Affine systems in $L_2(\mathbb{R}^d)$ II: dual systems

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ABSTRACT

The fiberization of affine systems via dual Gramian techniques, that was developed in previous papers of the authors, is applied here for the study of affine frames which have an affine dual system. Gramian techniques are also used to verify whether a dual pair of affine frames is a also a pair of bi-orthogonal Riesz bases. A general method for a painless derivation of a dual pair of affine frames from an arbitrary MRA is obtained via the *mixed extension principle*.

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1. Introduction and review of previous work

1.1. General

We continue in this paper our investigation of systems of functions in $L_2(\mathbb{R}^d)$. Previous papers on the matter include [RS1] where the framework of our *fiberization techniques* is established, [RS2] where Weyl-Heisenberg systems are analyzed, [RS3] where the theory of affine frames, and in particular tight affine frames, is developed, and [RS4] where that theory is invoked for the construction of multivariate tight affine frames generated by compactly supported splines of arbitrary smoothness. Other investigations and/or applications of these fiberization techniques can be found in [RS5-6] and [GR].

The present paper is devoted to *affine* systems, also known as *wavelet* systems. Our previous studies of this setup ([RS3,4], [GR]) were focused on *tight affine frames*. The reason for that is that the analysis in [RS3] led to simple 'extension principles' for constructing tight affine frames, and these simple principles led further to the constructions of a wealth of concrete tight wavelet frames.

In this paper, we focus on the theory of general affine systems, quasi-affine systems, and their dual counterparts. In what follows, we define some of the basics concerning function systems in $L_2 := L_2(\mathbb{R}^d)$, and review the main ingredients of our fiberization techniques. We then describe some of the highlights of [RS3,4], and summarize the main findings of the present paper.

Let X be a countable subset of L_2, referred to hereafter as a *system*. The system X can be used either for the reconstruction or for the decomposition of other functions. The relevant operators in this context are the **synthesis operator** $T := T_X$ defined by

$$T : \ell_2(X) \to L_2 : c \mapsto \sum_{x\in X} c(x)x,$$

and its adjoint, the **analysis operator** $T^* := T^*_X$ defined by

$$T^* : L_2 \to \ell_2(X) : f \mapsto (\langle f, x\rangle)_{x\in X}.$$

If either (hence both) the synthesis operator or the analysis operator is well-defined and bounded, we say that X is **a Bessel system**. A Bessel system X whose analysis operator is *bounded below* is called **a fundamental frame**; thus X is a fundamental frame if and only if there exist constants $c, C > 0$ such that

$$c\|f\|^2_{L_2} \le \sum_{x\in X} |\langle f, x\rangle|^2 \le C\|f\|^2_{L_2}, \quad \text{all } f \in L_2.$$

The sharpest possible C (c, respectively) is called **the upper (lower**, respectively**) frame bound**. Since almost all frames discussed in this paper are fundamental, we omit this adjective in any further reference to a frame X, and add special remarks to results that apply also to the non-fundamental case. Finally, a Bessel system X whose *synthesis* operator is bounded below is a **Riesz basis**.

If the analysis operator T^* is *unitary*, then the identity $TT^* = \mathrm{I}$ holds, and one can then use the same system X in both the analysis and synthesis steps. In this case X is called a **tight frame**. For a frame which is not tight, one needs to find another frame $\mathrm{R}X$ (with $\mathrm{R} : X \to L_2$ the association between the elements of X and those of the new frame), called **a dual frame of** X, such that $T_X T^*_{\mathrm{R}X} = \mathrm{I}$, i.e.,

$$\sum_{x\in X} \langle f, \mathrm{R}x\rangle x = f, \quad \text{all } f \in L_2.$$

By applying transposition to $T_X T^*_{\mathrm{R}X} = \mathrm{I}$, one obtains that $T_{\mathrm{R}X} T^*_X = \mathrm{I}$, hence duality is symmetric here. There might be many dual frames for a given X; however, there exists a unique dual frame $\mathrm{R}X$ for which the projector $T^*_{\mathrm{R}X} T_X$ is self-adjoint, hence orthogonal; we refer hereafter to this dual frame as the **minimal dual frame**. When $T^*_{\mathrm{R}X} T_X = I$, X is a fundamental Riesz basis. A fundamental Riesz basis has a unique dual system $\mathrm{R}X$ which is characterized by the bi-orthogonality relations

$$\langle x, \mathrm{R}x'\rangle = \delta_{x,x'}.$$

As the following simple proposition reveals, there is a certain amount of redundancy in the above definition of dual frames:

Proposition 1.1. *Let X be a Bessel system, and $\mathrm{R} : X \to L_2$ some map. Assume that $\mathrm{R}X$ is a Bessel system, too. Then the following conditions are equivalent:*
(a) X is a frame and $\mathrm{R}X$ is a frame dual to X.
(b) $T_X T^*_{\mathrm{R}X} = \mathrm{I}$.

□

Also, for the minimality of the dual frame, one really needs to prove the self-adjointness of R:

$$\langle \mathrm{R}x, x'\rangle = \langle x, \mathrm{R}x'\rangle, \quad \text{all } x, x' \in X$$

(cf. Proposition 4.1 here).

1.2. An overview of the fiberization techniques

A casual stride through some of the highlights of our previous papers, which is the goal here, is possible only if one forsakes rigorousness. In particular, we intentionally ignore all questions concerning functions that are only a.e. defined, operators that are only densely defined on their domain, invertibility of operators, etc. This approach here is the antipode of the original approach in our papers, where meticulous discussions of such fine points are included.

Let E^α, $\alpha \in \mathbb{R}^d$, be the **shift operator**:

$$E^\alpha : f \mapsto f(\cdot + \alpha),$$

and let L be **a d-lattice** in $\mathbb{R}^d$, i.e., the image of $\mathbb{Z}^d$ under some linear invertible map. Recall that the **dual lattice** $\widetilde{L}$ of L is the lattice defined by

$$\widetilde{L} := \{k \in \mathbb{R}^d : \langle k, l\rangle \in 2\pi\mathbb{Z}, \quad \text{all } l \in L\}.$$

For example, the dual lattice of $h\mathbb{Z}^d$ is $2\pi\mathbb{Z}^d/h$.

Next, assume that the system X is **shift-invariant** (with respect to the given lattice L), i.e., there exists a subset $F \subset X$, such that

$$X = (E^\alpha f : f \in F, \alpha \in L).$$

In [RS1], we associate the shift-invariant X with a collection of 'fiber operators' as follows. The fibers are indexed by $\omega \in \mathbb{R}^d$, and each fiber $J_\omega := J_\omega(X)$ is an operator from $\ell_2(F)$ into $\ell_2(\widetilde{L})$, hence is a matrix whose rows are indexed by $\widetilde{L}$ and whose columns are indexed by F. The $(l, f) \in \widetilde{L} \times F$ entry of J_ω is

$$d(L)\,\widehat{f}(\omega + l),$$

with $d(L)$ is some normalization constant (actually, $d(L) = (\det L)^{-1/2}$). We refer to the collection of $(J_\omega)_\omega$ as the **pre-Gramian fiberization of** X. In order to illustrate at this early point the role played by the various Gramian matrices, we associate the fibers $(J_\omega)_\omega$ with the norm functions

$$\mathcal{J} : \omega \mapsto \|J_\omega\|, \ \ \mathcal{J}^* : \omega \mapsto \|J_\omega^*\|,$$

where $\|J_\omega\|$, $\|J_\omega^*\|$ are the operator norms of J_ω, J_ω^*, and where J_ω^* is the adjoint matrix of J_ω (considered thus as an operator from $\ell_2(\widetilde{L})$ into $\ell_2(F)$). One has:

Result 1.2. ([RS1]) *The following three conditions are equivalent:*
(a) X *is a Bessel system.*
(b) $\mathcal{J} \in L_\infty(\mathbb{R}^d)$.
(c) $\mathcal{J}^* \in L_\infty(\mathbb{R}^d)$.
Moreover, $\|T_X\| = \|\mathcal{J}\|_{L_\infty(\mathbb{R}^d)} = \|\mathcal{J}^*\|_{L_\infty(\mathbb{R}^d)}$. □

Example 1.3. Let X be a **Principal Weyl-Heisenberg** (PWH) system (see [RS2]). This, by definition, means that for some $\phi \in L_2$, and two lattices K, $L \subset \mathbb{R}^d$,

$$X = \{E^k(e_\ell \phi) : \ k \in K, \ell \in L\}, \tag{1.4}$$

where $e_\ell \mapsto e^{-i\ell \cdot t}$, the exponential with frequency ℓ. We then observe that X is shift-invariant with respect to

$$F := \{e_\ell \phi : \ \ell \in L\}.$$

Indexing F by L, we obtain that the matrix $J_\omega(X)$ is indexed by $\widetilde{K} \times L$, with entries

$$d(K)\widehat{\phi}(\omega + \ell + k), \quad (k, \ell) \in \widetilde{K} \times L.$$

Now, consider another PWH system, viz.

$$X^* := \{E^\ell(e_k\phi) : \; k \in \widetilde{K}, \; \ell \in \widetilde{L}\}.$$

In [RS2], X^* is named the **adjoint system of** X. A computation similar to the previous one shows that $J_\omega(X^*)$ is indexed by $L \times \widetilde{K}$ with entries

$$d(\widetilde{L})\widehat{\phi}(\omega + \ell + k), \quad (\ell, k) \in L \times \widetilde{K}.$$

This means that

$$\frac{d(\widetilde{L})}{d(K)}\overline{J^*_\omega}(X) = J_\omega(X^*). \tag{1.5}$$

The connection expressed in (1.5) is the basis for the **duality principle of Weyl-Heisenberg systems**, [RS2] (many ingredients of that principle were independently discovered, using different techniques, by Janssen in [J], and by Daubechies, Landau and Landau in [DLL]). That duality principle (which is stated in Result 1.8) deals with the intimate relation that exists between a PWH system and its adjoint PWH system.

Let us now return to general shift-invariant systems X, and assume that $L := \mathbb{Z}^d$ (the treatment of general lattices is truly important only in the context of WH systems; in the wavelet case there is no loss in choosing the lattice canonically as we just did). The pre-Gramian matrices can be used to create self-adjoint non-negative fibers in two different ways:

$$G_\omega := J^*_\omega J_\omega, \quad \widetilde{G}_\omega := J_\omega J^*_\omega.$$

The collection $(G_\omega)_\omega$ is the **Gramian fiberization of** X while $(\widetilde{G}_\omega)_\omega$ is the **dual Gramian fiberization of** X. Note that the Gramian (dual Gramian, respectively) fibers are non-negative self-adjoint endomorphisms of $\ell_2(F)$ ($\ell_2(2\pi\mathbb{Z}^d)$, respectively). It is the operator $T^*_X T_X$ which is analysed via the the Gramian fibers, while the dual Gramian fibers are particularly useful in the analysis of the other operator, $T_X T^*_X$. A direct computation shows that

$$G_\omega(f, g) = \sum_{\alpha \in 2\pi\mathbb{Z}^d} \widehat{f}(\omega + \alpha)\overline{\widehat{g}(\omega + \alpha)}, \quad (f, g) \in F \times F. \tag{1.6}$$

The right hand side of (1.6) is the **bracket product** $[\widehat{f}, \widehat{g}]$ that was introduced (in a slightly different form) in [JM] and in the present form in [BDR1]. The dual Gramian fibers have the form

$$\widetilde{G}_\omega(\alpha, \beta) = \sum_{f \in F} \widehat{f}(\omega + \alpha)\overline{\widehat{f}(\omega + \beta)}, \quad (\alpha, \beta) \in 2\pi\mathbb{Z}^d \times 2\pi\mathbb{Z}^d.$$

As we did in the pre-Gramian discussion, we associate the Gramian matrices with their norm functions

$$\begin{aligned}
\mathcal{G} \; &: \; \mathbb{R}^d \to \mathbb{R}_+ \; : \; \omega \mapsto \|G_\omega\|, \\
\mathcal{G}^* \; &: \; \mathbb{R}^d \to \mathbb{R}_+ \; : \; \omega \mapsto \|\widetilde{G}_\omega\|, \\
\mathcal{G}^- \; &: \; \mathbb{R}^d \to \mathbb{R}_+ \; : \; \omega \mapsto \|G^{-1}_\omega\|, \\
\mathcal{G}^{*-} \; &: \; \mathbb{R}^d \to \mathbb{R}_+ \; : \; \omega \mapsto \|\widetilde{G}^{-1}_\omega\|.
\end{aligned}$$

The question whether X is a fundamental frame, a Riesz basis, a tight frame, or an orthonormal system can be completely settled with the aid of the above norm functions:

Result 1.7. [RS1] *Let X be a shift-invariant system associated with the Gramian norm functions as above. Then:*

(a) *X is a Riesz basis if and only if $\mathcal{G}, \mathcal{G}^- \in L_\infty$. Further, $\|T\|^2 = \|\mathcal{G}\|_{L_\infty}$ and $\|T^{-1}\|^2 = \|\mathcal{G}^-\|_{L_\infty}$. In particular, X is orthonormal if and only if G_ω is the identity matrix for (almost) every ω.*

(b) *X is a fundamental frame if and only if $\mathcal{G}^*, \mathcal{G}^{*-} \in L_\infty$. Further, $\|T^*\|^2 = \|\mathcal{G}^*\|_{L_\infty}$ and $\|T^{*-1}\|^2 = \|\mathcal{G}^{*-}\|_{L_\infty}$. In particular, X is a fundamental tight frame if and only if $\widetilde{G}_\omega$ is the identity matrix for (almost) every ω.*

This result was combined in [RS2] with (1.5) to yield the following:

Result 1.8. *Let X be a PWH system whose generator has norm 1, X^* its adjoint system.*

(a) *X^* is a frame if and only if X is a frame.*

(b) *X^* is a tight frame if and only if X is a tight frame.*

(c) *X^* is a Riesz basis if and only if X is a fundamental frame.*

(d) *X^* is an orthonormal basis if and only if X is a fundamental tight frame.*

(e) *'The dual of the adjoint is the adjoint of the dual': The generator of the minimal dual frame of X^* is the same as the generator of the minimal dual frame of X, up to a multiplicative constant.*

In addition to the Gramian matrices, it is also important to consider *mixed Gramian matrices*: this is the case when we multiply the pre-Gramians of two different shift-invariant systems (that are indexed, say, by the same index set). Such matrices are important when studying dual systems, and extensive discussions of that setup are provided in [RS1] for general SI systems, and in [RS2] for the special WH system. The affine version of such matrices will be central to our investigations in the present paper. But, first, let us introduce wavelets systems and their celebrated fiberization.

1.3. Fiberization of affine systems

The systems studied in this paper are known as *affine* or *wavelet*. To define an affine system, let $\Psi \subset L_2$ be a finite set of **mother wavelets**, and let D be a **dilation operator**:

$$D : f \mapsto |\det s|^{1/2} f(s\cdot).$$

Here, s is any fixed $d \times d$ *expansive* matrix, i.e., a matrix whose entire spectrum lies outside the closed unit disk. The **affine system generated by** Ψ is then the system

$$X := X(\Psi) := \cup_{k\in\mathbb{Z}} D^k E(\Psi),$$

where

$$E(\Psi) := \{E^\alpha \psi : \ \psi \in \Psi, \ \alpha \in \mathbb{Z}^d\}.$$

Throughout this article, we also assume that the matrix s has *integer* entries.

An affine system is invariant under the dilation operator D, but is not shift-invariant. At the same time, our fiberization techniques require X to be shift-invariant. We had circumvented this difficulty in [RS3] with the aid of the notions of the *truncated affine system* X_0 and the *quasi-affine system* X^q of an affine system:

Definition: the truncated affine system. Let X be an affine system generated by Ψ. The **truncated system** X_0 of X is the shift-invariant system

$$X_0 = \cup_{k \geq 0} D^k E(\Psi).$$

$\square$

Thus, the truncated affine system is obtained by removing all the *negative* dilation levels $D^k E(\Psi)$, $k < 0$, from X.

Definition: the quasi-affine system. Let X be an affine system generated by Ψ. The **quasi-affine system** X^q of X is the shift-invariant system

$$X^q = X_0 \bigcup (\cup_{k<0} |\det s|^{\frac{k}{2}} E(D^k \Psi)).$$

$\square$

Thus, the quasi-affine system is obtained from the affine system by modifying the system in the *negative* dilation levels. For each $k < 0$, the set $D^k E(\Psi)$ (which consists of the sparse $s^{-k}\mathbb{Z}^d$-shifts of $D^k \Psi$), is replaced by the denser $\mathbb{Z}^d$-shifts of the renormalized functions $|\det s|^{k/2} D^k \Psi$. Note that the normalization factor $|\det s|^{k/2}$ is < 1.

The main result of [RS3] is as follows.

Result 1.9. *Let X be an affine system, and X^q its quasi-affine counterpart. Then:*

(a) X is a Bessel system if and only if X^q is a Bessel system.

(b) X is a frame if and only if X^q is a frame. Moreover, the two frames have the same upper frame bound and the same lower frame bound.

In particular, X is a tight frame if and only if X^q is a tight frame.

We remark that the original theorem in [RS3] assumes the following very mild smoothness condition on Ψ:

$$\sum_{\psi \in \Psi} \sum_{k=0}^{\infty} c(\psi, k) < \infty, \tag{1.10}$$

where for every $k \in \mathbb{Z}_+$,

$$A_k := \{\alpha \in 2\pi\mathbb{Z}^d : \ |\alpha| > 2^k\},$$

and

$$c(\psi, k) := \| \sum_{\alpha \in A_k} |\widehat{\psi}(\cdot + \alpha)|^2 \|_{L_\infty([-\pi,\pi]^d)}.$$

It is elementary to prove that (1.10) is satisfied once $\widehat{\psi}(\omega) = O(|\omega|^{-\rho})$, as $\omega \to \infty$, for some $\rho > d/2$, and every $\psi \in \Psi$. This smoothness condition on Ψ was removed in [CSS].

Result 1.9 allows one to apply fiberization techniques to affine systems: first, one analyses quasi-affine systems (which *are* shift-invariant) using results like Result 1.7, and then transports the 'frame parts' of such results to affine systems using the above result. In order to obtain concrete results along these lines, one needs first to represent the quasi-affine system $X^q(\Psi)$ as a shift-invariant system $E(F)$, and then to compute the dual Gramian fibers. This technical step was carried out in [RS3]. It turns out that the (α, β)-entry of the dual Gramian fiber is

$$\widetilde{G}_\omega(\alpha,\beta) = \sum_{\psi\in\Psi} \sum_{k=\kappa(\alpha-\beta)}^{\infty} \widehat{\psi}(s^{*k}(\omega+\alpha))\overline{\widehat{\psi}(s^{*k}(\omega+\beta))},$$

where κ is the s^*-valuation function

$$\kappa(\omega) := \inf\{k \in \mathbb{Z} : \ s^{*k}\omega \in 2\pi\mathbb{Z}^d\}.$$

Anticipating an extensive use of such expressions in wavelet analysis, we introduced in [RS3] the following **affine product**:

$$\Psi[\omega,\omega'] = \sum_{\psi\in\Psi} \sum_{k=\kappa(\omega-\omega')}^{\infty} \widehat{\psi}(s^{*k}\omega)\overline{\widehat{\psi}(s^{*k}\omega')}.$$

Thus, in terms of the affine product, the dual Gramian entry is $\Psi[\omega+\alpha,\omega+\beta]$. It is worth noting that the diagonal entries of that dual Gramian thus have the form

$$\widetilde{G}_\omega(\alpha,\alpha) = \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}} |\widehat{\psi}(s^{*k}(\omega+\alpha))|^2 = \Psi[\omega+\alpha,\omega+\alpha] =: \Psi[\omega+\alpha].$$

In the sequel, we always refer to the above dual Gramian matrix as the 'dual Gramian of the affine X', though, strictly speaking, it is the dual Gramian of the quasi-affine X^q.

It is now easy to formulate results that analyse the affine X in terms of its dual Gramian fibers. For example, a complete characterization of the frame property of the affine X in terms of the above dual Gramian is obtained by combining part (b) of Result 1.7 together with Result 1.9. Of particular interest is the characterization of *tight frames*. One can write (in view of (b) of Result 1.7) an immediate result that simply requires the diagonal entries of the dual Gramians to be equal to 1 and requiring the off-diagonal elements to be equal to 0. However, there is a certain amount of redundancy in the so-obtained conditions. After removing those redundancies, one obtains the following characterization of tight affine frames. We note that this result was obtained independently by Bin Han in [H]:

Result 1.11. ([H], [RS3]) *Let X be an affine system generated by Ψ. Then X is a tight frame if and only if the following two conditions are valid for almost every $\omega \in \mathbb{R}^d$:*
(a) $\Psi[\omega] = 1$.
(b) $\Psi[\omega,\omega+\alpha] = 0$, *for every* $\alpha \in 2\pi(\mathbb{Z}^d\backslash(s^*\mathbb{Z}^d))$.

Example. In one variable and for dyadic dilations, these two conditions boil down to the requirements that for almost every ω and for every *odd* integer j,

$$\sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^k\omega)|^2 = 1, \quad \sum_{\psi\in\Psi}\sum_{k=0}^{\infty} \widehat{\psi}(2^k\omega)\overline{\widehat{\psi}(2^k(\omega+2\pi j))} = 0.$$

1.4. Fiberization of affine frames: MRA

The use of multiresolution analysis allows one to derive from the abstract results of the previous subsection useful algorithms for constructing tight and other affine frames. In [RS3], it is assumed that the MRA may be generated by a vector of scaling functions (i.e., it is an FSI MRA). However, for all practical considerations that we discuss here, it suffices to consider the PSI case, i.e., that of a singleton scaling function.

Our notion of MRA is a very weak (hence general) one: we are given a scaling function $\phi \in L_2$, and mean by that that ϕ satisfies the relation

$$\widehat{\phi}(s^* \cdot) = \tau_\phi \widehat{\phi},$$

with the 2π-periodic mask τ_ϕ assumed to be *bounded*. We then select the mother wavelets Ψ from V_1 (:= the s-dilate of V_0, the latter being the smallest closed L_2-space that contains the shifts of ϕ). Thus, each ψ satisfies a relation

$$\widehat{\psi}(s^* \cdot) = \tau_\psi \widehat{\phi},$$

for some 2π-periodic **wavelet mask** τ_ψ. Note that, importantly, no a-priory assumption on the number of mother wavelets is made.

We can then substitute the above relations into the affine product expression, combine that with Result 1.11, and obtain in this way a characterization of all tight affine frames that can be constructed by MRA (cf. Theorem 6.5 of [RS3]). From that characterization, it is easy to conclude the following **unitary extension principle**:

Result 1.12. *Let ϕ be a refinable function with bounded mask τ_ϕ, and assume that $\widehat{\phi}(0) = 1$. Let Ψ be a finite collection of functions defined by*

$$\widehat{\psi}(s^* \cdot) = \tau_\psi \widehat{\phi},$$

with each τ_ψ bounded and 2π-periodic. Then the affine system generated by Ψ is a tight frame if the following orthogonality conditions hold for almost every $\omega \in \mathbb{R}^d$, and for every $\alpha \in 2\pi(s^{*-1}\mathbb{Z}^d/\mathbb{Z}^d)$:

$$\tau_\phi(\omega)\overline{\tau_\phi(\omega+\alpha)} + \sum_{\psi\in\Psi} \tau_\psi(\omega)\overline{\tau_\psi(\omega+\alpha)} = \delta_{0,\alpha}.$$

Example. We consider again the univariate dyadic case. In that event, $2\pi((\mathbb{Z}/2)/\mathbb{Z}) = \{0, \pi\}$. Let τ be the vector whose first entry is τ_ϕ and other entries τ_ψ, $\psi \in \Psi$. There are then two conditions here. The first is that $\tau(\omega)$ has a unit length, and the other is that $\tau(\omega)$ is orthogonal to $\tau(\omega + \pi)$. □

Concrete constructions that use this extension principle are given in [RS3], [RS4] and [GR]. In [RS3], we selected the refinable function to be the univariate B-spline of order m, and found m compactly supported mother wavelets (in V_1, i.e., having the half integers as their knots) all symmetric (or anti-symmetric) that generate a tight frame. In [RS4] various multivariate constructions of compactly supported spline wavelets that are derived from a box spline are discussed (in fact, it is shown there that, essentially, such a tight frame can be derived from *any* given refinable box spline). For example, we show there how to construct a tight frame using the Zwart-Powell element (which is a C^1 piecewise-quadratic and has four direction mesh lines of symmetry; the wavelets have these four lines of symmetry/anti-symmetry, too). The constructions of [RS4] apply to dilation matrices that satisfy $s^k = aI$, for some integer k. For a general dilation matrix s, tight compactly supported affine frames of arbitrarily high smoothness were constructed in [GR]. Those frames are *not* piecewise-polynomial in general.

1.5. Layout of the current paper

We mentioned before that every frame can be complemented by a dual system (and that there may be many such dual systems). However, structural properties of the original system X may not be preserved in the dual system. In particular, it is well-known that an affine frame may not have a dual system which is affine, too. On the other hand, a tight frame obviously have an affine dual system (viz., the system itself). Thus, the affine frames that *have* an affine dual system can be considered as an intermediate case between the special tight systems and the most general affine frames. The focus in this paper is on such systems.

While extending the tight frame results of [RS3] to this more general setup, we had to deal with several problems of different nature:

(a): The systems that were analyzed in [RS1] were not only assumed to be shift-invariant, but, also, the fiberized operators were assumed to be *self-adjoint* (a condition that is certainly valid for $T_X T_X^*$ and $T_X^* T_X$); in fact, the main results of [RS1] fail to hold without this assumption. When dealing with a system-dual system relation, the operators that are fiberized are not self-adjoint. However, it turned out that only a small, light, fraction of the general fiberization results of [RS1] is really needed here, and that portion of the fiberization theory extends with ease to operators that are not self-adjoint. In §2.1, we establish those minor extensions that are required in this paper. The reader may skip this technical section without an essential loss of continuity. In the subsequent section, §2.2, we introduce the *mixed affine product* and describe the dual Gramian fibers of the mixed operators; these, too, can be considered as straightforward technical extensions of the [RS3] analysis.

(b): As explained before, it is the quasi-affine system which is really fiberized; conclusions concerning the affine X then follow with the aid of Result 1.9. This result, however, lacks assertions concerning possible connections between dual systems of affine frames and dual systems of their quasi-affine counterparts. This study is the subject of §3.1. In this context, one should keep in mind that 'wished for' statements of the form "each dual system of a quasi-affine frame is the quasi-affine counterpart of a system dual to the affine frame" simply do not make sense: as we said, dual systems of affine frames need not to be affine. The results of §3.1, though, show that an appropriately modified version of the above statement *is* valid (cf. Theorem 3.1).

(c): The foundation laid in §2.1 and §2.2, together with Proposition 1.1, allows us then, for given affine Bessel systems X and $\mathrm{R}X$, to study the question whether $\mathrm{R}X$ is a dual system to X via the fiberization of $T_{X^q}T^*_{(\mathrm{R}X)^q}$. This still requires one to check in advance that the systems in question are *Bessel*. No new techniques for verifying this property are given in the current paper. However, in addition to being characterized fully in [RS3] (cf. Theorem 5.5 there), simple sufficient conditions for the Bessel property of an affine X are known (see e.g. [RS6]). Roughly speaking, all these results say that a system X is Bessel if the corresponding mother wavelets are smooth (in a mild sense) and have zero-mean values. With all these in hand, we analyze, in §3.2, affine systems constructed by MRA. The analysis in this part extends the analysis of MRA constructions of tight affine frames that was made in §6 of [RS3].

(d): Given two dual (affine) frames, we attempt in §4 to verify whether these systems are bi-orthogonal Riesz bases. Dual Gramian techniques are applied in §4.1 to this end. These techniques differ from the Gramian approaches used in [RS6] and [RS3]. Finally, the study of bi-orthogonal wavelet constructions from MRA is the topic of §4.2.

2. The mixed Gramian matrices

2.1. Two technical lemmata

The Gramian matrices of a given shift-invariant system X were obtained as the product of the pre-Gramian matrix and its adjoint. For the intended simultaneous study of *two* systems, X and $\mathrm{R}X$, we need to multiply the pre-Gramian of one of these systems by the adjoint pre-Gramian of the other system. We call the resulting matrices the **mixed Gramian matrices**. We will then need some extensions of the results of [RS1] concerning the Gramian matrices to the new mixed setup. Had we assumed that the mixed matrices are also *non-negative and self-adjoint*, a complete generalization of the [RS1] results (hence of Result 1.7 in particular) would have been possible; unfortunately, in the context of system-dual system setup, such an assumption holds only for *minimal* dual systems, hence, in view of our stated objectives, is prohibitive.

In the absence of the self-adjointness assumption, we need to assume *in advance* that $E(\Phi)$ and $E(\mathrm{R}\Phi)$ are Bessel systems. Under this assumption, we prove in this section two simple lemmata that will be required in the next subsection.

We denote by S the Fourier transform of the operator $T_{E(\mathrm{R}\Phi)}T^*_{E(\Phi)}$ i.e.,

$$Sf := (T_{E(\mathrm{R}\Phi)}T^*_{E(\Phi)}f^\vee)\widehat{\ }, \tag{2.1}$$

where $f^\vee$ is the inverse Fourier transform of f. The previously defined dual Gramian is now replaced by **the mixed dual Gramian of** $E(\Phi)$ **and** $E(\mathrm{R}\Phi)$ (still denoted by $\widetilde{G}$) which is a matrix whose rows and columns are indexed by $2\pi\mathbb{Z}^d$, and whose (α,β)-entry is

$$\widetilde{G}_\omega(\alpha,\beta) = \sum_{\phi\in\Phi} \widehat{\mathrm{R}\phi}(\omega+\alpha)\overline{\widehat{\phi}}(\omega+\beta).$$

It is proved in [RS1], that, since $E(\Phi)$ and $E(\mathrm{R}\Phi)$ are assumed Bessel, each of the above entries is well-defined a.e., in the sense that the corresponding sum converges absolutely (and to a finite limit). The dual Gramian analysis in [RS1] is based on the observation that, for a.e. $\omega \in \mathbb{R}^d$,

$$(Sf)_{|_{\omega+2\pi\mathbb{Z}^d}} = \widetilde{G}_\omega(f_{|_{\omega+2\pi\mathbb{Z}^d}}). \tag{2.2}$$

Lemma 2.3. *Assume $E(\Phi)$ and $E(\mathrm{R}\Phi)$ are Bessel systems. Then $E(\mathrm{R}\Phi)$ is a system dual to $E(\Phi)$ if and only if $\widetilde{G} = \mathrm{I}$ for almost every ω.*

Proof: $E(\mathrm{R}\Phi)$ is dual to $E(\Phi)$ if and only if $T_{E(\mathrm{R}\Phi)}T^*_{E(\Phi)} = I$ (cf. Proposition 1.1) if and only if, with S as above, $S = I$. Now, if $S = I$, then $Sf = f$ a.e., for every $f \in L_2$, which, due to (2.2), readily implies that $\widetilde{G} = I$ a.e. Conversely, if $\widetilde{G} = I$ a.e., then, (2.2), $(Sf)_{|_{\omega+2\pi\mathbb{Z}^d}} = (f_{|_{\omega+2\pi\mathbb{Z}^d}})$ a.e., for every f, and hence $S = I$. □

The other result that we need is also very simple. The subspace H_r, $r \geq 0$, that appears in the following lemma is defined as:

$$H_r := \{f \in L_2 : \widehat{f}(\omega) = 0, \text{ for } |\omega| \leq r\}.$$

Lemma 2.4. *Assume that $E(\Phi)$ and $E(\mathrm{R}\Phi)$ are Bessel. Let $\widetilde{G}$ be their mixed dual Gramian. Assume further that, for some $r \geq 0$, the restriction of $T_{E(\mathrm{R}\Phi)}T^*_{E(\Phi)} - I$ to H_r has norm $\leq \varepsilon$, for some $\varepsilon > 0$. Then, for a.e. $\omega \in \mathbb{R}^d$, and for all $\alpha \in 2\pi\mathbb{Z}^d$, if $|\omega| > r$,*

$$|\widetilde{G}_\omega(\alpha, 0) - \delta_\alpha| \leq \varepsilon.$$

Proof: Fix $\alpha \in 2\pi\mathbb{Z}^d$. Consider the map

$$M : \Omega :\to \mathbb{C} : \omega \mapsto \widetilde{G}_\omega(\alpha, 0) - \delta_\alpha,$$

with Ω the complement in $\mathbb{R}^d$ of a ball centered at the origin, with radius r. Since we assume $E(\Phi)$ and $E(\mathrm{R}\Phi)$ to be Bessel, then, [RS1], $\omega \mapsto \widetilde{G}_\omega(\alpha, 0)$ is well-defined in the sense that the corresponding sum converges absolutely a.e. and therefore M is measurable (as the sum of measurable functions). Let

$$A := \{\omega \in \Omega : |M(\omega)| > \varepsilon\}.$$

Then A is measurable. Assume that, for some $\beta \in 2\pi\mathbb{Z}^d$, the intersection $A_\beta := A \cap (\beta + [-\pi, \pi]^d)$ has positive measure, and let f be the support function of A_β. Then, with S as in (2.1), and for every $\omega \in A_\beta$, $((S - I)f)_{|_{\omega+2\pi\mathbb{Z}^d}}$ is the vector $\gamma \mapsto \widetilde{G}_\omega(\gamma, 0) - \delta_\gamma$, and hence,

$$\|((S - I)f)_{|_{\omega+2\pi\mathbb{Z}^d}}\|_{\ell_2(2\pi\mathbb{Z}^d)} \geq |\widetilde{G}_\omega(\alpha, 0) - \delta_\alpha| > \varepsilon = \varepsilon f(\omega).$$

Squaring the last inequality and integrating it over A_β we easily obtain (after taking into account that $(S - I)f$ is supported only in $A_\beta + 2\pi\mathbb{Z}^d$) that

$$\|(S - I)f\| > \varepsilon\|f\|.$$

However, since $f^\vee \in H_r$, this contradicts the fact that, by assumption, $\|(S - I)g\| \leq \varepsilon\|g\|$, for every $g \in \widehat{H_r}$. Hence A_β is a null-set, and consequently A is a null-set, too, which is what we wanted to prove. □

We note that the above argument applies to any translation-invariant subspace of L_2 (one only needs to replace the above set Ω by the spectrum of that space).

2.2. Fiberization of quasi-affine systems

Let X be an affine system generated by Ψ, let $\mathrm{R}:\Psi\to L_2$ be some map, and let $\mathrm{R}X$ be the affine system generated by $\mathrm{R}\Psi$. We assume that X and $\mathrm{R}X$ are Bessel, and conclude (from Result 1.9) that the quasi-affine counterparts X^q and $(\mathrm{R}X)^q$ are Bessel, too. This ensures us that the mixed Gramian matrices of X^q and $(\mathrm{R}X)^q$ are well-defined (as discussed in the previous subsection). The details of these matrices are provided below, but we omit here the actual computations of these mixed dual Gramian fibers: the derivation follow *verbatim* that of [RS3] (where the case $\mathrm{R}X = X$ is studied).

First, we introduce the **mixed affine product**: given any finite Ψ, and $\mathrm{R}:\Psi\to L_2$, the mixed affine product

$$\Psi_{\mathrm{R}}[\,,\,]$$

is a map from $\mathbb{R}^d\times\mathbb{R}^d$ to $\mathbb{C}$ defined as

$$\Psi_{\mathrm{R}}[\,,\,]:(\omega,\omega')\mapsto\Psi_{\mathrm{R}}[\omega,\omega'] := \sum_{\psi\in\Psi}\sum_{k=\kappa(\omega-\omega')}^{\infty}\widehat{\psi}(s^{*k}\omega)\overline{\widehat{\mathrm{R}\psi}}(s^{*k}\omega'),\quad \omega,\omega'\in\mathbb{R}^d. \tag{2.5}$$

Here, the κ-function is defined as before:

$$\kappa:\mathbb{R}^d\to\mathbb{Z}:\omega\mapsto\inf\{k\in\mathbb{Z}: s^{*k}\omega\in 2\pi\mathbb{Z}^d\}. \tag{2.6}$$

One easily observes that $\Psi_{\mathrm{R}}[\,,\,]$ is s^*-invariant, i.e.,

$$\Psi_{\mathrm{R}}[s^*\omega,s^*\omega'] = \Psi_{\mathrm{R}}[\omega,\omega'],\quad \text{all } \omega,\omega'. \tag{2.7}$$

The mixed affine product is well-defined (i.e., absolutely convergent a.e.) if the systems generated by Ψ and $\mathrm{R}\Psi$ are known to be Bessel.

In a way entirely analogous to the computation in Proposition 5.1 of [RS3], we obtain the following:

Lemma 2.8. *Let X and $\mathrm{R}X$ be two affine Bessel systems generated by Ψ and $\mathrm{R}\Psi$, respectively. Let $\widetilde{G}_\omega$, $\omega\in\mathbb{R}^d$, be the ω-fiber of the mixed dual Gramian of the shift-invariant (Bessel) systems X^q and $(\mathrm{R}X)^q$. Then, the (α,β)-entry, $\alpha,\beta\in 2\pi\mathbb{Z}^d$, of this $\widetilde{G}_\omega$ is*

$$\widetilde{G}_\omega(\alpha,\beta) = \Psi_{\mathrm{R}}[\omega+\alpha,\omega+\beta].$$

Lemma 2.8 when combined with Lemma 2.3 provides a characterization of the property '$(\mathrm{R}X)^q$ is dual to X^q'. The requirement is that

$$\Psi_{\mathrm{R}}[\omega+\beta,\omega+\alpha] = \delta_{\alpha,\beta}$$

for almost every $\omega\in\mathbb{R}^d$, and every $\alpha,\beta\in 2\pi\mathbb{Z}^d$. A simple change of variables allows us to reduce the above condition to

$$\Psi_{\mathrm{R}}[\omega,\omega+\alpha] = \delta_{\alpha,0}.$$

Furthermore, if $\kappa(\alpha)=k<0$, we may use the s^*-invariance of the affine product to obtain that

$$\Psi_{\mathrm{R}}[\omega,\omega+\alpha] = \Psi_{\mathrm{R}}[\omega',\omega'+\alpha'],\quad (\omega',\alpha') := s^{*-k}(\omega,\alpha).$$

Since $\kappa(\alpha')=0$, we have obtained the following characterization:

Corollary 2.9. *Let X and $\mathrm{R}X$ be two affine Bessel systems generated by Ψ and $\mathrm{R}\Psi$ respectively. Then the following conditions are equivalent.*
(a) X^q is a frame and $(\mathrm{R}X)^q$ is a frame dual to X^q.
(b) For almost every $\omega \in \mathbb{R}^d$, and for every $\alpha \in 2\pi(\mathbb{Z}^d \backslash s^* \mathbb{Z}^d)$,

$$\Psi_{\mathrm{R}}[\omega, \omega] = 1, \quad \Psi_{\mathrm{R}}[\omega, \omega + \alpha] = 0.$$

3. Affine frames: dual systems

In the first subsection we establish connections between dual systems of affine frames and dual systems of their quasi-affine counterparts. In the second subsection we show how the vehicle of multiresolution analysis can be used in the derivation of 'extension principles' for the construction of affine frames together with their dual systems.

3.1. Dual systems of affine frames: quasi-affine analysis

Throughout this section, we assume that X is a fundamental affine frame generated by Ψ. Thus, by Result 1.9, the quasi-affine X^q is a fundamental frame, too. If, now, $\mathrm{R}X$ is yet another given affine system generated by $\mathrm{R}\Psi$, we would like to know whether X and $\mathrm{R}X$ form a pair of dual frames. The main tool in that analysis is the fiberization of the mixed dual Gramian of $X, \mathrm{R}X$ that was detailed in Lemma 2.8. One should, therefore, keep in mind that the relevant dual Gramian fibers do not represent the affine system, but, rather, its associated quasi-affine system (cf. Corollary 2.9); the ability to analyse the affine system via the fiberization of its quasi-affine counterpart is due to the established intimate relation between the two systems (Result 1.9). In the present context, we deal simultaneously with two affine systems, the original affine system and its dual system; therefore, we must first verify that there is a close relation not only between an affine system and its quasi-affine counterpart, but also between the dual systems of these affine and quasi-affine systems. We approach that problem with necessary care: as mentioned, the affine frame X may not have an affine dual system, and in such cases the notion of 'the quasi-affine system of the dual system of X' is nonsense. One should also keep in mind the fact that there may be many dual systems to the given affine one.

We now embark on the actual analysis. In that analysis, we assume to have in hand an affine Bessel system X, and another Bessel system $\mathrm{R}X$, and address the question whether $\mathrm{R}X$ is a system dual to X. We study this question by converting it to the quasi-affine domain: i.e., we would like to consider the related question whether X^q and $(\mathrm{R}X)^q$ are dual systems. The system X^q is well-defined, and in fact, thanks to Result 1.9, is guaranteed to be Bessel. With the added assumption that $\mathrm{R}X$ is affine, too, the system $(\mathrm{R}X)^q$ is also well-defined.

We use in the theorems below the notation

$$\mathbb{D} := \{f \in L_2(\mathbb{R}^d) : \widehat{f} \quad \text{satisfies (1.10)}\}.$$

Theorem 3.1. *Let X be a fundamental affine frame generated by $\Psi \subset \mathbb{D}$.*

(a) *There exists a bijective correspondence between all dual systems $\mathrm{R}X \subset \mathbb{D}$ of X that are affine and all dual systems $Y^q \subset \mathbb{D}$ of X^q that are quasi-affine. The correspondence is given by $\mathrm{R}X \mapsto (\mathrm{R}X)^q$.*

(b) *In particular, the unique dual basis of a fundamental affine Riesz basis $X \subset \mathbb{D}$ is affine and in $\mathbb{D}$ if and only if the quasi-affine (fundamental frame) X^q has a quasi-affine dual system (necessarily, unique) in $\mathbb{D}$.*

(c) *If the minimal dual system $\mathrm{R}^q X^q \subset \mathbb{D}$ of $X^q \subset \mathbb{D}$ is quasi-affine, then the minimal dual system $\mathrm{R}X$ of X is affine, and $(\mathrm{R}X)^q = \mathrm{R}^q X^q$.*

The fact that the map in (a) is well-defined (which is a part of the statement in (a)) can already be found in [CSS] (without the assumption (1.10)).

In the proof, we use the following lemma:

Lemma 3.2. *Let X and $\mathrm{R}X$ be two affine Bessel systems. If, for every $\varepsilon > 0$ there exists $r > 0$ such that, for every $f \in H_r$,*

$$\|(T_{(\mathrm{R}X)^q} T^*_{X^q} - I)f\| \leq \varepsilon \|f\|, \tag{3.3}$$

then $(\mathrm{R}X)^q$ is a system dual to X^q.

Proof: In order to show that $(\mathrm{R}X)^q$ is dual to X^q, we should show (Proposition 1.1) that $T_{(\mathrm{R}X)^q} T^*_{X^q}$ is the identity. By Lemma 2.3, this amounts (since X^q is shift-invariant) to showing that the mixed dual Gramian $\widetilde{G}$ of $(\mathrm{R}X)^q, X^q$ is the identity a.e. For that, it suffices to show that, given any $\varepsilon > 0$, $|\widetilde{G}_\omega(\beta, \alpha) - \delta_{\beta,\alpha}| \leq \varepsilon$ for every $\alpha, \beta \in 2\pi\mathbb{Z}^d$, and for almost every ω. Since $\widetilde{G}_\omega(\beta, \alpha) = \widetilde{G}_{\omega+\beta}(0, \alpha - \beta)$, we may assume without loss that $\beta = 0$. Now, fix any $\omega \in \mathbb{R}^d \backslash 0$, $\alpha \in 2\pi\mathbb{Z}^d$, and $\varepsilon > 0$. Choose r such that (3.3) holds for every $f \in H_r$, and choose k sufficiently large such $|s^{*k}\omega| > r$. By the s^*-invariance of the mixed affine product,

$$\widetilde{G}_\omega(0, \alpha) = \widetilde{G}_{s^{*k}\omega}(0, s^{*k}\alpha).$$

By Lemma 2.4, $|\widetilde{G}_{s^{*k}\omega}(0, s^{*k}\alpha) - \delta_\alpha| \leq \varepsilon$, hence also $|\widetilde{G}_\omega(0, \alpha) - \delta_\alpha| \leq \varepsilon$. Since ω, α are independent of ε, we obtain that $|\widetilde{G}_\omega(0, \alpha)| = \delta_\alpha$, and the desired result follows. □

Proof of Theorem 3.1. We first note the following: By Result 1.9, an affine system Y is Bessel if and only if its quasi-affine counterpart is so. Since, for any system Y discussed below, either Y or Y^q is assumed to be Bessel, we may assume, without loss, that all systems discussed are Bessel. We start with the proof of (a).

The map $\mathrm{R}X \mapsto (\mathrm{R}X)^q$ certainly maps injectively all affine dual systems of X to quasi-affine Bessel systems. We need to show that every system in the range of this map is not only quasi-affine, but is also dual to X^q. We further need to show that the map discussed is surjective.

We first show that, assuming $\mathrm{R}X$ is an affine dual system of X, $(\mathrm{R}X)^q$ is a (quasi-affine) dual system of X^q. Since $\mathrm{R}X$ is dual to X, $T_{\mathrm{R}X} T^*_X = I$. Now, let $T^*_{0-,r}$ be the restriction of the operator $T^*_{X \backslash X_0}$ to H_r (X_0 is the truncated affine system, as defined in §1.3). By Lemma 4.7 of [RS3], there exists, for any given $\varepsilon > 0$, r such that $\|T^*_{0-,r}\| < \varepsilon$ (cf. display (4.9) and display (4.10) in the proof of that lemma). Since $T_{\mathrm{R}X}$ is assumed bounded, it follows that, on H_r,

$$\|T_{\mathrm{R}X_0} T^*_{X_0} - I\| \leq \varepsilon'.$$

At the same time, Lemma 5.4 of [RS3] shows that, for large r, $\|T^*_{X^q\backslash X_0}\| \leq \varepsilon$ on H_r, and thus, on H_r,

$$\|T_{(\mathrm{R}X)^q}T^*_{X^q} - I\| \leq \varepsilon''.$$

Therefore, Lemma 3.2 implies that $(\mathrm{R}X)^q$ is a dual system for X^q.

We now show that every quasi-affine dual system Y^q of X^q gives rise to an affine dual of X. First, since Y^q is assumed to be quasi-affine, it is necessarily the quasi-affine counterpart of some affine system Y (as the suggestive notation Y^q indicates). Further, the correspondence between the elements of X^q and Y^q induces a similar correspondence $\mathrm{R} : X \to Y$. We need to prove that $T_{\mathrm{R}X}T^*_X = I$. For that we fix $\varepsilon > 0$, and invoke Lemma 5.4 of [RS3] to find a sufficiently large r such that $\|T^*_{X^q\backslash X_0}\| \leq \varepsilon$ on H_r. Since $T_{(\mathrm{R}X)^q}T^*_{X^q} = I$ by assumption, it follows that, on H_r, $\|T_{(\mathrm{R}X)_0}T^*_{X_0} - I\| \leq \varepsilon'$. Dilating this inequality, we obtain that, with $X_k := D^{-k}X_0$, $\|T_{(\mathrm{R}X)_k}T^*_{X_k} - I\| \leq \varepsilon'$ on $D^{-k}H_r$, and from that it follows that $\|T_{\mathrm{R}X}T^*_X - I\| \leq \varepsilon'$.

The statement in (b) is certainly a special case of (a). As to (c), if the minimal dual system of X^q is quasi-affine, then, by (a), it is of the form $(\mathrm{R}X)^q$, and with $\mathrm{R}X$ an affine system dual to X. Since $(\mathrm{R}X)^q$ is the *minimal* dual of X^q, the operator $T^*_{(\mathrm{R}X)^q}T_{X^q}$ is self-adjoint, *a fortiori* its truncated part $T^*_{(\mathrm{R}X)^q{}_0}T_{(X^q)_0}$ is self-adjoint. Now, let $x, x' \in X$. Then, for sufficiently large k, $D^kx, D^kx' \in X_0$. Furthermore, $\mathrm{R}X$ was defined to coincide with R^qX^q on X_0, and to commute with D, and hence

$$\begin{aligned}
\langle \mathrm{R}x, x'\rangle =& \langle D^k\mathrm{R}x, D^kx'\rangle\\
=& \langle \mathrm{R}D^kx, D^kx'\rangle\\
=& \langle \mathrm{R}^qD^kx, D^kx'\rangle\\
=& \langle D^kx, \mathrm{R}^qD^kx'\rangle\\
=& \langle D^kx, \mathrm{R}D^kx'\rangle\\
=& \langle D^kx, D^k\mathrm{R}x'\rangle\\
=& \langle x, \mathrm{R}x'\rangle.
\end{aligned}$$

This proves that $T^*_{\mathrm{R}X}T_X$ is self-adjoint, and hence $\mathrm{R}X$ is the *minimal* dual of X. □

By combining Corollary 2.9 with Theorem 3.1, we obtain the following result, which was independently proved by Bin Han in [H]:

Corollary 3.4. *Let $\Psi \subset L_2$ be finite, and $\mathrm{R} : \Psi \to L_2$ some map. Let X, $\mathrm{R}X$ be the affine systems generated by Ψ, $\mathrm{R}\Psi$. Assuming that Ψ and $\mathrm{R}\Psi$ satisfy (1.10), the following conditions are equivalent:*

(a) X is a fundamental frame, and $\mathrm{R}X$ is a dual system of X.

(b) X and $\mathrm{R}X$ are Bessel systems, and for almost every $\omega \in \mathbb{R}^d$, and for every $\alpha \in 2\pi(\mathbb{Z}^d\backslash s^*\mathbb{Z}^d)$,

$$\Psi_{\mathrm{R}}[\omega, \omega] = 1, \quad \Psi_{\mathrm{R}}[\omega, \omega + \alpha] = 0.$$

3.2. Multiresolution analysis: the mixed extension principle

In the previous parts of this paper, we established results concerning connections between affine systems and their quasi-affine counterparts, and derived from those results certain characterizations of and other conditions on affine frames. However, none of these results explicitly suggest new methods for constructing affine frames.

In this section, we re-examine the results of the previous sections under the additional assumption that the affine system is constructed by *Multiresolution Analysis* (MRA). This will lead to simple, useful principles for constructing affine frames. For simplicity, we assume throughout this section that the MRA is generated by a *single* scaling function. The analysis, however, can be adapted to cover the case of multiple scaling functions (as already indicated in §6 of [RS3] in the context of tight frames).

At the heart of each MRA sits a function $\phi \in L_2(\mathbb{R}^d)$ which is known as a *scaling function* or a *refinable function* or a *father wavelet*. In fact, there is no universal definition to that notion, and our definition here is on the weak side (which is an advantage: by assuming less, we allow in more scaling functions, hence obtain a wider spectrum of wavelet constructions). It is worth emphasizing that (a) in the definition below in the major assumption, while the two other conditions are mild and technical.

Definition 3.5. A function $\phi \in L_2(\mathbb{R}^d)$ is called a **scaling function** with respect to a given dilation matrix s if the following three conditions are satisfied:

(a) *Refinability*: there exists a 2π-periodic function $\tau_\phi \in L_\infty(\mathbb{T}^d)$ (called the **mask**) such that, a.e.,

$$\widehat{\phi}(s^* \cdot) = \tau_\phi \widehat{\phi}.$$

(b) $\widehat{\phi}$ is continuous at the origin and $\widehat{\phi}(0) = 1$.

(c) $\phi \in \mathbb{D}$, i.e., it satisfies (1.10).

Note that we have embedded, for technical convenience, our mild smoothness assumption (1.10) into the definition of 'scaling function'. Note also that, in contrast with most of the wavelet literature, we are not making *a-priori* any assumption on the shifts of the scaling function: these shifts may not be orthonormal, nor they need to form a Riesz basis, nor even a frame.

We denote by

$$V_0$$

the closed linear span of the shifts of ϕ and by

$$V_j$$

the s^j-dilate of V_0. The refinability assumption is equivalent to the nestedness assumption $V_j \subset V_{j+1}$, all j. We mention in passing two other popular conditions in the wavelet literature: the first is that $\cap_{j\in\mathbb{Z}} V_j = 0$. That condition was shown in §4 of [BDR2] to follow from the refinability assumption (together with the fact that $\phi \in L_2$). The second condition is that $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R}^d)$. That condition is not automatic but follows from our assumption (b) above (cf., again, §4 of [BDR2]. The corresponding results for FSI scaling functions can be found in [JS]).

Since we deal herein with *two* affine systems, we need two refinable functions to begin with. We thus assume to be given two such scaling functions ϕ and $\mathrm{R}\phi$ (with respect to the same dilation matrix s), and denote their corresponding shift-invariant spaces by V_0 and V_0^d, respectively.

In classical MRA constructions of affine systems, as initiated in [Ma] and [Me], one selects $|\det s| - 1$ (mother) wavelets Ψ from the space V_1 in some clever way, so that the space W_0 which is spanned by $E(\Psi)$ complements (in some suitable sense) V_0 in V_1; for example, W_0 may be the orthogonal complement of V_0 in V_1. Our MRA constructions in [RS3] deviates from this classical approach: while still selecting the mother wavelets Ψ from V_1, the shifts of those mother wavelets do not complement those of ϕ. In fact, in most of our constructions, the space W_0 is dense in V_1. At the same time, our results here and in [RS3] suggest that successful constructions of affine frames and their dual systems may be carried out under minimal assumptions on the scaling functions and/or their masks. More specifically, we allow the cardinality of the mother wavelet set Ψ to exceed the traditional number $|\det s| - 1$, and use the acquired degrees of freedom for the constructions of affine frames with desired properties.

For notational convenience, we set

$$\Psi' := \Psi \cup (\phi),$$

and abbreviate

$$\mathcal{Z} := 2\pi(s^{*-1}\mathbb{Z}^d/\mathbb{Z}^d). \tag{3.6}$$

The assumption $\Psi' \subset V_1$, is equivalent, [BDR1], to the equality

$$\widehat{\psi}(s^*\cdot) = \tau_\psi \widehat{\phi}, \quad \psi \in \Psi', \tag{3.7}$$

for some measurable $\tau := (\tau_\psi)_{\psi\in\Psi'}$ whose components are each 2π-periodic. The function τ_ϕ is the **refinement mask**, and the other τ_ψ's are the **wavelet masks**. Given the masks τ_ψ, $\psi \in \Psi'$, we define a (rectangular) matrix Δ whose rows are indexed by Ψ' and whose columns are indexed by $\mathcal{Z}$ as follows:

$$\Delta := (E^\nu \tau_\psi)_{\psi\in\Psi',\nu\in\mathcal{Z}}. \tag{3.8}$$

Based on the results of §3.1, we can extend the MRA constructions of tight affine frames (of [RS3]) to the system/dual system setup. Most of the derivation details follow *verbatim* those of [RS3], hence are omitted. In particular, the proof of the result below is identical to the proof of the corresponding result for the tight frames (§6 of [RS3]): The only difference is that we need to appeal to Corollary 3.4 of this paper, rather than to Corollary 5.7 of [RS3]. We refer to the construction principle that is established in the following result as *the mixed extension principle*.

Theorem 3.9. *Let ϕ and $\mathrm{R}\phi$ be two scaling functions corresponding to MRAs $(V_j)_j$ and $(V_j^d)_j$, respectively. Let Ψ be a finite subset of V_1, and let $\mathrm{R} : \Psi \to V_1^d$ be some map. Let Δ be the matrix (3.8) that corresponds to $\Psi' := \Psi \cup \phi$, and let Δ^d be the matrix of (3.8) that corresponds to $\Psi' := \mathrm{R}\Psi \cup \mathrm{R}\phi$. Finally, let X and $\mathrm{R}X$ be the affine systems generated by Ψ and $\mathrm{R}\Psi$, respectively. If*

(a) X and $\mathrm{R}X$ are Bessel, and

(b) $\Delta^*\Delta^d = I$, a.e.,

then X and $\mathrm{R}X$ are fundamental frames that are dual one to the other.

With Theorem 3.9 in hand, one can construct compactly supported spline frames with compactly supported dual (spline) frames by simply modifying the algorithm provided in [RS4], where compactly supported spline tight frames were constructed.

4. Affine Riesz bases

4.1. Riesz bases and minimal dual systems

In this subsection, we record some observations concerning minimal dual frames and Riesz bases that are not confined to affine systems, and then apply those to affine systems. We basically assume here, at the outset, that we are given two dual (affine) frames, and would like to verify whether these systems are bi-orthogonal, i.e., form bi-orthogonal Riesz bases. Of course, the initial assumption that we have two dual frames can be first verified by using the techniques and results of the previous section.

The spirit in this section is significantly different from previous approaches we chose for the analysis of affine Riesz bases (cf. [RS6] and §4 of [RS3]). Our previous techniques were based on a direct analysis of the synthesis operator T_X using 'Gramian methods'. However, we currently find that our 'frame techniques' (which analyse the analysis operator T_X^* using dual Gramian techniques) are so powerful that it does not make sense to start the Riesz analysis from 'scratch'.

For a given frame X, one needs to verify the *independence* property (i.e., the injectivity of T_X) of the system X in order to conclude that X is a Riesz basis. Establishing that independence property can, in general, be a formidable challenge; fortunately, the difficulty of that problem is greatly reduced if one knows that X is a frame, and, further, the minimal dual system is already known. That is a bit too pretentious to assume: the constructions of the previous section do provide us with a frame X and a dual frame $\mathrm{R}X$, but do not guarantee $\mathrm{R}X$ to be minimal. Thus, our analysis here is divided into two disjoint problems:

(a) Given a frame X and a dual frame $\mathrm{R}X$ of it, decide whether $\mathrm{R}X$ is the *minimal* dual of X, and

(b) Given a frame X and its minimal dual frame $\mathrm{R}X$, decide whether X is a Riesz basis.

We note that (surprisingly?) the more involved problem is the first one.

We start with the following general proposition that was mentioned in the introduction and was already used in the previous section:

Proposition 4.1. *Let X be a frame in a Hilbert space H. Let $\mathrm{R}X$ be a frame dual to X. Then $\mathrm{R}X$ is minimally dual if and only if*

$$\langle x, \mathrm{R}x' \rangle = \langle \mathrm{R}x, x' \rangle, \quad \text{all } x, x' \in X. \tag{4.2}$$

Proof: Assume first that $\mathrm{R}X$ is minimal. It is then well-known that R is the restriction to X of the linear map $(T_X T_X^*)^{-1}$. Since $T_X T_X^*$ is self-adjoint, so is its inverse, hence (4.2) follows.

Assume now that (4.2) holds. Then it is easy to check that, by using (4.2), $T_X^* T_{\mathrm{R}X} = T_{\mathrm{R}X}^* T_X$. On the other hand, $T_{\mathrm{R}X}^* T_X = (T_X^* T_{\mathrm{R}X})^*$. This implies that $T_X^* T_{\mathrm{R}X}$ is self-adjoint, hence that $\mathrm{R}X$ is minimal. □

We postpone further discussions of the minimality property to the second part of this subsection, and focus first on our second goal: verifying the independence of frames with the aid of their minimal dual frames:

Theorem 4.3. *Let X be a frame in a Hilbert space H and let $\mathrm{R}X$ be its minimal dual system. Then $\langle x, \mathrm{R}x\rangle \le 1$ of every $x \in X$. Furthermore, X is a Riesz system if and only if*

$$\langle x, \mathrm{R}x\rangle = 1, \quad \text{all } x \in X. \tag{4.4}$$

Proof: If X is a Riesz system then X and $\mathrm{R}X$ are bi-orthogonal hence (4.4) certainly holds. Conversely, using the minimality of the dual system $\mathrm{R}X$ together with Proposition 4.1, we obtain for an arbitrary $x' \in X$ that

$$1 = \langle x', \mathrm{R}x'\rangle = \sum_{x\in X} \langle x', \mathrm{R}x\rangle\langle x, \mathrm{R}x'\rangle = \sum_{x\in X} \langle x', \mathrm{R}x\rangle\langle \mathrm{R}x, x'\rangle = \sum_{x\in X} |\langle x, \mathrm{R}x'\rangle|^2 = 1 + \sum_{x\neq x'} |\langle x, \mathrm{R}x'\rangle|^2.$$

Hence, $\langle x, \mathrm{R}x'\rangle = \delta_{x,x'}$.

Finally, the above argument shows that, always, $\langle x, \mathrm{R}x\rangle$ is non-negative and that $\langle x, \mathrm{R}x\rangle \ge (\langle x, \mathrm{R}x\rangle)^2$. Therefore $\langle x, \mathrm{R}x\rangle \le 1$. □

In the case of affine frames of $L_2(\mathbb{R}^d)$, the above theorem can be slightly improved: if X is an affine system generated by Ψ, then a typical element $x \in X$ is of the form $x = D^k E^j \psi$. By definition, we have then that $\mathrm{R}x = D^k E^j \mathrm{R}\psi$. This implies that $\langle x, \mathrm{R}x\rangle = \langle \psi, \mathrm{R}\psi\rangle$ here, hence we obtain from Theorem 4.3 the following corollary:

Corollary 4.5. *Let X be an affine frame (not necessarily fundamental) generated by Ψ and assume that its minimal dual system is an affine system generated by $\mathrm{R}\Psi$. Then X is a Riesz basis if and only if*

$$\langle \psi, \mathrm{R}\psi\rangle = 1, \quad \text{all } \psi \in \Psi.$$

We now turn our attention to the second problem: characterizing the minimality of the dual frame. Our goal is to apply the basic Proposition 4.1 to affine systems. For that, we recall (from the introduction) the notion of the *bracket product*:

$$[f, g] := \sum_{\alpha\in 2\pi\mathbb{Z}^d} E^\alpha f \overline{E^\alpha g}, \quad f, g \in L_2.$$

While the affine product is the building block of the analysis operator of the affine system X, the bracket product is the building block of the synthesis operator of X.

Corollary 4.6. *Let X be an affine frame (not necessarily fundamental) generated by Ψ, and assume that $\mathrm{R}X$ is an affine frame dual to X which is generated by $\mathrm{R}\Psi$. Then $\mathrm{R}X$ is the minimal dual of X if and only if, for every $\psi, \psi' \in \Psi$, the following conditions hold:*

$$[D_*^k \widehat{\psi}, \mathrm{R}\widehat{\psi'}] = [D_*^k \mathrm{R}\widehat{\psi}, \widehat{\psi'}], \quad \text{all } k \le 0, \tag{4.7}$$

where $D_* f := |\det s^{*-1}|^{1/2} f(s^{*-1}\cdot)$.

Proof: We invoke Proposition 4.1, hence would like to show that (4.7) is equivalent to the equalities

$$\langle x, \mathrm{R}x'\rangle = \langle \mathrm{R}x, x'\rangle, \quad x, x' \in X. \tag{4.8}$$

A typical element $x \in X$ is of the form $D^k E^j \psi$, $\psi \in \Psi$, $j \in \mathbb{Z}^d$, $k \in \mathbb{Z}$. Since D is unitary, we have then $\langle x, \mathrm{R}x'\rangle = \langle D^k x, D^k \mathrm{R}x'\rangle$, and one easily conclude that we only need to check (4.7) for x and x' of the following form:

$$x = D^k E^j \psi, \quad x' = E^{j'}\psi', \quad \psi, \psi' \in \Psi,\ j, j' \in \mathbb{Z}^d,\ k \le 0.$$

Furthermore, $D^k E^j = E^{s^{-k}j} D^k$, and since $s^{-k}\mathbb{Z}^d$ is a sublattice of $\mathbb{Z}^d$ for $k \le 0$, and E^j is unitary we can further reduce the required condition to x, x' of the form

$$x = D^k \psi, \quad x' = E^j \psi', \quad \psi, \psi' \in \Psi,\ j \in \mathbb{Z}^d,\ k \le 0. \tag{4.9}$$

However, it is well-known and easy to check that the numbers

$$\langle f, E^j g\rangle, \quad j \in \mathbb{Z}^d$$

are the Fourier coefficients of the bracket product $[\widehat{f}, \widehat{g}]$. Using that final identity for the choice $f = D^k \psi$, $g = \mathrm{R}\psi'$, and then using that same identity for $f = D^k \mathrm{R}\psi$ and $g = \psi'$, one finds that the conditions required in (4.9) are equivalent to those set in (4.7). □

Remark: The conditions (4.7) can be replaced by

$$\langle D^k E^j \psi, \mathrm{R}E^{j'}\psi'\rangle = \langle D^k E^j \mathrm{R}\psi, E^{j'}\psi'\rangle, \quad k \ge 0, \quad j, j' \in \mathbb{Z}^d. \tag{4.10}$$

□

We conclude this subsection with the following characterization of affine Riesz bases. It is obtained by combining Corollary 3.4 with Corollaries 4.5 and 4.6. The result is in terms of system-dual system. We could also obtain a similar result intrinsically in terms of the given affine system X (without searching for a dual system first) using our characterization of affine frames in [RS3]. We forgo mentioning this additional result since the scope of this paper is limited to the study of system-dual system setups.

Theorem 4.11. *Let* $\Psi \subset L_2$*, and let* $\mathrm{R} : \Psi \to L_2$ *be some map. Assume that the affine systems* X *and* $\mathrm{R}X$ *generated by* Ψ *and* $\mathrm{R}\Psi$ *are Bessel and that* Ψ *and* $\mathrm{R}\Psi$ *satisfy (1.10). Then* X *and* $\mathrm{R}X$ *are bi-orthogonal Riesz bases if and only if the following four conditions hold:*
(a) $\langle \psi, \mathrm{R}\psi\rangle = 1$, all $\psi \in \Psi$.
(b) $[D_*^k \widehat{\psi}, \mathrm{R}\widehat{\psi'}] = [D_*^k \mathrm{R}\widehat{\psi}, \widehat{\psi'}]$*, for every* $\psi, \psi' \in \Psi$*, and every* $k \le 0$.
(c) $\Psi_{\mathrm{R}}[\omega, \omega] = 1$*, for a.e.* ω.

(d) $\Psi_{\mathrm{R}}[\omega,\omega+\alpha]=0$, *for every* $\alpha\in 2\pi(\mathbb{Z}^d\backslash s^*\mathbb{Z}^d)$ *and a.e.* ω.

Remark 4.12. *Note that our condition (b) above does not imply directly any bi-orthogonality relations between* X *and* $\mathrm{R}X$. *In fact, we could have completely trivialized the above theorem by strengthening condition (b) there to*

$$[D_*^k\widehat{\psi},\mathrm{R}\widehat{\psi'}]=[D_*^k\mathrm{R}\widehat{\psi},\widehat{\psi'}]=\delta_{\psi,\psi'}\delta_{k,0},\quad k\le 0. \tag{4.13}$$

Indeed, these stronger assumptions are equivalent to the bi-orthogonality of X_0 *and* $\mathrm{R}X_0$, *and, together with the Bessel assumptions, imply that* X_0 *and* $\mathrm{R}X_0$ *are Riesz bases, hence that* X *and* $\mathrm{R}X$ *are Riesz bases, too, by Theorem 4.3 of* [RS3]. *The extra conditions (a,c,d) become then redundant.* □

4.2. MRA: Riesz bases

In this subsection we assume that two Bessel systems, X and $\mathrm{R}X$, are constructed via MRA using the *mixed extension principle*, Theorem 3.9. By that theorem, the systems are guaranteed to be fundamental frames one dual to the another. We investigate here the question whether the two systems are *Riesz bases* using for that purpose the tools developed in the previous section.

We showed in the last subsection that if $\mathrm{R}X$ is already known to be the *minimal* dual of X, then the mere additional requirement we need in order to obtain that X is a Riesz basis is the condition $\langle\psi,\mathrm{R}\psi\rangle=1$, all $\psi\in\Psi$ (cf. Corollary 4.5). If the system was constructed according to the mixed extension principle, then these conditions are equivalent to a single condition on the scaling function. This result extends Corollary 6.9 in [RS3].

Corollary 4.14. *Let* X *and* $\mathrm{R}X$ *be two dual frames that were constructed according to the mixed extension principle. Assume further that*
(I) *The system* $\mathrm{R}X$ *is the minimal dual system of* X, *and*
(II) *The extension is square i.e., the cardinality of the mother wavelet set* Ψ *is* $|\det s|-1$.
Then the following statements are equivalent:
(a) X *and* $\mathrm{R}X$ *are bi-orthogonal Riesz bases.*
(b) $\langle\phi,\mathrm{R}\phi\rangle=1$, *with* ϕ *and* $\mathrm{R}\phi$ *the two underlying scaling functions.*

Proof: We adopt the notations used in Theorem 3.9 and in the discussion preceding that theorem. In particular, $\tau=(\tau_\psi)_{\psi\in\Psi'}$, and $\langle\tau,\mathrm{R}\tau\rangle=\sum_{\psi\in\Psi'}\tau_\psi\overline{\tau_{\mathrm{R}\psi}}$.

Since $\langle\tau,\mathrm{R}\tau\rangle=1$ a.e., we easily conclude, by integrating the equality

$$\widehat{\phi}\,\overline{\mathrm{R}\widehat{\phi}}=\langle\tau,\mathrm{R}\tau\rangle\;\widehat{\phi}\,\overline{\mathrm{R}\widehat{\phi}}=\sum_{\psi\in\Psi'}\widehat{\psi}(s^*\cdot)\overline{\mathrm{R}\widehat{\psi}(s^*\cdot)},$$

(and using Parseval's identity) that

$$|\det s|\langle\phi,\mathrm{R}\phi\rangle=\sum_{\psi\in\Psi'}\langle\psi,\mathrm{R}\psi\rangle.$$

Thus,

$$\sum_{\psi\in\Psi}\langle\psi,\mathrm{R}\psi\rangle = (|\det s|-1)\langle\phi,\mathrm{R}\phi\rangle. \tag{4.15}$$

Now, if X is a Riesz basis and $\mathrm{R}X$ is its dual basis, $\langle\psi,\mathrm{R}\psi\rangle = 1$, all ψ, and with the assumption that there are exactly $|\det s|-1$ wavelets, we obtain that $\langle\phi,\mathrm{R}\phi\rangle = 1$.

Conversely, assuming $\mathrm{R}X$ to be the minimal dual frame of X, we conclude from Theorem 4.3 that $\langle\psi,\mathrm{R}\psi\rangle \le 1$. Substituting that into (4.15), together with the assumption that $\langle\phi,\mathrm{R}\phi\rangle = 1$, we obtain (in view of the fact that $\#\Psi = |\det s|-1$) that $\langle\psi,\mathrm{R}\psi\rangle = 1$, all ψ. Therefore, X is a Riesz basis and $\mathrm{R}X$ is its dual basis, by Corollary 4.5. $\square$

Note that Corollary 4.14 assumes that $\mathrm{R}X$ is the minimal dual system. Checking the minimality of $\mathrm{R}X$ may not always be possible, even if one uses results like Corollary 4.6. Our last corollary provides a simple sufficient condition for the Riesz basis property for a pair of dual frames constructed by the (square) mixed extension principle. The result is independent of our other results in this paper, and is also well-known. However, the proof we provide here is rather simple. We need first the following lemma.

Lemma 4.16. *Assume that for some scaling functions $\phi,\mathrm{R}\phi\in L_2$ the shifts $E(\phi)$ are bi-orthogonal to the shifts $E(\mathrm{R}\phi)$. Suppose that Ψ and $\mathrm{R}\Psi$ are two mother wavelet sets that are constructed by the square version of the mixed extension principle (i.e., the number of mother wavelets is $|\det s|-1$), using the above scaling functions. Then*

$$[D_*^k\widehat{\psi},\mathrm{R}\widehat{\psi'}] = \delta_{\psi,\psi'}\delta_{k,0},\quad k\le 0. \tag{4.17}$$

Proof: We recall our notation $\Psi' = \phi\cup\Psi$, $\tau = \tau_\phi\cup\tau_\Psi$. We will show first that $Y := E(\Psi')$ are bi-orthogonal to $\mathrm{R}Y = E(\mathrm{R}\Psi')$. For that, it suffices to shows that the mixed Gramian of these two systems is the identity, i.e., that

$$J^*(\mathrm{R}Y)J(Y) = I.$$

A typical row in $J(Y)$ is of the form

$$E^\alpha\widehat{\Psi}' = E^\alpha\widehat{\phi}(s^{*-1}\cdot)\,E^\alpha\tau(s^{*-1}\cdot).$$

We first let J_1 be the block of $J(Y)$ corresponding to the $2\pi s^*\mathbb{Z}^d$ rows (remember that the rows of J are indexed by $2\pi\mathbb{Z}^d$), and let J_2 be the same block in $J(\mathrm{R}Y)$. Thus, with

$$\mathcal{Z}_1 := 2\pi(\mathbb{Z}^d/s^*\mathbb{Z}^d),$$

$$J^*(\mathrm{R}Y)J(Y) = \sum_{\nu\in\mathcal{Z}_1} E^\nu(J_2^*J_1).$$

However, with Δ and Δ^d the matrices from the extension principle, the refinablity of Ψ' and $\mathrm{R}\Psi'$, together with the 2π-periodicity of their masks imply that

$$\overline{J_2^*J_1} = (\Delta^d)_0(s^{*-1}\cdot)\ \mathrm{R}f\ f^*\ (\Delta_0)^*(s^{*-1}\cdot),$$

with Δ_0 the column indexed by 0 in Δ, and f, $\mathrm{R}f$ the (row) vectors

$$f(\alpha) := \widehat{\phi}(s^{*-1}\cdot+\alpha),\ \mathrm{R}f(\alpha) := \mathrm{R}\widehat{\phi}(s^{*-1}\cdot+\alpha), \quad \alpha\in 2\pi\mathbb{Z}^d.$$

The bi-orthogonality of $E(\phi)$ and $E(\mathrm{R}\phi)$ then implies that $\mathrm{R}f\,f^*=1$, and hence

$$\overline{J^*(\mathrm{R}Y)J(Y)} = \sum_{\nu\in\mathcal{Z}_1} E^\nu(\overline{J_2^*J_1}) = \sum_{\nu\in\mathcal{Z}_1} E^\nu((\Delta^d)_0(s^{*-1}\cdot)(\Delta_0)^*(s^{*-1}\cdot)) = \Delta^d\Delta^*(s^{*-1}\cdot) = I.$$

This implies that $E(\mathrm{R}\Psi)$ are bi-orthogonal to $E(\Psi)$, which is the case $k=0$ in (4.17). Furthermore, this also implies that $E(\mathrm{R}\Psi)$ are orthogonal to $E(\phi)$ hence to V_0. However, for $k<0$, $D^kE(\Psi)\subset V_0$, hence this set must be orthogonal to $E(\mathrm{R}\Psi)$, which is the case $k<0$ in (4.17). □

Using Lemma 4.16 and Remark 4.12 we obtain the following well-known result. This result was first established in [CDF] (for the case of compactly supported univariate biorthogonal wavelets), and was extended since then in various directions (cf. [RiS]).

Corollary 4.18. *Assume that ϕ and $\mathrm{R}\phi$ are two scaling functions and that $E(\phi)$ and $E(\mathrm{R}\phi)$ are bi-orthogonal Riesz bases. Suppose that X and $\mathrm{R}X$ are two affine systems that were constructed by the square version of the mixed extension principle (i.e., the number of mother wavelets is $|\det s|-1$). Then X and $\mathrm{R}X$ are fundamental affine Riesz bases in $L_2(\mathbb{R}^d)$ which are dual one to the another, provided that they are Bessel systems.*

References

[BDR1] C. de Boor, R.A. DeVore and A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$, Transactions of Amer. Math. Soc. **341** (1994), 787–806.

[BDR2] C. de Boor, R. DeVore and A. Ron, On the construction of multivariate (pre) wavelets, Constr. Approx. **9** (1993), 123–166.

[CDF] A. Cohen, I. Daubechies and J.C. Feauveau, Biothogonal bases of compactly supported wavelets, Comm. Pure. Appl. Math. **45** (1992), 485-560.

[CSS] C. K. Chui, X. L. Shi and J. Stöckler, Affine frames, quasi-affine frames, and their duals, preprint, 1996.

[D1] I. Daubechies, The wavelet transform, time-frequency localization and signal analysis, IEEE Trans. Inform. Theory **36** (1990), 961–1005.

[D2] I. Daubechies, Ten lectures on wavelets, CBMF conference series in applied mathematics, Vol **61**, SIAM, Philadelphia, 1992.

[DLL] I. Daubechies, H. J. Landau and Z. Landau, Gabor time-frequency lattices and the Wexler-Raz identity, J. Fourier Anal. Appl., **1** (1995), 437–478.

[GR] K. Gröchenig and A. Ron, Tight compactly supported wavelet frames of arbitrarily high smoothness, to appear in Proc. Amer. Math. Soc.
Ftp site: `ftp://ftp.cs.wisc.edu/Approx` file `cg.ps`.

[H] B. Han, On dual wavelet tight frames, ms., (1995).

[J] A.J.E.M. Janssen, Duality and biorthogonality for Weyl-Heisenberg frames, J. Fourier Anal. Appl., **1** (1995), 403–436.

[JM] R.Q. Jia and C.A. Micchelli, Using the refinement equation for the construction of pre-wavelets II: Powers of two, in Curves and Surfaces (P.J. Laurent, A. Le Méhauté and L.L. Schumaker eds), Academic Press, New York, 1991, 209-246.

[JS] R.Q. Jia and Z. Shen, Multiresolution and wavelets, Proc. Edinburgh Math. Soc. **37** (1994), 271–300.

[Ma] S.G. Mallat, Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathrm{R})$, Trans. Amer. Math. Soc. **315**(1989), 69–87.

[Me] Y. Meyer, "Ondelettes et Opérateurs I: Ondelettes", Hermann Éditeurs, 1990

[RiS] S.D. Riemenschneider and Z. Shen, Construction of biorthogonal wavelets in $L_2(\mathbb{R}^s)$, preprint (1997).

[RS1] A. Ron and Z. Shen, Frames and stable bases for shift-invariant subspaces of $L_2(\mathbb{R}^d)$, Canad. J. Math., **47** (1995), 1051-1094.
Ftp site: `ftp://ftp.cs.wisc.edu/Approx` file `frame1.ps`.

[RS2] A. Ron and Z. Shen, Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$, Duke Math. J., **xx** (1997), xxx-xxx.
Ftp site: `ftp://ftp.cs.wisc.edu/Approx` file `wh.ps`.

[RS3] A. Ron and Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator, J. Functional Anal., **xx** (1997), xxx-xxx.
Ftp site: `ftp://ftp.cs.wisc.edu/Approx` file `affine.ps`.

[RS4] A. Ron and Z. Shen, Compactly supported tight affine spline frames in $L_2(\mathbb{R}^d)$, Math. Comp., **xx** (1997), xxx-xxx.
Ftp site: `ftp://ftp.cs.wisc.edu/Approx` file `tight.ps`.

[RS5] A. Ron and Z. Shen, Frames and stable bases for subspaces of $L_2(\mathbb{R}^d)$: the duality principle of Weyl-Heisenberg sets, *Proceedings of the Lanczos International Centenary Conference*, Raleigh NC, 1993, D. Brown, M. Chu, D. Ellison, and R. Plemmons eds., SIAM Pub. (1994), 422–425

[RS6] A. Ron and Z. Shen, Gramian analysis of affine bases and affine frames, *Approximation Theory VIII, Vol. 2: Wavelets and Multilevel Approximation*, C.K. Chui and L.L. Schumaker eds, World Scientific Publishing, New Jersey, 1995, 375-382.