular. The $I - 1$ equations (4) are, therefore, linearly independent and hence determine the p_i, $i = 1, \cdots, I - 1$, uniquely.

The Corollaries to Lemmas 1 and 2 imply Theorem 1, which concludes the proof.

Lemmas 1 and 2 may be used to devise an efficient computational scheme for the evaluation of the interpolating function $u(x)$ for given u_i, p_0, p_I. In this scheme, one computes values $p_i = u'(x_i)$, $i = 1, \cdots, I - 1$, from equation (4). Since $u(x)$ equals a cubic polynomial $c_i(x)$ in each interval $[x_{i-1}, x_i]$, one

* This follows from Gershgorin's Circle Theorem, cf. [2, Thm. 3.3.(a), p. 11].

then uses equation (2) to compute $u(x) = c_i(x)$ from $c_i(x_k) = u_k$ and $c_i'(x_k) = p_k$, $k = i - 1, i$, for $x \in [x_{i-1}, x_i]$.

Theorem 1 has as a consequence

COROLLARY 1. $S(x; x_0, \cdots, x_I)$ is an $(I + 3)$-dimensional linear space.

PROOF. Equation (1) assigns to each $u(x) \in S(x; x_0, \cdots, x_I)$ a unique vector $\{u_0, \cdots, u_I, p_0, p_I\}$. Theorem 1 shows that equation 1 assigns, conversely, a unique $u(x) \in S(x; x_0, \cdots, x_I)$ to each vector $\{u_0, \cdots, u_I, p_0, p_I\}$.

COROLLARY 2. The set of $\phi_i(x) \in S(x; x_0, \cdots, x_I)$, $i = 0, \cdots, I + 2$, defined by the conditions

$$\phi_j(x_i) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}, \qquad \phi_j'(x_0) = \phi_j'(x_I) = 0, \quad \text{for} \quad i, j = 0, \cdots, I,$$

$$\phi_{I+1}(x_i) = \phi_{I+2}(x_i) = 0, \quad \text{for} \quad i = 0, \cdots, I, \tag{5}$$

$$\phi_{I+1}'(x_0) = \phi_{I+2}'(x_I) = 1, \qquad \phi_{I+1}'(x_I) = \phi_{I+2}'(x_0) = 0,$$

is a basis of the linear space $S(x; x_0, \cdots, x_I)$.

3. Bicubic Spline Interpolation. We are now ready to treat bicubic spline interpolation.

For the $(J + 3)$-dimensional linear space $S(y; y_0, \cdots, y_J)$, let $\{\psi_j(y)\}$, $j = 0, \cdots, J + 2$, denote the basis defined in Corollary 2 of Theorem 1. Consider the tensor product $T = S(x; x_0, \cdots, x_I) \otimes S(y; y_0, \cdots, y_J)$. T is the $(I + 3)(J + 3)$-dimensional linear space of all functions of the form

$$u(x, y) = \sum_{m=0}^{I+2} \sum_{n=0}^{J+2} \beta_{mn}\phi_m(x)\psi_n(y). \tag{6}$$

The ϕ_m and ψ_n are piecewise cubic and of class C^2 on $R: x_0 \leqq x \leqq x_I ; y_0 \leqq y \leqq y_J$. Therefore, any product or linear combination of the ϕ_m and ψ_n is piecewise bicubic and of class C^2, i.e., $u(x, y) \in C^2$ on R for any choice of the coefficients β_{mn}.* Conversely, every function, which is a bicubic polynomial in each of the rectangles $R_{ij} : x_{i-1} \leqq x \leqq x_i ; y_{j-1} \leqq y \leqq y_j$, and is of class C^2 on R, is in T.

THEOREM 2. Let there be given values

$$\begin{aligned} u_{ij} &= u(x_i, y_j), && i = 0, \cdots, I; j = 0, \cdots, J, \\ p_{ij} &= u_x(x_i, y_j), && i = 0, I; j = 0, \cdots, J, \\ q_{ij} &= u_y(x_i, y_j), && i = 0, \cdots, I; j = 0, J, \quad \text{and} \\ s_{ij} &= u_{xy}(x_i, y_j), && i = 0, I; j = 0, J. \end{aligned} \tag{7}$$

Then there exists exactly one piecewise bicubic function $u(x, y)$ of the form (6), which satisfies (7).

PROOF. Equations (5) (and their analogs for $\psi_n(y)$) imply that, for functions

* Clearly, the higher order partial derivatives u_{xxy}, u_{xyy}, u_{xxyy}, of u are continuous on R as well.

of the form (6), equations (7) are equivalent to

$$\begin{aligned} u_{ij} &= u(x_i, y_j) \\ &= \sum_{m=0}^{I+2} \sum_{n=0}^{J+2} \beta_{mn}\phi_m(x_i)\psi_n(y_j) = \beta_{ij}, \quad i = 0, \cdots, I; j = 0, \cdots, J, \end{aligned}$$

$$p_{ij} = u_x(x_i, y_j) = \sum\sum \beta_{mn}\phi_m'(x_i)\psi_n(y) = \begin{Bmatrix} \beta_{I+1,j}, & i = 0 \\ \beta_{I+2,j}, & i = I \end{Bmatrix}, \quad j = 0, \cdots, J,$$

$$q_{ij} = u_y(x_i, y_j) = \sum\sum \beta_{mn}\phi_m(x_i)\psi_n'(y_j) = \begin{Bmatrix} \beta_{i,J+1}, & j = 0 \\ \beta_{i,J+2}, & j = J \end{Bmatrix}, \quad i = 0, \cdots, I, \tag{8}$$

$$s_{ij} = u_{xy}(x_i, y_j) = \sum\sum \beta_{mn}\phi_m'(x_i)\psi_n'(y_j) = \begin{cases} \beta_{I+1,J+1}, & i = 0, j = 0 \\ \beta_{I+1,J+2}, & i = 0, j = J \\ \beta_{I+2,J+1}, & i = I, j = 0 \\ \beta_{I+2,J+2}, & i = I, j = J \end{cases}.$$

Since each β_{mn} occurs exactly once in the last members of the preceding $(I + 3)(J + 3)$ equations, and each of these equations is equivalent to one of the $(I + 3)(J + 3)$ conditions (7), the theorem follows.

4. Derivatives at Mesh-points. In §3, the existence and uniqueness of a piecewise bicubic function $u(x, y) \in C^2$ of the form (6) satisfying the conditions (7) was proved. In the following pages, an efficient computational scheme for the evaluation of $u(\bar{x}, \bar{y})$ defined by (6) and (7) at a point $(\bar{x}, \bar{y}) \in R$ is derived, which makes use of the piecewise polynomial character of $u(x, y)$. The procedure is a two-dimensional analog of the one described at the end of §2 for "linearized spline interpolation". The relevant equations are derived in the following Lemmas 3 and 4.

By definition, the interpolating function $u(x, y)$ equals a bicubic polynomial

$$c_{ij}(x, y) = \sum_{m,n=0}^{3} \gamma_{mn}^{ij}(x - x_{i-1})^m(y - y_{j-1})^n \tag{9}$$

in $R_{ij} : x_{i-1} \leqq x \leqq x_i ; y_{j-1} \leqq y \leqq y_j$.

LEMMA 3. Let u_{ij}, p_{ij}, q_{ij} and s_{ij} be given at the four corners of the rectangle R_{ij}. Then there exists exactly one bicubic polynomial $c_{ij}(x, y)$ (9) which assumes the given values. The matrix $\Gamma_{ij} = \| \gamma_{mn}^{ij} \|$ of coefficients in (9) is given in terms of the matrix K_{ij} of given values by the matrix equation

$$A(\Delta x_{i-1})K_{ij}A^T(\Delta y_{j-1}) = \Gamma_{ij}, \tag{10}$$

where

$$K_{ij} = \left| \begin{array}{c|c} B_{i-1,j-1} & B_{i-1,j} \\ \hline B_{i,j-1} & B_{ij} \end{array} \right| \quad \text{with} \quad B_{mn} = \begin{vmatrix} u_{mn} & q_{mn} \\ p_{mn} & s_{mn} \end{vmatrix},$$

and the matrix $A(h)$ is defined by

$$A(h) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/h^2 & -2/h & 3/h^2 & -1/h \\ 2/h^3 & 1/h^2 & -2/h^3 & 1/h^2 \end{vmatrix}.$$

Proof. The first part of the lemma is the special case $I = J = 1$ of Theorem 2. Since equation (10) is linear in K_{ij}, the second part of the lemma may be verified by computations showing its correctness for the sixteen basis functions $(x - x_{i-1})^m(y - y_{j-1})^n$, $m, n = 0, \cdots, 3$.

Lemma 4. If the values (7) are given, then, for $u(x, y)$ of the form (6), the values $p_{ij} = u_x(x_i, y_j)$, $(i = 1, \cdots, I - 1; j = 0, \cdots, J)$, $q_{ij} = u_y(x_i, y_j)$, $(i = 0, \cdots, I; j = 1, \cdots, J - 1)$, and $s_{ij} = u_{xy}(x_i, y_j)$, $(i = 1, \cdots, I - 1; j = 0, J$, and $i = 0, \cdots, I; j = 1, \cdots, J - 1)$, are uniquely determined by the following $2I + J + 5$ linear systems of altogether $3IJ + I + J - 5$ equations: for $j = 0, \cdots, J$,

$$\Delta x_{i-1}\, p_{i+1,j} + 2(\Delta x_{i-1} + \Delta x_i)p_{ij} + \Delta x_i\, p_{i-1,j} = 3\left[\frac{\Delta x_{i-1}}{\Delta x_i}(u_{i+1,j} - u_{ij}) + \frac{\Delta x_i}{\Delta x_{i-1}}(u_{ij} - u_{i-1,j})\right], \quad i = 1, \cdots, I - 1; \tag{11}$$

for $j = 0, J$,

$$\Delta x_{i-1}\, s_{i+1,j} + 2(\Delta x_{i-1} + \Delta x_i)s_{ij} + \Delta x_i\, s_{i-1,j} = 3\left[\frac{\Delta x_{i-1}}{\Delta x_i}(q_{i+1,j} - q_{ij}) + \frac{\Delta x_i}{\Delta x_{i-1}}(q_{ij} - q_{i-1,j})\right], \quad i = 1, \cdots, I - 1; \tag{12}$$

for $i = 0, \cdots, I$,

$$\Delta y_{j-1}\, q_{i,j+1} + 2(\Delta y_{j-1} + \Delta y_j)q_{ij} + \Delta y_j\, q_{i,j-1} = 3\left[\frac{\Delta y_{j-1}}{\Delta y_j}(u_{i,j+1} - u_{ij}) + \frac{\Delta y_j}{\Delta y_{j-1}}(u_{ij} - u_{i,j-1})\right], \quad j = 1, \cdots, J - 1; \tag{13}$$

for $i = 0, \cdots, I$,

$$\Delta y_{j-1}\, s_{i,j+1} + 2(\Delta y_{j-1} + \Delta y_j)s_{ij} + \Delta y_j\, s_{i,j-1} = 3\left[\frac{\Delta y_{j-1}}{\Delta y_j}(p_{i,j+1} - p_{i,j}) + \frac{\Delta y_j}{\Delta y_{j-1}}(p_{ij} - p_{i,j-1})\right], \quad j = 1, \cdots, J - 1. \tag{14}$$

Proof. Along each mesh-line $y = y_j$, $j = 0, \cdots, J$,

$$u(x, y) = v_j(x) \in S(x; x_0, \cdots, x_I), \quad \text{and} \quad u_x(x, y_j) = v_j'(x).$$

By the Corollary to Lemma 2 (in §2), the numbers $v_j'(x_i) = u_x(x_i, y_j) = p_{ij}$, $i = 1, \cdots, I - 1$, are uniquely determined if $v_j(x_i)$, $i = 0, \cdots, I$, and $v_j'(x_0)$, $v_j'(x_I)$ are known. Since $v_j(x_i) = u_{ij}$, $i = 0, \cdots, I$, and $v_j'(x_0) = p_{0j}$, $v_j'(x_I) = p_{Ij}$ are given for $j = 0, \cdots, J$, it follows from the Corollary to Lemma 2 that p_{ij}, $(i = 1, \cdots, I - 1; j = 0, \cdots, J)$ is uniquely determined by the $J + 1$ sets of $(I - 1)$ equations (11), given the values (7). By similar reasoning, equations (13) determine q_{ij}, $(i = 0, \cdots I; j = 1, \cdots, J - 1)$, uniquely, given the values (7). Along each mesh-line $y = y_j$, $j = 0, J$,

$$u_y(x, y_j) = \sum (\phi_m(x)(\sum \beta_{mn}\psi_n'(y_j))) = w_j(x) \in S(x; x_0, \cdots, x_I),$$

and $u_{xy}(x, y_j) = w_j'(x)$. Since $w_j'(x_i) = s_{ij}$, $i = 0, I$, and $w_j(x_i) = q_{ij}$, $i = 0, \cdots, I$, is given for $j = 0, J$, equations (12) determine s_{ij}, $(i = 1, \cdots, I - 1; j = 0, J)$, uniquely, given the values (7). Finally, for each $i = 0, \cdots, I$,

$$u_x(x_i, y) = z_i(y) \in S(y; y_0, \cdots, y_J), \quad \text{and} \quad u_{xy}(x_i, y) = z_i'(y).$$

For each $i = 0, \cdots, I$, $z_i(y_j) = u_x(x_i, y_j)$, $j = 0, \cdots, J$, is either given or can be uniquely determined from (11), and $z_i'(y_j)$, $j = 0, J$, is either given or can be uniquely determined from (12). We invoke the Corollary to Lemma 2 a last time to conclude that s_{ij}, $(i = 0, \cdots, I; j = 1, \cdots, J - 1)$, is uniquely determined by equations (14) with (11) and (12), given the values (7). This proves Lemma 4.

5. Computational Procedure. We are now ready to describe the computational procedure. First compute the values p_{ij}, q_{ij}, and s_{ij} from the given values (7) at all mesh-points (x_i, y_j), at which they are not given, using equations (11)–(14). The computation of these numbers p_{ij}, q_{ij} and s_{ij} from these equations can be done very efficiently and accurately by Gauss elimination, since the matrix of each of the $2I + J + 5$ systems of equations (11)–(14) is tridiagonal and strictly diagonally dominant. In solving such a linear system $B\underline{z} = \underline{d}$, $B = \| b_{ij} \|$ and tridiagonal, $\underline{z} = \{z_1, z_2, \cdots, z_n\}$, $\underline{d} = \{d_1, d_2, \cdots, d_n\}$, by Gauss elimination, one first computes quantities b_{ii}' by

$$b_{11}' = b_{11}, \quad b_{ii}' = b_{ii} - b_{i,i-1}b_{i-1,i}/b_{i-1,i-1}', \quad i = 2, \cdots, n. \tag{15}$$

One then computes a vector $\underline{d}' = \{d_1', d_2', \cdots, d_n'\}$ by

$$d_1' = d_1, \quad d_i' = d_i - b_{i,i-1}\, d_{i-1}'/b_{i-1,i-1}', \quad i = 2, \cdots, n, \tag{16}$$

and, finally, finds the solution by the recursion formula

$$z_n = d_n'/b_{nn}', \quad z_i = (d_i' - b_{i,i+1}z_{i+1})/b_{ii}', \quad i = n - 1, n - 2, \cdots, 1. \tag{17}$$

Since only two distinct matrices appear in equations (11)–(14), one has to use (15) only twice, and then solves each of the $2I + J + 5$ systems (11)–(14) in turn, using (16) and (17) only.

Having solved equations (11)–(14), and stored the results together with the given values (7), one then has the value of u, u_x, u_y, and u_{xy} available at every mesh-point (x_i, y_j) of the mesh. Now use equation (10) in each rectangle R_{ij} to compute the coefficients γ_{mn}^{ij} of the bicubic polynomial (9) in that rectangle from the values of u, u_x, u_y and u_{xy} at the four corners of R_{ij}. Once the coefficients γ_{mn}^{ij} of (9) are computed for each rectangle R_{ij}, the evaluation of the interpolating function $u(x, y)$ at a point $(\bar{x}, \bar{y}) \in R$ reduces to finding indices (i, j) such that $(\bar{x}, \bar{y}) \in R_{ij}$, followed by the evaluation of the bicubic polynomial (9).

The method of bicubic spline interpolation can be generalized, using tensor products, to functions of n independent variables of class C^2 on an n-dimensional hypercuboid, following the pattern outlined in §§3–5. This generalization will be presented elsewhere.

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