Box-Spline Tilings

Carl de Boor[1,2] and Klaus Höllig[2,3]

Abstract. We describe a simple method for generating tilings of $\mathbb{R}^d$. The basic tile is defined as

$$\Omega := \{x \in \mathbb{R}^d : |f(x)| < |f(x+j)| \quad \forall j \in \mathbb{Z}^d \backslash 0\},$$

with f a real analytic function for which $|f(x+j)| \to \infty$ as $|j| \to \infty$ for almost every x. We show that the translates of $\overline{\Omega}$ over the lattice $\mathbb{Z}^d$ form an essentially disjoint partition of $\mathbb{R}^d$. As an illustration of this general result, we consider in detail the special case $d = 2$ and

$$f(x) := (\xi^{\mathrm{t}} x)(\eta^{\mathrm{t}} x)$$

with ξ, η column vectors in $\mathbb{Z}^2$. Already this simple choice, which arises in box-spline theory, yields rather interesting partitions of $\mathbb{R}^2$.

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Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a real analytic function such that, for almost all x and for $j \in \mathbb{Z}^d$,

$$|f(x+j)| \to \infty \quad \text{as } |j| \to \infty. \tag{1}$$

Then the translates of the set

$$\Omega := \Omega(f) := \{x \in \mathbb{R}^d : |f(x)| < |f(x+j)| \ \ \forall j \in \mathbb{Z}^d \backslash 0\} \tag{2}$$

provide a tiling for $\mathbb{R}^d$, in the following sense.

Theorem. *The sets $\overline{\Omega} + j$, $j \in \mathbb{Z}^d$, form an essentially disjoint partition of $\mathbb{R}^d$, i.e.*

(i) $\overline{\Omega} \cap (\Omega + j) = \emptyset \quad \forall j \neq 0$;

(ii) $\operatorname{meas}\big(\mathbb{R}^d \backslash (\Omega + \mathbb{Z}^d)\big) = 0$;

(iii) $\operatorname{meas}(\Omega) = 1$.

Such sets Ω arise in box spline theory, in the characterization of functions of exponential type as limits of multivariate cardinal series (cf. the Appendix). In that setting, the functions f have the simple form

$$f_\Xi(x) = \prod_{\xi \in \Xi} \xi^{\mathrm{t}} x,$$

in which x, ξ are taken to be column matrices, Ξ is a multiset from $\mathbb{Z}^d \backslash 0$ which spans $\mathbb{R}^d$, and ξ^{t} denotes the transpose of ξ. Already for $d = 2$ and for Ξ consisting of just two vectors, even these very simple f give rise to surprisingly complex (and strangely beautiful) $\Omega = \Omega_\Xi$.

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(3)Figure. $\overline{\Omega}_\Xi$ for $\Xi = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

Proof of the theorem To prove (i), let $x = \lim x_n$ with $x_n \in \Omega$ and $x - j \in \Omega$. Then, the definition of Ω leads to the contradiction

$$1 > \frac{|f(x-j)|}{|f\big((x-j)+j\big)|} = \frac{|f(x-j)|}{|f(x)|} = \lim \frac{|f(x_n - j)|}{|f(x_n)|} \geq 1.$$

For the proof of (ii), we deduce from (1) that the function

$$j \mapsto f(x+j)$$

has a mimimum for almost all x. If this minimum is unique, then there exists j^* so that

$$|f(x+j^*)| < |f(x+j)| \quad \forall j \neq j^*,$$

and therefore $x \in \Omega - j^*$. Consequently, up to a set of measure zero, the set $\mathbb{R}^d \backslash (\Omega + \mathbb{Z}^d)$ lies in the union of the zero sets of the (countably many) functions

$$g(x) := |f(x+j)|^2 - |f(x+k)|^2, \quad j \neq k.$$

Since each such g is analytic, its zero set is of measure zero unless g vanishes identically. But, this latter possibility is excluded since $g = 0$ implies that f is periodic in the direction $j - k$ and this would contradict assumption (1).

For the proof of (iii), we conclude from (i) and (ii) that, up to a set of measure zero, $[0,1]^d$ is the disjoint union of the sets $[0,1]^d \cap (\Omega + j)$ with $j \in \mathbb{Z}^2$, while Ω is the disjoint union of the sets $\big([0,1]^d - j\big) \cap \Omega$ with $j \in \mathbb{Z}^2$, and

$$\text{meas}\big([0,1]^d - j\big) \cap \Omega = \text{meas}\big([0,1]^d \cap (\Omega + j)\big).$$

$\square$

(4)Figure. $\Omega_{\left[\begin{smallmatrix}1&3\\3&1\end{smallmatrix}\right]}$ rearranged to fill the unit square.

Special case In this paper, we limit ourselves to the very special case

$$f(x) = (\xi^{\mathrm{t}}x)(\eta^{\mathrm{t}}x) \quad x \in \mathbb{R}^2$$

with $\xi, \eta \in \mathbb{Z}^2$ linearly independent.

In this situation, it is convenient to introduce the new variables

$$(u, v) := \Xi^{\mathrm{t}}x = (\xi^{\mathrm{t}}x, \eta^{\mathrm{t}}x).$$

In these new coordinates, the definition of Ω becomes

$$\Omega(\Gamma) := \{(u, v) : |u||v| < |u + \alpha||v + \beta| \text{ for } (\alpha, \beta) \in \Gamma\backslash 0\}$$

with

$$\Gamma := \Xi^{\mathrm{t}}\mathbb{Z}^2$$

a sublattice of $\mathbb{Z}^2$.

(5)Figure. Sublattice Γ for $\Xi = \left[\begin{smallmatrix}1&3\\3&1\end{smallmatrix}\right]$.

The original Ω can always be recovered via the linear transformation

$$\Omega(\Xi) = (\Xi^{\mathrm{t}})^{-1}\Omega(\Gamma).$$

Therefore, in the new coordinates,

$$\text{meas}(\Omega) = |\det \Xi|. \tag{6}$$

Also, the tiling is now obtained by translating Ω over the *sublattice* Γ (rather than over $\mathbb{Z}^2$). On the other hand, we have gained much simplicity since now all possible Ω are intersections of some of the *same* sets $\Omega_{\alpha,\beta}$ with

$$\Omega_{\alpha,\beta} := \{(u, v) : |u||v| < |u + \alpha||v + \beta|\}$$

(see (7)Figure), different Ω being obtained from different choices of the sublattice Γ.

(7)Figure. $\Omega_{\alpha,\beta}$ for $\alpha = -1, 1, 2, 3$ and $\beta = -1, \ldots, 2$.

Symmetries We now investigate how many essentially different tiles we can obtain in this way. We begin by noting the following obvious symmetries.

(i) Since $\Gamma = -\Gamma$, we also have $\Omega = -\Omega$.

(ii) Γ does not change if Ξ' is multiplied from the right by a unimodular matrix, i.e. an integer matrix with determinant ± 1.

In particular, we may restrict attention to Ξ' of the form

$$\begin{bmatrix} p & a \\ 0 & \varepsilon \end{bmatrix} \qquad \text{with } p := |\det \Xi|/\varepsilon,\ \varepsilon := \gcd(\eta_1, \eta_2),$$

and $a \in [0, p[$. For, with σ the appropriate sign, $\eta^* := \sigma(\eta_2, -\eta_1)/\varepsilon \in \mathbb{Z}^2$ is carried by Ξ' to $(\sigma \det \Xi/\varepsilon, 0) = (p, 0)$, while the fact that η_1/ε and η_2/ε are relatively prime implies the existence of an integer vector y for which $\eta^{\mathrm{t}} y = \varepsilon$. Thus, for some choice of the integer c, Ξ' carries $\gamma := c\eta^* + y \in \mathbb{Z}^2$ to (a, ε) with $a \in [0, p[$. Consequently, $\left[\begin{smallmatrix} p & a \\ 0 & \varepsilon \end{smallmatrix}\right] = \Xi' \left[\eta^*, \gamma\right]$, with $\left[\eta^*, \gamma\right]$ necessarily unimodular.

(iii) The scaling

$$\Gamma \mapsto \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \Gamma$$

changes Ω correspondingly to

$$\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \Omega.$$

We consider such Ω obtainable one from the other by such scaling as essentially the same. This means that we may further restrict attention to Ξ' of the form $\Xi' = \left[\begin{smallmatrix} p & a \\ 0 & 1 \end{smallmatrix}\right]$ with $0 < a < p$ and $a \not| p$. In fact, since

$$\begin{bmatrix} p & p - a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$

it is sufficient to consider Ξ' of the form

$$\begin{bmatrix} p & a \\ 0 & 1 \end{bmatrix}, \qquad \text{with } 0 < a < p/2 \text{ and } a \not| p. \tag{8}$$

In particular, there is just one lattice of interest for each value of $p < 5$, and $p = 7$ is the first value for which there are, offhand, three lattices of interest.

The resulting lattices

$$\Gamma = \Gamma_{p,a} := \begin{bmatrix} p & a \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2, \quad 0 < a < p/2, \ a \not| p,$$

are indeed different one from the other in that, e.g., $(a, 1)$ is the only point in $\Gamma_{p,a}$ of the form $(b, 1)$ with $0 \le b < p$. This follows from the fact that

$$\min\{b > 0 : (b, 0) \in \Gamma_{p,a}\} = p. \tag{9}$$

The corresponding statement

$$\min\{b > 0 : (0, b) \in \Gamma_{p,a}\} = p \tag{10}$$

also holds since

$$(\Xi')^{-1} = \begin{bmatrix} 1/p & -a/p \\ 0 & 1 \end{bmatrix},$$

hence $(\Xi')^{-1}(0, b) = (-ba/p, b)$, and, since $a \not| p$, this is in $\mathbb{Z}^2$ iff $p|b$.

Bounds We conclude from (9) and (10) that

$$\Omega \subset \Omega_{\alpha,0} \cap \Omega_{-\alpha,0} \cap \Omega_{0,\alpha} \cap \Omega_{0,-\alpha}$$

with $\alpha = p$. The sets appearing on the right hand side are halfspaces (cf. (7)Figure); e.g.

$$\Omega_{\alpha,0} = \{(u, v) : u > -\alpha/2\}.$$

Consequently,

$$\Omega \subset (p/2)[-1, 1]^2. \tag{11}$$

Note that this bounding square has area p^2, while Ω has area p. This implies that $\Omega = [-1, 1]^2/2$ when $p = 1$. It indicates that, for large p, Ω is a rather small subset of this bounding square.

Certain lines are excluded from Ω. Since $|u + \alpha| = 0$ for $u = -\alpha$, Ω cannot contain any point (u, v) with $u = -\alpha$, for which $(\alpha, \beta) \in \Gamma$ for some β. This condition holds for every $\alpha \in \mathbb{Z}\backslash 0$, hence Ω meets none of the lines $u + \alpha = 0$ (therefore also none of the lines $v + \alpha = 0$) for $\alpha \in \mathbb{Z}\backslash 0$.

(12)Figure. Ω must lie inside such a set.

We conclude from (11) that, in constructing $\Omega = \bigcap_{j\in\Gamma} \Omega_j$, we only need to consider

$$j \in p[-1,1]^2. \tag{13}$$

For, if $(u,v) \in (p/2)[-1,1]^2$ and, e.g., $(\alpha,\beta) > 0$, then

$$|u+\alpha||v+\beta| < |u+\alpha+mp||v+\beta+np|$$

for any positive integers m and n. Consequently

$$x \in (p/2)[-1,1]^2 \cap \bigcap_{j\in\Gamma\cap[0,p]^2} \Omega_j \implies x \in \bigcap_{j\in\Gamma\cap\mathbb{Z}_+^2} \Omega_j.$$

Figures We conclude this note with pictures of the first few essentially different tilings obtained in this special case.

For every p, there is a lattice Γ generated by $(p,0)$ and $(1,1)$, viz. $\Gamma = \Gamma_{p,1}$. For $p = 1$, the corresponding tile is the centered square of side length 1. For $p = 2$, it is the centered diamond with side length 2, i.e., the diamond with vertices at the unit vectors. As p increases, the central portion of the confining set shown in (12)Figure is too small to contain all of Ω, and Ω sprouts four arms. The lattice is invariant under the map $(u,v) \mapsto (v,u)$ (in addition to the symmetry $\Gamma = -\Gamma$ observed earlier), hence so is Ω. The resulting four-fold symmetry implies that, in constructing Ω, only one of its four 'arms' need be calculated. The corresponding Ω all look similar, and the following figure gives a typical example.

(14)Figure. $\overline{\Omega}$ for $\Xi^{\mathrm{t}} = \begin{bmatrix} 8 & 1 \\ 0 & 1 \end{bmatrix}$.

The first tiling of a different kind occurs for $p = 5$. Since its lattice, $\Gamma_{5,2}$, is invariant under rotation of 90°, so is the tile.

(15)Figure. $\overline{\Omega}$ for $\Xi^{\mathrm{t}} = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$.

Here are the next few ‘unorthodox’ tiles.

(16)Figure. $\overline{\Omega}$ for $\Xi^{t} = \begin{bmatrix} 7 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 8 & 3 \\ 0 & 1 \end{bmatrix}.$

(17)Figure. $\overline{\Omega}$ for $\Xi^{t} = \begin{bmatrix} 9 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 10 & 3 \\ 0 & 1 \end{bmatrix}.$

Based on the above figures, one might conjecture that the set Ω is confined to the union

$$[-1,1] \times [-p/2, p/2] \ \cup \ [-p/2, p/2] \times [-1,1]$$

of the two central strips of (12)Figure. As (18)Figure shows, this is in general not true. In fact, rather complicated patterns develop as p increases. The smallest p for which we first encounter a disconnected tile is $p = 15$, and this is the tile shown in (18).

(18)Figure. A disconnected tile: $\Xi^{t} = \begin{bmatrix} 15 & 4 \\ 0 & 1 \end{bmatrix}.$

The next figure shows a more elaborate tile.

(19)Figure. Tiling for $\Xi^{t} = \begin{bmatrix} 17 & 5 \\ 0 & 1 \end{bmatrix}.$

As we mentioned in the beginning, we have considered in this paper a very special choice of f, motivated by results from box-spline theory. Our final figures give a hint of things to come [BH].

(20) Figure. The BUG: generating function $f(x,y) := x^3 + y^3 - 2xy$.

(21) Figure. NOVA: generating function $f(x,y) := x^3 + y^3 - x$.

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Appendix

We discuss briefly the connection to box-spline theory. Let $M : \mathbb{R}^d \mapsto \mathbb{R}$ be a bounded function with compact support and denote by

$$M_n := M * \ldots * M$$

the n-fold convolution of M. Further, denote by

$$S_n := \{ \sum_{j \in \mathbb{Z}^d} M_n(\cdot - j)c(j) : \ c \in \ell_2(\mathbb{Z}^d) \}$$

the linear span of the translates of M_n with square summable coefficients. We showed in [BHR] that a function $g \in L_2(\mathbb{R}^d)$ can be approximated by a sequence $g_n \in S_n$, $n \in \mathbb{N}$, if and only if the support of the Fourier transform of g is contained in the set

$$D(M) := \{ x \in \mathbb{R}^d : \ |\widehat{M}(x + 2\pi j)| < |\widehat{M}(x)|, \ j \in \mathbb{Z} \backslash 0 \}. \tag{22}$$

With minor modifications, this agrees with the definition of the basic tile in (2), i.e.

$$D(M) = 2\pi\Omega(f), \quad \text{with } f := 1/\widehat{M}(2\pi\cdot).$$

Thus the fundamental domain D generates a tiling of $\mathbb{R}^d$.

In the main application of this result, M is chosen as the centered box-spline. Its Fourier transform has the simple form (cf. [BH$_1$], [H])

$$\widehat{M}(x) := \widehat{M_\Xi}(x) := \prod_{\xi \in \Xi} \operatorname{sinc}(\xi^{\mathrm{t}} x/2)$$

where $\operatorname{sinc}(t) := \sin t/t$ and Ξ is a multiset of integer d-vectors. Because of periodicity, the factors $\sin(\xi^{\mathrm{t}}x/2)$ are irrelevant for the definition of the fundamental domain, hence

$$D(M_\Xi) = 2\pi\Omega(f_\Xi), \quad \text{with} \;\; f_\Xi := \prod_{\xi\in\Xi} \xi^{\mathrm{t}}x.$$

The simplest special case, when $d = 2$ and Ξ consists of just two vectors, is considered in the present paper.

References

[BH$_1$] C. de Boor and K. Höllig, B-splines from parallelepipeds, *J. d'Analyse Math.* **42** (1982/83), 99-115.

[BH$_2$] C. de Boor and K. Höllig, Tilings via generating functions, ms.

[BHR] C. de Boor, K. Höllig, and S. D. Riemenschneider, Convergence of Cardinal Series, *Proc. Amer. Math. Soc.* **98** (1986), 936-939.

[H] K. Höllig, Box-splines, in: *Approximation Theory V*, C.K. Chui, L.L. Schumaker and J. Ward, eds., Academic Press 1986, 71-95.