A NECESSARY AND SUFFICIENT CONDITION FOR THE LINEAR INDEPENDENCE OF THE INTEGER TRANSLATES OF A COMPACTLY SUPPORTED DISTRIBUTION

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Abstract

Given a multivariate compactly supported distribution ϕ, we derive here a necessary and sufficient condition for the global linear independence of its integer translates. This condition is based on the location of the zeros of $\widehat{\phi}$ = The Fourier-Laplace transform of ϕ. The utility of the condition is demonstrated by several examples and applications, showing in particular, that previous results on box splines and exponential box splines can be derived from this condition by a simple combinatorial argument.

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1. Introduction and Statement of Main Results

Let $\mathcal{E}'(\mathbb{R}^s)$ be the space of all s-dimensional complex valued distributions of compact support. For each $\phi \in \mathcal{E}'(\mathbb{R}^s)$ we denote

$$K_\phi = \{q : \mathbb{Z}^s \to \mathbb{C} \big| \sum_{\underline{\alpha} \in \mathbb{Z}^s} q(\underline{\alpha}) E^{\underline{\alpha}} \phi \equiv 0\} \ , \tag{1.1}$$

where $E^{\underline{\alpha}}$ is the shift in the $\underline{\alpha}$ direction, namely

$$E^{\underline{\alpha}} \phi = \phi(\cdot - \underline{\alpha}) \ . \tag{1.2}$$

The condition $K_\phi = 0$ is usually referred to as "global linear independence of the integer translates of ϕ" and is known to be a substantial condition when approximating a function by the linear span of $\{E^{\underline{\alpha}}\phi\}_{\underline{\alpha} \in \mathbb{Z}^s}$.

The main aim of this note is to establish a necessary and sufficient condition for the global linear independence of the integer translates of a $\phi \in \mathcal{E}'(\mathbb{R}^s)$. The methods here make an essential use of distribution theory. For background material we refer to [T], which is also the source for some of the notations. The utility of this necessary and sufficient condition is demonstrated by several examples and applications.

Given $\phi \in \mathcal{E}'(\mathbb{R}^s)$, it is well known that the Fourier transform of ϕ is extendible to an entire function on $\mathbb{C}^s$, which is denoted here by $\widehat{\phi}$. Define

$$N(\phi) = \{\underline{\theta} \in \mathbb{C}^s \big| \widehat{\phi}(\underline{\theta} + 2\pi\underline{\alpha}) = 0 \qquad \forall \underline{\alpha} \in \mathbb{Z}^s\} \ . \tag{1.3}$$

We claim

Theorem 1.1. Let $\phi \in \mathcal{E}'(\mathbb{R}^s)$ then

(a) $\{e^{i\underline{\theta}\cdot\underline{\alpha}}\}_{\underline{\alpha} \in \mathbb{Z}^s} \in K_\phi$ if and only if $\underline{\theta} \in N(\phi)$.

(b) $K_\phi = 0$ if and only if $N(\phi) = \emptyset$.

The approach taken here towards the proof of Theorem 1.1 yields some additional results which seem to be of independent interest. To present this approach denote by $\mathcal{Q}$ the Fréchet space of all sequences defined on $\mathbb{Z}^s$, equipped with the topology of pointwise convergence (i.e., uniform convergence on compact sets). A subspace $Q \subset \mathcal{Q}$ is termed *shift-invariant* if it is invariant under each $E^{\underline{\alpha}}$, $\underline{\alpha} \in \mathbb{Z}^s$. Obviously, every space of the type $K_\phi, \phi \in \mathcal{E}'(\mathbb{R}^s)$ is a closed shift invariant subspace of $\mathcal{Q}$. The result we need about shift invariant subspaces of $\mathcal{Q}$ is recorded in the following theorem which is due to Lefranc.

Theorem 1.2. (Lefranc, [Le]). Every non-trivial closed shift invariant subspace of Q contains an exponential $q(\underline{\alpha}) = \underline{z}^{\underline{\alpha}}$.

Here we used the standard notation $\underline{z}^{\underline{\alpha}} = \prod_{j=1}^{s} z_j^{\alpha_j}$.

Given a closed shift invariant subspace Q of $\mathcal{Q}$ it is natural to ask whether there always exists $\phi \in \mathcal{E}'(\mathbb{R}^s)$ such that $K_\phi = Q$. The next theorem, while providing an affirmative answer to this question, shows that such ϕ can be chosen as a linear combination of the Dirac distribution and its translates (see [T: Chap. 22,24]):

Theorem 1.3. Let Q be a subspace of $\mathcal{Q}$. Then Q is closed and shift invariant if and only if there exists $\phi \in \mathcal{E}'(\mathbb{R}^s)$ of order zero and with finite support such that $Q = K_\phi$.

As an illustration to the usefulness of Theorem 1.1 to the problem of the linear independence of the translates of a compactly supported distribution, we present here two of its applications.

Corollary 1.1. Let $\phi_1, \ldots, \phi_n \in \mathcal{E}'(\mathbb{R}^s)$, $k_1, \ldots, k_n \in \mathbb{N}$. Define $\phi, \psi \in \mathcal{E}'(\mathbb{R}^s)$ by their Fourier transform as follows:

$$\widehat{\phi} = \widehat{\phi}_1^{k_1} \cdots \widehat{\phi}_n^{k_n}$$

$$\widehat{\psi} = \widehat{\phi}_1 \cdots \widehat{\phi}_n \ .$$

Then $K_\phi = 0$ if and only if $K_\psi = 0$.

Corollary 1.1 is an immediate consequence of Theorem 1.1. (Indeed, we see that for $\underline{x} \in \mathbb{C}^s, \widehat{\phi}(\underline{x}) = 0$ if and only if $\widehat{\psi}(\underline{x}) = 0$, thus $N(\phi) = N(\psi)$.) In addition Theorem 1.1 shows that the set of exponentials in K_ϕ is identical to that of K_ψ.

The second example generalizes the known results about the global linear independence of the translates of a polynomial box spline , [J], and a real exponential box spline [R], [DM$_3$].

Theorem 1.4. Let $X = \{\underline{x}^1, \ldots, \underline{x}^n\}$ be a collection of non-trivial vectors in $\mathbb{Z}^s$, which span $\mathbb{R}^s$. For every $J = \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$ denote by X_J the matrix with columns $\underline{x}^{i_1}, \ldots, \underline{x}^{i_s}$. Assume that X is unimodular, i.e., for every J as above

$$\det X_J \in \{1, 0, -1\} \ .$$

Let $\phi_1, \ldots, \phi_n$ be univariate compactly supported distributions which satisfy

$$\widehat{\phi}_j(t) = 0 \qquad \text{only if} \qquad t \in 2\pi\mathbb{Z}\backslash 0 \ , \tag{1.4}$$

and define $\phi \in \mathcal{E}'(\mathbb{R}^s)$ via its Fourier transform by

$$\widehat{\phi}(\underline{x}) = \prod_{j=1}^{n} \widehat{\phi}_j(i\lambda_j + \underline{x}^j \cdot \underline{x}) \ ,$$

where $\lambda_1, \ldots, \lambda_n$ are arbitrary real numbers. Then $K_\phi = 0$.

2. Shift Invariant Subspaces of $\mathcal{Q}$.

Here we prove Theorems 1.1-1.3 and derive additional characterizations of the closed shift invariant subspaces of $\mathcal{Q}$.

Let $\mathcal{P}$ be the vector space consisting of all sequences in $\mathcal{Q}$ with finite support. Let π be the subspace of $\mathcal{P}$ consisting of those sequences with support contained in $\mathbb{Z}_+^s := \{\underline{\alpha} \in \mathbb{Z}^s \big| \alpha_1, \ldots, \alpha_s \geq 0\}$. Every $p \in \mathcal{P}$ induces a linear functional on $\mathcal{Q}$ defined by

$$p \cdot q = \sum_{\underline{\alpha} \in \mathbb{Z}^s} p(\underline{\alpha}) q(\underline{\alpha}) \ . \tag{2.1}$$

The same definition shows that every $q \in \mathcal{Q}$ can be viewed as a linear functional defined on $\mathcal{P}$. It is well known, (see e.g., [T;Th.22.1]), that $\mathcal{P}$ forms the topological dual of $\mathcal{Q}$ while $\mathcal{Q}$ forms the dual of $\mathcal{P}$. In case $p \cdot q = 0$ we write $p \perp q$. For subsets $Q \subset \mathcal{Q}$ and $P \subset \mathcal{P}$ we use the usual orthogonality symbol

$$P \perp Q \quad \Leftrightarrow \quad \{p \perp q \ \big| \ \forall p \in P \ , \ q \in Q\} \ . \tag{2.2}$$

Given $Q \subset \mathcal{Q}$ its orthogonal $Q^\perp$ in $\mathcal{P}$ is defined by

$$Q^\perp = \{p \in \mathcal{P} \ \big| p \perp q \qquad \forall q \in Q\} \ , \tag{2.3}$$

and a similar definition holds for the orthogonal $P^\perp$ in $\mathcal{Q}$ of a subset P of $\mathcal{P}$. Since $\mathcal{Q}$, as a Fréchet space, is locally convex we can appeal to the Hahn-Banach theorem to deduce:

Proposition 2.1. Every closed subspace Q of $\mathcal{Q}$ satisfies

$$(Q^\perp)^\perp = Q \ .$$

A subspace of π is termed *shift-invariant* if it is invariant under every $E^{\underline{\alpha}}$, $\underline{\alpha} \in \mathbb{Z}_+^s$. Note that π is isomorphic (as a vector space) to the space $\pi(X)$ of all polynomials in s variables. This isomorphism I is defined by

$$\pi \ni p \xrightarrow{I} \sum_{\underline{\alpha} \in \mathbb{Z}_+^s} p(\underline{\alpha}) \underline{x}^{\underline{\alpha}} \ .$$

Since $E^{\underline{\alpha}}$ is transformed by I to multiplication by $\underline{x}^{\underline{\alpha}}$, we conclude that a subspace $\mathcal{F}$ is a shift invariant subspace of π if and only if its image $\mathcal{F}(X)$ by I is an ideal of $\pi(X)$.

Proof of Theorem 1.2. Since Q is a non-trivial shift invariant subspace of $\mathcal{Q}$, then $Q^\perp$ is a proper shift invariant subspace of $\mathcal{P}$; therefore $Q^\perp$ contains no element with a one-point support. (Indeed, if $p \in Q^\perp$ and its support consists of a single point then, since $\{E^{\underline{\alpha}} p\}_{\underline{\alpha} \in \mathbb{Z}^s}$ span $\mathcal{P}$, we would get $Q^\perp = \mathcal{P}$). This implies that $(\pi \cap Q^\perp)(X)$ is an ideal of $\pi(X)$ which contains no monomials.

Let Z denote the set of all common zeros of the polynomials in $(\pi \cap Q^\perp)(X)$. Let $H = \cup_{j=1}^s \{\underline{z} \in \mathbb{C}^s \big| z_j = 0\}$. We contend that $Z \not\subset H$. Indeed, if $Z \subset H$, then Hilbert's Nullstellensatz (see e.g., [La;p.256]) would imply that a power of the monomial $x_1 x_2 \ldots x_s$ lies in $(Q^\perp \cap \pi)(X)$.

Let $\underline{z} \in Z \backslash H$. Then $p(\underline{z}) = 0$ for every $p(\underline{x}) \in (Q^\perp \cap \pi)(X)$, which is equivalent to the sequence $\underline{z}^{\underline{\alpha}}$ being orthogonal to $Q^\perp \cap \pi$. Let $p \in Q^\perp$ and choose $\underline{\beta} \in \mathbb{Z}^s$ such that $E^{\underline{\beta}} p \in Q^\perp \cap \pi$, then

$$p \cdot \underline{z}^{\underline{\alpha}} = E^{\underline{\beta}}(p) \cdot E^{\underline{\beta}}(\underline{z}^{\underline{\alpha}}) = \underline{z}^{-\underline{\beta}}(E^{\underline{\beta}}(p) \cdot \underline{z}^{\underline{\alpha}}) = 0 \ .$$

Therefore, by Proposition 2.1, $\underline{z}^{\underline{\alpha}} \in (Q^\perp)^\perp = Q$. □

Since for every $\phi \in \mathcal{E}'(\mathbb{R}^s)$ K_ϕ is a closed shift invariant subspace of $\mathcal{Q}$, Theorem 1.2 implies the following result of Dahmen and Micchelli (see [DM$_1$; Theorem 4.1]).

Corollary 2.1. (Dahmen and Micchelli). Let ϕ be a compactly supported continuous function. If K_ϕ is not trivial then it contains an exponential.

We now turn to the proof of Theorem 1.1. This proof utilizes the following lemma:

Lemma 2.1. Let $\phi \in \mathcal{E}'(\mathbb{R}^s)$ and $\underline{\theta} \in \mathbb{C}^s$. Denote

$$\phi_{\underline{\theta}} := \sum_{\underline{\alpha} \in \mathbb{Z}^s} e^{i\underline{\theta} \cdot \underline{\alpha}} E^{\underline{\alpha}} \phi \ . \tag{2.4}$$

Let $f \in \mathcal{D}(\mathbb{R}^s)$:= the space of all compactly supported infinitely differentiable functions. Then

$$\phi_{\underline{\theta}}(f) = \sum_{\underline{\alpha} \in \mathbb{Z}^s} \widehat{\phi}(\underline{\theta} - 2\pi\underline{\alpha}) \widehat{f}(-\underline{\theta} + 2\pi\underline{\alpha}) \ .$$

Proof. By Poisson's summation formula (see e.g., [Y;p.149])

$$\sum_{\underline{\alpha} \in \mathbb{Z}^s} g(\underline{\alpha}) = \sum_{\underline{\alpha} \in \mathbb{Z}^s} \widehat{g}(2\pi\underline{\alpha}) \quad , \quad \forall g \in \mathcal{D}(\mathbb{R}^s) \ . \tag{2.5}$$

Since ϕ is of compact support, it is continuous on $\mathbb{C}^\infty(\mathbb{R}^s)$ (see [T; pp.255-257]), hence the definition of $\phi_{\underline{\theta}}$ shows that

$$\phi_{\underline{\theta}}(f) = \phi\Big(\sum_{\underline{\alpha} \in \mathbb{Z}^s} e^{i\underline{\theta} \cdot \underline{\alpha}} f(\cdot + \underline{\alpha})\Big) \ . \tag{2.6}$$

Fix $\underline{x} \in \mathbb{R}^s$ and let $g(\underline{y}) = e^{i\underline{\theta} \cdot y} f(\underline{x} + \underline{y})$. Thus $g \in \mathcal{D}(\mathbb{R}^s)$ and the series $\sum_{\underline{\alpha} \in \mathbb{Z}^s} e^{i\underline{x} \cdot (2\pi\underline{\alpha} - \underline{\theta})} \widehat{f}(2\pi\underline{\alpha} - \underline{\theta})$ is convergent in the $C^\infty(\mathbb{R}^s)$ topology. Therefore (2.5) can be combined with (2.6) to yield

$$\begin{aligned}
\phi_{\underline{\theta}}(f) &= \phi\Big(\sum_{\underline{\alpha} \in \mathbb{Z}^s} e^{i\underline{x} \cdot (2\pi\underline{\alpha} - \underline{\theta})} \widehat{f}(2\pi\underline{\alpha} - \underline{\theta})\Big) \\
&= \sum_{\underline{\alpha} \in \mathbb{Z}^s} \widehat{f}(2\pi\underline{\alpha} - \underline{\theta}) \phi(e^{i\underline{x} \cdot (2\pi\underline{\alpha} - \underline{\theta})}) \\
&= \sum_{\underline{\alpha} \in \mathbb{Z}^s} \widehat{f}(2\pi\underline{\alpha} - \underline{\theta}) \widehat{\phi}(\underline{\theta} - 2\pi\underline{\alpha}) \ .
\end{aligned}$$

$\square$

Proof of Theorem 1.1. Let $\underline{\theta} \in \mathbb{C}^s$ and define $\phi_{\underline{\theta}}$ as in Lemma 2.1. By definition $e^{i\underline{\theta}\cdot\underline{\alpha}} \in K_\phi$ if and only if $\phi_{\underline{\theta}} \equiv 0$. But by Lemma 2.1 $\phi_{\underline{\theta}} \equiv 0$ if and only if $\widehat{\phi}(\underline{\theta} + 2\pi\underline{\alpha}) = 0$ for all $\underline{\alpha} \in \mathbb{Z}^s$, which is to say $\underline{\theta} \in N(\phi)$. This proves (a), while (b) follows from (a) and Theorem 1.2. $\square$

Next, we are interested in the characterization of the *tempered* sequences in K_ϕ. Recall that a sequence $q \in \mathcal{Q}$ is termed "tempered" if there exists a polynomial $p(\underline{x})$ such that

$$\left|q(\underline{\alpha})\right| \leq \left|p(\underline{\alpha})\right| \qquad \forall \underline{\alpha} \in \mathbb{Z}^s \ . \tag{2.7}$$

Given a tempered $q \in Q$, we define $\widehat{q}(\underline{x}) = \sum_{\underline{\alpha}\in\mathbb{Z}^s} q(\underline{\alpha}) e^{-i\underline{\alpha}\cdot\underline{x}}$. This sum is always convergent (in the distributional sense) to a tempered distribution which is periodic with respect to $2\pi\mathbb{Z}^s$.

Theorem 2.1. Let $\phi \in \mathcal{E}'(\mathbb{R}^s)$ and denote $N_{\mathbb{R}}(\phi) = N(\phi) \cap \mathbb{R}^s$. Then

(a) If $q \in K_\phi$ is tempered then supp $\widehat{q} \subset N_{\mathbb{R}}(\phi)$.

(b) $N_{\mathbb{R}}(\phi) = \emptyset$ if and only if K_ϕ contains no tempered sequence other than zero.

(c) If $K_\phi \neq \mathcal{Q}$ then $K_\phi \cap \ell_p = 0$ for $1 \leq p \leq 2$.

Proof. Assume that $q \in K_\phi$ is tempered. Application of Fourier transform to both sides of the equation $\sum\limits_{\underline{\alpha}\in\mathbb{Z}^s} q(\underline{\alpha}) E^{\underline{\alpha}}\phi \equiv 0$ yields

$$\widehat{\phi}(\underline{x})\widehat{q}(\underline{x}) = \sum_{\underline{\alpha}\in\mathbb{Z}^s} q(\underline{\alpha}) e^{-i\underline{\alpha}\cdot\underline{x}} \widehat{\phi}(\underline{x}) \equiv 0 \ . \tag{2.8}$$

To advance our proof we need

Lemma 2.2. Let $f \in C^\infty(\mathbb{R}^s)$, and let g be in $\mathcal{D}'(\mathbb{R}^s)$ (= the space of all complex valued s-dimensional distributions). If $fg \equiv 0$ then $\underline{x} \in$ supp g only if $f(\underline{x}) = 0$.

Proof of Lemma 2.2. Since $f \in C^\infty(\mathbb{R}^s)$, it is a multiplier in $\mathcal{D}'(\mathbb{R}^s)$ (see [T; p.250]), and hence fg is well defined. Assume $\underline{x}^\circ \in \mathbb{R}^s$ and $f(\underline{x}^\circ) \neq 0$. Since f is continuous, there exists a neighbourhood Ω of $\underline{x}^\circ$ such that $f(\underline{x})$ vanishes nowhere in the closure of Ω. Let $\varphi(\underline{x})$ be a test function in $\mathcal{D}(\mathbb{R}^s)$ with support contained in Ω. Let $\psi(\underline{x})$ be a test function such that $\psi(\underline{x}) = \frac{1}{f(\underline{x})}$ in Ω. Then

$$g(\varphi) = g(f\psi\varphi) = (fg)(\psi\varphi) = 0 \ .$$

Hence $\underline{x}^0 \notin$ supp g. $\square$

To prove part (a) of Theorem 2.1 assume that $\underline{\theta} \in$ supp $\widehat{q}$. Since $\widehat{q}$ is periodic it follows that $\underline{\theta} + 2\pi\underline{\alpha} \in$ supp $\widehat{q}$, $\underline{\alpha} \in \mathbb{Z}^s$. Thus, application of Lemma 2.2 (with $f = \widehat{\phi}, g = \widehat{q}$) shows, in view of (2.8), that $\widehat{\phi}(\underline{\theta} + 2\pi\underline{\alpha}) = 0$ for every $\underline{\alpha} \in \mathbb{Z}^s$, hence $\underline{\theta} \in N(\phi)$.

For (b) assume first that $N_{\mathbb{R}}(\phi) \neq \emptyset$ and let $\underline{\theta} \in N_{\mathbb{R}}(\phi)$. Then by Theorem 1.1(a), K_ϕ contains the exponential $e^{i\underline{\theta}\cdot\underline{\alpha}}$ which is bounded, and hence tempered, on $\mathbb{Z}^s$. Conversely, if K_ϕ contains a non-trivial tempered sequence $q \in \mathcal{Q}$, then by part(a) of the theorem supp $\widehat{q} \subset N_{\mathbb{R}}(\phi)$. Since supp $\widehat{q}$ is not empty neither is $N_{\mathbb{R}}(\phi)$.

Finally to see that (c) is valid, note that if $q \in \ell_p, 1 \leq p \leq 2$ then by part (a) above $\widehat{q}$ is a measurable function whose support lies in the set of all zeros of the entire function $\widehat{\phi}$. Since $K_\phi \neq \mathcal{Q}$ then $\phi \not\equiv 0$, hence $\widehat{\phi} \not\equiv 0$, and therefore $\widehat{\phi}$, as a real analytic function, cannot vanish identically on any subset of $\mathbb{R}^s$ of positive measure. We conclude that $\widehat{q} = 0$ a.e. $(\mathbb{R}^s)$ and consequently $q \equiv 0$. $\square$

We now prove Theorem 1.3. Note that this theorem allows to transfer the results of Theorem 2.1 to the shift invariant subspaces of $\mathcal{Q}$.

Proof of Theorem 1.3. Only the "only if" implication needs verification. Assume therefore that Q is a closed shift invariant subspace of $\mathcal{Q}$. Since $\pi(X)$ is Noetherian, the ideal $(Q^\perp \cap \pi)(X)$ is finitely generated, say by $\{p_j\}_{j=1}^n$. This means that the subspace $Q^\perp \cap \pi$ of π is spanned by $\{E^{\underline{\alpha}} p_j \,|\, \underline{\alpha} \in \mathbb{Z}_+^s, 1 \leq j \leq n\}$. But for every $p \in Q^\perp$ there exists $\underline{\beta} \in \mathbb{Z}^s$ such that $E^{\underline{\beta}} p \in \pi \cap Q^\perp$, thus we conclude that $Q^\perp$ is spanned by $\{E^{\underline{\alpha}} p_j \,|\, \underline{\alpha} \in \mathbb{Z}^s, 1 \leq j \leq n\}$. For every $1 \leq j \leq n$ denote

$$\phi_j = \sum_{\underline{\alpha} \in \mathbb{Z}^s} p_j(\underline{\alpha}) E^{-\underline{\alpha}} \delta$$

where δ is the Dirac distribution. Choose $\{\underline{\alpha}^j\}_{j=1}^n \subset \mathbb{R}^s$ such that $\{E^{\underline{\alpha}^j}(\mathbb{Z}^s)\}_{j=1}^n$ are pairwise disjoint sets and define

$$\phi = \sum_{j=1}^n E^{\underline{\alpha}^j} \phi_j \ .$$

Obviously ϕ is of order zero and of finite support. We contend that $K_\phi = Q$. Indeed, since the supports of $E^{\underline{\alpha}^j} \phi_j$ $j = 1, \ldots, n$ are pairwise disjoint it follows that $K_\phi = \cap_{j=1}^n K_{\phi_j}$. Moreover, it is easy to see that

$$q \in K_{\phi_j} \qquad \Leftrightarrow \qquad q \perp E^{\underline{\alpha}} p_j \quad , \quad \forall \underline{\alpha} \in \mathbb{Z}^s \ ,$$

thus

$$q \in K_\phi \qquad \Leftrightarrow \qquad q \perp E^{\underline{\alpha}} p_j \quad , \quad \underline{\alpha} \in \mathbb{Z}^s \ , \ 1 \leq j \leq n \ .$$

But $\{E^{\underline{\alpha}} p_j \,|\, \underline{\alpha} \in \mathbb{Z}^s \ , \ 1 \leq j \leq n\}$ span $Q^\perp$ and therefore Proposition 2.1 implies

$$q \in K_\phi \qquad \Leftrightarrow \qquad q \in (Q^\perp)^\perp = Q \ .$$

$\square$

Corollary 2.2. Let $\{\phi_t\}_{t \in T} \subset \mathcal{E}'(\mathbb{R}^s)$. Then there exists $\phi \in \mathcal{E}'(\mathbb{R}^s)$ of finite support and order zero such that

$$\bigcap_{t \in T} K_{\phi_t} = K_\phi \ .$$

Proof. Since every intersection of closed shift invariant subspaces of $\mathcal{Q}$ is again a closed shift invariant subspace of $\mathcal{Q}$, the claim follows directly from Theorem 1.3. $\square$

Corollary 2.3. Let $q \in \ell_p \backslash 0,\ 1 \leq p \leq 2$. Let $u \in \mathcal{Q}$ be arbitrary. Then, for every $\varepsilon > 0$ and a compact $M \subset \mathbb{Z}^s$, there exist $\{c_j\}_{j=1}^n \subset \mathbb{C}$ and $\{\underline{\alpha}^j\}_{j=1}^n \subset \mathbb{Z}^s$ such that

$$\left|u(\underline{\alpha}) - \sum_{j=1}^n c_j q(\underline{\alpha}^j + \underline{\alpha})\right| < \varepsilon \qquad \forall \underline{\alpha} \in M \ .$$

Proof. Let Q be the closure in $\mathcal{Q}$ of the linear span of $\{E^{\underline{\alpha}} q\}_{\underline{\alpha} \in \mathbb{Z}^s}$. Since Q is a closed shift invariant subspace of $\mathcal{Q}$ and $q \in Q \cap \ell_p$ we can combine Theorem 2.1(c) together with Theorem 1.3 to conclude that $\mathcal{Q} = Q$, and our claim thus follows from the nature of the topology of $\mathcal{Q}$. $\square$

In case $s = 1$ the closed shift invariant subspaces of $\mathcal{Q}$ have a very explicit and simple structure:

Corollary 2.4. Let $s = 1$ and let Q be a closed shift invariant subspace of $\mathcal{Q}$ then

(a) Q is of finite dimension.

(b) Q constitutes the kernel of some difference operator with constant coefficients.

Proof. Since the ring of all univariate polynomials is principal we can follow the proof of Theorem 1.3 to conclude that there exists $p \in \mathcal{P}$ such that $Q = K_\phi$, where $\phi = \sum_{\alpha \in \mathbb{Z}} p(\alpha) E^{-\alpha} \delta$. Assume supp $\phi \subset \{-n, -n+1, \ldots, n\}$ and let $\{q_j\}_{j=-n}^n$ be arbitrary complex numbers that satisfy $\sum\limits_{j=-n}^{n} q_j p(j) = 0$. It is easy to verify that $\{q_j\}_{j=-n}^n$ has at most one extension to an element $q \in K_\phi$ hence $\dim Q = \dim K_\phi \leq 2n$.

To prove (b) let E be the shift operator defined by $E\phi(\cdot) = \phi(\cdot - 1)$. By (a), E is a linear operator that maps the *finite dimensional* space Q into itself. Let $E_{|Q}$ be the restriction of E to Q. Let $\lambda_1, \ldots, \lambda_n$ be the spectrum of $E_{|Q}$ counting multiplicities. By the Cayley-Hamilton theorem $L := \prod_{j=1}^n (E_{|Q} - \lambda_j)$ annihilates Q. Since $\dim Q = \dim \ker L = n$ we conclude $Q = \ker L$. $\square$

Remark 2.1. The above corollary shows that for univariate ϕ, $K_\phi \cap \ell_p = 0$ for $1 \leq p < \infty$, thus contradicting Proposition 4.10.2 of [DM$_2$].

Corollary 2.5. Assume $s = 1$, $q \in \mathcal{Q}$ and q is *not* an exponential polynomial (i.e., a linear combination of products of exponentials by polynomials). Then the linear span of $\{E^\alpha q\}_{\alpha \in \mathbb{Z}}$ is dense in $\mathcal{Q}$.

Proof. By the assumption on q, it does not lie in any kernel of a difference operator with constant coefficients, therefore Corollary 2.4 shows that q does not lie in any proper closed shift invariant subspace of $\mathcal{Q}$. $\square$

Finally, we mention that Lefranc also proved in [Le] that every closed shift invariant subspace $Q \subset \mathcal{Q}$ contains a dense subset spanned by exponential polynomials. Some of the results here could be alternatively obtained with the aid of this stronger claim.

3. Examples and Applications.

Here we give several examples in which Theorem 1.1 is applied to solve the question of the linear independence of the translates of a $\phi \in \mathcal{E}'(\mathbb{R}^s)$.

Corollary 3.1. Let $\phi, \psi \in \mathcal{E}'(\mathbb{R}^s)$ and assume that

$$\widehat{\phi}(\underline{x}) = 0,\ \underline{x} \in \mathbb{C}^s \Rightarrow\ \widehat{\psi}(\underline{x}) = 0\ . \tag{3.1}$$

Then

$$K_\phi \neq 0 \quad \Rightarrow \quad K_\psi \neq 0\ ,$$

or equivalently

$$K_\psi = 0 \quad \Rightarrow \quad K_\phi = 0$$

Proof. By (3.1) $N(\phi) \subset N(\psi)$, hence the corollary follows directly from Theorem 1.1(b). □

Example 3.1. Let ϕ be a bivariate continuous function whose support is contained in the hexagon drawn below

Fig. 3.1

Assume that $\phi(0,0) \neq 0$ and denote

$$\psi = \overbrace{\phi * \phi * \cdots \phi}^{k \text{ times}}\ .$$

Then $K_\psi = 0$.

To prove this statement we first see that the assumptions on ϕ imply that $\sum_{\underline{\alpha}\in\mathbb{Z}^s} q(\underline{\alpha})E^{\underline{\alpha}}\phi(\underline{\beta}) = 0$ for some $\underline{\beta} \in \mathbb{Z}^s$, only if $q(\underline{\beta}) = 0$, and hence $K_\phi = 0$. The general case now follows from Corollary 3.1 since for $\underline{x} \in \mathbb{C}^s$

$$\widehat{\phi}(\underline{x}) = 0 \qquad \Leftrightarrow \qquad \widehat{\psi}(\underline{x}) = 0\ .$$

Our next example is closely related to the previous one:

Example 3.2. Let ρ_1, ρ_2, ρ_3 be univariate compactly supported distributions which satisfy for $j = 1, 2, 3$

$$\widehat{\rho}_j(t) = 0 \qquad \text{only if} \qquad t \in 2\pi\mathbb{Z}\backslash 0 \; . \tag{3.2}$$

Let k_1, k_2, k_3 be positive integers. Define $\phi \in \mathcal{E}'(\mathbb{R}^2)$ via its Fourier transform as follows

$$\widehat{\phi}(x_1, x_2) = \widehat{\rho}_1(x_1)^{k_1} \widehat{\rho}_2(x_2)^{k_2} \widehat{\rho}_3(x_1 + x_2)^{k_3} \; .$$

To analyze K_ϕ let

$$\rho(t) = \begin{cases} 1 & 0 \le t \le 1, \\ 0 & \text{otherwise} \; . \end{cases}$$

Then $\widehat{\rho}(t) = \int_0^1 e^{-itx} dx$ and so $\widehat{\rho}(t) = 0$ if and only if $t \in 2\pi\mathbb{Z}\backslash 0$. Define

$$\psi(x_1, x_2) = \rho(x_1) * \rho(x_2) * \rho(x_1 + x_2) \; .$$

It is well known (see [BH_1]) that $\psi(x_1, x_2)$ is the hat function (with support similar to that drawn in Fig. 3.1) and therefore $K_\psi = 0$. But since $\widehat{\psi}(\underline{x}) = \widehat{\rho}(x_1)\widehat{\rho}(x_2)\widehat{\rho}(x_1 + x_2)$ we see from (3.2) that

$$\widehat{\phi}(\underline{x}) = 0 \qquad \Rightarrow \qquad \widehat{\psi}(\underline{x}) = 0 \; ,$$

and consequently we can appeal to Corollary 3.1 to conclude $K_\phi = 0$.

We now prove Theorem 1.4. It should be mentioned that Example 3.2 is a special case of this theorem, yet the proof of the general case employs more delicate arguments than those needed for the simple example above.

Proof of Theorem 1.4. By Theorem 1.1 it is sufficient to show $N(\phi) = \emptyset$. Let $\underline{\theta} \in \mathbb{C}^s$. To prove that $\underline{\theta} \notin N(\phi)$ one needs to find $\underline{\alpha} \in \mathbb{Z}^s$ such that $\widehat{\phi}(\underline{\theta} + 2\pi\underline{\alpha}) \neq 0$, which is equivalent to

$$\widehat{\phi}_j(i\lambda_j + (\underline{\theta} + 2\pi\underline{\alpha}) \cdot \underline{x}^j) \neq 0 \qquad j = 1, \ldots, n \; . \tag{3.3}$$

Denoting

$$\nu_j = \frac{i\lambda_j + \underline{\theta} \cdot \underline{x}^j}{2\pi} \qquad j = 1, \ldots, n \; ,$$

we see in view of (1.4), that (3.3) is valid if

$$\nu_j + \underline{\alpha} \cdot \underline{x}^j \notin \mathbb{Z}\backslash 0 \qquad j = 1, \ldots, n \; . \tag{3.4}$$

Without loss of generality we can assume that $\{\underline{x}^1, \ldots, \underline{x}^n\}$ are ordered such that for some $0 \le \ell \le s \le m \le n$

(a) $\nu_j \in \mathbb{Z} \qquad j = 1, \ldots, \ell$,

(b) $\nu_j \notin \mathbb{Z} \qquad j = \ell + 1, \ldots, m$,

(c) $\underline{x}^j \in \text{ span } \{\underline{x}^1, \ldots, \underline{x}^\ell\} \qquad j = m + 1, \ldots, n$,

(d) $\{\underline{x}^1, \ldots, \underline{x}^s\}$ forms a basis to $\mathbb{R}^s$.

Denote by $\underline{\alpha}$ the unique solution of

$$\nu_j + \underline{\alpha} \cdot \underline{x}^j = 0 \qquad j = 1, \ldots, \ell \ , \tag{3.5}$$

$$\underline{\alpha} \cdot \underline{x}^j = 0 \qquad j = \ell + 1, \ldots, s \ . \tag{3.6}$$

Since $\{\nu_j\}_{j=1}^{\ell}$ are integers and $|\det X_{J_0}| = 1$ for $J_0 = \{1, \ldots, s\}$ we conclude $\underline{\alpha} \in \mathbb{Z}^s$. This $\underline{\alpha}$ clearly satisfies (3.4) for $j = 1, \ldots, \ell$. Let $\ell < j \leq m$, then $\nu_j \notin \mathbb{Z}$, while $\underline{\alpha} \cdot \underline{x}^j \in \mathbb{Z}$, hence $\nu_j + \underline{\alpha} \cdot \underline{x}^j \notin \mathbb{Z}$

Finally, let $m < j \leq n$. Since $\underline{x}^j \in \text{span}\ \{\underline{x}^1, \ldots, \underline{x}^\ell\}$, there exist $\beta_1, \ldots, \beta_\ell \in \mathbb{R}$ such that $\underline{x}^j = \sum\limits_{k=1}^{\ell} \beta_k \underline{x}^k$, hence by (3.5)

$$\nu_j + \underline{\alpha} \cdot \underline{x}^j = \nu_j + \sum_{k=1}^{\ell} \beta_k (\underline{\alpha} \cdot \underline{x}^k) = \nu_j - \sum_{k=1}^{\ell} \beta_k \nu_k = \frac{i(\lambda_j - \sum_{k=1}^{\ell} \beta_k \lambda_k)}{2\pi} \ .$$

Since $\lambda_1, \ldots, \lambda_n, \beta_1, \ldots, \beta_\ell$ are real, it follows that $\nu_j + \underline{\alpha} \cdot \underline{x}^j \notin \mathbb{Z} \backslash 0$. Consequently, $\underline{\alpha}$ satisfies (3.4) and hence (3.3). We conclude that $N(\phi) = \emptyset$ and application of Theorem 1.1(b) completes the proof. ▫

In our last example we prove the global linear independence of the integer translates of a bivariate function ϕ which was examined in [BH$_2$].

Example 3.3. Let τ be the characteristic function of the triangle with vertices $(0,0), (1,0), (1,1)$. Let ψ be a three directional polynomial box spline, namely

$$\widehat{\psi}(\underline{x}) = (\int_0^1 e^{-ix_1 t} dt)^{k_1} (\int_0^1 e^{-ix_2 t})^{k_2} (\int_0^1 e^{-i(x_1+x_2)t} dt)^{k_3} =: \ \widehat{\phi}_1^{k_1} \widehat{\phi}_2^{k_2} \widehat{\phi}_3^{k_3}$$

where k_1, k_2, k_3 are nonnegative integers (see [BH$_{1,2}$] for a discussion of polynomial box splines). Define

$$\phi = \psi * \tau \ .$$

We claim that $K_\phi = 0$. For this purpose first note that

$$\widehat{\phi}(\underline{0}) = \widehat{\tau}(\underline{0}) = \frac{1}{2} \neq 0 \ . \tag{3.7}$$

Note also that

$$K_{\tau * \phi_1} = K_{\tau * \phi_2} = K_{\tau * \phi_3} = 0 \ . \tag{3.8}$$

This can be easily seen when examining the supports of $\tau * \phi_j$, $j = 1, 2, 3$ which are drawn in Fig. 3.2

Supp $\tau * \phi_1$ Supp $\tau * \phi_2$ Supp $\tau * \phi_3$

Fig. 3.2

We see that for $1 \leq j \leq 3$, every integer translate of $\tau * \phi_j$ contains a triangle which is not covered by the other translates of $\tau * \phi_j$. This verifies (3.8).

Now assume that $\underline{\theta} \in N(\phi)$, i.e., $\widehat{\phi}(\underline{\theta} + 2\pi\underline{\alpha}) = 0$ for all $\underline{\alpha} \in \mathbb{Z}^2$. From (3.8) and Theorem 1.1(b) we know that $N(\tau * \phi_j) = \emptyset$, $j = 1, 2, 3$ and therefore

$$\underline{\theta} \notin N(\tau * \phi_j) \qquad j = 1, 2, 3 \ . \tag{3.9}$$

Since $\widehat{\phi} = \widehat{\tau}\widehat{\phi}_1^{k_1}\widehat{\phi}_2^{k_2}\widehat{\phi}_3^{k_3}$, we conclude that there exist distinct $\underline{\alpha}^1, \underline{\alpha}^2 \in \mathbb{Z}^2$ and distinct $j, k \in \{1, 2, 3\}$ such that

$$\widehat{\phi}_j(\underline{\theta} + 2\pi\underline{\alpha}^1) = \widehat{\phi}_k(\underline{\theta} + 2\pi\underline{\alpha}^2) = 0 \ . \tag{3.10}$$

But

$$\widehat{\phi}_1(\underline{x}) = 0 \qquad \Leftrightarrow \qquad x_1 \in 2\pi\mathbb{Z}\backslash 0 \ ,$$

$$\widehat{\phi}_2(\underline{x}) = 0 \qquad \Leftrightarrow \qquad x_2 \in 2\pi\mathbb{Z}\backslash 0 \ ,$$

$$\widehat{\phi}_3(\underline{x}) = 0 \qquad \Leftrightarrow \qquad x_1 + x_2 \in 2\pi\mathbb{Z}\backslash 0 \ ,$$

and hence (3.10) implies that $\underline{\theta} \in 2\pi\mathbb{Z}^2$. Denoting $\underline{\beta} = (2\pi)^{-1}\underline{\theta} \in \mathbb{Z}^2$ it follows from (3.7) that

$$\widehat{\phi}(\underline{\theta} - 2\pi\underline{\beta}) = \widehat{\phi}(\underline{0}) = \frac{1}{2} \neq 0 \ ,$$

contradicting the assumption $\underline{\theta} \in N(\phi)$. Consequently $N(\phi) = \emptyset$ and application of Theorem 1.1(b) yields $K_\phi = 0$ as claimed.

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