Box splines revisited: convergence and acceleration methods for the subdivision and the cascade algorithms

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Abstract. A general convergence analysis of the cascade algorithm, for the determination of a refinable function from its mask, is applied to box splines in which case certain difficulties adherent to the general case can be resolved completely and even elegantly. In the process, the understanding of the convergence of the adjoint process, subdivision, is also enhanced.

1. Introduction

Let $\varphi \in L_1(\mathbb{R}^d)$ be a compactly supported refinable function. The adjective **refinable** means the existence of a sequence $a \in \mathcal{Q}_1$ such that

$$\varphi = \mathcal{D}\varphi * a := \sum_{k \in \mathcal{Z}_1} \varphi(2(\cdot - k))a(k). \tag{1.1}$$

Here,

$$\mathcal{D} : f \mapsto f(2\cdot)$$

is **(dyadic) dilation**, with

$$\mathcal{Q}_k := \mathbb{C}^{\mathcal{Z}_k} \quad \text{and} \quad \mathcal{Z}_k := 2^{-k}\mathbb{Z}^d,$$

and, for any h and any discretely defined v,

$$h*v := \sum_{s \in \operatorname{supp} v} h(\cdot - s)v(s). \tag{1.2}$$

The mesh-function $a \in \mathcal{Q}_1$ in the **refinement equation** (1.1) is called the **mask** of φ, and we assume it throughout to be finitely supported. We also assume that $\hat{\varphi}(0) = 1$.

In most cases, the mask a is known explicitly, while the refinable function is only known implicitly, as the solution of (1.1). This raises the problem of computing φ from its mask a. To this end, we note that φ is a fixed point of the corresponding **cascade operator**

$$C : f \mapsto \mathcal{D}f*a,$$

hence is, in principle, constructible by the power method, i.e., as the limit of

$$C^k f = \mathcal{D}^k f * \mathcal{D}^{k-1}a * \cdots * \mathcal{D}^0 a =: \mathcal{D}^k f * a^{[k]} \tag{1.3}$$

as $k \to \infty$, assuming f is a suitable initial guess.

This process is called the **cascade algorithm**. We say that it **converges in the p-norm** (for some $p \in [1\,..\,\infty]$) **on the set of functions G at a rate** α (for some $\alpha > 0$) if, for every $g \in G$,

$$\|C^k g - \varphi\|_{L_p(\mathbb{R}^d)} \le \mathrm{const}_g\, 2^{-\alpha k}. \tag{1.4}$$

As (1.3) makes clear, the cascade iterations produce a function $C^k g$ that lies in the space $\mathcal{D}^k S(g)$, with

$$S(g) := \mathrm{span}\{g(\cdot - k) : k \in \mathbb{Z}^d\}$$

the **principal shift-invariant** (PSI) space generated by g. The iterations can be recast, [NRS], in the language of quasi-interpolation, the latter being a standard approach for approximating smooth functions from dilates of PSI spaces. In doing so, one identifies several conditions that are necessary for α-rate convergence starting with an initial seed g.

(1.5) Assumptions. *We consider the following four assumptions on the triplet (φ, g, α):*

(a) The refinable function φ lies in the Sobolev space $W_p^\alpha(\mathbb{R}^d)$.

(b) The Fourier transform $\widehat{a} := \sum_{k\in\mathcal{Z}_1} a(k)\mathrm{e}^{-\mathrm{i}k\cdot}$ of the mask a of φ has a zero of some order $m > \alpha$ at each point in $\{0, 2\pi\}^d\backslash 0$.

(c) For some $m > \alpha$, the PSI space $S(g)$ provides approximation order m in the p-norm, viz., for each sufficiently smooth f, as $k \to \infty$,

$$\mathrm{dist}_{L_p}(f, \mathcal{D}^k S(g)) = O(2^{-km}).$$

(d) The convolution operators $g*$ and $\varphi*$ coincide on the space Π_α of polynomials of degree $\le \alpha$. In other words, $\widehat{g} - \hat{\varphi}$ has a zero of order $m > \alpha$ at the origin.

The smoothness assumption (a) on φ implies (at least for $p = 2$, [R1]) that φ **satisfies the Strang-Fix (SF) conditions of order** $m := \lfloor \alpha + 1 \rfloor$:

$$\hat{\varphi} \text{ has a zero of order } m \text{ at each } \gamma \in 2\pi\mathbb{Z}^d\backslash 0.$$

In the context of the cascade iterations, the slightly stronger condition (b) needs to be imposed on the refinable φ. Condition (b) is usually referred to as "the SF condition of order m of the mask a of φ". The satisfaction of the SF condition by the mask a implies the satisfaction of the SF condition of the same order by the refinable φ, but not vice versa. That said, refinable functions that violate this converse implication are quite pathological: the simplest example is, in 1D, the support function $\chi_{[0..2]}$ of the interval $[0\,..\,2]$.

As to condition (c) above, it is well-known that, at least for a compactly supported $g \in L_p(\mathbb{R}^d)$, that condition is equivalent to g satisfying the SF condition of order m.

We will not prove formally the necessity of the four assumptions listed in (1.5) for the convergence of the cascade iterations at order α. We focus on the converse problem, i.e., whether these conditions are *sufficient*. To this end, we restrict *a priori* our attention to initial seeds g that satisfy (c), (d) above, and, for convenience, given a compactly supported refinable φ, a positive α, and $p \in [1\,..\,\infty]$, denote by

$$G_{p,\alpha}(\varphi)$$

the set of all compactly supported functions $g \in L_p(\mathbb{R}^d)$ that satisfy (c) and (d) above.

Now, it is quite easy to show that the four assumptions alone are *not* sufficient for an α-rate convergence. This leads us to the following problem:

(1.6)Problem. *Let φ be, as before, compactly supported and refinable. Given $\alpha > 0$, and $p \in [1\,..\,\infty]$, assume that φ satisfies (a) and (b) of (1.5). What additional conditions need one to assume on φ in order to obtain α-rate convergence of the cascade iterations to φ (in the p-norm) for every initial seed $g \in G_{p,\alpha}(\varphi)$?*

In this paper, after describing in some detail the general approach that was developed in [NRS] for the analysis of this problem, we will establish a complete solution to (1.6) in case φ is a *box spline*. To this end, we continue now with a more detailed description of the [NRS] approach, and the additional structure that is available in the box spline case.

Given a compactly supported function $g \in L_p(\mathbb{R}^d)$ (or, even better, in $G_{p,\alpha}(\varphi)$), we apply the cascade algorithm to functions of the form

$$f = g{*}u = \sum_{j\in\mathbb{Z}^d} g(\cdot - j)u(j) \qquad \text{for some } \ u \in \mathcal{Q} := \mathcal{Q}_0.$$

We assume the sequence u to have finite support (hence f is of compact support, too). The motivation here is that certain careful choices of u lead to functions f for which the convergence analysis of the cascade iteration is simpler. Specifically, our approach hinges on a decomposition of g into the sum $g = g{*}(u_1 + u_2)$ in a way that the cascade iterations converge at α-rate to φ on the initial seed $g{*}u_1$, and converge at that same rate to 0 on the initial seed $g{*}u_2$.

We begin by writing

$$C^k f = \mathcal{D}^k(g{*}u){*}a^{[k]} = \mathcal{D}^k g{*}\mathcal{C}^k u,$$

with

$$\mathcal{C}u := \mathcal{D}u{*}a \ = \sum_{j\in\mathcal{Z}_1} u(2(\cdot - j))a(j)$$

the corresponding **discrete cascade operator**, and so

(1.7) $$\mathcal{C}^k u = \mathcal{D}^k u{*}a^{[k]}.$$

In particular, the key quantity here, namely the mesh-function $a^{[k]} \in \mathcal{Q}_k$, equals

$$a^{[k]} = \mathcal{C}^{k-1}a = \mathcal{C}^k \delta_0,$$

with

$$\delta_j := (\delta_{jn} : n \in \mathbb{Z}^d)$$

the delta-sequence centered at $j \in \mathbb{Z}^d$.

The specific choices for u for which the cascade iterations are easily trackable are of the form

$$u = (\nu * \varphi)_{|}$$

for some distribution ν, and with

$$f_{|j} := f|_{\mathcal{Z}_j}, \quad f_{|} := f_{|0},$$

the restriction of f to the lattice $\mathcal{Z}_j$, respectively to the integer lattice

$$\mathcal{Z} := \mathcal{Z}_0 = \mathbb{Z}^d.$$

Then,

$$\mathcal{D}u = (\mathcal{D}(\nu * \varphi))_{|1} = 2^d (\mathcal{D}\nu * \mathcal{D}\varphi)_{|1},$$

and one derives by induction that, for $u = (\nu * \varphi)_{|}$,

$$C^k(g * u) = 2^{dk} \mathcal{D}^k g * (\mathcal{D}^k \nu * \varphi)_{|k}.$$

This places $C^k(g*u)$ into the dilate $\mathcal{D}^k S(g)$ of the PSI space $S(g)$ generated by g. More precisely, it identifies $C^k(g*u)$ as the quasi-interpolant

$$I_k \varphi := \mathcal{D}^k I(\mathcal{D}^{-k}\varphi)$$

to φ from $\mathcal{D}^k S(g)$, with the underlying quasi-interpolator I specified by the distribution ν in the sense that

$$Ih := g * (\nu * h)_{|}.$$

Convergence of $I_k h$ to h is well-understood (see, e.g., [BR]): we will show later that, once we are given a sequence $u = (\nu * \varphi)_{|}$, and once we adopt (1.5)Assumptions, we only need to assume further that $1 - \widehat{u}$ has a zero at the origin of order α in order to conclude that

$$\|\varphi - I_k \varphi\|_p = O(2^{-k\alpha}).$$

This clearly provides us with many choices of sequences $u = (\nu * \varphi)_{|}$ that yield convergence of $C^k(g*u)$ to φ to $O(2^{-k\alpha})$. But, it falls short of justifying the use of the initial seed $g = g * \delta_0$: While the choice $u := \delta_0$ trivially satisfies the "flatness condition" $1 - \widehat{u} = O(|\cdot|^\alpha)$, the quasi-interpolation argument further requires u to be of the form $u = (\nu * \varphi)_{|}$.

Now, as is pointed out in Proposition 3.2.7 of [R2] (from which the above discussion is taken which culminates there in Theorem 3.2.4), $u = (\nu * \varphi)_{|}$ for some smooth, compactly supported function ν iff $u * K_\varphi = 0$, with

$$K_\varphi := \{q \in \mathcal{Q} : \varphi * q = 0\}$$

the space of φ's **dependence relations** and, as before,

$$\mathcal{Q} = \mathcal{Q}_0 = \mathbb{C}^{\mathcal{Z}}.$$

Thus, if φ or, more precisely, $(\varphi(\cdot - j) : j \in \mathbb{Z}^d)$ is linearly independent, then the above quasi-interpolation approach allows the choice $u := \delta_0$, hence the cascade iterations converge as desired even when the initial seed is taken to be the function $g \in G_{p,\alpha}(\varphi)$ itself.

This paper concerns the contrary case. In that case, choosing an appropriate $u := (\nu * \varphi)_{|}$, we have that, with $v := \delta_0 - u$, $\widehat{v} = O(|\cdot|^\alpha)$. Thus, we "only" need to prove that, for any given finitely supported v, if $\widehat{v} = O(|\cdot|^\alpha)$ near the origin, then

$$\mathcal{C}^k v = O(2^{-\alpha k}), \tag{1.8}$$

hence

$$C^k(g * v) = O(2^{-\alpha k}),$$

too. Moreover, by restricting the support of the distribution ν to a small neighborhood of the origin, we can ensure that the support of the sequence v is not only finite but lies in a well-defined finite subset $\Omega \subset \mathcal{Z}$. Let $U_\alpha \subset \mathcal{Q}$ be the space of all sequences v supported in Ω and satisfying $1 - \widehat{v} = O(|\cdot|^\alpha)$. Our sought-for result is that (1.8) holds for every $v \in U_\alpha$. The quest for this result is the core of our analysis in this paper.

The main tool in this search is Theorem 3.3 of [NRS], which is called there the **Double-Tree Theorem**. The theorem is stated and proved (in the full generality of [NRS]) in §4 of the current article. As for now, we will briefly discuss some of the pertinent ingredients of that tool.

The Double-Tree Theorem relies on identifying a space $V \subset \mathcal{Q}$ of finitely supported sequences that is shift-invariant and $\mathcal{C}$-invariant (we explain that latter notion later), and such that (1.8) is valid for every $v \in V \cap U_\alpha$. It then extends the validity of (1.8) from $V \cap U_\alpha$ to all of U_α by examining the iterations of the subdivision operator $\mathcal{S}$ on a suitably defined orthogonal complement $V^\perp \subset \mathcal{Q}$:

$$V^\perp := \{q \in \mathcal{Q} : \sum_j v(j)\overline{q(j)} = 0, v \in V\}. \tag{1.9}$$

Here, $\mathcal{S}$ is the **subdivision operator**

$$\mathcal{S} : \mathcal{Q} \to \mathcal{Q} : q \mapsto \sum_{j \in \mathbb{Z}^d} \tilde{a}(\frac{\cdot}{2} - j)q(j) = \mathcal{D}^{-1}(\tilde{a} * q), \tag{1.10}$$

with $\tilde{a}$ the "**involution**" or **adjoint** (or, more flippantly, the **flip**) of a, i.e.,

$$\tilde{a}(j) := \overline{a(-j)}, \qquad j \in \mathbb{Z}^d. \tag{1.11}$$

The Double-Tree Theorem extends (1.8) from $V \cap U_\alpha$ to all of U_α, upon assuming that the iterations of $\mathcal{S}$ converge suitably fast to 0 on $V^\perp$; see §4 for the details. The application of the Double-Tree Theorem envisioned in [NRS] corresponds to the choice

$$V := \{(\nu * \varphi)_{|} : \operatorname{supp} \nu \text{ compact}\},$$

and the entire analysis of convergence is then shifted, thanks to the Double-Tree Theorem, to the study of the action of subdivision on $V^\perp$. In [NRS], this is brought to a satisfactory conclusion only in case K_φ is *finite-dimensional*, using the natural embedding of $V^\perp$ in K_φ that exists in that case.

In this paper, we take for φ a box spline M_Ξ; see the Appendix for the definition of M_Ξ and a list of its pertinent properties. In the box spline case, it is possible to have an infinite-dimensional K_{M_Ξ}, hence the analysis of [NRS] cannot be applied here verbatim. Our successful analysis of this case relies on the special structure of the mask a_Ξ of the box spline: it allows us to choose for the space $V \subset \mathcal{Q}$ (that serves as the cornerstone of the [NRS]-approach) one that is larger than the space V of "quasi-interpolation sequences" that was detailed above. Specifically, in the box spline case, we have available the following alternative choice for V, namely the shift-invariant space V_m generated by the convolution products

$$u_Y := (\delta_0 - \delta_\xi) * \cdots * (\delta_0 - \delta_\zeta), \quad Y := [\xi, \ldots, \zeta] \subset \Xi, \ \#Y = m,$$

indexed by the m-column submatrices Y of Ξ. For this choice of V, we will show (with relative ease) that (1.8) holds for every $v \in V$, provided that $\alpha \le m$. This means that in the box spline case convergence of the cascade iterations is guaranteed on a far larger domain than the one captured by the quasi-interpolation approach of [NRS].

In the above discussion, the space $V := V_m$ depends on the choice of the integer m: a larger m results in a smaller V, leading thereby to a larger $V^\perp$ in the Double-Tree Theorem (hence to a more demanding condition on the subdivision side). Moreover, the analysis works only if we assume that

$$m \le m(\Xi) + 1 := \min\{\#Y : \operatorname{rank}(\Xi \backslash Y) < d\}. \tag{1.12}$$

We note that this limitation is natural since the number $m(\Xi) + 1/p$ captures the smoothness of M_Ξ in the L_p-norm ($M_\Xi \in W_p^\alpha(\mathbb{R}^d)$ for every $\alpha < m(\Xi) + 1/p$), hence is an upper bound on the rate of convergence of the cascade iterations in the p-norm.

Our analysis in this article shows that, with m restricted as above, we have one of the two cases:

Case I: The space $V^\perp$ is a **nilpotent space** of $\mathcal{S}$ (in the sense that $\mathcal{S}^k(V^\perp) = 0$ for some finite k). In this case, the Double-Tree Theorem applies to yield that (1.8) extends from $V \cap U_\alpha$ to U_α.

Case II: The space $V^\perp$ is not a nilpotent subspace of $\mathcal{S}$. In this case, our analysis shows that, for $p = \infty$, the largest α for which (1.8) can hold is $\alpha = m - 1$.

Identifying the minimal m for which $V^\perp$ is not nilpotent is an entirely algebraic task. To this end, we define, given the direction matrix Ξ (see the Appendix) of the box spline M_Ξ, and $t \in [0 \,.\,.\, 2\pi)^d \backslash 0$, the submatrix Ξ_t of Ξ by

$$\Xi_t := [\xi \in \Xi : \mathrm{e}_t(\xi) = 1],$$

with

$$\mathrm{e}_t : \mathbb{R}^d \to \mathbb{C} : x \mapsto \exp(\mathrm{i} \sum_k x(k) t(k)).$$

We are only interested in the case when Ξ_t is of full rank d, and mention that, regardless of the choice of Ξ, only finitely many $t \in [0\,..\,2\pi)^d \backslash 0$ satisfy this full-rank assumption. Moreover, among those finitely many t that correspond to a full rank Ξ_t, we are interested only in those whose corresponding exponential, e_t, is **malignant** (with respect to Ξ) in the sense that its corresponding exponential sequence $\mathrm{e}_{t|} = (\mathrm{e}_t(j) : j \in \mathbb{Z}^d)$ is *not* a nilpotent sequence of the subdivision $\mathcal{S}$.

Our precise result reads as follows:

(5.1)Theorem. *Let $\varphi = M_\Xi$ be the box spline with direction matrix Ξ of full rank. Let*

(i) m_0 be the minimum over all $\#(\Xi \backslash \Xi_t)$ as $t \in [0\,..\,2\pi)^d \backslash 0$ ranges over all vectors for which Ξ_t is of full rank and $\mathrm{e}_{t|}$ is malignant with respect to Ξ (setting $m_0 := \#\Xi + 1$ in case there is no such t);

(ii) $m_p := m(\Xi) + 1/p$, with $p \in [1\,..\,\infty]$ given and fixed, and $m(\Xi)$ defined as in (1.12); and

(iii) m_s be the largest integer m for which (b) of (1.5) is valid.

Then, given $\alpha > 0$, the cascade iterations converge to M_Ξ at rate α in the p-norm on every initial seed $g \in G_{p,\alpha}(M_\Xi)$, provided that $\alpha < m_p$, and that $\alpha \le \min\{m_0, m_s\}$.

Discussion. The highlight of the above result is that all its parameters are algebraic. It even avoids the need to identify the spectrum of a related finite-rank operator. Ideally, the convergence rates of the cascade iteration should only be saturated by the smoothness of the refinable function, i.e., by m_p here. The theorem identifies two "obstacles" on the road to this optimal convergence rate. One is a possible suboptimal SF order of the mask a_Ξ (i.e., the situation when $m_s < m(\Xi) + 1$). The hampering of the cascade convergence rate by such suboptimal SF orders is not peculiar to box splines (see the discussion around (1.5)Assumptions). On the other hand, our theorem here identifies the remaining obstacle for optimal convergence rates as the existence of malignant exponentials. We note that the existence of such malignant exponentials associated with Ξ is rare. Thus, optimal convergence rates for box splines are the rule, not the exception. $\square$

Example. As an illustration, let us consider the case of linear or sublinear convergence (i.e., $\alpha \le 1$ in (5.1)Theorem). The theorem makes three assumptions to this end. The first is that the Fourier series of the mask a_Ξ of the box spline vanishes on $\pi(\mathbb{Z}^d \backslash 2\mathbb{Z}^d)$, or, equivalently, that the sum of the values of the restriction of a_Ξ to any coset $\gamma + \mathbb{Z}^d$, $\gamma \in \{0, 1/2\}^d$, is 1. This condition is a standard necessary condition for convergence analysis of cascade and subdivision algorithms.

The second assumption is that the box spline has positive p-smoothness. In the case $1 \le p < \infty$ this means that the set Ξ is of full rank d. For $p = \infty$, the assumption means that Ξ remains of full rank even after removing from it (any) one of its vectors.

The most interesting condition is the third one, concerning the existence of malignant exponentials. Note that, in the notations of (5.1)Theorem, if $m_0 \ge 1$, then the existence of malignant exponentials does not hamper sublinear convergence. However, our theorem fails to establish positive convergence rates once $m_0 = 0$. This case can happen only when there exists $t \in \mathbb{R}^d \backslash \mathbb{Z}^d$ for which Ξt is an integer vector, which is exactly the case when $\Xi\mathbb{Z}^\Xi$ is a proper sublattice of $\mathbb{Z}^d$. The condition $\Xi\mathbb{Z}^\Xi = \mathbb{Z}^d$ was identified in [BHR: (VII.23)] as a sufficient condition (once the continuity of M_Ξ and the satisfaction of the SF conditions of order 1 are assumed) for the L_∞-convergence of the subdivision algorithm. Our results, when restricted to this special case, produce a stronger version: convergence happens even when $\Xi\mathbb{Z}^\Xi$ is a proper sublattice, provided that none of the non-constant exponentials $\mathrm{e}_{t|}$ in the joint kernel of the first-order difference operators ∇_ξ, $\xi \in \Xi$, is malignant. $\square$

The paper is laid out as follows. In §2, subdivision applied to exponentials is studied for general φ, then specialized to box splines; this reveals the side effects the box spline suffers from the existence of malignant exponentials. The basic facts about the cascade tree and its dual are stated and proved in §3. Discussion and proof of the Double-Tree Theorem are brought in §4. §5 is devoted to a proof of (5.1)Theorem. This is followed by an appendix which recalls, for the reader's convenience, information about box splines from [BHR].

2. Subdivision applied to exponentials

We consider, first for general refinable φ and then for $\varphi = M_\Xi$, the action of the subdivision operator

$$\mathcal{S}v := \mathcal{D}^{-1}(\tilde{a}{*}v) = \mathcal{D}^{-1}\tilde{a}{*}\mathcal{D}^{-1}v : j \mapsto \sum_{\nu\in\mathcal{Z}} \overline{a}(\nu - \frac{j}{2})v(\nu)$$

on $\mathrm{e}_{t|}$, the 'discrete exponential' with 'frequency' t, aware of the fact that $\mathrm{e}_{t|} = \mathrm{e}_{s|}$ if $t - s \in 2\pi\mathbb{Z}^d$ hence restricting attention to $t \in [0\,..\,2\pi)^d$.

Note that $\mathcal{S}$ is upsampling, preceded by convolution with the (properly dilated) adjoint mask.

For the upsampling, observe that, with

$$\Gamma := \{0, 1/2\}^d,$$

for any sequence $c \in \mathcal{Q}_1$,

$$\mathcal{Q} \ni \mathcal{D}^{-1}c = 2^{-d}c(\cdot/2)(\sum_{\gamma\in\Gamma} \mathrm{e}_{2\pi\gamma})_|.$$

With this, consider subdivision applied to the exponential $v = \mathrm{e}_{\eta|}$. We have

$$\mathcal{S}\mathrm{e}_{\eta|} = 2^{-d}\tilde{a}(\cdot/2) * (\sum_{\gamma\in\Gamma} \mathrm{e}_{2\pi\gamma+\eta/2})_| = 2^{-d}\sum_{\gamma\in\Gamma} \widehat{a}(4\pi\gamma + \eta)\mathrm{e}_{2\pi\gamma+\eta/2|}. \tag{2.1}$$

Now, for a finite $T \subset [0\,..\,2\pi)^d$, let

$$\mathrm{spect}(\sum_{t\in T} c(t)\mathrm{e}_{t|}) := \{t : c(t) \neq 0\}$$

denote the **spectrum** of such an **exponential sum**. Then we now know that

$$\operatorname{spect} \mathcal{S}v \subset 2\pi\Gamma + (\operatorname{spect} v)/2 \subset [0\,..\,2\pi)^d, \quad \forall v \in \operatorname{Exp}_T := \operatorname{ran}[\mathrm{e}_{t|} : t \in T].$$

Hence, if $\operatorname{spect} \mathcal{S}v$ and $\operatorname{spect} \mathcal{S}w$ have a point in common, say the point s, then $2\pi\gamma_v + s_v/2 = s = 2\pi\gamma_w + s_w/2$ (for some $\gamma_r \in \Gamma$ and some $s_r \in \operatorname{spect} r$, $r = u, w$), hence $s_v - s_w = 0 \bmod 2\pi$, therefore $s_v = s_w$. Put the other way,

$$\operatorname{spect} v \cap \operatorname{spect} w = \emptyset \quad \Longrightarrow \quad \operatorname{spect} \mathcal{S}v \cap \operatorname{spect} \mathcal{S}w = \emptyset.$$

By the linear independence (over $\mathcal{Z}$) of exponentials with different frequencies in $[0\,..\,2\pi)^d$, we conclude that, given $v \in \operatorname{Exp}_T$,

$$\mathcal{S}^n v = 0 \quad \Longrightarrow \quad \forall \{t \in \operatorname{spect} v\} \;\; \mathcal{S}^n \mathrm{e}_{t|} = 0.$$

Now assume that Exp_T is an invariant subspace of $\mathcal{S}$. Then, for any t in

$$T_0 := \{t \in T : \exists n \;\; \mathcal{S}^n \mathrm{e}_{t|} = 0\},$$

also $\operatorname{spect} \mathcal{S}\mathrm{e}_{t|} \subset T_0$, while the converse is obviously true. In particular, for any t in

$$T_1 := T \backslash T_0,$$

also $\operatorname{spect} \mathcal{S}\mathrm{e}_{t|} \cap T_1 \neq \emptyset$. Consequently,

$$(\operatorname{spect} \mathcal{S}\mathrm{e}_{t|} \cap T_1 : t \in T_1)$$

is a partition of T_1 into non-empty sets, hence, since T_1 is finite, must consist of 1-sets. So, the prescription

$$\operatorname{spect} \mathcal{S}\mathrm{e}_{t|} \cap T_1 =: \{\sigma t\}, \quad \forall t \in T_1,$$

defines a *permutation* on T_1. Thus, each $t \in T_1$ has an **order**, n_t say, namely the length of the orbit under σ to which it belongs, i.e., the smallest n for which $\sigma^n t = t$, with $n_t \leq \#T_1$ trivially. More than that, with (2.1),

$$\mathcal{S}\mathrm{e}_{t|} \;\in\; 2^{-d}\widehat{a}(2\sigma t)\mathrm{e}_{\sigma t|} + \operatorname{Exp}_{T_0}, \quad \forall t \in T_1, \tag{2.2}$$

with $\widehat{a}(2\sigma t) \neq 0$. Hence any nonzero eigenvalue λ of $\mathcal{S}$ as a map on the smallest $\mathcal{S}$-invariant space that contains $\mathrm{e}_{t|}$ must have an eigenvector of the form

$$v = \sum_{s \in T} \mathrm{e}_{s|} c(s),$$

in which $c(s) \neq 0$ for $s \in \{t, \sigma t, \ldots, \sigma^{n_t - 1} t\}$, hence (assuming, without loss, that $c(t) = 1$) λ^{n_t} must be the coefficient of $\mathrm{e}_{t|}$ in $\mathcal{S}^{n_t}\mathrm{e}_{t|}$. But this says, with (2.2), that

$$\lambda^{n_t} = \prod_{j=1}^{n_t} 2^{-d}\widehat{a}(2^j t), \tag{2.3}$$

using the fact that $\sigma^{-1} t = 2t \bmod 2\pi$ hence, as $\sigma^{n_t} t = t$, also

$$2\sigma^j t = 2\sigma^{j-n_t} t = 2^{1+j-n_t} t \bmod 4\pi, \quad j = 1, \ldots, n_t,$$

while $\widehat{a}$ is 4π-periodic.

The discussion so far was generic, in the sense that it applies to any refinable function and any finite-dimensional exponential space that is invariant under the corresponding subdivision operator. Our interest is in the particular case when the refinable function is a box spline, and the invariant subspace contains a malignant exponential. In this case, the above discussion leads to the following result:

(2.4)Proposition. *Let $\varphi = M_\Xi$ be the box spline with direction matrix Ξ, of full rank. Let $\mathrm{e}_{t|}$ be malignant with respect to Ξ, i.e., $t \in [0\,..\,2\pi)^d\backslash 0$, Ξ_t is of full rank, and $\mathcal{S}^n\mathrm{e}_{t|} \neq 0$ for all $n \in \mathbb{N}$. Then the convergence rate of the cascade algorithm in the ∞-norm cannot exceed*

$$m := \#(\Xi\backslash\Xi_t).$$

Proof: Suppose that the cascade iterations converge at rate $\alpha > 0$ on $G_{\infty,\alpha}(\varphi)$. Choose a compactly supported continuous $g \in G_{\infty,\alpha}(\varphi)$ whose shifts are ∞-stable (equivalently, these shifts form a Riesz basis in $L_2(\mathbb{R}^d)$, [BDeR]). Then

$$\|(C^k g) * \mathrm{e}_{t|}\|_{L_\infty} = \|g * \mathcal{S}^k\mathrm{e}_{t|}\|_{L_\infty} \sim \|\mathcal{S}^k\mathrm{e}_{t|}\|_{\ell_\infty}.$$

On the other hand, by (6.7)Proposition, $0 = \varphi * \mathrm{e}_{t|}$, hence

$$\|(C^k g) * \mathrm{e}_{t|}\|_{L_\infty} = \|(C^k g - \varphi) * \mathrm{e}_{t|}\|_{L_\infty} \le \mathrm{const}\|C^k g - \varphi\|_{L_\infty},$$

with const dependent only on $\operatorname{supp} g$, hence independent of k. Thus

$$\|\mathcal{S}^k\mathrm{e}_{t|}\|_{\ell_\infty} \le \mathrm{const}\|C^k g - \varphi\|_{L_\infty}, \tag{2.5}$$

hence our desired result will follow once we estimate (from below) the spectral radius of the restriction of $\mathcal{S}$ to the smallest $\mathcal{S}$-invariant space that contains $\mathrm{e}_{t|}$.

By (6.7)Proposition, the smallest $\mathcal{S}$-invariant space containing $\mathrm{e}_{t|}$ is finite-dimensional, hence (2.3) is available to us. We need to estimate the value $|\lambda|$ in (2.3). For this evaluation in our special situation, we infer from (6.5) and (6.3) that, for $a = a_\Xi$,

$$\widehat{a} = 2^d \prod_{\xi\in\Xi} \frac{1 + \mathrm{e}_{\xi/2}}{2}. \tag{2.6}$$

With that, (2.3) gives, since our t is in T_1 and with $n := n_t$,

$$\lambda^n = \prod_{j=1}^n 2^{-d}\widehat{a}(2^j t) = \prod_{j=1}^n \prod_{\xi\in\Xi} \frac{1 + \mathrm{e}_\xi(2^j t/2)}{2} = \prod_{\xi\in\mathrm{Z}} \prod_{j=1}^n \frac{1 + \mathrm{e}_\xi(2^j t/2)}{2},$$

with

$$\mathrm{Z} := \Xi\backslash\Xi_t = [\xi \in \Xi : \mathrm{e}_\xi(t) \neq 1].$$

For $\xi \in \mathrm{Z}$, we have $\mathrm{e}_\xi(t) \neq 1$, hence

$$\prod_{j=1}^n \frac{1 + \mathrm{e}_\xi(2^j t/2)}{2} = 2^{-n} \sum_{r=0}^{2^n-1} \mathrm{e}_\xi(rt) = 2^{-n}\frac{\mathrm{e}_\xi(2^n t) - 1}{\mathrm{e}_\xi(t) - 1} \ = \ 2^{-n},$$

the last equality since $\xi \in \mathbb{Z}^d$ and, by choice of n, $2^n t = t \bmod 2\pi$. Therefore, altogether, $\lambda^n = (2^{-n})^{\#\mathrm{Z}} = 2^{-mn}$, hence

$$|\lambda| = 2^{-m}.$$

We conclude that

$$\|\mathcal{S}^k\mathrm{e}_{t|}\|_{\ell_\infty} \neq o(2^{-km}),$$

hence, with (2.5), that the convergence rate of the cascade iterations in the ∞-norm is not faster than m. $\square$

We close this section with an additional technical property concerning the subdivision operator. This property is well-known for convolution operators and, since the extension from convolution operators to subdivision operators involve routine arguments, we only sketch the proof, detailing the parts that are less routine.

(2.7)Lemma. *Let $W \subset \mathcal{Q}$ be finite-dimensional, $\mathcal{S}$-invariant, and shift-invariant. Then W is $\mathcal{S}$-nilpotent (i.e., $\mathcal{S}^k W = 0$ for some k) if and only if every sequence $\mathrm{e}_{t|} \in W$, $t \in \mathbb{C}$, is $\mathcal{S}$-nilpotent.*

Proof: The "only if" implication is trivial. For the proof of the "if" implication, we first note that, since W is shift-invariant and finite-dimensional, it is spanned by sequences of the form $(\mathrm{e}_t q)_|$, with $t \in \mathbb{C}$ and q a polynomial. It is sufficient, therefore, to prove that any such sequence in W is $\mathcal{S}$-nilpotent. We prove this claim by negation: we assume that there exists in W a sequence as above that is not $\mathcal{S}$-nilpotent, and seek a contradiction. Among all those sequences that violate the nilpotency property, we choose one whose polynomial factor is of (necessarily positive) minimal degree.

Now, the key in the proof is the similarity between the subdivision $\mathcal{S}$ and the more standard convolution operators. Recall that a convolution operator $b*$ (with b, say, some finite sequence) satisfies

$$b * (\mathrm{e}_t q)_| = q(\cdot)(b * \mathrm{e}_{t|}) + (\mathrm{e}_t r)_|,$$

with r some polynomial of degree $< \deg q$. Using this in the derivation of (2.1) but applied there to $v = (\mathrm{e}_t q)_|$ rather than just to $\mathrm{e}_{t|}$, one derives an analogous property for subdivision, viz.,

$$\mathcal{S}(\mathrm{e}_t q)_| = q(\cdot/2)\mathcal{S}\mathrm{e}_{t|} + \text{l.o.t},$$

with "l.o.t" a linear combination of exponential polynomials of the form $(\mathrm{e}_\eta r)_|$, with $\eta \in \mathbb{C}$ (actually, $\eta \in t/2 + 2\pi\Gamma$), and with the η-dependent r a polynomial of degree $< \deg q$. Now, if we assume that $(\mathrm{e}_t q)_|$ lies in W, then the shift-invariance, finite dimensionality and $\mathcal{S}$-invariance of W can be combined to yield that the above-mentioned summands $(\mathrm{e}_\eta r)_|$ in $\mathcal{S}(\mathrm{e}_t q)_|$ are in W, too, hence, by the minimality assumption on $\deg q$, are $\mathcal{S}$-nilpotent. Thus, $\mathcal{S}(\mathrm{e}_t q)_| - q(\cdot/2)\mathcal{S}\mathrm{e}_{t|}$ is nilpotent, too.

The above argument, with some trivial modifications, can be extended to show that, for every k, $\mathcal{S}(q\,\mathcal{S}^k \mathrm{e}_{t|}) - q(\cdot/2)\,\mathcal{S}^{k+1}\mathrm{e}_{t|}$ is $\mathcal{S}$-nilpotent, hence, by induction, that $\mathcal{S}^k(q\mathrm{e}_t)_| - q(\cdot/2^k)\mathcal{S}^k\mathrm{e}_{t|}$ is $\mathcal{S}$-nilpotent, too. This completes the proof since, by assumption, $\mathcal{S}^k\mathrm{e}_{t|} = 0$ for some k. □

3. The cascade tree and its dual

In this section, we give, for completeness, the definition of the cascade tree and derive its basic properties. The cascade tree together with its dual, the subdivision tree, are used in the proof of the Double-Tree Theorem in the next section. The cascade tree also forms the backbone for the convergence analysis of subdivision schemes via the computation of its joint spectral radius, [J], [HJ].

With χ_A the support function of the domain A, let

$$|_A : f \mapsto \chi_A f$$

and recall the translation map

$$E^y : f \mapsto f(\cdot - y).$$

Then

$$E^y|_A E^{-y} = |_{A+y},$$

hence, since

$$(\mathcal{Z} + \gamma : \gamma \in \Gamma_k := [0 \mathinner{\ldotp\ldotp} 1)^d \cap \mathcal{Z}_k)$$

is a partition of $\mathcal{Z}_k$, we have

$$|_{\mathcal{Z}_k} = \sum_{\gamma \in \Gamma_k} E^\gamma |_{\mathcal{Z}} E^{-\gamma}.$$

Therefore, since $\mathcal{C}^k(\mathcal{Q}) \subset \mathcal{Q}_k$, we have, on $\mathcal{Q}$,

$$\mathcal{C}^k = |_{\mathcal{Z}_k} \mathcal{C}^k = \sum_{\gamma \in \Gamma_k} E^\gamma \mathcal{C}_{k,\gamma},$$

with

$$\mathcal{C}_{k,\gamma} : f \mapsto (E^{-\gamma} \mathcal{C}^k f)_{|}, \qquad \gamma \in \Gamma_k.$$

Claim. *$\mathcal{C}_{k,\gamma} = \mathcal{C}_{\gamma_k} \cdots \mathcal{C}_{\gamma_1}$, with*

$$\gamma =: \sum_{j=1}^k \gamma_j 2^{j-k}$$

and

$$\mathcal{C}_\varepsilon := \mathcal{C}_{1,\varepsilon}, \qquad \varepsilon \in \Gamma = \Gamma_1 = \{0, 1/2\}^d.$$

Proof: Let $\varepsilon \in \Gamma$ and $\gamma \in \Gamma_k$ and consider $\mathcal{C}_\varepsilon \mathcal{C}_{k,\gamma} = |_{\mathcal{Z}} E^{-\varepsilon} \mathcal{C} |_{\mathcal{Z}} E^{-\gamma} \mathcal{C}^k$. For any $q \in \mathcal{Q}_k$ (hence, in particular, for $q = \mathcal{C}^k f$ for any $f \in \mathcal{Q}$),

$$(\mathcal{C}|_{\mathcal{Z}} E^{-\gamma} q)(x) \; = \; \sum \{ q(2(x + \gamma/2 - s)) a(s) : s \in \mathcal{Z}_1, 2(x - s) \in \mathcal{Z} \},$$

with the sum nonempty iff $x - s \in \mathcal{Z}_1$ for some $s \in \mathcal{Z}_1$ iff $x \in \mathcal{Z}_1$. Hence

$$\mathcal{C}|_{\mathcal{Z}} E^{-\gamma} q \; = \; |_{\mathcal{Z}_1} E^{-\gamma/2} \mathcal{C} q.$$

Therefore, altogether,

$$\mathcal{C}_\varepsilon \mathcal{C}_{k,\gamma} = |_{\mathcal{Z}} E^{-\varepsilon} |_{\mathcal{Z}_1} E^{-\gamma/2} \mathcal{C}^{k+1} = |_{\mathcal{Z}} |_{\mathcal{Z}_1 - \varepsilon} E^{-\varepsilon - \gamma/2} \mathcal{C}^{k+1} = |_{\mathcal{Z}} E^{-\varepsilon - \gamma/2} \mathcal{C}^{k+1} = \mathcal{C}_{k+1, \varepsilon + \gamma/2},$$

with $|_{\mathcal{Z}} |_{\mathcal{Z}_1 - \varepsilon} = |_{\mathcal{Z}}$ since $\mathcal{Z} \subset \mathcal{Z}_1 = \mathcal{Z}_1 - \varepsilon$. □

It follows that $\mathcal{C}^k$ can be completely understood if one understands the maps $\mathcal{C}_{k,\gamma}$, $\gamma \in \Gamma_k$, all of which are maps on the same k-independent space $\mathcal{Q}$. Further, each map $\mathcal{C}_{k,\gamma}$ gives rise to exactly 2^d maps at the next level in this **cascade tree**, namely

$$\mathcal{C}_\varepsilon \mathcal{C}_{k,\gamma} = \mathcal{C}_{k+1,\varepsilon+\gamma/2}, \qquad \varepsilon \in \Gamma.$$

In particular, if U is a linear subspace of $\mathcal{Q}$ invariant under each $\mathcal{C}_\varepsilon$, $\varepsilon \in \Gamma$, then it is invariant under every node of the cascade tree. We call any such U **invariant with respect to the cascade tree** or, simply if slightly misleadingly, **$\mathcal{C}$-invariant**, for short.

Since, as we assume, the mask, a, is finitely supported,

$$U = \mathcal{Q}_\Omega := \{q \in \mathcal{Q} : \operatorname{supp} q \subset \Omega\}$$

is a $\mathcal{C}$-invariant subspace with the choice

$$\Omega = 2A, \quad \text{ with } \ A \supseteq \operatorname{conv}(\operatorname{supp} a - \Gamma)$$

a bounded convex set, since then, for $u \in U$ and $\varepsilon \in \Gamma$,

$$\operatorname{supp} \mathcal{C}_\varepsilon u \ \subseteq \ A + \operatorname{supp} a - \varepsilon \ \subseteq \ A + A \ = \ \Omega$$

(using the assumed convexity of A). This shows, more generally, that $\mathcal{C}_\varepsilon$ is **contractive** in the sense that $\operatorname{supp} \mathcal{C}_\varepsilon u$ is much smaller than $\operatorname{supp} u$ in case $\operatorname{supp} u$ is much larger than

$$\Omega_0 := 2 \operatorname{conv}(\operatorname{supp} a - \Gamma).$$

For example, with

$$\Omega_r := \Omega_0 - r \operatorname{conv}(\Gamma), \tag{3.1}$$

we have

$$\mathcal{C}_\varepsilon(\mathcal{Q}_{\Omega_{2r}}) \subset \mathcal{Q}_{\Omega_r} \tag{3.2}$$

since, for $u \in \mathcal{Q}_{\Omega_{2r}}$,

$$\operatorname{supp} \mathcal{C}_\varepsilon u \subset (\Omega_0 - 2r \operatorname{conv}(\Gamma))/2 + \operatorname{supp} a - \varepsilon \subset \Omega_0/2 - r \operatorname{conv}(\Gamma) + \Omega_0/2 = \Omega_r$$

(using again the convexity of Ω_0).

With the cascade tree defined, define the corresponding **subdivision tree** as the tree formed by the adjoints, i.e., by

$$\mathcal{S}_{k,\gamma} := \mathcal{C}_{k,\gamma}^* = \mathcal{C}_{\gamma_1}^* \cdots \mathcal{C}_{\gamma_k}^*, \quad \gamma \in \Gamma_k, \ k = 0, 1, \ldots,$$

with the adjoint taken with respect to the standard inner product $\langle \cdot, \cdot \rangle$ on ℓ_2. This makes sense since we assume the mask to be finitely supported.

Indeed, for $u, v \in \ell_2$, and also for any $u, v \in \mathcal{Q}$ with one of them finitely supported,

$$(3.3)\quad \langle \mathcal{C}_0 u, v\rangle = \sum_j \sum_k u(2(j-k))a(k)\overline{v(j)} = \sum_k \sum_j u(2k)a(j-k)\overline{v(j)} = \langle u, \mathcal{D}^{-1}(\tilde{a}{*}v)\rangle,$$

showing $\mathcal{C}_0^*$ to equal the subdivision operator $\mathcal{S}$ defined in (1.10). Since $\mathcal{C}_\varepsilon = \mathcal{C}_0 E^{-2\varepsilon}$, $\varepsilon \in \Gamma$, this implies that

$$(3.4)\qquad \mathcal{S}_\varepsilon = \mathcal{C}_\varepsilon^* = E^{2\varepsilon}\mathcal{S} = \mathcal{S}E^{\varepsilon}, \qquad \varepsilon \in \Gamma,$$

hence that

$$(3.5)\qquad \mathcal{S}_{k,\gamma} = E^{2^k\gamma}\mathcal{S}^k.$$

We will actually consider these operators only on $\mathcal{Q}_\Omega$ for some bounded set Ω, hence $\mathcal{Q}_\Omega$ is trivially in ℓ_2.

4. The double-tree theorem

We continue to have φ refinable with finite mask a, and let

$$\Omega := \Omega_2$$

(see (3.1)), hence $\mathcal{Q}_\Omega$ is finite-dimensional, and invariant under each of the $\mathcal{C}_\varepsilon$, $\varepsilon \in \Gamma$. This implies that *the action of any $\mathcal{S}_\varepsilon q = \mathcal{C}_\varepsilon^* q$ on $\mathcal{Q}_{\Omega_4}$ only depends on $q|_\Omega$*. Indeed, for any q vanishing on Ω,

$$\mathcal{C}_\varepsilon^* q : \mathcal{Q}_{\Omega_4} \to \mathbb{C} : v \mapsto \langle \mathcal{C}_\varepsilon v, q\rangle$$

is the zero map since, by (3.2), $\mathcal{C}_\varepsilon(\mathcal{Q}_{\Omega_4}) \subset \mathcal{Q}_{\Omega_2} = \mathcal{Q}_\Omega$, hence $\mathcal{C}_\varepsilon^* q$ must vanish on Ω_4. In the same way,

$$(4.1)\qquad \chi_{\Omega_2} q = 0 \quad \Longrightarrow \quad \chi_{\Omega_{2^{k+1}}}\mathcal{C}_{k,\gamma}^* q = 0, \qquad \gamma \in \Gamma_k,\ k = 2, 3, \ldots .$$

Thus, an assumption like (4.4) below is, at least, not impossible.

For any $\mathcal{C}$-invariant linear subspace U of $\mathcal{Q}_\Omega$, we set

$$\|\mathcal{C}^k\|_{p,U}^p \ := \ \sup_{u \in U} \frac{\sum_{\gamma\in\Gamma_k} \|\mathcal{C}_{k,\gamma}u\|_U^p/2^{dk}}{\|u\|_U^p},$$

with $\|\cdot\|_U$ any convenient norm on U. A particularly suitable norm might be the p-norm. With that choice,

$$\sum_{\gamma\in\Gamma_k} \|\mathcal{C}_{k,\gamma}u\|^p = \|\mathcal{C}^k u\|_p^p,$$

hence then

$$\|\mathcal{C}^k\|_{p,U} = \|\mathcal{C}^k : U \subset \ell_p(\mathcal{Z}\cap\Omega) \to \ell_p(\mathcal{Z}_k \cap \Omega)\|/2^{dk/p}.$$

But, in the proof below, we work with a more convenient choice for $\|\cdot\|_U$, knowing the statement $\|\mathcal{C}^k\|_{p,U} = O(2^{-\alpha k})$ to be independent of the norm on U since U is finite-dimensional.

(4.2)Double-Tree Theorem ([NRS: Theorem 3.3]). *Let U, V be $\mathcal{C}$-invariant subspaces of $\mathcal{Q}$, with $U \subset \mathcal{Q}_\Omega$, $\Omega := \Omega_2$, and V shift-invariant and spanned by finitely supported sequences. If, for some $\alpha > 0$ and some $1 \le p \le \infty$,*

$$\|\mathcal{C}^k\|_{p,U\cap V} = O(2^{-\alpha k}) \tag{4.3}$$

and

$$\|\mathcal{S}^k\| := \sup_{0\ne w\in W} \|\chi_{\Omega_{2^{k+1}}} \mathcal{S}^k w\|_\infty / \|w\|_W = O(2^{-\alpha k}) \tag{4.4}$$

in some, hence every, norm on the finite-dimensional linear space

$$W := RV^\perp,$$

with

$$V^\perp := \{q \in \mathcal{Q} : \langle v, q\rangle = 0, v \in V\}, \qquad R : \mathcal{Q} \to \mathcal{Q}_\Omega : f \mapsto \chi_\Omega f,$$

then

$$\forall\{\beta < \alpha\} \quad \|\mathcal{C}^k\|_{p,U} = O(2^{-\beta k}). \tag{4.5}$$

Proof: For completeness, particularly since we need to refer later to a certain proof detail, we give here a version of the proof in [NRS]. For it, we found it convenient to replace the condition $\|\mathcal{S}^k\|_{\infty,V^\perp} = O(2^{-\alpha k})$ (which is (3.5) there) by the more explicit condition (4.4).

As a start, observe that (4.3) and (4.4) imply, for any $\beta < \alpha$, the existence of some k_0 so that

$$\|\mathcal{C}^{k_0 k}\|_{p,U\cap V} \le 2^{-\beta k_0 k}, \quad k = 1, 2, \ldots, \tag{4.6}$$

and

$$\|\mathcal{S}^{k_0 k}\| \le 2^{-\beta k_0 k}, \quad k = 1, 2, \ldots,$$

due to the fact that, whatever the const hiding behind the O in (4.3) and (4.4), there is k_0 so that

$$\text{const} 2^{-\alpha k_0} \le 2^{-\beta k_0}.$$

More than that, for any particular positive constant K, we can so choose k_0 that also

$$\|\mathcal{S}^{k_0 k}\| \le 2^{-\beta k_0 k} K, \quad k = 1, 2, \ldots. \tag{4.7}$$

At the same time, if we can prove from this that

$$\|\mathcal{C}^{k_0 k}\|_{p,U} = O(2^{-\beta k_0 k}), \tag{4.8}$$

then we are done since it is not hard to see that $\|\mathcal{C}^{k_0}\|_{p,U} \le \|\mathcal{C}\|_{p,U}^{k_0}$.

For notational simplicity, we hide the constant k_0 by using

$$\tilde{\mathcal{C}}_{k,\gamma} := \mathcal{C}_{k_0 k,\gamma}, \quad \tilde{\mathcal{S}}_{k,\gamma} := \tilde{\mathcal{C}}^*_{k,\gamma}, \qquad \gamma \in \tilde{\Gamma}_k := \Gamma_{k_0 k}.$$

The cascade tree for $\tilde{\mathcal{C}}$ uses dilation by

$$\lambda := 2^{k_0},$$

and, correspondingly,

$$\|\tilde{\mathcal{C}}^k u\|^p := \lambda^{-dk} \sum_{\gamma \in \tilde{\Gamma}_k} \|\tilde{\mathcal{C}}_\gamma u\|_U^p.$$

In these terms, (4.6) becomes

$$\|\tilde{\mathcal{C}}^k u\| \le \lambda^{-\beta k} \|u\|_U, \quad u \in U \cap V. \tag{4.9}$$

We now use (4.9) and (4.7) to prove, by induction on k, that, for every $u \in U$,

$$\|\tilde{\mathcal{C}}^k u\| \le \|\tilde{\mathcal{C}}\|_{p,U}\, k\, \lambda^{-\beta(k-1)} \|u\|_U, \tag{4.10}$$

it being evidently true for $k = 1$ by the very definition of $\|\tilde{\mathcal{C}}\|_{p,U}$.

For this, we deduce from the discussion in §3 that

$$\tilde{\mathcal{C}}^k = \sum_{\gamma \in \tilde{\Gamma}_k} E^\gamma \tilde{\mathcal{C}}_{k,\gamma},$$

with

$$\tilde{\mathcal{C}}_{k,\gamma} := \tilde{\mathcal{C}}_{\gamma_k} \cdots \tilde{\mathcal{C}}_{\gamma_1} \quad \text{and} \quad \tilde{\mathcal{C}}_\varepsilon := \tilde{\mathcal{C}}_{1,\varepsilon},$$

and

$$\gamma =: \sum_{j=1}^{k} \gamma_j \lambda^{j-k}.$$

Therefore, for any $u \in U$,

$$\|\tilde{\mathcal{C}}^k u\|^p = \lambda^{-dk} \sum_{\gamma \in \tilde{\Gamma}_k} \|\tilde{\mathcal{C}}_{k,\gamma} u\|_U^p = \lambda^{-d} \sum_{\varepsilon \in \tilde{\Gamma}} \lambda^{-d(k-1)} \sum_{\gamma \in \tilde{\Gamma}_{k-1}} \|\tilde{\mathcal{C}}_{k-1,\gamma} \tilde{\mathcal{C}}_\varepsilon u\|_U^p.$$

In other words,

$$\|\tilde{\mathcal{C}}^k u\|^p = \lambda^{-d} \sum_{\varepsilon \in \tilde{\Gamma}} \|\tilde{\mathcal{C}}^{k-1} \tilde{\mathcal{C}}_\varepsilon u\|^p.$$

We can bring (4.9) to bear on this only in case $\tilde{\mathcal{C}}_\varepsilon u \in U \cap V$, hence use now

$$\|\tilde{\mathcal{C}}^{k-1} \tilde{\mathcal{C}}_\varepsilon u\| \le \|\tilde{\mathcal{C}}^{k-1} P \tilde{\mathcal{C}}_\varepsilon u\| + \|\tilde{\mathcal{C}}^{k-1} Q \tilde{\mathcal{C}}_\varepsilon u\|,$$

with P denoting the orthoprojector onto $U \cap V$ and $Q := 1 - P$ the complementary projector.

Correspondingly, we define the norm on U as follows. Choose some finite subset B of $V^{\perp}$ for which RB is a basis for $W = RV^{\perp}$ (as we may since W is finite-dimensional). Then, with

$$B^* u := (\langle u, b \rangle : b \in B),$$

necessarily

$$U \cap V = U \cap \ker B^*.$$

We set

$$\|u\|_U := \|Pu\|_p + \|B^* u\|_\infty$$

and see that this is a norm on U since it is a seminorm (as a sum of seminorms), while $\|u\|_U = 0$ implies $Pu = 0$ and $u \in U \cap V$, the latter by choice of B, therefore also $u = Pu$, and so $u = 0$. Note that

$$\|Pu\|_U = \|Pu\|_p \le \|u\|_U, \quad \|Qu\|_U = \|B^* u\|_\infty \le \|u\|_U, \quad u \in U. \tag{4.11}$$

Then,

$$\|\tilde{\mathcal{C}}^k u\| \le \|\tilde{\mathcal{C}}^{k-1} P \tilde{\mathcal{C}} u\| + \|\tilde{\mathcal{C}}^{k-1} Q \tilde{\mathcal{C}} u\|, \tag{4.12}$$

with

$$\|\tilde{\mathcal{C}}^{k-1} A \tilde{\mathcal{C}} u\|^p := \lambda^{-d} \sum_{\varepsilon \in \tilde{\Gamma}} \|\tilde{\mathcal{C}}^{k-1} A \tilde{\mathcal{C}}_\varepsilon u\|^p = \lambda^{-d} \sum_{\varepsilon \in \tilde{\Gamma}} \lambda^{-d(k-1)} \sum_{\gamma \in \tilde{\Gamma}_{k-1}} \|\tilde{\mathcal{C}}_{k-1,\gamma} A \tilde{\mathcal{C}}_\varepsilon u\|_U^p,$$

for $A := P, Q$.

With (4.9) and (4.11), we obtain

$$\|\tilde{\mathcal{C}}^{k-1} P \tilde{\mathcal{C}}_\varepsilon u\| \le \lambda^{-\beta(k-1)} \|P \tilde{\mathcal{C}}_\varepsilon u\|_U \le \lambda^{-\beta(k-1)} \|\tilde{\mathcal{C}}_\varepsilon u\|_U,$$

hence,

$$\|\tilde{\mathcal{C}}^{k-1} P \tilde{\mathcal{C}} u\|^p \le \lambda^{-d} \sum_{\varepsilon \in \tilde{\Gamma}} \lambda^{-\beta(k-1)p} \|\tilde{\mathcal{C}}_\varepsilon u\|_U^p = \lambda^{-\beta(k-1)p} \|\tilde{\mathcal{C}} u\|^p,$$

giving, finally,

$$\|\tilde{\mathcal{C}}^{k-1} P \tilde{\mathcal{C}} u\| \le \lambda^{-\beta(k-1)} \|\tilde{\mathcal{C}}\|_{p,U} \|u\|_U. \tag{4.13}$$

For a bound on $\|\tilde{\mathcal{C}}^{k-1} Q \tilde{\mathcal{C}} u\|$, we use the induction hypothesis, (4.10), to get

$$\|\tilde{\mathcal{C}}^{k-1} Q \tilde{\mathcal{C}}_\varepsilon u\| \le \|\tilde{\mathcal{C}}\|_{p,U} (k-1) \lambda^{-\beta(k-2)} \|Q \tilde{\mathcal{C}}_\varepsilon u\|_U. \tag{4.14}$$

With (4.11),

$$\|Q \tilde{\mathcal{C}}_\varepsilon u\|_U = \|B^* \tilde{\mathcal{C}}_\varepsilon u\|_\infty = \max_{b \in B} |\langle \tilde{\mathcal{C}}_\varepsilon u, b \rangle|, \tag{4.15}$$

while

$$\langle \tilde{\mathcal{C}}_\varepsilon u, b\rangle = \langle u, \tilde{\mathcal{S}}_\varepsilon b\rangle = \langle u, R\tilde{\mathcal{S}}_\varepsilon b\rangle.$$

Now, recalling the abbreviation $W := RV^\perp$, we denote by R^{-1} the (linear) right inverse of R that satisfies $R^{-1}(Rb) = b$, all $b \in B$. Let

$$T_\varepsilon := W \to W : w \mapsto R\tilde{\mathcal{S}}_\varepsilon R^{-1} w.$$

Since, with (3.5),

$$\tilde{\mathcal{S}}_\varepsilon = \mathcal{S}_{k_0,\varepsilon} = E^{\lambda\varepsilon}\mathcal{S}^{k_0},$$

we conclude from (4.7) that

$$\|T_\varepsilon w\|_\infty \le \lambda^{-\beta} K \|w\|_W, \qquad w \in W, \tag{4.16}$$

with respect to whatever norm on W and positive constant K was used when k_0 was chosen at the beginning of the proof. Specifically, we now reveal this norm to be

$$\|w\|_W := \|B^{-*}w\|_1, \quad w \in W,$$

with

$$B^- := \{c^- \in \mathcal{Q} : c \in B\}$$

chosen dual to B, i.e., so that

$$w = \sum_{c\in B} c\,\langle w, c^-\rangle, \qquad w \in W.$$

Then, the $B \times B$ matrix defined by $\mathcal{T}(c,b) := \langle R\tilde{\mathcal{S}}_\varepsilon b, c^-\rangle$, $c, b \in B$, represents T_ε with respect to the basis RB of W, hence

$$\|\mathcal{T}\|_1 = \|T_\varepsilon\|_{W,W} \le \|T_\varepsilon\|_{W,\infty} \|\mathrm{id}_{|W}\|_{\infty,W}.$$

Thus, with the choice

$$K := 1/\|\mathrm{id}_{|W}\|_{\infty,W},$$

we conclude that $\|\mathcal{T}\|_1 \le \lambda^{-\beta}$.

With this,

$$|\langle \tilde{\mathcal{C}}_\varepsilon u, b\rangle| = |\langle u, R\tilde{\mathcal{S}}_\varepsilon b\rangle| = |\sum_{c\in B} \langle u, c\rangle \mathcal{T}(c,b)| \le \|B^* u\|_\infty \|\mathcal{T}\|_1 \le \|u\|_U \lambda^{-\beta},$$

the last inequality also using (4.11). This, together with (4.14) and (4.15), gives

$$\|\tilde{\mathcal{C}}^{k-1} Q \tilde{\mathcal{C}} u\| = (\lambda^{-d} \sum_{\varepsilon\in\tilde{\Gamma}} \|\tilde{\mathcal{C}}^{k-1} Q \tilde{\mathcal{C}}_\varepsilon u\|)^{1/p} \le \|\tilde{\mathcal{C}}\|_{p,U} (k-1) \lambda^{-\beta(k-2)} \lambda^{-\beta} \|u\|_U. \tag{4.17}$$

Thus, on using (4.13) and (4.17) in (4.12), we obtain

$$\|\tilde{\mathcal{C}}^{k-1}\tilde{\mathcal{C}} u\| \le \lambda^{-\beta(k-1)} \|\tilde{\mathcal{C}}\|_{p,U} \|u\|_U + \|\tilde{\mathcal{C}}\|_{p,U} (k-1) \lambda^{-\beta(k-2)} \lambda^{-\beta} \|u\|_U,$$

which is (4.10), i.e., what we had to show. □

Remark. While we have stated and proved the Double-Tree Theorem in its full [NRS] generality, we will only use it for the case when $V^\perp$ is $\mathcal{S}$-nilpotent, i.e., $\mathcal{S}^r V^\perp = 0$ for some r. In this case, the proof of the theorem can be simplified as follows: if we assume that the parameter k_0 that appears in the proof satisfies $k_0 \ge r$, then, in the notations of the proof, $V^\perp \subset \ker \tilde{\mathcal{S}}$, which implies (cf. (3.4)) that $\tilde{\mathcal{S}}_\varepsilon V^\perp = 0$, for every $\varepsilon \in \Gamma$. From that we conclude that $Q\tilde{\mathcal{C}}_\varepsilon u$ in (4.12) equals 0, hence that $\tilde{\mathcal{C}}_\varepsilon$ maps U into $U \cap V$. In this case, assumption (4.3) delivers directly the necessary bound. Moreover, the conclusion (4.5) is valid then for $\beta = \alpha$, and not only for $\beta < \alpha$. □

5. Convergence of the cascade algorithm for box splines

We are ready for a proof of our main result (relying throughout this section on the Appendix to supply whatever specific information concerning the box spline M_Ξ is needed).

(5.1)Theorem. *Let $\varphi = M_\Xi$ be the box spline with direction matrix Ξ of full rank. Let*

(i) m_0 be the minimum over all $\#(\Xi\backslash\Xi_t)$ as $t \in [0\,..\,2\pi)^d\backslash 0$ ranges over all vectors for which Ξ_t is of full rank and $\mathrm{e}_{t|}$ is malignant with respect to Ξ (setting $m_0 := \#\Xi + 1$ in case there is no such t);

(ii) $m_p := m(\Xi) + 1/p$, with $p \in [1\,..\,\infty]$ given and fixed, and $m(\Xi)$ defined as in (1.12); and

(iii) m_s be the largest integer m for which (b) of (1.5) is valid.

Then, given $\alpha > 0$, the cascade iterations converge to M_Ξ at rate α in the p-norm on every initial seed $g \in G_{p,\alpha}(M_\Xi)$, provided that $\alpha < m_p$, and that $\alpha \le \min\{m_0, m_s\}$.

Proof: Fix $p \in [1\,..\,\infty]$, and let $g \in G_{p,\alpha}(M_\Xi)$, with $\alpha < m(\Xi) + 1/p$. Then $M_\Xi \in W_p^\alpha(\mathbb{R}^d)$, hence, as outlined in §1 and also discussed in [R2], we already know that, for any compactly supported distribution ν, if $\nu{*}M_\Xi$ is continuous, and if $1-\widehat{g\nu}$ has a zero of order α at the origin, the cascade algorithm on $g{*}v$, $v := (\nu{*}M_\Xi)_|$, converges in the p-norm to M_Ξ at a rate α. In order to draw a conclusion about the rate of convergence of $(C^k g)_k$ to M_Ξ, we examine the convergence rate to 0 of the cascade iterations $C^k(g * (\delta_0 - v)))$, with v as above. We note that the Fourier transform $\widehat{M_\Xi}$ of the box spline (see (6.2)) has a zero of order $m(\Xi)+1$ at each $\omega \in 2\pi\mathbb{Z}^d\backslash 0$. Moreover, Poisson's summation formula yields that

$$\widehat{v} = \sum_{\omega \in 2\pi\mathbb{Z}^d} (\widehat{\nu}\widehat{M_\Xi})(\cdot + \omega),$$

and consequently $1-\widehat{v}$ has a zero of order $\le m(\Xi)+1$ at the origin if and only if $1-\widehat{\nu}\widehat{M_\Xi}$ has such zero at the origin. Since α here is $< m(\Xi)+1$, we can then appeal to (d) of (1.5)Assumptions to conclude that, since $1-\widehat{\nu}\widehat{g}$ has a zero of order α at the origin, so does $1-\widehat{v}$. Thus, the sequence $u := \delta_0 - v$ has α zero moments: $u \perp (\Pi_{<\alpha})_|$ (with $\Pi_{<\alpha}$ the space of polynomials of degree $< \alpha$). The proof will be complete once we determine the convergence rate to 0 of the cascade iterations starting with the initial seed $g{*}u$. We consider now two different cases:

Case I: $\alpha \le m(\Xi)$. In this case, we show that

$$\|\mathcal{C}^k\|_{p,U} = O(2^{-mk}), \tag{5.2}$$

with $U := U_m$ a suitable space of compactly supported sequences perpendicular to $\Pi_{<m}$, and with m the least integer $\ge \alpha$ (hence $m \le \min\{m_0, m_s, m(\Xi)\}$).

For its proof by the Double-Tree theorem, we choose (as outlined already in §3) the bounded convex subset $\Omega := \Omega_2$ of $\mathbb{R}^d$. Then $\mathcal{Q}_\Omega$ is $\mathcal{C}$-invariant. The assumption $m \le m_s$ guarantees that $\mathcal{C}$ also leaves invariant the orthogonal complement U of $(\Pi_{<m})_|$ in $\mathcal{Q}_\Omega$.

We choose $V := V_m$ to be the shift-invariant space (in $\mathcal{Q}$) generated by the convolution product sequences

$$u_Y := (\delta_0 - \delta_\xi) * \cdots * (\delta_0 - \delta_\zeta), \quad Y := [\xi, \ldots, \zeta] \subset \Xi,\ \#Y = m.$$

We claim first that

$$\|\mathcal{C}^k u_Y\|_\infty \le 2^{-mk}, \quad Y \subset \Xi,\ \#Y = m. \tag{5.3}$$

Indeed, consider $\mathcal{C}^k u$ for the particular mesh function

$$u = u_\xi := \delta_0 - \delta_\xi,$$

for some $\xi \in \Xi$. Then (see (1.7) and (6.3), (6.5), and (6.6)),

$$\mathcal{C}^k u_\xi = \mathcal{D}^k u_\xi * a_\Xi^{(1/2^k)} = \mathcal{D}^k u_\xi * b_{[\xi]}^{(1/2^k)} * a_{\mathrm{Z}}^{(1/2^k)},$$

with

$$\mathrm{Z} \cup \xi := \Xi.$$

Since

$$b_{[\xi]}^{(1/2^k)} = 2^{-k} \sum_{j=0}^{2^k-1} \delta_{-j\xi/2^k},$$

it follows that $\mathcal{D}^k u_\xi * b_{[\xi]}^{(1/2^k)} = 2^{-k} u_\xi$, hence

$$\mathcal{C}^k u_\xi = 2^{-k} u_\xi * a_{\mathrm{Z}}^{(1/2^k)}.$$

Since $\mathcal{D}^k(u*v) = (\mathcal{D}^k u)*(\mathcal{D}^k v)$, it follows that, correspondingly,

$$\mathcal{C}^k u_Y = 2^{-k\#Y} u_Y * a_{\Xi\backslash Y}^{(1/2^k)}, \quad Y \subset \Xi. \tag{5.4}$$

(This establishes the fact, of use later, that $\mathcal{C}^k u_Y$ lies in the span of the shifts of u_Y, hence that our chosen V is not only shift-invariant but also $\mathcal{C}$-invariant.)

Note that $\#Y = m \le m(\Xi)$ hence $\mathrm{Z} := \Xi \backslash Y$ is of full rank. We claim that, for any $\mathrm{Z} \subset \Xi$ of full rank, $a_{\mathrm{Z}}^{(h)}$ is bounded independently of h. From the definition (6.5),

$$a_{\mathrm{Z}}^{(h)} = h^{\#\mathrm{Z}-d} \sum_{j \in \mathrm{m}_h^{\#\mathrm{Z}}} \delta_{\mathrm{Z}j}.$$

In particular,

$$\|a_{\mathrm{Z}}^{(h)}\|_\infty = 1$$

if Z is a basis for $\mathbb{R}^d$. Also,

$$a_{\mathrm{Z}\cup\xi}^{(h)} = b_{[\xi]}^{(h)} * a_{\mathrm{Z}}^{(h)},$$

while

$$\|b_{[\xi]}^{(h)}\|_1 := \sum_{j \in \mathrm{m}_h} b_{[\xi]}^{(h)}(j) = h\#\mathrm{m}_h = 1.$$

Hence

$$\|a_{\mathrm{Z}\cup\xi}^{(h)}\|_\infty \le \|a_{\mathrm{Z}}^{(h)}\|_\infty.$$

This proves (4.3) for $p = \infty$. For $p < \infty$, since $m \le m(\Xi)$ (which is necessarily the case for $p = \infty$, but need not to be the case for smaller p), then (4.3) extends to $p \in [1\,..\,\infty]$, since, always,

$$\|\mathcal{C}^k\|_{p,U\cap V} \le \|\mathcal{C}^k\|_{\infty,U\cap V}.$$

Case II: $\alpha > m(\Xi)$. In this case, $m_s = m(\Xi) + 1 \le m_0$ and $p < \infty$. We choose $m := m(\Xi) + 1$. We deal with this case by modifying the proof of the case $p = \infty$. In the current case, Y has $m(\Xi) + 1$ columns, hence it is possible that

$$\mathrm{Z} := \Xi \backslash Y$$

is *not* of full rank. In that case, Z is guaranteed to have at least rank $d - 1$. We use now (5.4), which tells us that

$$\|\mathcal{C}^k u_Y\|_{p,U} \le \mathrm{const} 2^{-k\#Y} \|\mathcal{C}_{\mathrm{Z}}^k \delta_0\|_{p,U}.$$

Here, we use the facts that (i) the sequence u_Y in the right-hand-side of (5.4) has bounded ℓ_1-norm, and (ii) $a_{\mathrm{Z}}^{(h)} = \mathcal{C}_{\mathrm{Z}}^k \delta_0$, with $\mathcal{C}_{\mathrm{Z}}$ the discrete cascade operator associated with M_{Z}, and with $h = 2^{-k}$. Straightforward modifications to the $p = \infty$ proof yield then that

$$\|\mathcal{C}_{\mathrm{Z}}^k \delta_0\|_{\infty,U} = \|a_{\mathrm{Z}}^{(h)}\|_\infty = 2^k.$$

However, when estimating $\|\mathcal{C}_{\mathrm{Z}}^k \delta_0\|_{p,U}$ for $p < \infty$, we can take advantage of the fact that, since $\mathcal{C}_{\mathrm{Z}}^k \delta_0$ is supported on a hyperplane, its support contains $O(2^{k(d-1)})$ points, hence that, since the (p, U)-norm is normalized by $2^{kd/p}$, we get an extra $2^{k/p}$ factor to spare. Hence

$$\|\mathcal{C}_{\mathrm{Z}}^k \delta_0\|_{p,U} \le \mathrm{const} 2^{k(1-1/p)}.$$

Thus our estimate here is

$$\|\mathcal{C}^k u_Y\|_{p,U} \le \text{const} 2^{-k(m(\Xi)+1)} 2^{k(1-1/p)} = \text{const} 2^{-k(m(\Xi)+1/p)},$$

which is what we need since $\alpha < m(\Xi) + 1/p$.

It remains to verify (4.4). For this, observe that

$$V^\perp = \cap_{Y\subset\Xi, \#Y=m} \ker \nabla_Y,$$

with

$$\nabla_Y := \prod_{\xi\in Y} \nabla_y$$

and

$$\nabla_y : \mathcal{Q} \to \mathcal{Q} : q \mapsto q - q(\cdot - y).$$

Since $m \le m(\Xi) + 1$, $V^\perp$ necessarily lies in

$$\Delta(\Xi) := \cap_{\mathrm{Z}\subset\Xi, \mathrm{rank}(\Xi\backslash \mathrm{Z})<d} \ker \nabla_{\mathrm{Z}}.$$

Now (cf. [BHR: (II.49)Theorem]),

$$\Delta(\Xi) = \bigoplus (\mathrm{e}_t P_t)_{|}, \tag{5.5}$$

with the sum ranging over all $t \in [0\,.\,.\,2\pi)^d$ for which

$$\Xi_t := [\xi \in \Xi : \mathrm{e}_t(\xi) = 1]$$

is of full-rank d, and $P_t := P_{t,\Xi}$ certain finite-dimensional subspaces spanned by homogeneous polynomials. In particular, $\Delta(\Xi)$ is finite-dimensional, hence, so is $V^\perp$. We next observe that, since $m \le m(\Xi)+1$, [BDyR: Lemma 7.15] implies that, for every t as above, we have $V^\perp \cap (\mathrm{e}_t P_t)_| \subset (\mathrm{e}_t \Pi_{<m})_|$. In particular, this is the case for $t = 0$. Also, the shift-invariance of $V^\perp$ ensures that

$$V^\perp = V^\perp \cap \Delta(\Xi) = \bigoplus (V^\perp \cap (\mathrm{e}_t P_t)_|).$$

We write now

$$V^\perp = K_0 + K_1,$$

with $K_0 := V^\perp \cap P_{0|}$, and K_1 the sum of all other summands, i.e., $K_1 := \bigoplus_{t\ne 0}(V^\perp \cap (\mathrm{e}_t P_t)_|)$. The space $V^\perp$ is $\mathcal{S}$-invariant, and this readily implies that K_1 is also $\mathcal{S}$-invariant. Moreover, $K_0 \subset \Pi_{<m|}$, and, thanks to (b) of (1.5)Assumptions (viz., to our assumption that $m \le m_s$), $\Pi_{<m|}$ is also $\mathcal{S}$-invariant, hence so is K_0. Since our use of the Double-Tree Theorem is done with respect to the space U that contains only sequences with m vanishing moments, we have that $K_0 \subset U^\perp$. Now, checking the sole use made of (4.4) in

the proof of the Double-Tree Theorem, it concerns bounding $\langle u, \tilde{\mathcal{S}}_\varepsilon b\rangle$ for $b \in B$, with B chosen in $V^\perp$ so as to provide a suitable semi-norm

$$u \mapsto \|B^* u\|_\infty = \max_{b\in B} |\langle u, b\rangle|$$

on U. Since U is perpendicular to K_0, we may assume without loss that B was chosen from K_1, hence may assume without loss that $V^\perp = K_1$, as we do from now on.

We will finally invoke the remaining assumption, $m \le m_0$, and show that K_1 is a nilpotent subspace of $\mathcal{S}$. This will certainly settle (4.4), hence will bring the proof to its conclusion. Since K_1 is shift-invariant, finite-dimensional, and $\mathcal{S}$-invariant, it is sufficient to prove (in view of Lemma 2.7) that each exponential $\mathrm{e}_{t|} \in K_1$ is $\mathcal{S}$-nilpotent. By the definition of K_1, each such exponential is annihilated by ∇_Y, $Y \subset \Xi$, $\#Y = m$. Note that $\nabla_Y \mathrm{e}_{t|} = 0$ iff $\nabla_y \mathrm{e}_{t|} = 0$ for some $y \in Y$. Also, $\nabla_y \mathrm{e}_{t|} = 0$ for $y \in \Xi$ if and only if $y \in \Xi_t$. Now, if $\mathrm{e}_{t|}$ is malignant, then $\#(\Xi\backslash\Xi_t) \ge m_0 \ge m$, hence, by choosing Y to be any m-submatrix of $\Xi\backslash\Xi_t$, we obtain ∇_Y that does not annihilate $\mathrm{e}_{t|}$. Thus, if $\mathrm{e}_{t|} \in K_1$, it cannot be malignant, hence must be $\mathcal{S}$-nilpotent. Invoking Lemma 2.7, we conclude that K_1 is $\mathcal{S}$-nilpotent. □

6. Appendix: Box splines

From [BHR], we recall the following basic box spline facts.

The **box spline**, M_Ξ, associated with the matrix $\Xi \in (\mathbb{Z}^d\backslash 0)^n$, is the distribution

$$M_\Xi : f \mapsto \int_{[0..1)^\Xi} f(\Xi t)\,\mathrm{d}t,$$

where, here and below,

$$A^\Xi$$

denotes the set of sequences, indexed by the columns ξ of Ξ, with entries in A. In particular,

$$M_{[\,]} = \delta_0,$$

and

$$M_{\Xi\cup\zeta} = \int_0^1 M_\Xi(\cdot - t\zeta)\,\mathrm{d}t.$$

$M_\Xi \in C^s$, with $s+2$ the minimum over $\#\mathrm{Z}$ with $\Xi\backslash\mathrm{Z}$ not of full rank. Thus, $s = m(\Xi)-1$, with $m(\Xi)$ the largest m for which $\Xi\backslash\mathrm{Z}$ is of full rank for all $\#\mathrm{Z} \le m$. Moreover, as any continuous compactly supported piecewise-polynomial, the derivatives of M_Ξ of order $m(\Xi)-1$ are all in Lip_1, hence $M_\Xi \in W^\alpha_\infty$ for every $\alpha < m(\Xi)$. Since the box spline is C^∞ on the complement of a compact subset of a finite union of hyperplanes, this further implies that $M_\Xi \in W^\alpha_p$ for every $1 \le p < \infty$ and $\alpha < m(\Xi) + 1/p$.

The **discrete box spline**, $b_\Xi^{(h)}$, is defined for any $h \in 1/\mathbb{N}$ as the distribution

$$b_\Xi^{(h)} : f \mapsto h^{\#\Xi} \sum_{j \in \mathrm{m}_h^\Xi} f(\Xi j)$$

with

$$\mathrm{m}_h := \{0, h, \ldots, 1-h\}.$$

We denote by $b_\Xi^{(h)}$ also the corresponding discretely defined function,

$$b_\Xi^{(h)} =: \sum_t b_\Xi^{(h)}(t)\delta_t.$$

Since

$$\lim_{h\to 0} \sum_{\mathrm{m}_h^\Xi} f(\Xi j) = \int_{[0..1)^\Xi} f(\Xi t)\,\mathrm{d}t,$$

$b_\Xi^{(h)}$ converges, pointwise on $C(\mathbb{R}^d)$, to M_Ξ as $h \to 0$, thus justifying the name.

Its Fourier transform (see [BHR: (VI.9)]) is

$$\widehat{b_\Xi^{(h)}} = \widehat{M_\Xi}/\widehat{M_\Xi}(h\cdot). \tag{6.1}$$

Since [BHR: (I.17)]

$$\widehat{M_\Xi} = \prod_{\xi\in\Xi} \frac{1 - \mathrm{e}_{-\xi}}{\mathrm{i}\xi\cdot} \tag{6.2}$$

(with $\xi\cdot : \mathbb{R}^d \to \mathbb{R} : x \to \sum_k \xi(k)x(k)$), this says that

$$\widehat{b_\Xi^{(h)}} = h^{\#\Xi} \prod_{\xi\in\Xi} \frac{1 - \mathrm{e}_{-\xi}}{1 - \mathrm{e}_{-h\xi}}.$$

In particular (cf. [BHR: (VI.12]),

$$\widehat{b_\Xi^{(1/2)}} = \prod_{\xi\in\Xi} \frac{1 + \mathrm{e}_{-\xi/2}}{2}. \tag{6.3}$$

We note that

$$b_{\mathrm{Y}}^{(h)} * b_{\mathrm{Z}}^{(h)} = b_{\mathrm{Y}\cup\mathrm{Z}}^{(h)}.$$

Also, for $h, h' \in 1/\mathbb{N}$,

$$b_\Xi{}^{(hh')} = b_\Xi{}^{(h)} * b_\Xi{}^{(h')}(\cdot/h), \tag{6.4}$$

by (6.1).

The box spline M_Ξ is refinable; precisely (see, e.g., [BHR: p. 141]), for any $h \in 1/\mathbb{N}$,

$$M_\Xi = M_\Xi(\cdot/h) * a_\Xi^{(h)},$$

with

$$a_\Xi^{(h)} := b_\Xi{}^{(h)}/h^d. \tag{6.5}$$

More than that, by (6.4),

$$a_\Xi^{(hh')} = b_\Xi^{(hh')}/(hh')^d = b_\Xi^{(h')}(\cdot/h) * b_\Xi^{(h)}/(hh')^d = a_\Xi^{(h')}(\cdot/h) * a_\Xi^{(h)}.$$

Therefore, in particular, with

$$a = a_\Xi := a_\Xi^{(1/2)},$$

we have

$$a^{[k]} = \mathcal{D}^{k-1}a * \mathcal{D}^{k-2}a * \cdots * \mathcal{D}^0 a = a_\Xi^{(1/2^k)}. \tag{6.6}$$

We also need the following

(6.7) Proposition. *If* $t \in [0\,.\,.\,2\pi)^d \backslash 0$, *and*

$$\Xi_t = [\xi \in \Xi : \mathrm{e}_t(\xi) = 1]$$

is of full rank, then $M_\Xi * \mathrm{e}_{t|} = 0$, *and the smallest* $\mathcal{S}$*-invariant space containing* $\mathrm{e}_{t|}$ *is finite-dimensional.*

A direct proof of the first claim is given in [BHR: proof of (II.55)]. As to the second claim, any such 'discrete exponential' $\mathrm{e}_{t|}$ lies in the space $\Delta(\Xi)$ defined in (5.5), and this space is finite-dimensional (as already observed in §5). More than that,

$$\Delta(\Xi) = W^\perp,$$

with W the shift-invariant subspace of $\mathcal{Q}$ spanned by $\{u_Y : Y \subset \Xi, \operatorname{rank}(\Xi\backslash Y) < d\}$, hence, by (5.4), $\mathcal{C}$-invariant, and this implies that $\Delta(\Xi)$ is also $\mathcal{S}$-invariant.

References

[BDeR] C. de Boor, R. DeVore, and A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$, *J. Funct. Anal.* **119(1)** (1994), 37–78.

[BDyR] C. de Boor, N. Dyn, and A. Ron, Polynomial interpolation to data on flats in $\mathbb{R}^d$, *J. Approx. Theory* **105(2)** (2000), 313–343.

[BHR] C. de Boor, K. Höllig, and S. D. Riemenschneider, "Box Splines", Springer-Verlag, New York, 1993.

[BR] C. de Boor and A. Ron, The exponentials in the span of the multi-integer translates of a compactly supported function, *J. London Math. Soc.* **45(3)** (1992), 519–535.

[HJ] Bin Han and Rong-Qing Jia, Multivariate refinement equations and convergence of subdivision schemes, *SIAM J. Math. Anal.* **29** (1998), 1177–1199.

[J] Rong-Qing Jia, Subdivision schemes in L_p spaces, *Adv. Comput. Math.* **3** (1995), 309–341.

[N] Marian Neamtu, On discrete simplex splines and subdivision, *J. Approx. Theory* **70(3)** (1992), 358–374.

[NRS] M. Neamtu, A. Ron, and Zuowei Shen, The cascade and subdivision algorithms, ms. 27 July, 1998.

[R1] A. Ron, Smooth refinable functions provide good approximation orders, *SIAM J. Math. Anal.* **28** (1997), 731–748.

[R2] A. Ron, Wavelets and their associated operators - A survey, *in* "Approximation Theory IX, Vol. 2: Computational Aspects" pp. 283–317, Vanderbilt University Press, Nashville TN, 1998.