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CONSTRUCTIVE APPROXIMATION

Convergence of Bivariate Cardinal Interpolation

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Abstract. We give necessary and sufficient conditions for the convergence of cardinal interpolation with bivariate box splines as the degree tends to infinity.

1. Introduction and Statement of Main Results

For a set of vectors $\Xi = \{\xi_1, \ldots, \xi_n\}$ with $\xi_\nu \in \mathbf{Z}^m$, the box spline M_Ξ is the functional on $C_0(R^m)$ defined by [1], [2]

$$(1)\qquad \langle M_\Xi, \phi\rangle := \int_{[-\frac{1}{2}, \frac{1}{2}]^n} \phi\Big(\sum_{\nu=1}^n \lambda_\nu \xi_\nu\Big)\, d\lambda.$$

As becomes apparent from its Fourier transform

$$(2)\qquad \hat{M}_\Xi(y) = \prod_{\nu=1}^n \frac{\sin(\xi_\nu \cdot y/2)}{\xi_\nu \cdot y/2},$$

the box spline is a natural generalization of the univariate cardinal spline.

Motivated by I. J. Schoenberg's beautiful results [7–9], we have studied cardinal interpolation for box splines. The first question is whether the interpolation problem is correct; i.e., whether there exists, for any continuous bounded function f, a unique bounded spline

$$I_\Xi f \in S_\Xi := \operatorname{span}\{M_\Xi(\cdot - j) : j \in \mathbf{Z}^m\}$$

that interpolates f at the lattice points, i.e.,

$$I_\Xi f(k) = f(k), \qquad k \in \mathbf{Z}^m.$$

Clearly a necessary condition is that the translates of the box spline M_Ξ be linearly independent. *We conjecture that this is also sufficient or, equivalently, that*

$$(3)\qquad P_\Xi(x) := \sum_{j\in\mathbf{Z}^m} M_\Xi(j) e^{ijx} > 0$$

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holds if and only if the box splines $M_\Xi(\cdot - j)$, $j \in \mathbf{Z}^m$, form a basis for S_Ξ. So far, the positivity of P has been proved only in the bivariate case [3, Theorem 4].

In this paper we continue our investigation in [3] concerning the convergence of bivariate cardinal spline interpolation as the degree tends to infinity. We obtain the bivariate analogue of the following result, which is due to F. B. Richards, I. J. Schoenberg and S. D. Riemenschneider.

Theorem [5,6,8].

(i) *If the Fourier transform of f is a tempered distribution with* $\operatorname{supp} \hat{f} \subset (-\pi, \pi)$, *then the (univariate) cardinal spline interpolants $I_m f$ of degree m converge locally uniformly to f; i.e., for any $a > 0$,*

$$\| f - I_m f \|_{\infty, [-a,a]} \to 0, \qquad m \to \infty.$$

(ii) *If a sequence of (univariate) cardinal splines s_m of degree m converges uniformly to a bounded function f on $\mathbf{R}$ as $m \to \infty$, then* $\operatorname{supp} \hat{f} \subseteq [-\pi, \pi]$.

Up to symmetry, bivariate cardinal interpolation is correct iff the vectors in Ξ are chosen from the set $\{(1, 0), (0, 1), (1, 1)\}$. We assume from now on that Ξ is of this form and refer to it by $n = (n_1, n_2, n_3) \in \mathbf{Z}_+^3$, where n_ν is the multiplicity of the corresponding vector in Ξ.

One might expect that $(-\pi, \pi)^2$ plays the role of the interval $(-\pi, \pi)$ in the bivariate analogue of the above theorem. However, the situation is more complicated. There is a continuum of different fundamental domains and the convergence of I_n depends on just how the components of n go to infinity.

Denote by n' the "middle" component of n, i.e., the second number in any ordering of n_1,n_2,n_3. We write

$$n \to N$$

if a sequence $n(m)$, $m \in \mathbf{N}$, satisfies

(n1) $$n'(m) \to \infty \text{ as } m \to \infty,$$

(n2) $$\lim_{m \to \infty} \frac{n(m)}{n'(m)} = N \in [0, \infty]^3.$$

We assume further that

(n3) $$|n| := n_1 + n_2 + n_3 \le c(n')^c,$$

where c is some positive constant. Examples of admissible sequences are

$$n(m) = (m,2m,3m) \qquad \text{with} \qquad N = (\tfrac{1}{2},1,\tfrac{3}{2}),$$

$$n(m) = (1,m,m^2) \qquad \text{with} \qquad N = (0,1,\infty).$$

The assumption (n3) excludes degenerate cases such as $n(m) = (1,m,m!)$.

The role of the interval $(-\pi,\pi)$ is played by certain domains Ω_N corresponding to the limit of the sequence n. For $N \in [0,\infty)^3$ they are defined by

(4) $$\Omega_N := \{2\pi x\colon 0 \le a_{N,j}(x) < 1 \qquad \text{for} \qquad j \in J\},$$

where $J := \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ and for $x = (u,v)$, $j = (k,\ell)$,

$$a_{N,j}(x) := \left(\frac{u}{u+k}\right)^{N_1}\left(\frac{v}{v+\ell}\right)^{N_2}\left(\frac{u+v}{u+v+k+\ell}\right)^{N_3}. \tag{5}$$

Clearly, the set Ω_N is bounded by the curves $\Gamma_{N,j} := \{2\pi x\colon a_{N,j}(x) = 1\}$, $j \in J$. If one of the components of N equals ∞, the sets Ω_N as well as the curves $\Gamma_{N,j}$ have to be interpreted as the appropriate limits (cf. Proposition 2). A qualitatively correct picture of Ω_N is given in Fig. 1. Figure 2 shows a few special cases. Of particular interest is the symmetric case $N = (1,1,1)$.

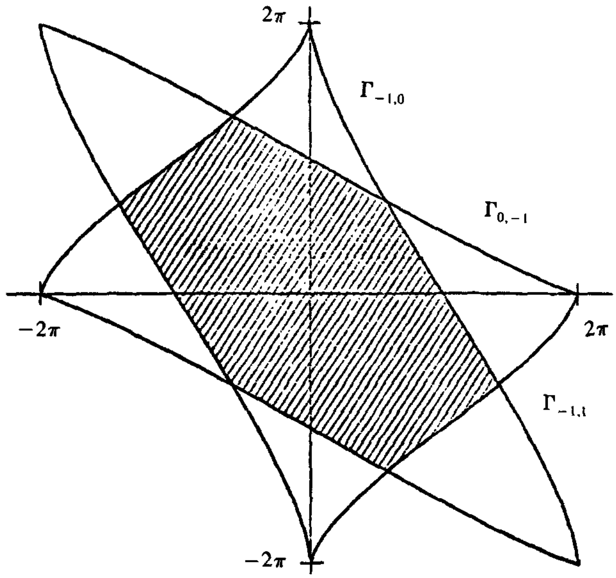

Fig. 1

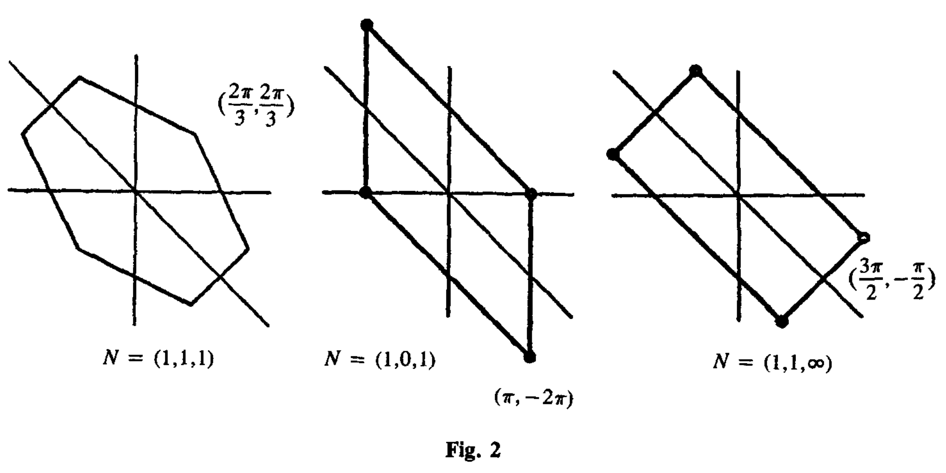

Fig. 2

A detailed discussion of the properties of the sets Ω_N is given in [3]. We merely note that they are *fundamental* domains, i.e., up to a set of measure zero, their translates $2\pi j + \Omega_N$, $j \in \mathbf{Z}^2$, form a partition of $\mathbf{R}^2$.

Our first result is an extension of Theorem 5.2 of [3] to include interpolation of data with power growth as was done in [7] for the univariate case.

Theorem 1. *Assume that the Fourier transform of f is a tempered distribution with* $\operatorname{supp} \hat{f} \subset \Omega_N$. *If the sequence n satisfies* (n1)–(n3), *then, for any* $\alpha \in \mathbf{Z}^2_+$, *the partial derivative $D^\alpha I_n f$ of the cardinal interpolant converges locally uniformly on $\mathbf{R}^2$ to $D^\alpha f$ as $n \to N$.*

As for the univariate case, the converse of the above theorem holds with "$\subset \Omega_N$" replaced by "$\subseteq \overline{\Omega}_N$":

Theorem 2. *Assume that the sequence n satisfies* (n1)–(n3). *If a sequence of cardinal splines $s_n \in S_n$ converges locally uniformly to f and if $|s_n(x)| \leq b(1 + |x|)^b$ for all n and some $b > 0$, then* $\operatorname{supp} \hat{f} \subseteq \overline{\Omega}_N$.

We may relax the assumption (n2). Clearly any subsequence of n also satisfies (n1) and (n3). If $\{N_\alpha\}$ are the limit points of the sequence n/n', then one has to replace the set Ω_N in the theorems by $\bigcap_\alpha \Omega_{N_\alpha}$. Figure 3 shows the intersection $\Omega_\cap$ and the union $\Omega_\cup$ of all possible limit sets.

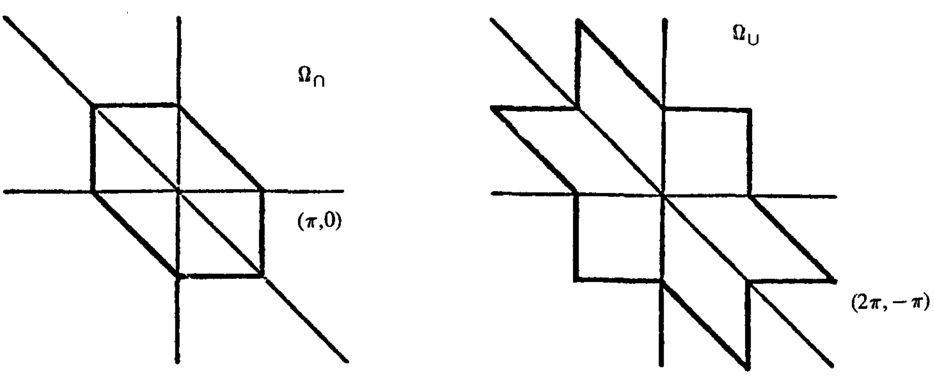

Fig. 3

2. Proofs

We assume throughout that the sequence n satisfies (n1)–(n3). We denote by c various positive generic constants that do not depend on n. These constants may change even within the same line. Further, we set

$$d_n(x) := \operatorname{dist}(x, \partial\Omega_n) \tag{1}$$

and denote by χ_n the characteristic function of the set Ω_n.

Denote by $L_n \in S_n$ the fundamental spline that interpolates the data $\delta_{0,k}$, $k \in \mathbf{Z}^2$. It is easily seen [3] that L_n decays exponentially at infinity. Therefore, if we assume, for example, that

$$|f(x)| \leq c(1+|x|)^c, \tag{2}$$

then we can write the cardinal interpolant in Lagrange form

$$I_n f = \sum_{j \in \mathbf{Z}^2} f(j) L_n(\cdot - j). \tag{3}$$

The proofs of Theorems 1 and 2 are based on the following estimate for the Fourier transform of L_n, which will be derived at the end of this section.

Theorem 3. *For any $\epsilon > 0$ and $\alpha \in \mathbf{Z}_+^2$ there exists n_0' such that for $n' \geq n_0'$ and $d_N(x) > \epsilon$*

$$|D^\alpha(\hat{L}_n(x) - \chi_N(x))| \leq (1 + c d_N(x))^{-n'}. \tag{4}$$

Proof of Theorem 1. Denote by $\mathcal{S}$ [4] the space of rapidly decreasing test functions $\varphi \in C^\infty(\mathbf{R}^2)$. The assumption $\hat{f} \in \mathcal{S}'$ and $\operatorname{supp} \hat{f} \subset \Omega_N$ implies (2) and hence the representation (3) is valid for the cardinal interpolant.

Set

$$\hat{f}_K := \sum_{|j| < K} f(j) e^{-ij\cdot}.$$

Since Ω_N is a fundamental domain that contains $\operatorname{supp} \hat{f}$, the values

$$f(-j) = (2\pi)^{-2} \langle \hat{f}, e^{ij\cdot} \rangle$$

are the Fourier coefficients of $\hat{f}$. Therefore $\hat{f}_K$ converges in $\mathcal{S}'$ to the periodic extension of $\hat{f}$:

$$\hat{f}^\circ := \sum_{j \in \mathbf{Z}^2} \hat{f}(\cdot + 2\pi j).$$

This means that there exists $\gamma \in \mathbf{Z}_+^2$ such that for any $\psi \in \mathcal{S}$,

$$|\langle \hat{f}^\circ - \hat{f}_K, \psi \rangle| = o(1) \cdot \|\psi\|_\gamma, \qquad \text{as } K \to \infty, \tag{5}$$

where

$$\|\psi\|_\gamma := \max_{\alpha, \beta \leq \gamma} \sup_{y \in \mathbf{R}^2} |y^\alpha D^\beta \psi(y)|.$$

Note that (5) implies

$$|\langle \hat{f}_K, \psi \rangle| \leq c \|\psi\|_\gamma, \tag{5'}$$

uniformly in K. Putting $\varphi(y) := (iy)^\sigma e^{ixy}$, we can write

$$(2\pi)^2 (D^\alpha I_n f)(x) = \lim_{K \to \infty} \int \hat{f}_K \hat{L}_n \varphi = \lim_{K \to \infty} \langle \hat{f}_K, \hat{L}_n \varphi \rangle$$

since f is of power growth and $\hat{L}_n$ decays exponentially away from Ω_N; hence $\hat{L}_n\varphi \in \mathcal{S}$. Further, with w a cut-off function in $\mathcal{S}$ satisfying

$$\operatorname{supp} w \subseteq B_\epsilon(\Omega_N) \quad \text{and} \quad w = 1 \text{ on } B_{\epsilon/2}(\Omega_N)$$

and $\epsilon > 0$ to be chosen below (and $B_r(G)$ the open ball of radius r around G), we have

$$(2\pi)^2(D^\alpha f)(x) = \langle \hat{f}, w\varphi \rangle,$$

since $\operatorname{supp} \hat{f} \subseteq \Omega_N$. Thus

$$(2\pi)^2 D^\alpha(f - I_n f)(x) = \langle \hat{f}, w\varphi \rangle + \lim_{K\to\infty} \langle \hat{f} - \hat{f}_K, w\hat{L}_n\varphi \rangle + \lim_{K\to\infty} \langle \hat{f}_K, (1-w)\hat{L}_n\varphi \rangle.$$

Since $\operatorname{supp} \hat{f} \subseteq \Omega_N$, the first term can be estimated using Theorem 3 with $\epsilon := \operatorname{dist}(\operatorname{supp} \hat{f}, \partial\Omega_N)/2$. Since $(2\pi j + \operatorname{supp} \hat{f}) \cap \operatorname{supp} w = \emptyset$ for $j \neq 0$ and $\operatorname{supp} \hat{f} \cap \operatorname{supp}(1-w) = \emptyset$, the second term equals

$$\lim_{K\to\infty} \langle \hat{f}^\circ - \hat{f}_K, w\hat{L}_n\varphi \rangle = 0.$$

As to the third term, note that $\operatorname{dist}(\operatorname{supp}(1-w), \Omega_N) \geq \epsilon/2$, which by (4) implies

$$\|(1-w)\hat{L}_n\varphi\|_\gamma \to 0 \quad \text{as} \quad n \to N.$$

It follows that

$$D^\alpha(f - I_n f)(x)$$

tends to zero, uniformly for bounded x (cf. definition of φ). ■

Proof of Theorem 2. Let $\varphi \in \mathcal{S}$ and assume that $\operatorname{supp} \varphi \cap \overline{\Omega}_N = \emptyset$. If the sequence $s_n \in S_n$ converges locally uniformly to f, then (2) holds and we have

$$\langle f, \hat{\varphi} \rangle = \lim_{n\to N} \left(\lim_{K\to\infty} \sum_{|j| \leq K} \langle s_n(j) L_n(\cdot - j), \hat{\varphi} \rangle \right).$$

Let $\Delta := \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2$ denote the Laplace operator. Since $\hat{L}_n$ together with all its derivatives vanishes at infinity, we have

$$\begin{aligned} \langle s_n(j) L_n(\cdot - j), \hat{\varphi} \rangle &= \langle s_n(j) e^{-ij\cdot} \hat{L}_n, \varphi \rangle \\ &= \langle (1 + |j|^2)^{-2-b/2} s_n(j) e^{-ij\cdot}, (1-\Delta)^{2+b/2}(\hat{L}_n\varphi) \rangle, \end{aligned}$$

and we assume without loss of generality that b is an even integer. Applying Theorem 3 with $\epsilon = \operatorname{dist}(\operatorname{supp} \varphi, \overline{\Omega}_N) > 0$ we have, for sufficiently large n',

$$|\langle \ldots \rangle| \leq c(1 + |j|^2)^{-2} [\sup_j (1 + |j|^2)^{-b/2} |s_n(j)|] (1 + c\epsilon)^{-n'} \|\varphi\|_\gamma.$$

It follows that $\langle \hat{f}, \varphi \rangle = 0$. ■

For the proof of Theorem 3 we make use of the following precise estimates for $\hat{L}_n$ and the numbers $a_{n,j}$ which were derived in [3].

Theorem 4 [3, Theorem 5.2].

$$|\hat{L}_n(x) - \chi_n(x)| \leq c[1 + cd_n(x)]^{-n'}. \tag{6}$$

Proposition 1. [3, Prop. 5.2, Lemmas 6.5, 6.6]. *Set* $J' := \{\pm(1,1), \pm(2,-1), \pm(-1,2)\}$. *Then for* $2\pi x \in \Omega_n$, *we have*

$$|a_{n,j}(x)| \leq \begin{cases} (1 + c \operatorname{dist}(2\pi x, \Gamma_{n,j} \cap \Omega_n))^{-n'}, & j \in J \\ (1 + c \operatorname{dist}(2\pi x, -j/2))^{-n'}, & j \in J' \\ (1 + c|j|)^{-n'}, & j \in \mathbf{Z}^2 \setminus \{J \cup J' \cup 0\}. \end{cases} \tag{7}$$

Proposition 2. [3, Prop. 5.1]. Ω_n *depends continuously on* n *in the Hausdorff topology.*

The reader who compares these statements with those in [3] will notice that we have changed the notation slightly. Note that the estimate (6) is stronger than the assertion of Theorem 3 for $\alpha = 0$, because the constants c in (6) do not depend on the distance of x to $\partial\Omega_n$.

We need the analogue of estimate (7) for the derivatives of $a_{n,j}$.

Lemma 1. *For any* $\delta > 0$, *there exist constants* c_δ, c *and* $n_0(\delta)$ *such that for all* $n' \geq n_0$ *and* $2\pi x \in \Omega_n$,

$$|D^\alpha a_{n,j}(x)| \leq c_\delta(1 + c\delta)^{n'} \begin{cases} [1 + c \operatorname{dist}(2\pi x, \Gamma_{n,j} \cap \Omega_n)]^{-n'}, & j \in J \\ [1 + c \operatorname{dist}(2\pi x, -j/2)]^{-n'}, & j \in J' \\ [1 + c|j|]^{-n'}, & j \in \mathbf{Z}^2 \setminus \{J \cup J' \cup 0\}. \end{cases} \tag{8}$$

The proof of this lemma is technical and we postpone it until the end.

Proposition 3. *Let* $x' = x + j$, *with* $j \in \mathbf{Z}^2 \setminus 0$ *and* $2\pi x \in \Omega_n$. *Then for some* c *and for any* $\delta > 0$, *there exist constants* c_δ *and* $n_0(\delta)$ *such that for all* $n' \geq n_0$,

$$|D^\alpha a_{n,j}(x)| \leq c_\delta(1 + c\delta)^{n'}[1 + cd_n(2\pi x')]^{-n'}. \tag{9}$$

For $\alpha = 0$ this is Proposition 5.4 in [3]. There we bounded the terms in square brackets on the right-hand side of (8) by $[1 + cd_n(2\pi x')]$, which appears on the right-hand side of (9). Clearly, the case $\alpha \neq 0$ can be treated in the same way.

Proof of Theorem 3. Since

$$\frac{1}{\hat{L}_n(2\pi x)} = \frac{P_n(2\pi x)}{\hat{M}_n(2\pi x)} = \sum_{j \in \mathbf{Z}^2} a_{n,j}(x),$$

we have, for $|\alpha| = 1$,

$$(2\pi)D^\alpha \hat{L}_n(2\pi x) = -\hat{L}_n(2\pi x)^2 \sum_{j \neq 0} D^\alpha a_{n,j}(x).$$

For arbitrary $\alpha \neq 0$ it follows that

(10) $$(2\pi)^{|\alpha|} D^{\alpha}\hat{L}(2\pi x) =$$

$$\sum_{j\neq 0} \sum_{\beta_1+\beta_2 < \alpha} c_{\beta} D^{\beta_1}\hat{L}_n(2\pi x) D^{\beta_2}\hat{L}_n(2\pi x) D^{\alpha-\beta_1-\beta_2} a_{n,j}(x).\cdot$$

Let us first assume that $2\pi x \in \Omega_n$. We claim that for any $\delta > 0$ there exists c_δ such that

(11) $$|D^{\alpha}\hat{L}_n(2\pi x)| \leq c_\delta(1+c\delta)^{n'}[1+cd_n(2\pi x)]^{-n'}, \qquad 2\pi x \in \Omega_n.$$

For $\alpha = 0$ this is a weaker statement than the assertion of Theorem 4. Using induction on $|\alpha|$ it is sufficient to show that

$$\sum_{j\neq 0} D^{\alpha} a_{n,j}(x)$$

can be bounded by the right-hand side of (11). Lemma 1 yields

$$|D^{\alpha} a_{n,j}(x)| \leq c_\delta(1+c\delta)^{n'} \cdot \begin{cases} [1+cd_n(2\pi x)]^{-n'}, & j \in J \cup J', \\ (1+c|j|)^{-n'}, & j \in \mathbf{Z}^2 \backslash \{J \cup J' \cup 0\}. \end{cases}$$

Summing this inequality over $j \in \mathbf{Z}^2 \backslash 0$ finishes the proof of (11).

Second, let $x' = x+j$, with $2\pi x \in \Omega_n$. Then, writing

$$\hat{L}_n(2\pi x') = \frac{\hat{M}_n(2\pi x)}{P_n(2\pi x')} \frac{\hat{M}_n(2\pi x')}{\hat{M}_n(2\pi x)} = \hat{L}_n(2\pi x) a_{n,j}(x),$$

we see that

$$D^{\alpha}\hat{L}_n(2\pi x') = \sum_{\beta \leq \alpha} c_\beta D^{\beta}\hat{L}_n(2\pi x) D^{\alpha-\beta} a_{n,j}(x).$$

Therefore, by (11) and Proposition 3, $D^{\alpha}\hat{L}_n$ can be estimated by

(12) $$|D^{\alpha}\hat{L}_n(2\pi x')| \leq c_\delta(1+c\delta)^{n'}(1+cd_n(2\pi x'))^{-n'},$$
$$x' = x+j, \qquad 2\pi x \in \Omega_n.$$

Theorem 3 easily follows from the estimates (11) and (12): Let $\epsilon > 0$ and assume that $d_N(2\pi x) > \epsilon$. We choose n_0' so that $\mathrm{dist}(\partial\Omega_n, \partial\Omega_N) < \epsilon/2$ for $n' > n_0'$. Now (11) and (12) give (4), since we can choose δ sufficiently small. ■

Proof of Lemma 1. In proving (8) we make use of the symmetries of the mesh. If A is a linear transformation that leaves the set J invariant, we have

$$a_{n,Aj}(Ax) = a_{\tilde{n},j}(x),$$

where $\tilde{n}$ is the appropriate permutation of n. Similarly,

$$A\Omega_n = \Omega_{\tilde{n}}.$$

From this one can check (cf. [3, Section 3]) that one may assume, by changing n if necessary, that $x =: (u,v)$ lies in the first quadrant. Further, since the roles of u and v may be interchanged and $2\pi x \in \Omega_n \subset \Omega$ (cf. Fig. 3), we shall assume throughout this proof that

$$0 \le v \le u \le \tfrac{1}{2}. \tag{13}$$

By definition (5) of $a_{n,j}$ in Section 1, we have

$$|D^\alpha a_{n,j}(u,v)| \le \tag{14}$$
$$c|n|^{|\alpha|} \sum \left[\frac{|u|^{n_1-\beta_1}}{|u+k|^{n_1+\beta_4}} \frac{|v|^{n_2-\beta_2}}{|v+\ell|^{n_2+\beta_5}} \frac{|u+v|^{n_3-\beta_3}}{|u+v+k+\ell|^{n_3+\beta_6}} \right],$$

where the sum is taken over all β that satisfy

$$\left\{ \begin{aligned} & 0 \le \beta_\nu \le n_\nu, \qquad \nu = 1,2,3, \\ & \sum_{\nu=1}^{6} \beta_\nu = \alpha_1 + \alpha_2, \\ & \beta_\nu = -\beta_{3+\nu} \text{ in case } \begin{cases} k = 0 & \nu = 1 \\ \ell = 0 & \nu = 2 \\ k+\ell = 0 & \nu = 3. \end{cases} \end{aligned} \right. \tag{15}$$

This last restriction comes from the fact that for $k = 0$, for example, the factor $(u/(u+k))^{n_1}$ is equal to 1, and hence does not figure in the differentiation. To estimate the individual summands in (14), we consider four cases. Unless $(k,\ell) \in \{(0,-1),(-1,1)\}$ [cases (ii)(b),(c) below], we bound each summand [. . .] on the right-hand side of (14) by

$$c_\delta(1+c\delta)^{n'} \max\{(1+c|j|)^{-n'},\ |a_{n,j}(x)|\}. \tag{16}$$

Case (i). $v \le \delta < \frac{1}{4}$, $u \le \frac{1}{8}$: Using the inequality

$$\left|\frac{p}{p+q}\right| \le (1+c_\epsilon|q|)^{-1}, \qquad q \in \mathbf{Z}\backslash 0, \qquad |p| < \tfrac{1}{2} - \epsilon, \tag{17}$$

we obtain the estimate

$$|[. . .]| \le c(1+c|k|)^{-n_1+\beta_1}(1+c|\ell|)^{-n_2+\beta_2}(1+c|k+\ell|)^{-n_3+\beta_3},$$

with c involving terms such as $|u+k|^{-(\beta_1+\beta_4)}$, $k \ne 0$, which are bounded. Since at most one of the components of n is less than n', this implies (8).

Case (ii). $v \le \delta < \frac{1}{4}$, $u > \frac{1}{8}$: We consider several subcases.

(a) $(k,\ell) \notin \{(0,1),(-1,1)\}$: We have

$$|[. . .]| \le c(1+c|\ell|)^{-n_2+\beta_2}\left|\frac{u}{u+k}\right|^{n_1}\left|\frac{u+v}{u+v+k+\ell}\right|^{n_3}. \tag{18}$$

This can be estimated as before unless $k = -1$ or $k+\ell = -1$. If $k = -1$ and $\ell \ne 0,1$, we can write the right side of (18) as

$$c(1+c|\ell|)^{-n_2+\beta_2}|a_{n,(-1,0)}(u,v)|\left|\frac{u+v-1}{u+v-1+\ell}\right|^{n_3}.$$

The last factor is less than $(1+c|\ell|)^{-n_3}$ and, since for $2\pi(u,v)\in\Omega_n$, $|a_{n,(-1,0)}(u,v)| < 1$, (8) follows. If $(k,\ell) = (-1,0)$ it is easily seen that the left-hand side of (18) can be bounded by $c|a_{n,(-1,0)}(u,v)|$. The cases $k+\ell = -1$, $k \neq 0,-1$ are treated similarly.

(b) $(k,\ell) = (0,-1)$: Set $n_\beta := (n_1,\ n_2-\beta_2,\ n_3)$. By Proposition 2, there exists $n_0 = n_0(\alpha,\delta)$ such that the boundaries of Ω_n and Ω_{n_β} are within δ of the boundary of the limit set Ω_N for $n' \geq n_0$ and all β_2 satisfying (15). Moreover, $\frac{1}{2}n' \leq n'_\beta \leq 2n'$, $n' \geq n_0$.

For $u+v \leq \frac{1}{2}$ (and $n' \geq n_0$) we obtain, using also Proposition 1,

$$\begin{aligned}|[\ldots]| &\leq c|a_{n_\beta,(0,-1)}(u,v)|\\ &\leq c[1+c\,\mathrm{dist}(2\pi x,\ \Gamma_{n_\beta,(0,-1)}\cap\Omega_{n_\beta})]^{-n'_\beta}\\ &\leq c(1+c\delta)^{n'}[1+c\,\mathrm{dist}(2\pi x,\ \Gamma_{n,(0,-1)}\cap\Omega_{n_\beta})]^{-n'}.\end{aligned}$$

For the last inequality we have used the fact that

$$\mathrm{dist}(\Gamma_{n_\beta,(0,-1)}\cap\Omega_{n_\beta},\ \Gamma_{n,(0,-1)}\cap\Omega_n) \leq \delta.$$

For $\frac{1}{2} < u+v \leq \frac{1}{2}+\delta$ we have

$$\begin{aligned}|[\ldots]| &\leq c\left|\frac{v}{1-v}\right|^{n_2-\beta_2}\left|\frac{u+v}{1-u-v}\right|^{n_3}\\ &\leq c\left|\frac{v}{1-v}\right|^{n_2-\beta_2}\min\left\{\left|\frac{u+v}{1-u-v}\right|^{n_3},\left|\frac{1-u}{u}\right|^{n_1}\right\},\end{aligned}$$

where we have used the fact that $a_{n,(-1,0)}(u,v) < 1$. By our assumptions on u and v, the minimum can be estimated by $(1+c\delta)^{n'}$. Therefore, if $n_2 \geq cn'$, inequality (8) follows. If $\lim_{n\to N} n_2/n' = 0$, the curve $\Gamma_{n,(0,-1)}\cap\overline{\Omega}_n$ converges to the segment $\{2\pi(u,v)\colon u+v = \frac{1}{2},\ (u,v) > (0,0)\}$. Therefore, we may assume that

$$\mathrm{dist}(2\pi x,\ \Gamma_{n,(0,-1)}\cap\Omega_n) \leq c\delta$$

for $n' \geq n_0$, and (8) follows.

(c) $(k,\ell) = (-1,1)$: We have

$$\begin{aligned}|[\ldots]| &\leq c\left|\frac{u}{1-u}\right|^{n_1}\left|\frac{v}{1+v}\right|^{n_2-\beta_2}\\ &\leq c(1+c)^{n_2-\beta_2}[1+c(\tfrac{1}{2}-u)]^{-n_1}.\end{aligned}$$

Since $\Gamma_{n,(-1,1)}$ does not intersect the square $[0,\pi]\times[-\pi,0]$, this implies (8).

Case (iii). $\delta \leq v$, $u < \frac{1}{2}-\delta$: Since $v \leq u$, we have

$$|[\ldots]| \leq c_\delta\left|\frac{u}{u+k}\right|^{n_1}\left|\frac{v}{v+\ell}\right|^{n_2}\left|\frac{u+v}{u+v+k+\ell}\right|^{n_3} \leq c_\delta|a_{n,j}(x)|.$$

Case (iv). $\delta \leq v$, $u \geq \frac{1}{2}-\delta$: We have

$$|[\ldots]| \le c_\delta |a_{n,j}(x)| \left|\frac{u+v}{u+v+k+\ell}\right|^{\beta_6}$$
$$\le c_\delta |a_{n,j}(x)| \left|\frac{u+v}{1-u-v}\right|^{\beta_6}.$$

Assume, for example, that $n_1 = \min(n_1,n_2)$. Since $2\pi x \in \Omega_n$,

$$|a_{n,(-1,0)}(x)| = \left(\frac{u}{1-u}\right)^{n_1}\left(\frac{u+v}{1-u-v}\right)^{n_3} < 1.$$

From this and the fact that $u \ge \frac{1}{2} - \delta$, $\beta_6 \le n_3$, we have

$$\left|\frac{u+v}{1-u-v}\right|^{\beta_6} \le \left|\frac{1-u}{u}\right|^{(n_1/n_3)\beta_6} \le (1+c\delta)^{n'}.$$

Combining the above estimates yields

$$|[\ldots]| \le c_\delta |a_{n,j}(x)|(1+c\delta)^{n'}. \qquad \blacksquare$$

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