FRAMELETS: MRA-BASED CONSTRUCTIONS OF WAVELET FRAMES

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ABSTRACT

We discuss wavelet frames constructed via multiresolution analysis (MRA), with emphasis on *tight* wavelet frames. In particular, we establish general principles and specific algorithms for constructing framelets and tight framelets, and we show how they can be used for systematic constructions of spline, pseudo-spline tight frames and symmetric biframes with short supports and high approximation orders. Several explicit examples are discussed. The connection of these frames with multiresolution analysis guarantees the existence of fast implementation algorithms, which we discuss briefly as well.

AMS (MOS) Subject Classifications: Primary 42C15, Secondary 42C30

Key Words: unitary extension principle, oblique extension principle, framelets, pseudo-splines, frames, tight frames, fast frame transform, multiresolution analysis, wavelets.

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1. Introduction

Although many compression applications of wavelets use wavelet bases, other types of applications work better with redundant wavelet families, of which wavelet frames are the easiest to use. The redundant representation offered by wavelet frames has already been put to good use for signal denoising, and is currently explored for image compression. Motivated by these and other applications, we explore in this article the theory of wavelet frames. We are interested here in wavelet frames and their construction via multiresolution analysis (MRA); of particular interest to us are *tight* wavelet frames. We restrict our attention to wavelet frames constructed via MRA, because this guarantees the existence of fast implementation algorithms. We shall explore the 'power of redundancy' to establish general principles and specific algorithms for constructing framelets and tight framelets. In particular, we shall give several systematic constructions of spline and pseudo-spline tight frames and symmetric bi-frames with short supports and high approximation orders. Before we state our main results, we start by reviewing some concepts concerning wavelet frames and their structure.

1.1. Wavelet frames

Our discussions here concern *dyadic* systems; more general wavelet frames are discussed in §5.

Basic notations: $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L_2(\mathbb{R}^d)$, i.e.,

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(y)\overline{g(y)}\, dy,$$

which can be extended to other f and g, e.g. when $fg \in L_1(\mathbb{R}^d)$. We normalize the Fourier transform as follows: $\widehat{f}(\omega) := \int_{\mathbb{R}^d} f(y) e^{-i\omega\cdot y}\, dy$. Given a function $\psi \in L_2(\mathbb{R}^d)$, we set $\psi_{j,k} : y \mapsto 2^{jd/2}\psi(2^j y - k)$. If the function ψ_i already carries an enumerative index, we write $\psi_{i,j,k}$ instead.

Let Ψ be a finite subset of $L_2(\mathbb{R}^d)$. **The** *dyadic* **wavelet system generated by the** *mother wavelets* Ψ is the family

$$X(\Psi) := \{\psi_{j,k} : \ \psi \in \Psi,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}.$$

Such a wavelet system $X(\Psi)$ can be used in order to represent other functions in $L_2(\mathbb{R}^d)$. Useful in this context is the **decomposition operator** (known also as the 'analysis operator'):

$$T^* : f \mapsto (\langle f, g\rangle)_{g\in X(\Psi)}.$$

The system $X(\Psi)$ is a **Bessel system** if the analysis operator is bounded, i.e., for some $C_1 > 0$, and for every $f \in L_2(\mathbb{R}^d)$,

$$\sum_{g \in X(\Psi)} |\langle f, g\rangle|^2 \le C_1 \|f\|^2_{L_2(\mathbb{R}^d)}.$$

For wavelet systems $X(\Psi)$, it is easy to satisfy this basic and natural requirement: if each of the mother wavelets has at least one vanishing moment, i.e., $\widehat{\psi}(0) = 0, \quad \text{for all} \quad \psi \in \Psi$, then $X(\Psi)$ is a Bessel system if the functions in Ψ satisfy some mild smoothness conditions (see e.g. [CS], [RS2]).

A Bessel system $X(\Psi)$ is a **frame** if the analysis operator is bounded below, i.e., if there exists $C_2 > 0$ such that, for every $f \in L_2(\mathbb{R}^d)$,

$$\sum_{g \in X(\Psi)} |\langle f, g\rangle|^2 \ge C_2 \|f\|^2_{L_2(\mathbb{R}^d)}.$$

This imposes more stringent conditions on $X(\Psi)$. A special case is provided by **tight frames**: this is the case when $X(\Psi)$ is a frame with equal frame bounds, i.e. $C_1 = C_2$; after a re-normalization of the $g \in X(\Psi)$, one then has

$$\sum_{g \in X(\Psi)} |\langle f, g\rangle|^2 = \|f\|^2_{L_2(\mathbb{R}^d)}, \quad \text{for all} \quad f \in L_2(\mathbb{R}^d).$$

This tight frame condition is equivalent to the *perfect reconstruction property*

$$f = \sum_{g \in X(\Psi)} \langle f, g\rangle\, g, \quad \text{for all} \quad f \in L_2(\mathbb{R}^d).$$

We are interested in the study of wavelet frames that are derived from a *multiresolution analysis* (MRA). Although some of our results and observations cover the case of vector MRA, we shall restrict our attention to the scalar case. We expect that a full description of the vector case will have additional features linked to the more complex analysis of approximation order (see e.g.[Pl], [PR]). Our scalar MRA setup follows [RS3] and represents an extension of the original MRA setup ([Ma], [Me], [D1]).

Let $\phi \in L_2(\mathbb{R}^d)$ be given and let $V_0 := V_0(\phi)$ be the closed linear span of its shifts, i.e., V_0 is the smallest closed subspace of $L_2(\mathbb{R}^d)$ that contains $E(\phi) := \{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$. Let $\mathcal{D}$ be the operator of dyadic dilation: $(\mathcal{D}f)(y) := \sqrt{2^d}\, f(2y)$, and set $V_j := \mathcal{D}^j V_0, \quad j \in \mathbb{Z}$. The function ϕ is said to generate the (stationary) MRA $(V_j)_j$ if the sequence $(V_j)_j$ is nested:

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots, \tag{1.1}$$

and, if, in addition, the union $\cup_j V_j$ is dense in $L_2(\mathbb{R}^d)$. (The MRA condition (1.1) is equivalent to the inclusion $V_0 \subset V_1$.) The generator ϕ of the MRA is known as a **scaling function** or a **refinable function**. Finally, the MRA is **local** if it is generated by a *compactly supported* refinable function. (The MRA condition in [Ma], [Me], [D2] also required that ϕ and its shifts constitute a Riesz basis of V_0, which is not required in [RS3] or here.)

(1.2) Definition: MRA constructions of wavelet systems [RS3]. *A wavelet system $X(\Psi)$ is said to be MRA-based if there exists an MRA $(V_j)_j$ such that the condition $\Psi \subset V_1$ holds. If, in addition, the system $X(\Psi)$ is a frame, we refer to its elements as* **framelets**. The notions of *mother framelets*, *tight framelets*, etc., have then their obvious meaning.

Some historical pointers: The concept of frames was first introduced by Duffin and Schaeffer in [DS]. Examples of univariate wavelet frames can already be found in the work of Daubechies, Grossmann and Meyer, [DGM]; necessary and sufficient conditions for mother wavelets to generate frames are implicit in e.g. [Me] and [D2]. Characterizations of univariate tight wavelet frames are implicit in the work of Wang and Weiss, [FGWW, HW] An explicit characterization of tight wavelet frames (in the multivariate case) was obtained by Han [H]. Independently of these, Ron and Shen gave in [RS3] a general characterization of all wavelet frames, and specialized this to the case of tight wavelet frames. Furthermore, applying its general theory, [RS3] also provided a complete characterization of all framelets. Note that [RS3] included a mild decay condition on $\widehat{\Psi}$ in one of its basic theorems (Theorem 5.5 of [RS3]); it was then shown by [CSS] that this theorem could also be proved without this decay assumption, effectively removing the decay constraint for all consequent results derived from Theorem 5.5 in [RS3], including the characterization of tight frames and framelets. More recently, several articles proved again some of those results without the decay constraint; see e.g. [Bo], [CH], [P1]. Finally, band limited tight framelets are also constructed by Benedetto and Li in [BL] (also see [BT]).

Several questions arise naturally:

(I) Under what conditions (on the MRA $(V_j)_j$ and the mother wavelets Ψ) does one obtain *framelets*, or, better, *tight framelets*?

(II) Can one construct (tight) framelets from any MRA? In particular, can one construct framelets from the MRA induced by a univariate B-spline or a multivariate box spline ϕ?

As to (I), we first briefly review the characterization of framelets given in [RS3]. For this, we start with recalling some basic facts from the theory of *shift-invariant spaces*. Suppose that $(V_j)_j$ is an MRA induced by a refinable function ϕ. Let $\Psi = (\psi_1, \ldots, \psi_r)$ be a finite subset of V_1 (these ψ_ℓ will be our mother wavelets in the MRA-based construction). Then (see [BDR1,2]), there exist 2π-periodic measurable functions τ_i, $i = 1, \ldots, r$ (referred to hereafter as the **wavelet masks**) such that, for every i,

$$\widehat{\psi_i} = (\tau_i \widehat{\phi})(\frac{\cdot}{2}).$$

Moreover, since $\phi \in V_1$ (by assumption), there also exists a 2π-periodic τ_0 (referred to as the **refinement mask**) such that $\widehat{\phi} = (\tau_0 \widehat{\phi})(\cdot/2)$; this τ_0 completely determines ϕ and therefore the underlying MRA. For notational convenience, we will occasionally list the refinable function together with the mother wavelets in the **parent wavelet vector**

$$F := (\psi_0, \psi_1, \ldots, \psi_r) := (\phi, \psi_1, \ldots, \psi_r).$$

Similarly, we introduce the notation $\tau := (\tau_0, \ldots, \tau_r)$ for the **combined MRA mask** that completely determines F.

In all examples considered in this article, the vector τ consists of *trigonometric polynomials*. In that case the parent vector F is necessarily of compact support. For the development of the theory, though, we assume only the following milder conditions:

"assum **(1.3) Assumptions.** All MRA-based constructions that are considered in this article are assumed to satisfy the following:
(a) Each mask τ_i in the combined MRA mask τ is measurable and (essentially) bounded.
(b) The refinable function ϕ satisfies $\lim_{\omega\to 0}\widehat{\phi}(\omega)=1$.
(c) The function $[\widehat{\phi},\widehat{\phi}] := \sum_{k\in 2\pi\mathbb{Z}^d}|\widehat{\phi}(\cdot+k)|^2$ is essentially bounded.

Note that the MRA does not determine ϕ and τ_0 uniquely. For example, if α is a 2π-periodic function which is non-zero a.e., and if the function φ defined by $\widehat{\varphi}(\omega)=\alpha(\omega)\widehat{\phi}(\omega)$ lies in $L_2(\mathbb{R}^d)$, then φ is refinable with mask $t_0(\omega)=\alpha(2\omega)\tau_0(\omega)/\alpha(\omega)$, and generates the same MRA as ϕ does. Incidentally, this remark shows that Assumptions (1.3) depend on the refinable function representing the MRA: for example, this little manipulation could transform an unbounded τ_0 into a bounded t_0.

The characterization in [RS3] of tight framelets involves a special 2π-periodic function Θ:

(1.4) Definition. *Let $\tau=(\tau_0,\ldots,\tau_r)$ be as above. Set*

$$\tau_+ := (\tau_1,\ldots,\tau_r),\quad |\tau_+(\omega)|^2 := \sum_{i=1}^{r}|\tau_i(\omega)|^2.$$

Given a combined MRA mask τ and the corresponding wavelet system $X(\Psi)$, define the **fundamental function Θ of the parent wavelet vector** *by*

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$$\Theta(\omega) := \sum_{j=0}^{\infty}|\tau_+(2^j\omega)|^2\prod_{m=0}^{j-1}|\tau_0(2^m\omega)|^2. \tag{1.5}$$

The definition of Θ implies the following important identity, (which is valid a.e.):

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$$\Theta(\omega) = |\tau_+(\omega)|^2 + |\tau_0(\omega)|^2\Theta(2\omega). \tag{1.6}$$

(Note that this identity was not featured in [RS3], it will be crucial in this paper.)
In our statements below, we use the following weighted semi-inner product (here $w\ge 0$, and $u,v\in\mathbb{C}^{r+1}$):

$$\langle u,v\rangle_w := wu_0\overline{v_0}+\sum_{i=1}^{r}u_i\overline{v_i}.$$

We also need to single out the following set (which is determined only up to a null set):

$$\sigma(V_0) := \{\omega\in[-\pi,\pi]^d:\ \widehat{\phi}(\omega+2\pi k)\neq 0,\ \text{for some } k\in\mathbb{Z}^d\}.$$

The set $\sigma(V_0)$ is the **spectrum** of the shift-invariant space V_0; it is independent of the choice of the generator ϕ of V_0, and plays an important role in the theory of shift-invariant spaces (cf. [BDR2,3]). The values assumed by τ outside the set $\sigma(V_0)$ affect neither the MRA nor the resulting wavelet system $X(\Psi)$. In almost every example of interest, the spectrum $\sigma(V_0)$ coincides (up to a null set) with the cube $[-\pi,\pi]^d$. In particular, whenever ϕ is compactly supported, we automatically have $\sigma(V_0)=[-\pi,\pi]^d$.

The following characterization of [RS3] answers question (I) for the tight frames:

“thmtet **Proposition 1.7** ([RS3]). *Assume that the combined MRA mask $\tau = (\tau_0, \ldots, \tau_r)$ is bounded. Assume that $\widehat{\phi}$ is continuous at the origin and $\widehat{\phi}(0) = 1$. Define Θ as in* (1.5). *Then the following conditions are equivalent:*

(a) *The corresponding wavelet system $X(\Psi)$ is a tight frame.*

(b) *For almost all $\omega \in \sigma(V_0)$, the function Θ satisfies:*

(b1) $\lim_{j \to -\infty} \Theta(2^j \omega) = 1$.

(b2) *If $\nu \in \{0, \pi\}^d \backslash 0$ and $\omega + \nu \in \sigma(V_0)$, then*

“extraIDtwo (1.8)
$$\langle \tau(\omega), \tau(\omega + \nu) \rangle_{\Theta(2\omega)} = 0.$$

This leads to several solutions to question (II) as described below:

1.2. Extension principles

Proposition 1.7 states mathematically how all the masks “work together” to make the whole family a tight frame. We have one single family of 2^d equations ((1.5) and (1.8)) that the masks have to satisfy jointly. In practical constructions, this leads to a “shared responsibility” which allows more flexibility. In the original construction of compactly supported orthonormal wavelets [D1], the refinement mask for ϕ had to satisfy a conjugate quadrature filter (CQF) conditions as well as stability properties. This excluded symmetric or antisymmetric wavelets, as well as spline wavelets (except for Haar wavelet). Many subsequent constructions sought to remedy this by relaxing some restrictions: in [CDF], symmetry was obtained at the cost of dropping orthogonality; in their construction two compactly supported dual refinable functions were needed, only one of which could be spline; in [CW] similar non-orthogonal dual symmetric, spline wavelet bases were given, but only one of them could be compactly supported; in [DGH], symmetry, orthonormality and compact support were combined at the price of having multiwavelets, or vector MRA; in [DGH], it was shown that this could be done with spline vector MRA. In this paper, we are relaxing the non-redundancy condition, which makes it possible to start from refinable ϕ that satisfy no other conditions than those in Assumptions 1.3.

At first sight, it is not clear how to use Proposition 1.7 for the practical construction of tight framelets; one needs to select simultaneously the combined MRA mask τ and the fundamental MRA function Θ, making sure that they satisfy the requirements (1.5) and (1.8); and this is nontrivial to solve. The problem simplifies drastically when one restricts to the case $\Theta = 1$ on $\sigma(V_0)$, the choice made in [RS3].

“thmuep **Proposition 1.9: The Unitary Extension Principle (UEP),** [RS3]. *Let τ be the combined MRA that satisfies Assumptions* (1.3). *Suppose that, for almost all $\omega \in \sigma(V_0)$, and all $\nu \in \{0, \pi\}^d$,*

(1.10)
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$$\sum_{i=0}^{r} \tau_i(\omega) \overline{\tau_i}(\omega + \nu) = \begin{cases} 1, & \nu = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Then the resulting wavelet system $X(\Psi)$ is a tight frame, and the fundamental function Θ equals 1 *a.e. on $\sigma(V_0)$.*

The proof of the UEP in [RS3] is based on Proposition 1.7. A 'stand-alone' proof of the UEP can be obtained by following the arguments we use in the proof of Lemma 2.4 of the current article. The UEP was then used in [RS3] as follows: Given τ_0, identify $\tau_1, \ldots, \tau_r$ such that the "unitarity condition" (1.10) holds, thus obtaining a tight wavelet frame. Note that when (1.10) holds, $\sum_{\nu\in\{0,\pi\}^d} |\tau_0(\omega+\nu)|^2 \le 1$ for almost every ω. Therefore, $\sum_{\nu\in\{0,\pi\}^d} |\tau_0(\omega+\nu)|^2 \le 1$ is a necessary condition to use the UEP.

The UEP proved to be a very useful tool to construct tight framelets, including univariate compactly supported spline tight frames [RS3,6], multivariate compactly supported boxlets, [RS5], and various other tight framelets and bi-framelets in [RS6]. On a more theoretical level, this extension principle was used in [GR] in order to construct, for any dilation matrix and any spatial dimension, compactly supported tight frames of arbitrarily high smoothness. Recently, the UEP was used in [CH], [P1,2] and [S] in the context of univariate strongly local constructions of framelets. We revisit these latter constructions at the end of this section.

However, these constructions have limitations. In all the constructions of spline framelets listed above, at least one of the wavelets has only 1 vanishing moment, and none of these frames has approximation order higher than 2. In this paper, we show how to overcome or circumvent these shortcomings. One option is to change the underlying MRA. In [RS3–6], spline MRAs were used; by leaving the spline framework, considering "pseudo-splines" as in §3.1 below, the same approach as in [RS3–6] leads to tight wavelet frames (bi-framelets) with higher approximation order, and with very short support. This was also discovered, simultaneously and independently, in [S] (see §4 of that paper). Another approach is to revisit Proposition 1.7 and extract more flexible construction rules. To replace the UEP, we formulate the more general Oblique Extension Principle or OEP, as another consequence of Proposition 1.7:

"thmoep **Proposition 1.11: The Oblique Extension Principle (OEP).** *Let τ be the combined mask of an MRA that satisfies Assumptions* (1.3). *Suppose that there exists a 2π-periodic function Θ that satisfies the following:*

(i) *Θ is non-negative, essentially bounded, continuous at the origin, and $\Theta(0) = 1$.*

(ii) *If $\omega \in \sigma(V_0)$, and if $\nu \in \{0,\pi\}^d$ is such that $\omega+\nu \in \sigma(V_0)$, then*

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$$\langle \tau(\omega), \tau(\omega+\nu)\rangle_{\Theta(2\omega)} = \begin{cases} \Theta(\omega), & \text{if } \nu = 0,\\ 0, & \text{otherwise.}\end{cases} \tag{1.12}$$

Then the wavelet system $X(\Psi)$ defined by τ is a tight wavelet frame.

There are several ways in which Proposition 1.11 can be proved. One approach is to build, like for Proposition 1.9, a stand-alone proof by copying the arguments for Lemma 2.4. Another approach is to follow the proof of Corollary 5.3: to show that the Θ here is the fundamental function associated with τ, and then to invoke Proposition 1.7. This also shows, incidentally, that the existence of Θ satisfying (i) and (ii) is also a necessary condition for $X(\Psi)$ to be a tight frame. It is more surprising that Proposition 1.11 can also be derived from Proposition 1.9:

Proof: Setting $\vartheta := \Theta^{1/2}$, we define a function φ via $\widehat{\varphi} := \vartheta\widehat{\phi}$. Since ϑ is bounded, φ lies in $L_2(\mathbb{R}^d)$. Consider the combined mask t with

$$t_0 := \frac{\vartheta(2\cdot)\tau_0}{\vartheta}, \quad t_i := \frac{\tau_i}{\vartheta},\ i > 0.$$

From (1.12), we obtain that $|t(\omega)|^2 = 1$, a.e. on $\sigma(V_0)$, hence t is well-defined and bounded, and t_0 is the refinement mask of φ. Moreover, since $\Theta(0) = 1$, we obtain that $\widehat{\varphi}$ is continuous at 0 and $\widehat{\varphi}(0) = 1$. Apply now Proposition 1.9 to t, and observe that the tight wavelet frame obtained from the combined vector t is the same as the wavelet system induced by the combined vector τ. $\square$

We thus see that Proposition 1.9 and Proposition 1.11 are equivalent. It follows that every OEP construction can be obtained also from the UEP, and vice versa, by replacing the generator of the MRA by another (carefully chosen) generator of the same MRA. Although the UEP construction suffices, in principle, to construct all MRA-based tight wavelet frames, the OEP greatly facilitates the search for new constructions in practice. Indeed, by choosing Θ and τ to be trigonometric polynomials that satisfy the OEP conditions we naturally obtain a *local* tight wavelet frame. If we attempt to construct the same system by the UEP, then the refinable function is generally not compactly supported, the corresponding masks are not trigonometric polynomials, and it is impossible to predict when we nevertheless will still obtain compactly supported mother wavelets.

Moreover, as we shall see in §3, constructing the τ_i's and Θ simultaneously is less daunting than it looks. Given τ_0, one needs to choose Θ and τ_i such that (1.12) holds. More explicitly, given a (trigonometric polynomial) τ_0 with $\tau_0(0) = 1$, we shall identify (trigonometric polynomials) τ_i and Θ such that the identity (1.12) holds for every $\omega \in [-\pi, \pi]^d$ and every $\nu \in \{0, \pi\}^d$. Then $X(\Psi)$ will be a local MRA-based tight wavelet frame (provided that Θ is non-negative and $\Theta(0) = 1$). We refer to such constructions as **strongly local**.

The remainder of this paper is organized as follows:

We first elaborate (in §2) on three basic properties of MRA-based wavelet systems: the approximation order of the underlying MRA, the approximation order of the wavelet system, and the vanishing moments of the mother wavelets. This analysis allows us to understand better the relative merit of various possible constructions.

We then turn our attention (in §3) to several systematic univariate constructions. One effort is directed at constructing refinable functions whose derived frame system has a high approximation order. A different effort yields spline frames with high approximation orders. We also discuss briefly general techniques for constructing frames from any given MRA.

In §4, we give the analysis of the implementation algorithm: *the fast framelet transform.* Though essentially identical to the widely used fast wavelet transform, the interpretation of the results of the framelet transform turns out to be somewhat different.

We conclude this article (§5) with the analysis of wavelet frames that are not necessarily tight, or dilations that are not necessarily dyadic, and correspondingly more flexible characterizations. A highlight in this section is the (systematic) construction of univariate (symmetric) spline framelets with optimal approximation order, and very short support;

the systems in that construction are always generated by two mother wavelets, and a specific construction in this class is detailed in §6.

Several authors used the results of [RS3] and obtained UEP-based constructions that are related to some of ours. Particularly, univariate UEP-based framelet systems that are generated by 2 or 3 mother wavelets were studied in [CH], [P1,2] and [S]. More recently, Chui, He and Stöckler completed an independent article [CHS] in which several results overlap ours. Neither group of authors was aware of the other's work before it was completed; the two papers are published here consecutively.

2. Approximation orders and vanishing moments for wavelet frames

"Good" wavelet systems are characterized by several desirable properties, which may compete with each other. Generally speaking, these properties can be grouped into four categories:

(I): The invertibility and redundancy of the representation. The system is required to be orthonormal, or bi-orthogonal, or a tight frame, or a frame. And, there must be a fast algorithm that implements the decomposition and the reconstruction.

(II): The space-frequency localization of the system. This is usually measured by the smoothness of the mother wavelet Ψ and the smoothness of its Fourier transform. If Ψ is compactly supported (or band-limited) one would measure the size of $\operatorname{supp}\Psi$ ($\widehat{\Psi}$, respectively).

(III): Approximation properties of $X(\Psi)$. The three pertinent notions here are the approximation order of the underlying MRA, the number of vanishing moments of the mother wavelets, and the approximation order of the system itself. These properties are investigated in the current section (for tight framelets), and in §5.2 (for the more general bi-framelets).

(IV): Miscellaneous properties. Most of these properties are motivated by the actual applications; they include the symmetry of the mother wavelets, the 'translation-invariance' of the system, or optimality with respect to certain cost functions.

In this section we concentrate on the *approximation* properties of the system.

(2.1) Definition: approximation orders and vanishing moments. Let ϕ be a refinable function that generates a multiresolution analysis $(V_j)_j$. Let Ψ be a finite collection of *mother wavelets* in V_1, and let $X(\Psi)$ be the induced wavelet system. We say that:

(a) **The refinable function** ϕ (or, more correctly, the MRA) **provides approximation order** m, if, for every f in the Sobolev space $W_2^m(\mathbb{R}^d)$,

$$\operatorname{dist}(f, V_n) := \min\{\|f-g\|_{L_2(\mathbb{R}^d)} : \ g \in V_n\} = O(2^{-nm}).$$

(b) **The wavelet system has vanishing moments of order** m_0 if, for each mother wavelet $\psi \in \Psi$, the Fourier transform $\widehat{\psi}$ of ψ has a zero of order m_0 at the origin.

(c) Assuming that $X(\Psi)$ is a tight frame, we define the *truncated representation* Q_n by

$$Q_n : f \mapsto \sum_{\psi\in\Psi, k\in\mathbb{Z}^d, j<n} \langle f, \psi_{j,k}\rangle \, \psi_{j,k}.$$

We say that **the tight frame $X(\Psi)$ provides approximation order** m_1 if, for every f in the Sobolev space $W_2^{m_1}(\mathbb{R}^d)$,

$$\|f - Q_n f\|_{L_2(\mathbb{R}^d)} = O(2^{-nm_1}).$$

□

It is customary to label the largest possible number for which these statements can be made as "the" approximation order of ϕ or of the MRA, etc.

(2.2) Remarks.

(1) Note that the approximation orders provided by ϕ are completely determined by the MRA $(V_j)_j$. Thus, two refinable functions that generate the same MRA provide the same approximation order. The study of the approximation order provided by the refinable function ϕ is a special case of the well-understood topic of the *approximation order of shift-invariant spaces*, [BDR1].

(2) Since the operator Q_n maps into V_n, it is obvious that the approximation order of the wavelet system cannot exceed the order provided by the MRA. If the system $X(\Psi)$ is orthonormal, the two orders coincide, since then Q_n is the *orthogonal projector* onto V_n, hence $\|f - Q_n f\|_{L_2(\mathbb{R}^d)} = \operatorname{dist}(f, V_n)$ for every $f \in L_2(\mathbb{R}^d)$. The same is not true for tight frames. In particular we shall see that, in contrast with the approximation order provided by ϕ (that depends only on the choice of the MRA), the approximation order of the wavelet system depends on the choice of the mother wavelets.

In the analysis below, we use the following **bracket product**, [JM], [BDR1]:

$$[f, g] := \sum_{k\in 2\pi\mathbb{Z}^d} f(\cdot + k)\overline{g(\cdot + k)}.$$

We quote briefly some basic results concerning the approximation orders provided by shift-invariant spaces. Given any function $\phi \in L_2(\mathbb{R}^d)$, it is known [BDR1], that ϕ provides approximation order m if and only if the function

"deflam
$$\Lambda_\phi := \left(1 - \frac{|\widehat{\phi}|^2}{[\widehat{\phi}, \widehat{\phi}]}\right)^{1/2} \tag{2.3}$$

has a zero of order m at the origin. Under certain conditions on ϕ (e.g., if ϕ is compactly supported and $\widehat{\phi}(0) \neq 0$), this requirement is equivalent to the Strang-Fix (SF) conditions, meaning that Λ_ϕ has a zero of order m at $\omega = 0$ if and only if '$\widehat{\phi}$ has a zero of order m at each $k \in 2\pi\mathbb{Z}^d\backslash 0$' (see [BDR1] for more results and analysis.) If $\widehat{\phi}(0) = 1$, and ϕ is refinable with refinement mask τ_0, then the SF conditions are implied (but not vice versa) by the requirement that 'τ_0 has a zero of order m at each of the points in $\{0, \pi\}^d\backslash 0$'.

In this section we explore the connections between the well-understood approximation order provided by the refinable function on the one hand, and the vanishing moments of the mother wavelets, as well as the approximation order of the frame system itself on the other hand. We start by the following lemma, which rewrites $Q_n f$ in MRA terms:

"lemqzero **Lemma 2.4.** *Let $X(\Psi)$ be an MRA tight frame system and Θ the corresponding fundamental function. Then the truncated operator Q_n satisfies*

$$\widehat{Q_n f} = \Big([\widehat{f}(2^n\cdot), \widehat{\phi}]\, \widehat{\phi}\, \Theta\Big)\Big(\frac{\cdot}{2^n}\Big), \quad f \in L_2(\mathbb{R}^d).$$

In particular, $\widehat{Q_0 f} = [\widehat{f}, \widehat{\phi}]\widehat{\phi}\,\Theta$, for every $f \in L_2(\mathbb{R}^d)$.

Proof: We start the proof by observing that

$$(Q_1 - Q_0)f = \sum_{i=1}^{r} \sum_{k\in\mathbb{Z}^d} \langle f, \psi_{i,0,k}\rangle\, \psi_{i,0,k}.$$

As shown in [RS1], this is equivalent to

"qone
$$\widehat{Q_1 f} - \widehat{Q_0 f} = \sum_{i=1}^{r} [\widehat{f}, \widehat{\psi}_i]\, \widehat{\psi}_i = \sum_{i=0}^{r} \Theta_i [\widehat{f}, \widehat{\psi}_i]\widehat{\psi}_i - \Theta\, [\widehat{f}, \widehat{\phi}]\widehat{\phi}, \tag{2.5}$$

where $\psi_0 := \phi$, $\Theta_0 := \Theta$ and $\Theta_i = 1$, $i = 1, \ldots, r$. Using the relation

"temprel
$$\widehat{\psi}_i = (\tau_i \widehat{\phi})(\cdot/2), \tag{2.6}$$

we further obtain that

$$[\widehat{f}, \widehat{\psi}_i] = \sum_{\nu\in\{0,\pi\}^d} (\overline{\tau_i}\xi)(\frac{\cdot}{2} + \nu),$$

where

$$\xi := [\widehat{f}(2\cdot), \widehat{\phi}] = \sum_{k\in 2\pi\,\mathbb{Z}^d} \widehat{f}(2(\cdot + k))\overline{\widehat{\phi}(\cdot + k)}.$$

Substituting this into (2.5), invoking again (2.6), and changing the order of the summation, we obtain

$$\begin{aligned}\widehat{Q_1 f} - \widehat{Q_0 f} &= \widehat{\phi}(\cdot/2) \sum_{\nu\in\{0,\pi\}^d} \xi(\cdot/2 + \nu) \sum_{i=0}^{r} \Theta_i \tau_i(\cdot/2)\overline{\tau_i(\cdot/2 + \nu)} - [\widehat{f}, \widehat{\phi}]\, \widehat{\phi}\, \Theta. \\ &= \Big([\widehat{f}(2\cdot), \widehat{\phi}]\, \widehat{\phi}\, \Theta\Big)\, (\frac{\cdot}{2}) - [\widehat{f}, \widehat{\phi}]\, \widehat{\phi}\, \Theta.\end{aligned}$$

The last equality follows from (1.12) if $\omega/2 \in \sigma(V_0)$; if $\omega/2 \notin \sigma(V_0)$ it follows from the fact that $\widehat{\phi}(\omega/2) = 0$. (The MRA tight frame must satisfy (1.12) by Proposition 1.7 and (1.6).)

Since $Q_n = \mathcal{D}^n Q_0 \mathcal{D}^{-n}$, we easily conclude that, for every n,

$$\widehat{Q_n f} - \widehat{Q_{n-1} f} = \Big([\widehat{f}(2^n\cdot), \widehat{\phi}]\, \widehat{\phi}\, \Theta\Big)\, (\frac{\cdot}{2^n}) - \Big([\widehat{f}(2^{n-1}\cdot), \widehat{\phi}]\, \widehat{\phi}\, \Theta\Big)\, (\frac{\cdot}{2^{n-1}}),$$

implying, for $j < n$, $\widehat{Q_n f} = \widehat{Q_j f} - \Big([\widehat{f}(2^j\cdot), \widehat{\phi}]\, \widehat{\phi}\, \Theta\Big)\, (\frac{\cdot}{2^j}) + \Big([\widehat{f}(2^n\cdot), \widehat{\phi}]\, \widehat{\phi}\, \Theta\Big)\, (\frac{\cdot}{2^n})$. It remains to show that the sequence $(P_j f)$ defined by

$$\widehat{P_j f} := \widehat{Q_j f} - \Big([\widehat{f}(2^j\cdot), \widehat{\phi}]\, \widehat{\phi}\, \Theta\Big)\, (\frac{\cdot}{2^j})$$

converges to 0 when $j \to -\infty$. This is a simple consequence of the weak compactness of the unit ball of $L_2(\mathbb{R}^d)$. (See, e.g., [BDR3] for this argument, which uses $\cap_j V_j = \{0\}$. Every MRA automatically satisfies this latter condition, as proved in [BDR3] as well.) □

The bracket product $[\widehat{\phi},\widehat{\phi}]$ and the difference $1-[\widehat{\phi},\widehat{\phi}]$ are known to play a role in MRA analysis. For instance, the orthogonal projection $\mathcal{P}_0$ of f onto V_0 satisfies (with the convention that $0/0 := 0$) [BDR1], $\widehat{\mathcal{P}_0 f} = \frac{[\widehat{f},\widehat{\phi}]}{[\widehat{\phi},\widehat{\phi}]}\widehat{\phi}$. Clearly, when $\Theta = 1$ and $\sigma(V_0) = [-\pi,\pi]^d$, $Q_0 = \mathcal{P}_0$ if and only if $1-[\widehat{\phi},\widehat{\phi}] = 0$; the latter is a well-known characterization of the orthonormality of $E(\phi)$. Lemma 2.4 (as well as Theorem 2.8 below) shows that even when $\Theta \neq 1$, the difference $1-\Theta[\widehat{\phi},\widehat{\phi}]$ continues to play a central role in the characterization of the approximation order provided by more general wavelet systems. Even more to the point, the lemma and theorem connect MRA-based wavelet systems with *quasi-interpolation*, [BR]: quasi-interpolation is the art of assigning suitable dual functionals to a given set of 'approximating' functions. The fundamental function Θ can be recognized to be a specific quasi-interpolation rule. Indeed, our proof of Theorem 2.8 below invokes the following result of Jetter and Zhou concerning quasi-interpolation:

“jz **Result 2.7** [JZ1-2]**.** *Let $\phi,\zeta \in L_2(\mathbb{R}^d)$, and $\widehat{\phi}(0) \neq 0$. Consider the approximation operators $(Q_n)_n$ where $Q_n = \mathcal{D}^n Q_0 \mathcal{D}^{-n}$, and*

$$\widehat{Q_0 f} = [\widehat{f},\widehat{\zeta}]\widehat{\phi}.$$

Assume that $[\widehat{\phi},\widehat{\phi}]$ is bounded. Then $(Q_n)_n$ provides approximation order m if and only if the following two conditions hold:
(a) $[\widehat{\phi},\widehat{\phi}] - |\widehat{\phi}|^2 = O(|\cdot|^{2m})$.
(b) $1-\overline{\widehat{\zeta}}\widehat{\phi} = O(|\cdot|^m)$.

“characao **Theorem 2.8.** *Let $X(\Psi)$ be an MRA tight frame system and Θ be the corresponding fundamental function. Assume that Assumptions (1.3) are satisfied; and the underlying refinable function provides approximation order $m < \infty$. Then the approximation order provided by the framelet system coincides with each of the following (equal) numbers*
(i) $\min\{m, m_1\}$, *with m_1 the order of the zero of $1-\Theta\,[\widehat{\phi},\widehat{\phi}]$ at the origin.*
(ii) $\min\{m, m_2\}$, *with m_2 the order of the zero of $\Theta - \Theta(2\cdot)|\tau_0|^2$ at the origin.*
(iii) $\min\{m, m_3\}$, *with m_3 the order of the zero of $1-\Theta\,|\widehat{\phi}|^2$ at the origin.*
Here, ϕ is the refinable function, and τ_0 is its mask.

Proof: We first prove that the approximation order provided by the frame system is $\min\{m, m_3\}$, and invoke to this end Result 2.7. In view of Lemma 2.4, our case here corresponds to the case $\widehat{\zeta} = \Theta\widehat{\phi}$ in Result 2.7, hence we need to check the zero order of $[\widehat{\phi},\widehat{\phi}] - |\widehat{\phi}|^2$ and of $1-\Theta|\widehat{\phi}|^2$. The latter order is m_3. As to the former, since $\widehat{\phi}$ is bounded above as well as away of zero in a neighborhood of the origin, the characterization of the approximation orders provided by ϕ (cf. [BDR1], or derive it directly from the characterization mentioned in the discussion around (2.3)) is given as half the order of the zero of $[\widehat{\phi},\widehat{\phi}] - |\widehat{\phi}|^2$ at the origin. Thus, Result 2.7 implies indeed that the frame system provides approximation order $\min\{m, m_3\}$.

Assuming ϕ to provide approximation order m, we obtain (again from either [BDR1] or directly from the discussion around (2.3)) that, since $\widehat{\phi}(0) = 1$, then, near the origin,

$$[\widehat{\phi},\widehat{\phi}] - |\widehat{\phi}|^2 = O(|\cdot|^{2m}).$$

In particular, $m_1 = m_3$ whenever one of these numbers is $< 2m$. Consequently, $\min\{m, m_1\} = \min\{m, m_3\}$.

Finally, since

“one (2.9)
$$|\tau_0|^2 |\widehat{\phi}|^2 = |\widehat{\phi}|^2(2\cdot),$$

we obtain that

$$[\Theta - \Theta(2\cdot)|\tau_0|^2]|\widehat{\phi}|^2 = \Theta|\widehat{\phi}|^2 - \Theta(2\cdot)|\widehat{\phi}(2\cdot)|^2.$$

Since $1 - \Theta|\widehat{\phi}|^2$ has a zero of exactly order m_3 at the origin, $1 - \Theta|\widehat{\phi}|^2 = q + o(|\cdot|^{m_3})$ near the origin where q is some homogeneous polynomial of total degree m_3. Hence, near the origin,

$$\Theta|\widehat{\phi}|^2 - \Theta(2\cdot)|\widehat{\phi}(2\cdot)|^2 = q(2\cdot) - q(\cdot) + o(|\cdot|^{m_3}).$$

Since $q(2\cdot) - q(\cdot)$ is a nonzero homogeneous polynomial of total degree m_3, whenever $m_3 > 0$ (which is the case, because $\Theta|\widehat{\phi}|^2(0) = 1$), we see that $\Theta|\widehat{\phi}|^2 - \Theta(2\cdot)|\widehat{\phi}(2\cdot)|^2$ has a zero of exactly order m_3 at the origin. The conclusion that $m_2 = m_3$ now follows from the fact that the order of the zero of $[\Theta - \Theta(2\cdot)|\tau_0|^2]|\widehat{\phi}|^2$ at the origin is exactly m_2. □

For a given refinable function ϕ, Theorem 2.8 (iii) suggests that in order to construct tight framelets that provide high approximation order, we should choose Θ as a suitable approximation, at the origin, to $1/|\widehat{\phi}|^2$. For example, if ϕ is a B-spline of order m, then $|\widehat{\phi}(\omega)| = \left|\frac{\sin(\omega/2)}{\omega/2}\right|^m$. Thus, we should choose Θ as a 2π-periodic function which approximates the function

$$\left|\frac{\omega/2}{\sin(\omega/2)}\right|^{2m}$$

at the origin. We shall revisit this issue in §3.3.

(2.10) Discussion: approximation orders vs. vanishing moments. If the behaviors of Θ and $|\widehat{\phi}|^2$ are not “matched” near the origin, then Theorem 2.8 shows that the approximation order of the framelet system can lag significantly behind the approximation order provided by the refinable function. On the other hand, the approximation order of the framelet system turns out to be strongly connected, perhaps in a somewhat surprising way, to the number of vanishing moments of the wavelets.

Since $\widehat{\psi_i} = (\tau_i \widehat{\phi})(\cdot/2)$, and since $\widehat{\phi}(0) = 1$, the vanishing moments of ψ_i are determined completely by the order of the zero (at the origin) of τ_i. This means that the MRA-based wavelet system $X(\Psi)$ has vanishing moments of order m_0 if and only if $|\tau_+|^2 = O(|\cdot|^{2m_0})$, near the origin. On the other hand, if our system is a tight framelet, it must satisfy the OEP conditions, and thus $|\tau_+|^2 = \Theta - \Theta(2\cdot)|\tau_0|^2$. It follows that the index m_2 of Theorem 2.8 (ii) is exactly equal to $2m_0$. This proves part of the following theorem:

“momao **Theorem 2.11.** *Let $X(\Psi)$ be an MRA tight frame system. Assume that the system has vanishing moments of order m_0, and that the refinable function ϕ provides approximation order m. Then:*

(a) ϕ satisfies the SF conditions of order m_0, i.e. $\widehat{\phi}$ vanishes at each $\omega \in 2\pi\mathbb{Z}^d\backslash 0$ to order m_0.

(b) *The approximation order of the tight frame system is* $\min\{m, 2m_0\}$.

Proof: Because of the remarks above, we need prove only (a).

Let $\nu \in \{0,\pi\}^d\backslash\{0\}$. If $X(\Psi)$ has vanishing moments of order m_0, then $|\tau_+|^2 = O(|\cdot|^{2m_0})$ (near the origin), hence, for every $i \ge 1$,

$$\tau_i = O(|\cdot|^{m_0}). \tag{2.12}$$

Let $j \in 2\pi\,\mathbb{Z}^d$. Since, thanks to the OEP conditions, $\langle\tau, \tau(\cdot+\nu)\rangle_{\Theta(2\cdot)}\widehat{\phi}(\cdot+\nu+j) = 0$, (on $\sigma(V_0)$, hence in a neighborhood of the origin), we obtain from (2.12) that $\Theta(2\cdot)\tau_0\tau_0(\cdot+\nu)\widehat{\phi}(\cdot+\nu+j) = O(|\cdot|^{m_0})$. Since $\Theta(0) = \tau_0(0) = 1$, we conclude that

$$\widehat{\phi}(2\cdot+2\nu+2j) = \tau_0(\cdot+\nu)\widehat{\phi}(\cdot+\nu+j) = O(|\cdot|^{m_0}), \quad \nu \in \{0,\pi\}^d\backslash 0,\ j \in 2\pi\,\mathbb{Z}^d.$$

A routine argument can then be used to prove that the last relation holds for $\nu = 0$ as well (provided then that $j \ne 0$). □

(2.13) Remark. Part (a) of the above result states, essentially, that the approximation order provided by ϕ is $\ge m_0$. For an MRA-based framelet with exactly m_0 vanishing moments, the approximation order of the framelet is therefore always between m_0 and $2m_0$.

In the theory of MRA-based orthonormal wavelets, the approximation order of the MRA, the approximation order of the wavelet system and the number of vanishing moments of the wavelets are always equal. (Note that this is no longer true for bi-orthogonal bases.) It is therefore customary to inspect only one of those quantities; most of the wavelet literature picks the number of vanishing moments as the focal property.

In contrast, these three parameters need not coincide in the context of framelets. A natural question then arises: which parameter should we attempt to maximize in actual constructions? The answer usually depends on the application:

The approximation order of the MRA is clearly important since it provides an upper bound for the approximation order of *any* framelet system derived from that MRA. Similarly, the approximation order of the framelet system is very important since the wavelet expansion must be truncated in any practical implementation. MRAs or framelet expansions of low approximation orders transfer to their high frequency scales information about the function/image/signal that could have been faithfully represented in the (sparser) low frequency scales of more appropriate framelet expansions.

A further evaluation of the difference between the approximation order of the MRA and that of the framelet system is as follows. The redundancy of the tight framelet system entails that a given $f \in L_2(\mathbb{R}^d)$ can be represented in many different ways as a convergent sum

$$f = \sum_{g\in X(\Psi)} c(g)\, g. \tag{2.14}$$

“tempo

The tight framelet representation

$$f = \sum_{g\in X(\Psi)} \langle f, g\rangle\, g \tag{2.15}$$

“temp

is one of many. One of its major advantages (over other representations of f as linear combinations of $X(\Psi)$) is that it is implemented by a fast transform, the fast frame transform. Now, assume that f is, say, a very smooth function. Then, a high approximation order of the MRA guarantees that *some* of the (2.14) representations of f are sparse and compact. Some other (2.14) representations of f may be dense and inefficient. A high approximation order of the framelet system ensures that the specific representation (2.15) is a good one, i.e., it is (asymptotically) as compact and as effective as the best possible (2.14) representation of f.

It might be worthwhile to mention that not every application requires high approximation orders of the framelet system. For example, in novel image compression algorithms that are currently under development, one uses the representation (2.15) as a springboard for finding the sparsest (2.14) representation of f. In this and similar applications the properties of the representation (2.15) are less crucial, since this representation is only an intermediate one. More important then is the ability to find a compact representation among all of those of the type (2.14), and this latter property is more connected to the approximation order of the MRA itself.

And, what about the impact of vanishing moments? A high number of vanishing moments is important for algorithms that involve the manipulation of the wavelet coefficients. For instance, wavelet representations of one-dimensional piecewise-smooth functions become sparser when the number of vanishing moments increases. On the other hand, in some applications, mother wavelets with *varying* vanishing moments may be preferred, since they can serve, e.g., as 'multiple detectors'. In other applications, the coefficients associated with the mother wavelet that has the highest vanishing moments can be used to capture the essential information about the object, while the other coefficients simply aid in the reconstruction process.

Let us illustrate this discussion by comparing several framelets. The first two examples, constructed by an application of the UEP, are borrowed from [RS3]:

"splinetwors **Example 2.16** (Figure 1)**.** Take $\tau_0(\omega) = (1+e^{-i\omega})^2/4$. Then ϕ is the B-spline function of order 2, i.e. the hat function. Let

$$\tau_1(\omega) := -\frac{1}{4}(1-e^{-i\omega})^2 \quad \text{and} \quad \tau_2(\omega) := -\frac{\sqrt{2}}{4}(1-e^{-i2\omega}).$$

The corresponding $\{\psi_1, \psi_2\}$ generates a tight framelet. The framelet has $m_0 = 1$ vanishing moments (though one of the wavelets has 2 vanishing moments); the approximation order of the MRA is 2. The approximation order of the framelet system equals $2 = \min(m, 2m_0)$.

"splinefourrs **Example 2.17** (Figure 2). Take $\tau_0(\omega) = (1+e^{-i\omega})^4/16$. Then ϕ is the B-spline function of order 4 which is a piecewise cubic polynomial. Let

$$\begin{aligned} \tau_1(\omega) &:= \tfrac{1}{4}(1-e^{-i\omega})^4, & \tau_2(\omega) &:= -\tfrac{1}{4}(1-e^{-i\omega})^3(1+e^{-i\omega}), \\ \tau_3(\omega) &:= -\tfrac{\sqrt{6}}{16}(1-e^{-i\omega})^2(1+e^{-i\omega})^2, & \tau_4(\omega) &:= -\tfrac{1}{4}(1-e^{-i\omega})(1+e^{-i\omega})^3. \end{aligned}$$

The corresponding $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ generates a tight framelet that has vanishing moments of order $m_0 = 1$. For this ϕ we have $m = 4$. The approximation order of the framelet system is $2 = \min(m, 2m_0)$**.**

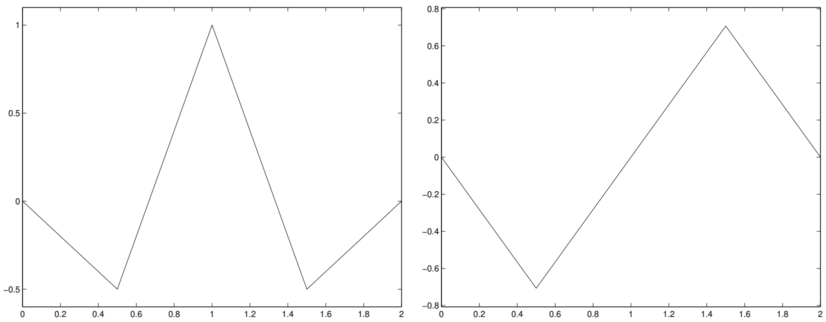

Figure 1. The graphs of the wavelet functions ψ_1 and ψ_2 derived from the B-spline function of order 2 in Example 2.16. $\{\psi_1, \psi_2\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$ and has vanishing moments of order 1. The framelet system provides approximation order 2, which is optimal for a piecewise-linear system.

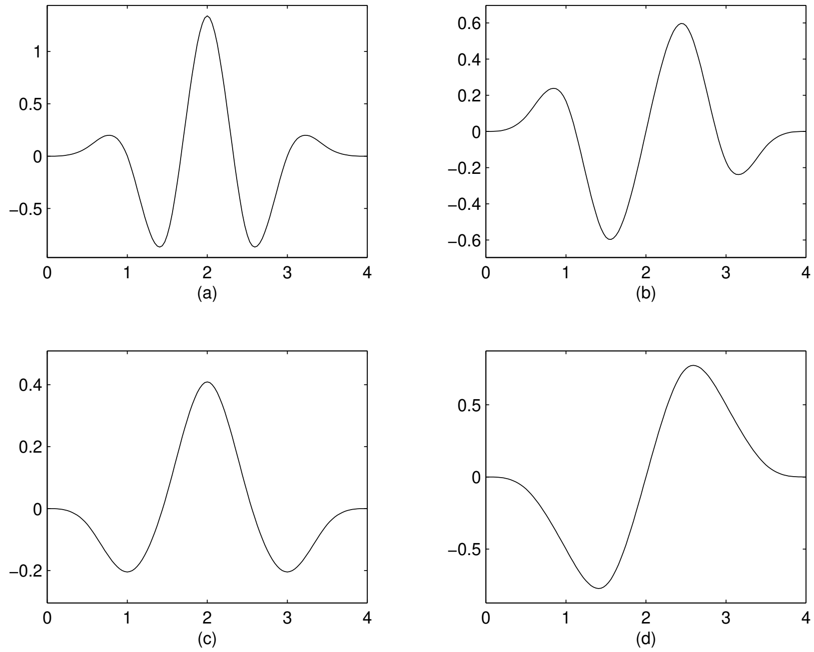

Figure 2. The graphs of the wavelet functions $\psi_1, \psi_2, \psi_3, \psi_4$ derived from the B-spline function of order 4 in Example 2.17; together, the four wavelets generate a tight framelet. Wavelet (d) has only one vanishing moment, hence the approximation order is 2, which is suboptimal since the corresponding MRA provides approximation order 4.

The next two examples are linear, respectively cubic spline framelets constructed by using the OEP, as described below. We list here τ_0, Θ and the τ_j, and revisit these examples later.

“splinetwonew **Example 2.18** (Figure 3)**.** Take $\tau_0(\omega) = (1 + e^{-i\omega})^2/4$ and $\Theta(\omega) = (4 - \cos\omega)/3$. Let

$$\tau_1(\omega) := -\frac{1}{4}(1 - e^{-i\omega})^2 \quad \text{and} \quad \tau_2(\omega) := -\frac{\sqrt{6}}{24}(1 - e^{-i\omega})^2(e^{-i\omega} + 4e^{-i2\omega} + e^{-i3\omega}).$$

The set $\{\psi_1, \psi_2\}$ generates a tight framelet and has vanishing moments of order 2. Both ψ_1 and ψ_2 are symmetric and their graphs are given in Figure 3. Even though $2m_0 = 4$, we still have $m = 2$, so that $\min(m, 2m_0)$ equals 2; this system has thus the same approximation order as in Example 2.16.

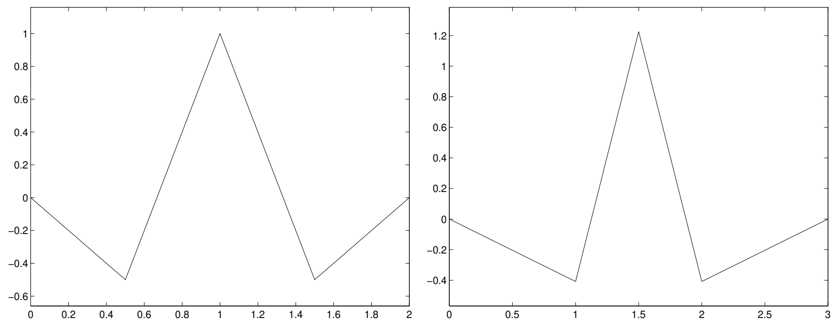

Figure 3. The graphs of the symmetric wavelet functions ψ_1 and ψ_2 derived from the B-spline function of order 2 in Example 2.18. $\{\psi_1, \psi_2\}$ generates a tight framelet, and each of the wavelets has two vanishing moments, and hence the approximation order of the system is $\min\{4, 2\} = 2$; the higher number of vanishing moments than in Example 2.16 leads to sparser wavelet coefficients but does not improve the decay of the error $\|Q_n f - f\|$ for the truncated reconstruction.

“splinefournew **Example 2.19** (Figure 4). Take $\tau_0(\omega) = (1 + e^{-i\omega})^4/16$ and

$$\Theta(\omega) = 2452/945 - 1657/840\cos(\omega) + 44/105\cos(2\omega) - 311/7560\cos(3\omega).$$

Let

$$\begin{aligned}
\tau_1(\omega) &= t_1\,(1 - e^{-i\omega})^4\Big[1 + 8e^{-i\omega} + e^{-i2\omega}\Big],\\
\tau_2(\omega) &= t_2\,(1 - e^{-i\omega})^4\Big[1 + 8e^{-i\omega} + (7775/4396\,t - 53854/1099)e^{-i2\omega} + 8e^{-i3\omega} + e^{-i4\omega}\Big],\\
\tau_3(\omega) &= t_3\,(1 - e^{-i\omega})^4\Big[1 + 8e^{-i\omega} + (21 + t/8)(e^{-i2\omega} + e^{-i4\omega}) + t\,e^{-i3\omega} + 8e^{-i5\omega} + e^{-i6\omega}\Big],
\end{aligned}$$

where $t_3 = \sqrt{32655}/20160$, $t = 317784/7775 + 56\sqrt{16323699891}/2418025$, and

$$t_1 = \frac{\sqrt{11113747578360 - 245493856965\,t}}{62697600}, \qquad t_2 = \sqrt{1543080 - 32655\,t}/40320.$$

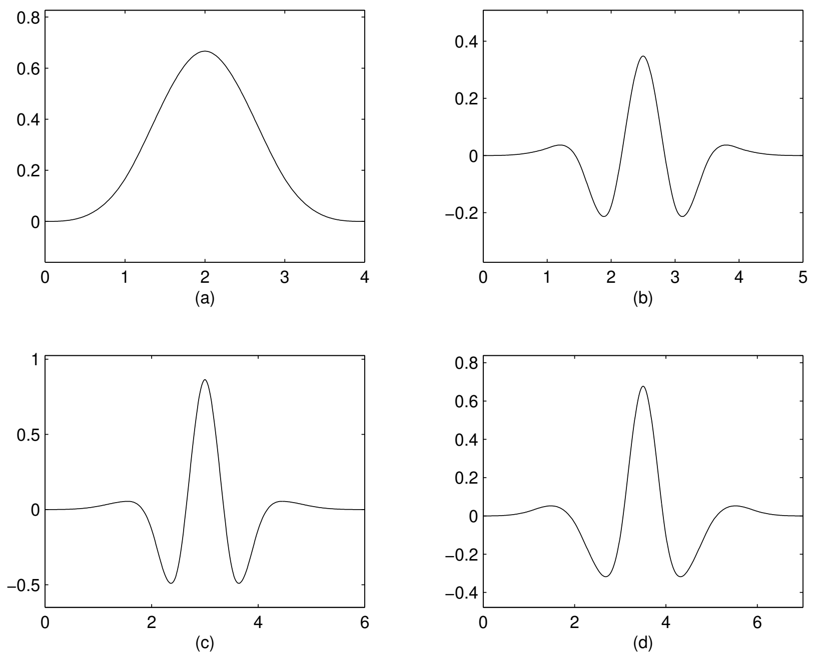

Figure 4. (b), (c) and (d) are the graphs of the symmetric mother wavelets derived from the cubic B-spline function (a) in Example 2.19. All the mother wavelets have 4 vanishing moments, hence the approximation order of the system is $\min\{4, 8\} = 4$.

The above masks satisfy the OEP conditions, hence lead to a tight framelet. All the wavelets here have 4 vanishing moments hence $m_0 = 4$. The mother wavelets ψ_1, ψ_2, ψ_3 are symmetric. Note that for this ϕ the approximation order of the MRA is $m = 4$. The approximation order of the framelet system is $4 = \min(m, 2m_0)$. The three filters are of size $7, 9, 11$**.**

A fifth example is constructed by using the UEP, now starting from a different, non-spline MRA; this construction will also be revisited in more detail in §3.1.

“fifth **Example 2.20** (Figure 5). In this case we have one scaling function and three wavelets. The filters τ_0 and $\tau_j, j = 1, 2, 3$ are obtained by spectral factorization, i.e. by “taking a square root”. In particular, we have $|\tau_0(\omega)|^2 = \cos^8(\omega/2)\left(1 + 4\sin^2(\omega/2)\right)$, $\tau_1(\omega) = e^{i\omega}\tau_0(\omega+\pi)$, $\tau_2(\omega) = \frac{\sqrt{5}}{2}\,\sin^2(\omega)$, and $\tau_3(\omega) = e^{i\omega}\tau_2(\omega)$. The wavelets in this system have 2 vanishing moments, so that $m_0 = 2$. The approximation order of the MRA is $m = 4$; the approximation order of the framelet is thus $\min(m, 2m_0) = 4$**.**

For these five examples, as well as for the bi-framelet of §6, and for three benchmark wavelet bases (not frames - we used here the Haar basis and the two bi-orthogonal wavelet bases known as (5,3) and (9,7)), we provide, for a very smooth function f, the error $\|Q_n f - f\|$, for increasing n. The results are listed in Table 1 (courtesy of Steven Parker of UW-Madison). For each system we also list three indices in the header of the column: the first is the number of vanishing moments of the system, the second is the approximation order of the system, and the third is the approximation order of the underlying MRA

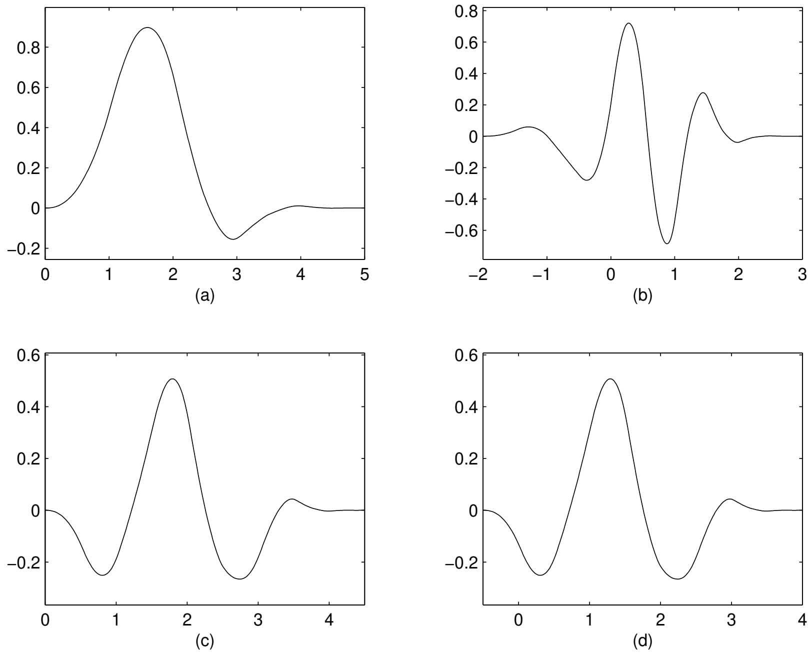

Figure 5. (b), (c) and (d) are the graphs of the mother wavelets in Example 2.20 of the Type I pseudo-spline tight framelets derived from the scaling function, pseudo-spline $(4, 1)$, shown in (a). (See §3.1 for details)

level	HAAR (1,1,1)	(5,3) (2,2,2)	(9,7) (4,4,4)	Ex. 2.16 (1,2,2)	Ex. 2.17 (1,2,4)	Ex. 2.18 (2,2,2)	Ex. 2.19 (4,4,4)	Ex. 2.20 (2,4,4)	Bi-frame (4/2,4,4)
2	6.36e-01	1.57e-01	9.15e-02	3.26e-01	5.62e-01	1.67e-01	5.44e-02	1.93e-01	1.20e-01
3	3.72e-01	4.83e-02	8.15e-03	1.14e-01	1.86e-01	4.24e-02	9.90e-04	1.64e-02	4.11e-03
4	1.92e-01	1.27e-02	5.54e-04	3.12e-02	5.01e-02	1.23e-02	3.20e-05	1.11e-03	9.96e-05
5	9.63e-02	3.20e-03	3.53e-05	7.97e-03	1.28e-02	3.17e-03	2.08e-06	7.12e-05	2.98e-06
6	4.75e-02	8.02e-04	2.20e-06	2.00e-03	3.20e-03	8.00e-04	1.31e-07	4.47e-06	1.32e-07
7	2.30e-02	2.00e-04	1.35e-07	5.01e-04	8.00e-04	2.00e-04	8.38e-09	2.80e-07	8.09e-09
8	1.07e-02	4.94e-05	8.07e-09	1.25e-04	1.98e-04	4.94e-05	5.81e-10	1.75e-08	5.77e-10
9	4.60e-03	1.18e-05	4.49e-10	3.06e-05	4.71e-05	1.18e-05	4.98e-11	1.09e-09	4.98e-11
10	1.53e-03	2.35e-06	2.47e-11	7.06e-06	9.41e-06	2.35e-06	5.54e-12	6.78e-11	5.56e-12
	1.07	2.00	3.99	1.96	1.99	2.00	4.08	3.95	4.13

Table 1. The errors $\|Q_n f - f\|$ for nine different systems, for increasing n (=level); the last row gives the slope of $-log_2\|Q_n f - f\|$ as a function of n, computed by linear regression.

(the last system is a bi-frame, meaning that the decomposition masks are different from the reconstruction masks: the former has four vanishing moments while the latter only two vanishing moments). At the bottom of Table 1 we give the numerical estimate of the decay rate of $\|Q_n f - f\|$ in n; this clearly is (approximately) equal, in all cases, to the approximation order of the system, and depends only marginally on the other two indices. Let us look at some particular comparisons. For the linear splines in Examples 2.16 and 2.18, the increase in the number of vanishing moments from Examples 2.16 to 2.18 does not improve the approximation order of the framelet. What this means is that the estimates of the sizes of the wavelet coefficients, as given by e.g. $\max_{i,k} |\langle f, \psi_{i,j,k}\rangle|$, will decay faster as j increases for Example 2.18 than for Example 2.16, but that the truncated wavelet expansions, using coefficients up to level j only, will exhibit comparable errors. For the cubic splines in Examples 2.17 and 2.19, the number of vanishing moments increases from 1 (for Example 2.17) to 4 (for Example 2.19); this is reflected by an increase in the approximation order of the corresponding framelets, from 2 to 4. In Example 2.20 we have only 2 vanishing moments, but the framelet approximation order is 4, and the decay of $\|Q_n f - f\|$ is comparable to that for Example 2.19, even though the decay of the wavelet coefficients will be less fast.

Let us proceed now with a more systematic tour.

3. A tour through univariate constructions of tight framelets

We restrict our attention here to *strongly local MRA-based constructions*. Constructions are typically guided by a desire for some of the following properties for the mother wavelets:
(i) Short filter/support.
(ii) High smoothness.
(iii) High approximation orders of the refinable function.
(iv) High approximation orders for the framelet system.
(v) High order of vanishing moments.
(vi) Small number of mother wavelets (equivalently: low order of oversampling).
(vii) Symmetry (or anti-symmetry) of the wavelets.

The constructions of [RS3-6] are optimal with respect to properties (i-iii) and (vii): they involve tight and other spline framelets with very small support. However, the approximation order of these framelet systems is 2 (which is optimal only in the case of the piecewise-linear tight framelet), because the number of vanishing moments is always 1. Moreover, the number of mother wavelets increases together with the underlying smoothness.

In order to improve the approximation order of the framelet system or the number of vanishing moments without changing the underlying MRA, one has to increase the support of the mother wavelets. Let us examine, as a major example, the case of the spline MRAs. In this case the refinable function ϕ is the B-spline of order m (with m some fixed positive integer) whose mask is

$$\tau_0 = \left(\frac{1+e^{-i\omega}}{2}\right)^m,$$

for which [RS3,5,6] use the UEP to construct a tight framelet. Since $1 - |\tau_0|^2 = O(|\cdot|^2)$ around the origin, Theorems 2.8 and 2.11 show why the approximation order of the resulting wavelet system cannot exceed 2 (regardless of the value of m). We attain better framelet approximation order via the OEP (see below), by choosing a *trigonometric polynomial* Θ; since $|\tau_+|^2 = \Theta - \Theta(2\cdot)|\tau_0|^2$, we necessarily obtain mother wavelets with longer support.

Let us examine another property of the framelet system, viz., the number of mother wavelets. Using any of the extension principles, we have two requirements to fulfill:

$$\Theta(2\cdot)|\tau_0|^2 + \sum_{i=0}^{r} |\tau_i|^2 = \Theta, \quad \text{and} \quad \Theta(2\cdot)\tau_0\overline{\tau_0(\cdot+\pi)} + \sum_{i=1}^{r} \tau_i\overline{\tau_i(\cdot+\pi)} = 0.$$

So far we have not specified r. Without imposing special conditions on the refinable function, we will need at least *two* mother wavelets in order to satisfy the above. A rigorous statement to that extent is found at the end of this section. (One needs great care when stating such results: after all, an orthonormal wavelet system can be derived from *any* local MRA, without any further conditions on the compactly supported refinable function [BDR3]. The single mother wavelet, however, may decay then at a very low rate, in stark contrast with the compact support of the refinable function.) Moreover, if we impose also the symmetry requirements (vii), then it may reasonably be expected that we need, at least for generic refinable functions, *three* mother wavelets. We shall therefore consider cases where r can be as large as 3. For simplicity, we restrict ourselves to $r = 3$, and provide a method to reduce the number of mother wavelets from 3 to 2, if desired. (This reduction usually comes at a price: the filters may be longer and/or have less symmetry.) There may, of course, be situations where one wishes to consider larger r, but we shall not do so here.

We advocate the use of systems in which the approximation order of the framelet systems matches, or at least does not lag significantly behind, the approximation order of the MRA itself, and this principle guides us throughout this section.

(3.1) Discussion: MRAs of approximation order 4. As an illustration for the above, let's consider several MRAs whose approximation order is 4. The orthonormal system of that order involves 8-tap filters [D1], and the mother wavelets have relatively low smoothness. Symmetry of the mother wavelets can be obtained by switching to a bi-orthogonal system, such as the 7/9 bi-orthogonal wavelets. In all these cases, the system provides approximation order 4, and the vanishing moments are of order 4, as well.

In [RS3,5] two different tight cubic spline framelets are constructed. One of them involves four mother wavelets each associated with a 5-tap filter. The approximation order of the system is 2 and the vanishing moment order is 1; the corresponding τ_0, τ_j were given in Example 2.17 above. The smoothness is maximal (for 5-tap filters). In order to increase the approximation order of the system from 2 to 4 we must use longer filters, regardless of whether we stay with a spline MRA or not.

In our first stop on the tour in this section, we will change the MRA (to a pseudo-spline MRA of type $(4,1)$, see below) and obtain three mother wavelets with associated filters of length $6, 5, 5$. We also construct from the same MRA a system with two mother wavelets with filters of length 6 and 14. The approximation order of the tight framelet is 4 in both cases, but the vanishing moments are only of order 2. In our second stop, we construct spline framelets of any order with any number of vanishing moments. In that construction, the number of wavelets is either 3 (with short filters) or 2 (with longer filters). In the former case, we achieve approximation order 4 (and vanishing moments 2) with three 7-tap filters, and in the latter case the two filters are of sizes 7 and 17.

It turns out that one can find ("by hand") tight spline framelets that have even shorter filters; examples of the results of such (*ad-hoc*) constructions within the cubic spline MRA, yielding two mother wavelets with 9- and 11-tap filters, with 4 vanishing moments, are given in the Appendix. Note that other framelet constructions with short support and few wavelets are given in [CH], [P1-2] and [S].

It is clear that one has to consider trade-offs when deciding which of these framelets, all of which have approximation order 4, one should use. Since gain in vanishing moments carries a price (in filter size), one should consider it only if the corresponding faster decay of wavelet coefficients is sought; if the most important feature is the order of approximation, then there is no need to look for higher numbers of vanishing moments than half the desired approximation order. The same applies to the gain in smoothness; the switch from pseudo-splines of $(4, 1)$ to splines of order 4 yields smoother mother wavelets, with longer associated filters, for the same approximation order. Which one is preferred is dictated by whether short filters or smooth wavelets are most desirable for the application at hand.

Wavelet mask construction: All the constructions in this section use the following approach. Suppose that we are given a refinable function with mask τ_0, and that we have chosen the fundamental MRA function to be some 2π-periodic Θ, such that the OEP condition is satisfied:

$$\Theta - \Theta(2\cdot)|\tau_0|^2 \geq 0.$$

Let's assume, in addition, that

$$A := \Theta - \Theta(2\cdot)|\tau_0|^2 - \Theta(2\cdot)|\tau_0(\cdot + \pi)|^2 \geq 0.$$

This extra condition will make it easy to find wavelet masks. Choose t_2, t_3 to be two 2π-periodic trigonometric polynomials such that

$$|t_2|^2 + |t_3|^2 = 1, \quad t_2\overline{t_2(\cdot + \pi)} + t_3\overline{t_3(\cdot + \pi)} = 0.$$

A standard choice for such t_2, t_3 is

$$t_2(\omega) = \frac{\sqrt{2}}{2}, \quad t_3(\omega)_: = \frac{\sqrt{2}}{2}e^{i\omega}.$$

Define ϑ and a to be square roots of Θ and A respectively. The three wavelet masks are then

$$\tau_1 := e_1\vartheta(2\cdot)\overline{\tau_0(\cdot + \pi)},$$

$$\tau_i := t_i a, \quad i = 2, 3,$$

where $e_1(\omega) = e^{i\omega}$. It is easy to check that the combined mask $\tau := (\tau_0, \ldots, \tau_3)$ satisfies the OEP conditions (cf. Proposition 1.11). Assuming that all the side-conditions of the OEP are satisfied (to be checked in individual constructions), we thus obtain a tight framelet.

One can reduce the number of mother wavelets to two by defining

$$\tau_1 := e_1\vartheta(2\cdot)\overline{\tau_0(\cdot + \pi)},$$

$$\tau_2 := \tau_0\, a(2\cdot).$$

Then $\tau = (\tau_0, \tau_1, \tau_2)$ satisfies the OEP conditions with a new fundamental function $\Theta - A$.

In the case where one uses the UEP rather than the OEP, $\Theta = 1$, and hence one uses the assumption that

$$A := 1 - |\tau_0|^2 - |\tau_0(\cdot + \pi)|^2 \geq 0\ .$$

Let a be the square root of A. One can then define three wavelet masks by

$$\tau_1 := e_1 \overline{\tau_0(\cdot + \pi)},$$

$$\tau_2 := \frac{a}{\sqrt{2}}, \tau_3 := e_1 \tau_2\ .$$

The reduction from three to two mother wavelets can still be carried out, but one then joins again the OEP case, now with the new fundamental function $1 - A$.

This section is organized as follows. First, in §3.1, we use the UEP approach just sketched to construct univariate tight framelets based on a new class of refinable functions, *pseudo-splines*, a class that ranges from B-splines at one end, to the refinable functions constructed in [D1] at the other end. This yields the pseudo-spline wavelets of Type I; a variant on the construction gives pseudo-spline wavelets of Type II. The main advantage of this construction is the ability to increase the approximation order (as compared to a spline system in [RS3]) of the system, while keeping the filters very short (although not as short as in the [RS3] construction). We also illustrate (Type III) the reduction to tight framelets that have only two mother wavelets.

In §3.2 we use the OEP approach sketched above to give a systematic construction of tight spline framelets, starting from B-splines of arbitrary order. Once again, each system is generated by two or three mother wavelets, and the wavelets, in general, are not symmetric. We obtain in this way, from any B-spline MRA, tight spline framelets of optimal approximation order. The filters, however, are longer than their pseudo-spline counterparts. The same construction can also yield tight spline framelets with maximal number of vanishing moments, by requiring then even longer filters.

In this era of `Matlab, Maple` and `Singular` (cf. [GPS]), one can also construct systems by ad-hoc methods, if the approximation order is not too large. In the Appendix, we present a variety of spline systems that were computed in this way. All the systems have the maximal number of vanishing moments (the approximation order of the system is, *a fortiori*, also maximal). Some of the systems are generated by two (not symmetric) mother wavelets, and others by three (symmetric) mother wavelets. In all examples the corresponding wavelet masks are shorter than the spline-masks in §3.2 (but still longer than the non-spline masks in §3.1).

All the above constructions have their bi-framelet counterparts, which can be a way to recover symmetry when an associated tight framelet uses non-symmetric wavelets. This is illustrated in §5; note, however, that at least one of the bi-framelet constructions in §5 cannot be regarded as a 'symmetrization' of a tight framelet construction.

3.1. Pseudo-spline tight framelets

Let $\ell < m$ be two non-negative integers. We denote

$$|\tau_0^{m,\ell}(\omega)|^2 := \cos^{2m}(\omega/2)\sum_{i=0}^{\ell}\binom{m+\ell}{i}\sin^{2i}(\omega/2)\cos^{2(\ell-i)}(\omega/2).$$

Since $|\tau_0^{m,\ell}|^2$ is non-negative, it is, by spectral factorization, the square of some trigonometric polynomial $\tau_0^{m,\ell}$. It is easy to prove that the corresponding refinable function $\phi_{m,\ell}$ lies in $L_2(\mathbb{R})$. Moreover, the shifts $E(\phi_{m,\ell})$ of $\phi_{m,\ell}$ form a Riesz basis for $V_0(\phi_{m,\ell})$. We refer to this refinable function as **a pseudo-spline of order m and type ℓ, or, in short, of type (m,ℓ)**. Fixing m, we note that a pseudo-spline of type 0 is an mth order B-spline, while the pseudo-spline of type $m-1$ coincides with the refinable functions of orthonormal shifts that were constructed in [D1]. $\tau_0^{m,\ell}$ is the mask of a filter with $m+\ell+1$ non-zero coefficients. The smoothness of $\phi_{m,\ell}$ increases with m and decreases with ℓ. For example, a straightforward computation (based on the transfer operator) shows that the $L_2(\mathbb{R})$-smoothness exponent of $\phi_{m,1}$ is

$$\alpha(m,1) := m - \log_2\sqrt{(m+2)}.$$

(I.e., $\phi_{m,1} \in W_2^{\alpha}(\mathbb{R})$ for every $\alpha < \alpha(m,1)$, but $\phi_{m,1} \notin W_2^{\alpha(m,1)}(\mathbb{R})$.) In the case $m = 4$ and $\ell = 1$ (which is of possible practical interest), we obtain that the smoothness parameter is $4 - \log_2\sqrt{6} \approx 2.71$, hence that $\phi_{4,1} \in C^2(\mathbb{R})$. We note that $\alpha(4,0) = 3.5$.

Next, we note that $|\tau_0^{m,\ell}|^2$ consists of the first $\ell+1$ terms in the binomial expansion of

$$1 = (\cos^2(\omega/2) + \sin^2(\omega/2))^{m+\ell}.$$

Thus, $|\tau_0^{m,\ell}(\omega)|^2 + |\tau_0^{m,\ell}(\omega+\pi)|^2 \le 1$ and therefore we can use the UEP. Also, $1 - |\tau_0^{m,\ell}|^2 = O(|\cdot|^{2\ell+2})$. This means that, in view of Theorems 2.8 and 2.11, all tight framelets that are extracted from the (m,ℓ)-pseudo-spline via the UEP will satisfy:

(a) The approximation order provided by the refinable function is m.
(b) The approximation order of the framelet system is $\min\{m, 2\ell+2\}$.
(c) The order of the vanishing moments is $\ell+1$.

For example, in the case $m = 4$ and $\ell = 1$, we obtain optimal approximation order 4, but we must have at least one wavelet in the system with only two vanishing moments.

We propose two simple UEP-based constructions of pseudo-spline tight framelets.

Type I Pseudo-spline tight framelets. This is a straightforward application of the principle above. Given $\tau_0 := \tau_0^{m,\ell}$, we define

$$\tau_1 := \tau_1^{m,\ell} := e_1\overline{\tau_0^{m,\ell}(\cdot+\pi)}.$$

where, as before, $e_1(\omega) = e^{i\omega}$. As in Mallat's construction, [Ma], $\tau_0\overline{\tau_0(\cdot+\pi)} + \tau_1\overline{\tau_1(\cdot+\pi)} = 0$. It also follows that

$$A := 1 - |\tau_0|^2 - |\tau_1|^2 = \sum_{i=\ell+1}^{m-1}\binom{m+\ell}{i}\cos^{2m+2\ell-2i}(\omega/2)\sin^{2i}(\omega/2).$$

Since A is a non-negative π-periodic trigonometric polynomial, we can find a π-periodic trigonometric polynomial a such that $A = |a|^2$. We then define $\tau_2 = \frac{a}{\sqrt{2}}$, and $\tau_3 := e_1\overline{\tau_2(\cdot+\pi)} = e_1\overline{\tau_2}$, to conclude that $\tau := (\tau_0, \ldots, \tau_3)$ satisfies the UEP. Hence, the resulting wavelet system is a tight frame. Note that each mask corresponds to an $(m+\ell+1)$-tap filter.

The case $m = 4$, $\ell = 1$ is depicted in Figure 5. In this case the filters are slightly shorter compared with the general case; one is 6-tap, and the others are 5-tap (this simplification happens because

$$A = 10\cos^6(\omega/2)\sin^4(\omega/2) + 10\cos^4(\omega/2)\sin^6(\omega/2) = 10\cos^4(\omega/2)\sin^4(\omega/2);$$

a similar reduction occurs in general provided that $l = m - 3$.) The approximation order of the system is 4 (optimal), one of the wavelets has 4 vanishing moments, while the two others have 2 vanishing moments. The L_2-smoothness parameter is 2.71.

Type II Pseudo-spline tight framelets. We proceed as in the Type I case to obtain τ_1 and A as before. We then split $A = A_1 + A_1(\cdot+\pi)$, with A_1 defined as the sum of the first $\frac{m-\ell-1}{2}$ terms in the definition of A. (We assume tacitly that $m+\ell$ is odd; the construction can be easily adapted to the even case, splitting the middle term evenly between A_1 and $A_1(\cdot+\pi)$.) Choosing τ_2 to be a square root of A_1, and $\tau_3 := e_1\overline{\tau_2(\cdot+\pi)}$, we obtain again a combined mask $\tau = (\tau_0, \ldots, \tau_3)$ that satisfies the UEP. Hence the resulting wavelet system is a tight frame. The wavelets for the case $m = 4$ and $\ell = 1$ are given in Figure 6.

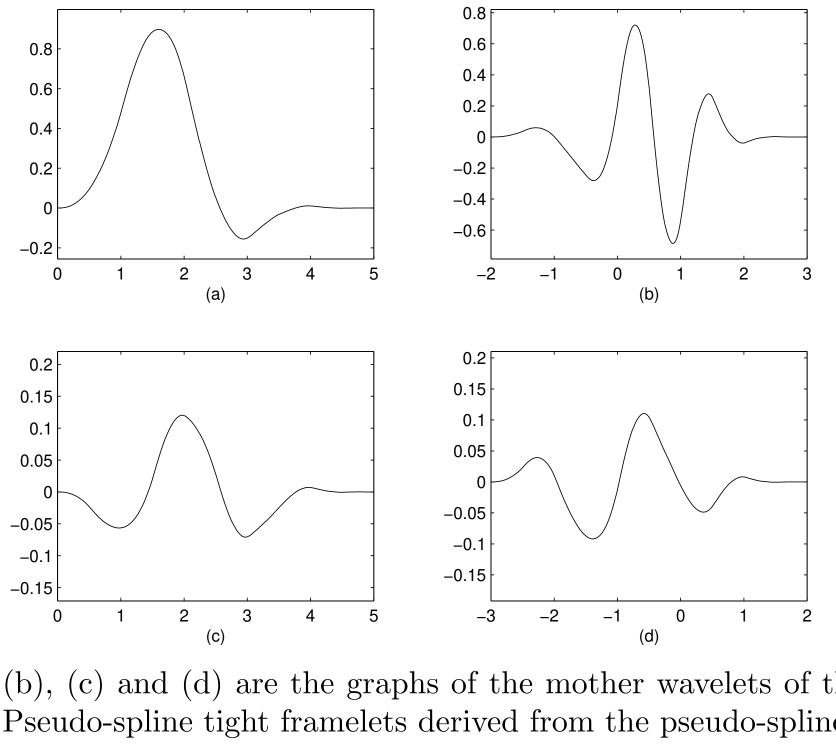

Figure 6. (b), (c) and (d) are the graphs of the mother wavelets of the Type II Pseudo-spline tight framelets derived from the pseudo-spline (4,1) (a).

Remarks.

1. The above constructions of pseudo-spline tight framelets, published here for the first time, have been in use for various applications since 1997. In particular, N. Stefansson used them, with excellent results, in signal compression experiments.

2. The papers [CH] by Chui and He and [P1,2] by Petukhov present general methods for solving the equations arising from the UEP method if $r = 2$, seeking to find two appropriate τ_1 and τ_2 where τ_0 is given such that $|\tau_0(\omega)|^2 + |\tau_0(\omega+\pi)|^2 \le 1$. (If τ_0 is symmetric, they also show how to handle the case when three symmetric τ_1, τ_2, τ_3 are desired.) Applying their general method to the pseudo-spline τ_0 would lead to $\tau_1', \tau_2', \tau_3'$ that are closely related to the τ_i given here. One could also use these methods to obtain two τ_1'', τ_2''. Either of these tight framelets will have the same approximation order as given here.

Type III Pseudo-spline tight framelets. Applying the "reduction" technique sketched above, one can define a tight pseudo-framelet with only two mother wavelets, corresponding to $\Theta := 1 - A$. Note that since $A = O(|\cdot|^{2\ell+2})$ around the origin, these type III framelets provide the same approximation orders (and have the same number of vanishing moments) as their type I and II counterparts. However, the second mother wavelet now has a very long filter: $3(m+\ell)+1$ in general, 14 in the more fortunate $(4,1)$-case.

3.2. A systematic construction of spline framelets of high approximation order

We shall here apply the OEP construction. Let ϕ be a B-spline of order m, then

$$\tau_0(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^m,$$

and

$$|\widehat{\phi}(\omega)|^2 = \frac{\sin^{2m}(\omega/2)}{(\omega/2)^{2m}}.$$

To construct tight framelets having approximation order 2ℓ, one needs to find $\Theta := \Theta_{m,\ell}$ of the form

$$\Theta(\omega) = 1 + \sum_{j=1}^{\ell-1} c_j \sin^{2j}(\omega/2) \tag{3.2}$$

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such that, at the origin,

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$$1 - \Theta|\widehat{\phi}|^2 = O(|\cdot|^{2\ell}). \tag{3.3}$$

In other words, $\Theta_{m,\ell}$ must approximate the function $1/|\widehat{\phi}|^2$ at the origin to order ℓ. Such a Θ can be determined uniquely as shown in the next lemma.

“stetml **Lemma 3.4.** *Let ϕ be the given B-spline of order m; let ℓ be an integer $\ell \le m$. Then there is a unique positive trigonometric polynomial of minimal degree*

$$\Theta(\omega) = 1 + \sum_{j=1}^{\ell-1} c_j \sin^{2j}(\omega/2)$$

satisfying, at the origin:

$$1 - \Theta|\widehat{\phi}|^2 = O(|\cdot|^{2\ell}).$$

Proof: The key in the proof is that the (uniquely determined) coefficients (c_j) in the definition of Θ are *non-negative.* From (3.3), we have

$$\Theta(\omega) = \left(\frac{\omega/2}{\sin\omega/2}\right)^{2m} [1 + O(|\omega|^{2\ell})].$$

Since

$$\arcsin\omega = \omega + \sum_{j=1}^{\infty} \frac{(2j-1)!!}{(2j)!!(2j+1)}\omega^{2j+1},$$

we have

$$\frac{\omega/2}{\sin(\omega/2)} = \frac{\arcsin(\sin(\omega/2))}{\sin(\omega/2)} = 1 + \sum_{j=1}^{\infty} \frac{(2j-1)!!}{(2j)!!(2j+1)} \sin^{2j}(\omega/2), \qquad \omega \to 0.$$

Therefore, Θ is the unique trigonometric polynomial of minimum degree in (3.2) such that

$$\Big(1 + \sum_{j=1}^{\infty} \frac{(2j-1)!!}{(2j)!!(2j+1)} y^j\Big)^{2m} = 1 + \sum_{j=1}^{\ell-1} c_j y^j + O(|y|^\ell), \qquad y \to 0.$$

It follows from the above equation that the $c_j, j \in \mathbb{N}$ are positive. In particular, $\Theta(\omega) > 0$ for all $\omega \in \mathbb{R}$. □

To apply the approach sketched earlier, we need to check that A is positive:

“keylem **Proposition 3.5.** *For integers ℓ, m with $\ell \le m$, let Θ be the trigonometric polynomial given in Lemma 3.4. Then the trigonometric polynomial*

$$A := \Theta - \Theta(2\cdot)(\cos^{2m}(\cdot/2) + \sin^{2m}(\cdot/2))$$

is non-negative. Furthermore, $A = O(|\cdot|^{2\ell})$ near the origin.

Proof: We start by writing A as a homogeneous polynomial of degree $n := m + 2\ell - 2$ in the arguments $x := \cos^2(\omega/2)$ and $y := \sin^2(\omega/2)$; this can be done by multiplying each term $\sin^{2j}(\omega/2)$ in Θ by $(\cos^2(\omega/2) + \sin^2(\omega/2))^{n-j} = (x+y)^{n-j}$. We thus replace y^j by

$$y^j(x+y)^{n-j} = \sum_{i=0}^{n} d_i(j) y^i x^{n-i}, \quad \text{with} \quad d_i(j) := \begin{cases} 0, & i < j, \\ \binom{n-j}{i-j}, & \text{otherwise.} \end{cases} \tag{3.6}$$

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In $\Theta(2\cdot)$, we replace each $\sin^{2j}(\omega) = (4xy)^j$ term by $2^{2j} y^j x^j (x+y)^{2\ell-2j-2}$.

Let $p(x, y)$ be the homogeneous polynomial in x, y (of degree n) that is obtained from this conversion of Θ. Then

$$p(x,y) = \sum_{i=0}^{n} d_i y^i x^{n-i}, \quad d_i := \sum_{j=0}^{\ell-1} c_j d_i(j).$$

We make the following straightforward observations:

(i) Since $d_i(j)$ and $c_j \geq 0$, for all i, j, it follows that $d_i \geq 0$, for all i.

(ii) Since, for each j, and for each $i < \frac{n}{2}$, $d_i(j) \leq d_{i+1}(j)$, we have

$$d_i \leq d_{i+1}, \quad i < \frac{n}{2}.$$

(iii) Since, for each j, and for each $i < \frac{n}{2}$, $d_i(j) \leq d_{n-i}(j)$, we have

$$d_i \leq d_{n-i}, \quad i < \frac{n}{2}.$$

(iv) One calculates that, for every j, $2d_{\ell-2}(j) \leq d_{\ell-1}(j)$. Therefore,

$$2d_{\ell-2} \leq d_{\ell-1}.$$

Let $q(x, y)$ be the polynomial (of degree $2\ell - 2$) that was obtained from $\Theta(2\cdot)$. Then $q(x, y) = q(y, x)$, and the representation of A is of the form

$$p(x,y) - q(x,y)(x^m + y^m) =: \sum_{i=0}^{n} b_i y^i x^{n-i}.$$

We prove the Proposition by showing that each b_i is non-negative. Since $q(x, y)(x^m + y^m)$ is symmetric, and in view of observation (iii) above, it suffices to show that $b_i \geq 0$ for $i \leq \frac{n}{2}$.

Now the condition $1 - \Theta|\widehat{\phi}|^2 = O(|\cdot|^{2\ell})$ is equivalent (cf. Theorem 2.8) to the condition

$$\Theta - \Theta(2\cdot)\cos^{2m}(\cdot/2) = O(|\cdot|^{2\ell}).$$

(This shows that $A = O(|\cdot|^{2\ell})$ near the origin.) Rewritten in terms of the polynomials p, q, this last condition says that $p(x, y) - q(x, y)x^m$ is divisible by y^ℓ. It follows that the terms in $q(x, y)$ in y^i, with $i < \ell$, must match up exactly with corresponding terms in $p(x, y)$. By the symmetry $q(x, y) = q(y, x)$, this determines all the coefficients in q; consequently,

$$q(x,y) = \sum_{i=0}^{\ell-1} d_i y^i x^{2\ell-2-i} + \sum_{i=0}^{\ell-2} d_i x^i y^{2\ell-2-i},$$

and $b_i = 0$, $i = 0, \ldots, \ell - 1$. Let $\ell \leq i \leq \frac{n}{2}$; then (with $d_k := 0$ for negative k), $b_i = d_i - (d_{2\ell-2-i} + d_{i-m})$. From observation (ii), $d_i \geq d_\ell$, while, since $2\ell-2-i, i-m \leq \ell-2$, the same observation yields that $d_{2\ell-2-i} + d_{i-m} \leq 2d_{\ell-2}$. Altogether, $b_i \geq d_\ell - 2d_{\ell-2} \geq 0$, by observation (iv). □

Proposition 3.5 and Lemma 3.4 show that we can use our general ansatz, and obtain a systematic construction of tight framelets (with two or three mother wavelets) with $\ell \leq m$ vanishing moments, for an arbitrary mth order B-spline.

Remark. The arguments given here for the construction of tight framelets can be expanded easily to "bi-framelets", where one needs to identify τ_i and τ_i^d, $i = 1, \ldots, r$, so that the resulting framelets are symmetric for both pseudo-spline and spline MRAs. Again, the general case requires an appropriate function Θ (which no longer needs to be positive); all the equations are the expected bi-orthogonal generalizations of our tight frame equations here (see §5). Because Θ is less constrained, the construction is much easier; in fact, it turns out [DH] that one can obtain dual framelets from *any* two refinable functions, i.e., for any pair of τ_0, τ_0^d.

(3.7) Example: spline framelets with approximation order 4**.** For the mth order B-spline with $m \geq 4$, take

$$\Theta(\omega) := 1 + \frac{m \sin^2(\omega/2)}{3}.$$

Then

$$\Theta(\omega)\frac{\sin^{2m}(\omega/2)}{(\omega/2)^{2m}} = 1 + O(|\omega|^4)$$

around the origin. We define

$$|\tau_1(\omega)|^2 := (1 + \frac{m \sin^2(\omega)}{3}) \sin^{2m}(\omega/2).$$

Then, in the notations of the lemma above,

$$A(\omega) = (x+y)^{m+2} + \frac{m}{3}y(x+y)^{m+1} - (x^2 + (2 + \frac{4m}{3})xy + y^2)(x^m + y^m).$$

This expression is indeed divisible by y^2, and is a non-negative linear combination of the various monomials involved.

For the benchmark case of $m = 4$ and $\ell = 1$, the type I construction yields three 7-tap filters, longer than the $(6, 5, 5)$-tap filters of the corresponding pseudo-spline construction. The approximation order is (the optimal) 4 in both cases. The two wavelets of type III now have filters of lengths 7 and 17. The case $m = 4, \ell = 4$ yields wavelets with four vanishing moments and with filters of lengths 11.

We have shown here how to construct tight spline framelets with 2 and 3 mother wavelets. A natural question is whether we can construct tight spline framelets with a single generator. A partial negative answer is given in the following result.

"notonewav **Theorem 3.8.** *All the constructions of strongly local MRA-based tight frames that are derived from a B-spline of order $m > 1$ must have at least two mother wavelets.*

Proof: The mask τ_0 of the mth order B-spline satisfies $|\tau_0(\omega)|^2 = \cos^{2m}(\omega/2)$. Suppose we used the OEP conditions to construct a strongly local tight frame based on a single wavelet mask τ_1; that is, τ_1 as well as the fundamental function Θ are trigonometric polynomials. Recall (see the proof of Proposition 1.11) that, equivalently, we could have applied the UEP with respect to the refinement mask whose square is

$$\frac{\Theta(2\cdot)|\tau_0|^2}{\Theta}.$$

But that implies that this latter refinement mask is CQF, i.e., $\frac{\Theta(2\cdot)|\tau_0|^2}{\Theta} + \frac{\Theta(2\cdot)|\tau_0|^2(\cdot+\pi)}{\Theta(\cdot+\pi)} = 1$, or, equivalently,

$$\Theta(2\cdot)(t + t(\cdot+\pi)) = \Theta\Theta(\cdot+\pi), \quad t := \Theta(\cdot+\pi)|\tau_0|^2.$$

Comparing the degrees of the two sides of the last equality, we conclude that, for some positive constant c,

“one
$$\Theta(2\cdot)c = \Theta\Theta(\cdot+\pi), \text{ and } t + t(\cdot+\pi) = c. \tag{3.9}$$

Because $|\tau_0|^2|\tau_0|^2(\cdot+\pi) = 4^{-m}|\tau_0|^2(2\cdot+\pi)$, we conclude from the first equality in (3.9), that

“two
$$t\,t(\cdot+\pi) = c4^{-m}t(2\cdot+\pi). \tag{3.10}$$

Suppose that $t(\omega) = \sum_{j=j_1}^{k_1} \alpha(j)e^{ij\omega}$. From (3.9) we conclude that $\alpha(0) = c/2$, and that $\alpha(2j) = 0$ for any $j \neq 0$. Thus, $k_1 \geq 0$. If $k_1 = 0$ then (by comparing the constant term on both sides of (3.10)) $(c/2)^2 = c4^{-m}c/2$, a contradiction.

Thus, $k_1 > 0$. Let k_2 be the degree of the second highest non-zero term of t. If $k_2 > 0$, we are led to a contradiction (since the coefficient of $e^{i(k_1+k_2)\omega}$ in the left-hand side of (3.10) is then non-zero, while the same coefficient in the right hand-side of (3.10) is zero). Thus, $k_2 = 0$. Similar arguments hold for the negative frequency contributions to t. We conclude, therefore, that t is a linear combination of (at most) three exponentials, hence can have at most a double zero at any given point. This implies that $m = 1$, since t has a zero of order $2m$ at π. □

Remarks.

1. The argument of this proof is instructive for non-spline MRA as well. If we have a strongly local MRA-based tight framelet with only one mother wavelet, then (3.9) still holds, ensuring that $|\tilde{\tau}_0|^2 = \Theta(2\cdot)|\tau_0|^2/\Theta$ is a trigonometric polynomial, which satisfies the CQF constraint $|\tilde{\tau}_0|^2 + |\tilde{\tau}_0|^2(\cdot+\pi) = 1$. In summary, all the strongly local tight framelet constructions in one variable that lead to a single mother wavelet can be equivalently done by a (strongly local) standard CQF construction.

2. Examples of exponential decay orthogonal spline wavelets constructed in [B] and [L] confirm that the assumption of the compactly supported mother wavelets is needed in the above Proposition.

4. The fast framelet transform

We assume in this section that the reader is familiar with the details of the *fast wavelet transform*. Our goal is to highlight the subtle difference between that widely used transform and its newer sibling, the fast framelet transform. Substantial frame software is currently under development and will be made available to the public as a part of the Software Distribution Center of the *Wavelet Center for Ideal Data Representation* (`www.waveletidr.org`).

Let $f \in L_2(\mathbb{R}^d)$; the function f is held fixed throughout the discussion. Assume that we are given information about f on some uniform grid, a grid which, for notational convenience, we assume to be the integer lattice $\mathbb{Z}^d$. The function f is thus assumed to be 'given to us' in terms of the discrete values

$$(F_{0,0}(k))_{k\in\mathbb{Z}^d}.$$

Concrete assumptions on the exact nature of $F_{0,0}$ are made in the sequel. As a general rule, $F_{0,0}(k)$ is a local average of the values of f around the point k.

Let $X(\Psi)$ be an MRA-based wavelet system associated with the combined mask $\tau = (\tau_0, \ldots, \tau_r)$. As before, the refinable function is denoted by ψ_0 as well as by ϕ. We denote by $x = (x_0, \ldots, x_r)$ the filters associated with $(\tau_0, \ldots, \tau_r)$.

The discussion of the fast framelet transform is made into three parts: (i) the decomposition algorithm, (ii) the reconstruction algorithm, and (iii) the interpretation of the wavelet coefficients that were obtained in (i).

The **analysis/decomposition** step of the fast framelet transform is identical to that of the fast wavelet transform, with the only change that we do not necessarily have $2^d - 1$ high pass filters. This step consists of the convolution of $(F_{0,j})$ $(j \le 0)$ with each of the filters x_i followed by the downsampling $_\downarrow$:

$$F_{i,j-1} \longleftarrow (x_i * F_{0,j})_\downarrow, \quad i = 0, \ldots, r.$$

The following simple observation is the basis for the *interpretation* of $F_{i,j}$-sequences. (No special assumptions on $X(\Psi)$ are required here; we also omit the straightforward proof.)

"fastdecom **Proposition 4.1.** *Assume that*

$$F_{0,0}(k) = \langle f, \psi_{0,0,k}\rangle, \quad k \in \mathbb{Z}^d.$$

Then

$$F_{i,j}(k) = \langle f, \psi_{i,j,k}\rangle, \quad i = 0, \ldots, r,\ j \le 0,\ k \in \mathbb{Z}^d.$$

□

Suppose now that the sequence $F_{0,0}$ does not satisfy the assumptions of this proposition. For example, suppose that $F_{0,0}$ comprises the coefficients that *synthesize* f, i.e., suppose that

$$\phi *' F_{0,0} := \sum_{k\in\mathbb{Z}^d} F_{0,0}(k)\,\phi(\cdot - k)$$

either coincides with f, or provides a good approximation to f. Concretely, let us assume that $\phi *' F_{0,0}$ is the *orthogonal projection* $\mathcal{P}f$ of f onto V_0. If the shifts of ϕ are orthonormal (an assumption that is natural in the construction of orthonormal $X(\psi)$), we still have $F_{0,0}(k) = \langle f, \psi_{0,0,k}\rangle =: \widetilde{F}_{0,0}(k)$. However, for other tight framelets, this is not the case: the analysis of §2 shows that if $\phi *' F_{0,0}$ is the orthogonal projection of f onto V_0, then the Fourier series of $F_{0,0}$ is the function

$$\frac{[\widehat{f}, \widehat{\phi}]}{[\widehat{\phi}, \widehat{\phi}]},$$

whereas the Fourier series for $\widetilde{F}_{0,0}$ is $[\widehat{f}, \widehat{\phi}]$. Thus, if we denote by a the Fourier coefficients of $[\widehat{\phi}, \widehat{\phi}]$, we have that

$$\widetilde{F}_{0,0} = F_{0,0} * a. \tag{4.2}$$

Since we do not assume the shifts of ϕ to be linearly independent, we might have many representations of the orthogonal projection $\mathcal{P}f$ in the form $\mathcal{P}f = \phi *' F_{0,0}$; we stress that (4.2) holds for *every* such $F_{0,0}$. We recall also that

$$a(k) = \langle \phi, \phi(\cdot - k)\rangle, \quad k \in \mathbb{Z}^d.$$

Thus, in case $F_{0,0}$ is comprised of the coefficients of the orthogonal projection as above, we can simply convolve it with a, obtain in this way the inner products $\widetilde{F}_{0,0}$ required in Proposition 4.1, and proceed to decompose $\widetilde{F}_{0,0}$. A similar analysis can be carried out if the data $F_{0,0}(k)$ correspond to averages of the type $F_{0,0}(k) = \langle f, g(\cdot - k)\rangle$, with respect to some "measurement function" g. One then computes $\widetilde{F}_{0,0}(k)$ as the inner products with $\phi(\cdot - k)$ of the function $\widetilde{f}$ in V_0 characterized by $\langle \widetilde{f}, g(\cdot - k)\rangle = F_{0,0}(k)$. The Fourier series c and $\widetilde{c}$ of $F_{0,0}$ and $\widetilde{F}_{0,0}$ are then related by $\widetilde{c}[\widehat{\phi}, \widehat{g}] = c[\widehat{\phi}, \widehat{\phi}]$.

Let us discuss now the reconstruction process. As in the fast wavelet transform, the reconstruction employs the filters

$$\overline{x_i}, \quad i = 0, \ldots, r,$$

whose Fourier series are $\overline{\tau_i}$, $i = 0, \ldots, r$. I.e., if x_i is real-valued,

$$\overline{x_i}(k) = x_i(-k).$$

If τ satisfies the assumptions of the UEP, then the reconstruction process is identical to that of the fast wavelet transform: each sequence $F_{i,j}$ is upsampled, and subsequently convolved with $\overline{x_i}$:

"recon
$$F_{i,j} \mapsto \overline{x_i} * (F_{i,j\uparrow}). \tag{4.3}$$

We then have the perfect reconstruction formula $F_{0,j+1} = \sum_{i=0}^{r} \overline{x_i} * (F_{i,j\uparrow})$, and hence the reconstruction step is as follows:

“recons
$$F_{0,j+1} \longleftarrow \sum_{i=0}^{r} \overline{x_i} * (F_{i,j\uparrow}), \quad j = j_0, \ldots, -1. \tag{4.4}$$

Note that the perfect reconstruction property is purely technical. It does not require the sequences $(F_{i,j})_i$ to carry any useful information; it only requires that τ satisfies the conditions of the UEP (Proposition 1.9), and that $(F_{i,j})_i$ are obtained from $F_{0,j+1}$ via the frame decomposition algorithm.

If the system $X(\Psi)$ is constructed via the Oblique Extension Principle, then we need to modify slightly the reconstruction process.

“fastoep **Proposition 4.5.** *Let $X(\Psi)$ be a tight framelet that is constructed via the OEP, based on a combined mask τ (where x_i is the filter associated to each mask τ_i) and a fundamental function Θ (whose Fourier coefficients form a sequence b). Let $F_{i,j}$, $i = 0, \ldots, r$, $j = 0, -1, \ldots, j_0$, be obtained from $F_{0,0}$ via the decomposition algorithm. Then, for each $j < 0$,*

$$b * F_{0,j+1} = \overline{x_0} * ((b * F_{0,j})_\uparrow) + \sum_{i=1}^{r} \overline{x_i} * (F_{i,j\uparrow}).$$

The proposition, thus, entails that the reconstruction can be done as follows:

(i) $F_{0,j_0} \longleftarrow b * F_{0,j_0}$.

(ii) Continue as in (4.4).

(iii) Keep in mind that the reconstructed $F^R_{0,j}$ differs from the decomposed $F^D_{0,j}$ (i.e., we do not satisfy the perfect reconstruction formula). Precisely, $F^R_{0,j} = F^D_{0,j} * b$. Since convolution with b amounts to local averaging, the reconstructed $F^R_{0,0}$ is a somewhat blurred version of the original $F_{0,0}$.

Note that, again, the reconstruction algorithm does not require us to have any special interpretation for the sequences $F_{i,j}$. We only need to know that τ, Θ satisfy the assumption of Proposition 1.11, and that $F_{i,j}$ were obtained by the decomposition algorithm.

We summarize the discussion above in

The fast framelet transform. Let $X(\Psi)$ be a tight framelet constructed by the OEP, and associated with the filters $(x_i)_i$, the refinable function ϕ, and the fundamental MRA function Θ. Let $a(k) := \langle \phi, \phi(\cdot - k) \rangle$, $k \in \mathbb{Z}^d$, and let b be the Fourier coefficients of Θ. Then:

input $F_{0,0} : \mathbb{Z}^d \to \mathbb{C}$.

(1) Decomposition:

`if` $f = \phi *' F_{0,0}$:

$F_{0,0} \longleftarrow a * F_{0,0}$

`end`

`% at this point we assume` $F_{0,0}(k) = \langle f, \psi_{0,0,k} \rangle$.

`for` $j = -1, -2, \ldots, j_0$

```
 for i = 1,...,r
  F_{i,j} = (x_i * F_{i,j+1})↓
 end
end
%at this point we obtain that F_{i,j}(k) = ⟨f, ψ_{i,j,k}⟩
```

(2) Reconstruction:

```
F_{0,j_0} ⟵ b * F_{0,j_0}
for j = j_0,...,-1
 F_{0,j+1} = Σ_{i=0}^r conj(x_i) * (F_{i,j}↑)
end
if Θ ≠ 1, deconvolve b from F_{0,0}, end
if F_{0,0} was convolved with a during the decomposition
 deconvolve a from F_{0,0}, end
```

We remark that the sequence $\delta - a * b$ has at least as many vanishing moments as the mother wavelets Ψ have (cf. Theorem 2.8). Thus, $a * b$ is a low-pass filter and its deconvolution has a sharpening effect on $F_{0,0}$. If f is known to be a smooth function, the deconvolution of $a * b$ may be then unnecessary because $a * b * F_{0,0}$ is already a high order approximation of $F_{0,0}$.

5. Bi-framelets

In this section, we discuss general MRA-based wavelet frames. Two major generalizations are: (i) we reconstruct bi-framelets, and not only tight framelets, and (ii) we allow the dilation operator to be based on any expansive matrix s with integer entries: given a $d \times d$ matrix s with integer entries whose entire spectrum lies outside the closed unit disk, we redefine the dilation operator $\mathcal{D}$ to be

$$(\mathcal{D}f)(y) = |\det s|^{1/2} f(sy).$$

Correspondingly, the wavelet $\psi_{i,j,k}$ is now defined by

$$\psi_{i,j,k} = \mathcal{D}^j(\psi_i(\cdot - k)) = |\det s|^{j/2} \psi_i(s^j \cdot -k).$$

The notion of a wavelet bi-frame is as follows: let $\Psi = (\psi_1, \ldots, \psi_r)$ and $\Psi^d = (\psi_1^d, \ldots, \psi_r^d)$ be two sequences of mother wavelets. We say that the pair of systems

$$(X(\Psi), X(\Psi^d))$$

is a **bi-frame** if each of the two systems is Bessel, and we have the perfect reconstruction formula

$$f = \sum_{i,j,k} \langle f, \psi_{i,j,k}^d \rangle \, \psi_{i,j,k}. \quad \text{for all} \quad f \in L_2(\mathbb{R}^d).$$

The definition implies that each of the two systems, is, in particular, a frame. Also, the roles of $X(\Psi)$ and $X(\Psi^d)$ in the above definition are interchangeable.

We discuss here MRA-based constructions of such bi-frames (i.e., each of the two systems is a framelet) and will refer to such constructions as **bi-framelets**. Note that the refinement mask τ_0 of a given refinable function now satisfies

$$\widehat{\phi}(s^*\cdot) = \tau_0\widehat{\phi},$$

and, similarly, the mother wavelets are determined from their masks by the relation

$$\widehat{\psi}_i(s^*\cdot) = \tau_i\widehat{\phi}.$$

Throughout the present section, we impose a smoothness condition on the refinable functions ϕ, ϕ^d, viz. condition (4.6) of [RS3]. This condition is so mild (it is being satisfied, e.g., by the support function of the unit cube), that we forgo mentioning it explicitly in the stated results.

5.1. Mixed extension principles

Suppose that $X(\Psi)$ and $X(\Psi^d)$ are two MRA-based wavelet systems that correspond to the combined (bounded) mask vectors $\tau = (\tau_0, \ldots, \tau_r)$, and $\tau^d = (\tau_0^d, \ldots, \tau_r^d)$. Let ϕ and ϕ^d be the corresponding refinable functions and let $(V_j)_j$ and $(V_j^d)_j$, respectively, be the corresponding MRAs.

Associated with the combined masks τ and τ^d is the following **mixed fundamental function of the parent vectors**:

$$\Theta_M(\omega) := \sum_{j=0}^{\infty} \tau_+(s^{*j}\omega)\overline{\tau_+^d(s^{*j}\omega)} \prod_{m=0}^{j-1} \tau_0(s^{*m}\omega)\overline{\tau_0^d(s^{*m}\omega)}.$$

(Here, $\tau_+\overline{\tau_+^d} := \sum_{i=1}^r \tau_i\overline{\tau_i^d}$.) The function Θ_M is well-defined (a.e.), whenever the two systems $X(\Psi)$ and $X(\Psi^d)$ are both Bessel (indeed, the Bessel property implies, [RS3], that the fundamental functions Θ and Θ^d of each system are finite a.e., while by Cauchy-Schwartz, $\Theta_M^2 \leq \Theta\Theta^d$. Thus the sum that defines Θ_M converges absolutely to an a.e. finite limit). Note that the definition of Θ_M implies the following analogue of (1.6):

"btetrelation (5.1)
$$\Theta_M(\omega) = \tau_+(\omega)\overline{\tau_+^d(\omega)} + \tau_0(\omega)\overline{\tau_0^d(\omega)}\Theta_M(s^*\omega).$$

Invoking Corollary 2 of [RS4], we may follow the argument in the proof of Theorem 6.5 of [RS3] to obtain the following result.

"bthmtet **Proposition 5.2.** *Assume that the combined MRA masks $\tau = (\tau_0, \ldots, \tau_r)$ and $\tau^d = (\tau_0^d, \ldots, \tau_r^d)$ are bounded. Assume also that $\widehat{\phi}$ and $\widehat{\phi}^d$ are continuous at the origin and $\widehat{\phi}(0) = \widehat{\phi}^d(0) = 1$, and that the corresponding wavelet systems $X(\Psi)$ and $X(\Psi^d)$ are Bessel systems. Then the following conditions are equivalent:*
(a) The system pair $(X(\Psi), X(\Psi^d))$ is a bi-framelet.

(b) For $\omega \in \sigma(V_0) \cap \sigma(V_0^d)$, the mixed fundamental MRA function Θ_M satisfies:
(b1) $\lim_{j\to-\infty} \Theta_M(s^{*j}\omega) = 1$.

(b2) If $\nu \in \mathbb{Z}^d/(s^*\mathbb{Z}^d)$, if $\omega+\nu \in \sigma(V_0)\cap\sigma(V_0^d)$, then

$$\langle \tau(\omega), \tau^d(\omega+\nu)\rangle_{\Theta_M(s^*\omega)} = 0.$$

With this, we have the following result, which extends the Mixed Unitary Extension Principle of [RS4].

“bthmoep **Corollary 5.3: The Mixed Oblique Extension Principle (MOEP).** *Let τ and τ^d be the combined masks of the wavelet systems $X(\Psi)$ and $X(\Psi^d)$, respectively. Assume that Assumptions* (1.3) *are satisfied by each system and that both $X(\Psi)$ and $X(\Psi^d)$ are Bessel systems. Suppose that we were able to find a 2π-periodic function Θ that satisfies the following:*
(i) *Θ is essentially bounded, continuous at the origin, and $\Theta(0) = 1$.*
(ii) *If $\omega \in \sigma(V_0)\cap\sigma(V_0^d)$, and $\nu \in \mathbb{Z}^d/(s^*\mathbb{Z}^d)$ such that $\omega+\nu \in \sigma(V_0)\cap\sigma(V_0^d)$, then*

“genext
$$\langle \tau(\omega), \tau^d(\omega+\nu)\rangle_{\Theta(s^*\omega)} = \begin{cases} \Theta(\omega), & \text{if } \nu = 0,\\ 0, & \text{otherwise.}\end{cases} \tag{5.4}$$

Then $(X(\Psi), X(\Psi^d))$ is a bi-framelet.

Proof: By Proposition 5.2, one needs to show only that Θ coincides with the mixed fundamental function Θ_M on $\sigma(V_0)\cap\sigma(V_0^d)$. Let $\omega \in \mathbb{R}^d$. We consider two different cases:

(a) For some j, $s^{*j}\omega \notin \sigma(V_0)\cap\sigma(V_0^d)$. In this case, we choose $j \ge 0$ to be minimal with respect to the above property, and iterate j times with the case $\nu = 0$ in (5.4) to obtain

$$\Theta(\omega) = \Theta(s^{*j}\omega)\prod_{m=0}^{j-1}\tau_0(s^{*m}\omega)\overline{\tau_0^d(s^{*m}\omega)} + \sum_{k=0}^{j-1}\tau_+(s^{*k}\omega)\overline{\tau_+^d(s^{*k}\omega)}\prod_{m=0}^{k-1}\tau_0(s^{*m}\omega)\overline{\tau_0^d(s^{*m}\omega)}.$$

Since $s^{*j}\omega \notin \sigma(V_0)\cap\sigma(V_0^d)$, we must have that $\tau_0(s^{*(j-1)}\omega)\overline{\tau_0^d(s^{*(j-1)}\omega)} = 0$. Now, we can repeat the same argument with Θ replaced by Θ_M (since Θ_M always satisfies (5.1) which is identical to the case $\nu = 0$ of (5.4)). Thus, $\Theta(\omega) = \Theta_M(\omega)$, since each coincides with

$$\sum_{k=0}^{j-1}\tau_+(s^{*k}\omega)\overline{\tau_+^d(s^{*k}\omega)}\prod_{m=0}^{k-1}\tau_0(s^{*m}\omega)\overline{\tau_0^d(s^{*m}\omega)}.$$

(b) In the other case, we can also iterate (5.4) j times, where j now is an arbitrary integer, and obtain the same relation as before. This time, the second term converges absolutely as $j \to \infty$, thanks to (iii), to the mixed fundamental function Θ_M (see the discussion above (5.1)). It remains to show that the first term converges to 0. For this, for a given $\omega \in \sigma(V_0)\cap\sigma(V_0^d)$, one first finds ω_1 and ω_2 in $\omega + 2\pi\mathbb{Z}^d$, such that $\widehat{\phi}(\omega_1)\overline{\widehat{\phi}^d(\omega_2)} \neq 0$. Then,

$$\Theta(s^{*j}\omega)\prod_{m=0}^{j-1}\tau_0(s^{*m}\omega)\overline{\tau_0^d(s^{*m}\omega)} = \frac{\Theta(s^{*j}\omega)\widehat{\phi}(s^{*j}\omega_1)\overline{\widehat{\phi}^d(s^{*j}\omega_2)}}{\widehat{\phi}(\omega_1)\overline{\widehat{\phi}^d(\omega_2)}}.$$

This completes the proof, since the right hand side converges to 0, for a.e. $\omega \in \sigma(V_0) \cap \sigma(V_0^d)$ (due to the facts that Θ is bounded and ϕ and ϕ^d are in $L_2(\mathbb{R}^d)$). $\square$

5.2. Approximation orders

With $(X(\Psi), X(\Psi^d))$ a given pair of bi-framelets, we define the corresponding *truncated representation* Q_n by

$$Q_n : f \mapsto \sum_{\psi\in\Psi, k\in\mathbb{Z}^d, j<n} \langle f, \psi^d_{j,k}\rangle\, \psi_{j,k}.$$

We note that the roles of Ψ and Ψ^d are not interchangeable in this definition, since the interchange of the Ψ and Ψ^d may lead to a different approximation order. We refer to the system $X(\Psi^d)$ as the dual system. An argument similar to the one used in the proof of Lemma 2.4 leads to the following result:

“blemone **Lemma 5.5.** *Let $(X(\Psi), X(\Psi^d))$ be a bi-framelet system. Let ϕ, ϕ^d be the two underlying refinable functions. Then*

$$\widehat{Q_n f} = \Big([\widehat{f}(s^{*n}\cdot), \widehat{\phi}^d]\, \widehat{\phi}\, \Theta_M\Big)(s^{*-n}\cdot), \quad f \in L_2(\mathbb{R}^d).$$

In particular, $\widehat{Q_0 f} = [\widehat{f}, \widehat{\phi}^d]\widehat{\phi}\,\Theta_M$ for every $f \in L_2(\mathbb{R}^d)$.

Assume further that the dilation matrix s is scalar, $s = \lambda I$, for some integer $\lambda > 1$. We say that the bi-framelet systems $X(\Psi)$ and $X(\Psi^d)$ provide **approximation order** m_1 if, for every f in the Sobolev space $W_2^{m_1}(\mathbb{R}^d)$,

$$\|f - Q_n f\|_{L_2(\mathbb{R}^d)} = O(\lambda^{-nm_1}).$$

The following result can be proven similarly to Theorem 2.8. In fact, it extends to the more general isotropic dilation case.

“bcharacao **Theorem 5.6.** *Let $(X(\Psi), X(\Psi^d))$ be a bi-framelet system. Let ϕ, ϕ^d be the two underlying refinable functions. Assume that ϕ provides approximation order m. Then the approximation order provided by the truncated representation Q_n coincides with each of the following (equal) numbers.*

(i) $\min\{m, m_1\}$, *with m_1 the order of the zero of $1 - \Theta_M\, [\widehat{\phi}, \widehat{\phi}^d]$ at the origin.*
(ii) $\min\{m, m_2\}$, *with m_2 the order of the zero of $\Theta_M - \Theta_M(s^*\cdot)\tau_0\overline{\tau_0^d}$ at the origin.*
(iii) $\min\{m, m_3\}$, *with m_3 the order of the zero of $1 - \Theta_M\, \widehat{\phi}\overline{\widehat{\phi}^d}$ at the origin.*

Next, we discuss the related notion of vanishing moments. We say that the bi-framelet pair has vanishing moments of order m_4 if, for $i = 1, \ldots, r$, each $\widehat{\psi}_i\overline{\widehat{\psi}_i^d}$ has a zero of order $2m_4$ at the origin. If the bi-framelet is constructed via the MOEP and has moments of order m_4, then

$$\Theta_M - \Theta_M(s^*\cdot)\tau_0\overline{\tau_0^d} = \tau_+\overline{\tau_+^d} = O(|\cdot|^{2m_4}),$$

near the origin. Thus:

"bmomao **Proposition 5.7.** *Let $(X(\Psi), X(\Psi^d))$ be a bi-framelet system. Assume that the bi-framelet has vanishing moments of order m_4, that the system $X(\Psi^d)$ has m_0 vanishing moments, and that the refinable function ϕ provides approximation order m. Then:*

(a) ϕ satisfies the SF conditions of order m_0, i.e. $\widehat{\phi}$ vanishes at each $\omega \in 2\pi\,\mathbb{Z}^d\backslash 0$ to order m_0.

(b) The approximation order m' of the (Q_n) satisfies $\min(m, 2m_4) \le m' \le m$; in particular, if $2m_4 \ge m$, then $m' = m$.

5.3. Constructions

The construction of a bi-framelet is, in fact, simpler than its tight framelet counterpart. Since there is no need to take the square root of Θ_M in MOEP (instead, one needs only to factor it), it is no longer necessary to require that Θ_M be nonnegative. This gives us more choices for Θ_M and more alternatives in the construction. Indeed, in the current section, (very) short symmetric spline bi-framelets (with only 2 generators!) of desirable vanishing moments are constructed.

On the other hand, by modifying the tight framelet constructions, one can get bi-framelet constructions that yield symmetric mother wavelets. If the refinable function itself is symmetric (for example, if ϕ is a B-spline), we may not change the MRA (and hence we will have then that $\phi = \phi^d$). Only the wavelet masks will be modified then. To capture symmetry, the key is to adhere to *real* (up to a linear phase) factorizations of the underlying trigonometric polynomials. If the refinable function ϕ is not symmetric (which is the case of all pseudo-splines of positive type), we will alter the underlying MRA first, i.e., we will choose a real factorization of $|\widehat{\phi}|^2$ into $\widehat{\phi}_1\widehat{\phi}_1^d$.

Here, we give some examples of such constructions. Using the MOEP, one can design many other examples, suited to particular applications.

Pseudo-spline bi-framelets With $t := |\tau_0^{m,\ell}|^2$ and A as in §3.1, we choose any real factorizations $t = \tau_0\tau_0^d$ and $A = 2\tau_2\tau_2^d$. We define

$$\tau_{j+1} := e_1\tau_j^d(\cdot + \pi), \quad \tau_{j+1}^d := e_1\tau_j(\cdot + \pi), \quad j = 0, 2.$$

Assuming that ϕ, ϕ^d lie in $L_2(\mathbb{R})$, and that each of the above wavelet masks has at least one vanishing moment, we obtain in this way a bi-framelet. We can choose, e.g., for an even m, $\tau_0(\omega) := \cos^m \frac{\omega}{2}$, and

$$\tau_0^d(\omega) := \cos^m(\omega/2)\sum_{i=0}^{\ell}\binom{m+\ell}{i}\sin^{2i}(\omega/2)\cos^{2(\ell-i)}(\omega/2).$$

(Warning: m, l need to be such that ϕ^d lies in $L_2(\mathbb{R})$! This arises also in the construction of biorthogonal wavelet bases, see e.g. [CDF].) As to τ_2 and τ_2^d, one can choose any (real) factorization of $1 - \tau_0\overline{\tau_0^d} - \tau_0(\cdot + \pi)\overline{\tau_0^d(\cdot + \pi)}$ with $\tau_2(0) = \tau_2^d(0) = 0$.

Example: bi-framelets of type $(4, 1)$**.** For the type $(4, 1)$ we have that

$$t(\omega) = \cos^8(\omega/2)(1 + 4\sin^2(\omega/2)).$$

$\tau_0^d =$	$a/16*$	[0	0	1	4	6	4	1	0	0]
$\tau_1^d =$	$a/16*$	[1	−1	−5	10	−5	−1	1	0	0]
$\tau_2^d =$	$s/8*$	[0	0	−1	0	2	0	−1	0	0]
$\tau_3^d =$	$s/8*$	[0	0	1	−2	0	2	−1	0	0]
$\tau_0 =$	$a/16*$	[0	−1	−1	5	10	5	−1	−1	0]
$\tau_1 =$	$a/16*$	[0	1	−4	6	−4	1	0	0	0]
$\tau_2 =$	$s/32*$	[0	−1	−2	1	4	1	−2	−1	0]
$\tau_3 =$	$s/32*$	[0	1	0	−3	0	3	0	−1	0]

Table 2. The six masks of the second pseudo-spline bi-framelets of type $(4,1)$. Here, $a = \sqrt{2}$ and $s = \sqrt{5}$. Based on signal compression experiments that were done at UW, we recommend to use τ^d for decomposition and τ for reconstruction.

We split t to obtain

$$\tau_0(\omega) = \cos^4(\omega/2), \quad \tau_0^d(\omega) = \cos^4(\omega/2)(1 + 4\sin^2(\omega/2)).$$

One checks then that $\phi^d \in L_2(\mathrm{IR})$ (in fact, $\phi^d \in C^1(\mathrm{IR})$). Also, in this case $A(\omega) = \frac{5}{8}\sin^4\omega$, hence we can choose

$$\tau_2(\omega) = \tau_2^d(\omega) = \frac{\sqrt{5}}{4}\sin^2(\omega).$$

Note that all the filters obtained, with the exception of τ_0^d, are 5-tap. The system provides approximation order 4, and has 2 vanishing moments.

Of course, the above factorization is one of many. The masks of another bi-framelet of type (4,1) are listed in Table 2 (courtesy of Narfi Stefansson, UW-Madison).

Spline bi-framelets. Let $\phi = \phi^d$ be a B-spline of order m, then

$$\tau_0(\omega) = \tau_0^d(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^m.$$

For a given ℓ, let Θ and A be the trigonometric polynomials given in Lemma 3.4 and Proposition 3.5 respectively in §3.2. We can choose now any real factorization to $\Theta(2\cdot) = tt^d$ and $A = 2aa^d$. Define

$$\tau_1 := e_1 t\overline{\tau_0(\cdot + \pi)}, \quad \tau_1^d := e_1\, t^d\overline{\tau_0(\cdot + \pi)},$$

$$\tau_2 = a, \quad \tau_2^d = a^d,$$

and

$$\tau_3 = e_1 a, \quad \tau_3^d = e_1 a^d.$$

Then the systems corresponding to $\tau := (\tau_0, \ldots, \tau_3)$ and $\tau^d := (\tau_0^d, \ldots, \tau_3^d)$ form bi-framelets, provided that $a(0) = a^d(0) = 0$ (that latter assumption is needed in order to satisfy condition (iii) of Corollary 5.3).

Example: Spline bi-framelets generated by two (short) mother wavelets. An interesting case of the above general approach goes as follows. Let $\tau_0 = \tau_0^d$ be the mask of the m order B-spline ϕ. We choose the trigonometric polynomial Θ such that (say, for an even m) $1 - \Theta|\widehat{\phi}|^2 = O(|\cdot|^{2\ell})$, $\ell > m/2$ (cf. §3.2). We define

$$\tau_1(\omega) = e^{i\omega} \sin^m(\omega/2), \quad \tau_1^d(\omega) = e^{i\omega}\Theta(2\omega)\sin^m(\omega/2).$$

Since $A = O(|\cdot|^{2\ell})$ near the origin, the corresponding trigonometric polynomial A must be divisible by $\sin^{2\ell}(\omega/2)$. Since $2\ell > m$, by the assumption, we may split A into $A(\omega) = 2aa^d$, with $a(\omega) = \sin^m(\omega/2)$. Continuing as in the general construction detailed above, we obtain

$$\tau_2(\omega) = \sin^m(\omega/2), \quad \tau_3(\omega) = e^{i\omega}\sin^m(\omega/2).$$

The dual system is then

$$\tau_2^d(\omega) = a^d(\omega), \quad \tau_3^d(\omega) = e^{i\omega}a^d(\omega).$$

Because these τ_j, τ_j^d, $j = 0, 1, 2, 3$ satisfies (5.4), and $\tau_1 = \tau_3$, we can also define a system with 2 wavelets instead of 3 by putting:

$$\widetilde{\tau}_0 = \tau_0, \quad \widetilde{\tau}_1 = \tau_1 \quad \widetilde{\tau}_2 = \tau_2.$$

The dual system is then

$$\widetilde{\tau}_0^d(\omega) = \tau_0, \quad \widetilde{\tau}_1^d(\omega) = \tau_1^d + \tau_3^d, \quad \widetilde{\tau}_2^d(\omega) = \tau_2.$$

These $\widetilde{\tau}_j$, $\widetilde{\tau}_j^d$, $j = 0, 1, 2$ also satisfy (5.4). Note that $\psi_2 = \psi_1(\cdot - 1/2)$.

For example, if we choose $m = l = 4$, then all the wavelets and the dual wavelets have four vanishing moments (and, of course, they are all symmetric). The filters for the system $X(\Psi)$ are then all of length 5. The dual system $X(\Psi^d)$ has still a low pass filter of length 5, while the high pass filters are 17-tap and 15-tap.

General constructions of bi-framelets with two or three (symmetric) wavelets. Let ϕ and ϕ^d be two univariate symmetric refinable functions with (bounded) masks τ_0 and τ_0^d, respectively. Let Θ be a bounded real-valued 2π-periodic function, $\Theta(0) = 1$. Assume that

$$A := \Theta - \Theta(2\cdot)(\tau_0\overline{\tau_0^d} + \tau_0(\cdot + \pi)\overline{\tau_0^d(\cdot + \pi)})$$

is real and has (at least) a double zero at the origin. Let tt^d be any real factorization of $\Theta(2\cdot)$, and let $2aa^d$ be any real factorization of A in a way that $a(0) = a^d(0) = 0$. Note that if A and Θ are trigonometric polynomials, we can choose all the factors to be trigonometric polynomials, too. We can then define the wavelet masks exactly as in the spline bi-framelet discussion (since we do not need to require A to be positive any more). We obtain in this way a bi-framelet system, provided that $X(\Psi)$ and $X(\Psi^d)$ are Bessel. There are three (symmetric) mother wavelets in each system.

We can modify the above construction and obtain systems generated by *two* mother wavelets, following the general recipe of §3.2:

$$\tau_1 = e_1 t\overline{\tau_0^d(\cdot+\pi)}, \quad \tau_2 = \tau_0 a(2\cdot),$$

while

$$\tau_1^d = e_1 t^d\overline{\tau_0(\cdot+\pi)}, \quad \tau_2^d = \tau_0^d a^d(2\cdot).$$

We then obtain two symmetric generators for each system.

Finally, if ϕ or ϕ^d is not symmetric, the above constructions still work, but the resulting mother wavelets may not be symmetric (and, of course, we need not require that the relevant factorizations be *real*).

In [DH] it is shown that one can, in fact, construct bi-framelets from any pair of refinable functions ϕ, ϕ^d (with compact support).

6. An especially attractive construction

As we said a few times before, the choice of the "right" framelet system should really depend on the application. However, we can still point at a few constructions that stand out, even in the packed group of "useful framelets". We present in this section one such example. The highlight of this construction is that we obtain maximal approximation order, maximal smoothness, maximal vanishing moments and relatively short filters in one example. Importantly, the example belongs to one of our systematic methods, which means that similar constructs, for other approximation orders, are possible.

In the example here, we choose the construction of a spline bi-framelet with two short filters from the previous section. We choose the MRA which is generated by the cubic B-spline ϕ, and correspondingly, we choose Θ to be

$$\Theta(\omega) = 1 + \frac{4}{3}\sin^2(\omega/2) + \frac{62}{45}\sin^4(\omega/2).$$

According to the theory in this paper, the total number (of the decomposition and the reconstruction masks) of vanishing moments of any bi-framelet system that is based in these ϕ and Θ is 6. The general approach for this type of construction entails that we put a maximum number of vanishing moments, i.e., 4, into the decomposition filters, hence only 2 vanishing moments into the reconstruction masks. Thus, we enjoy an optimal approximation order of the framelet system (4), an optimal number of vanishing moments in the decomposition masks (which is where we really need those moments), and relatively very short high-pass filters: $(5, 5)$ in the decomposition, and $(13, 11)$ in the reconstruction (a total of 34 non-zero coefficients. In comparison, the cubic spline tight framelet of Example A.2, which also has 4 vanishing moments, and which is an ad-hoc construction, involves a total of 40 non-zero coefficients. And, the bi-framelet here is *not* an ad-hoc construction!).

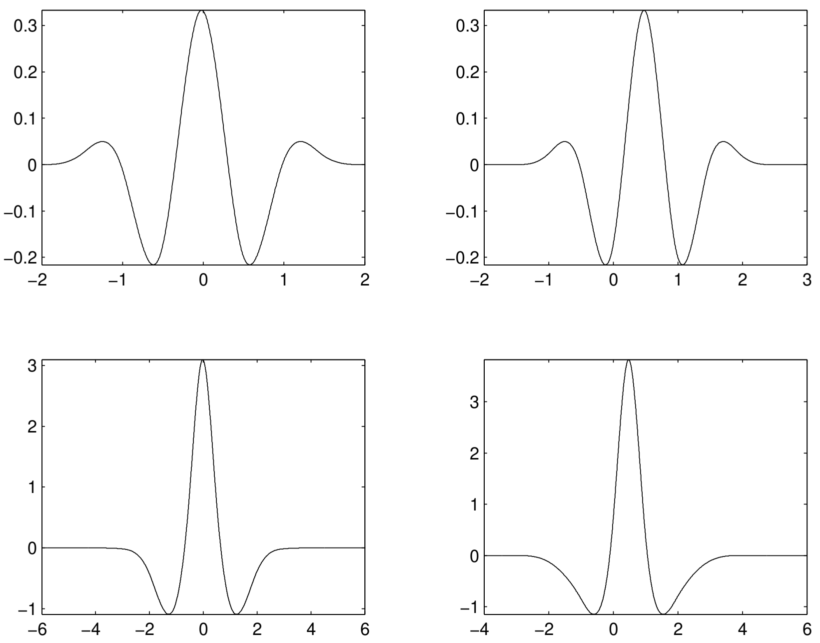

Figure 7. The graphs of the two decomposition wavelets of the example in §6 are depicted in the first row. Both are obtained by applying a 4-order difference to the cubic B-spline. The two reconstruction wavelets are depicted in the second row.

	τ_0	τ_1	τ_2	τ_0^d	τ_1^d	τ_2^d
6		-0.0053819444	-0.0430555556			
5		-0.0215277778	-0.1722222222			
4		-0.1621527778	-0.3930555556			
3		-0.5409722222	-0.7111111111			0.0625
2	0.0625	-0.5987847222	-0.1527777778	0.0625	0.0625	-0.2500
1	0.2500	0.2236111111	2.9444444444	0.2500	-0.2500	0.3750
0	0.3750	2.2104166667	-0.1527777778	0.3750	0.3750	-0.2500
-1	0.2500	0.2236111111	-0.7111111111	0.2500	-0.2500	0.0625
-2	0.0625	-0.5987847222	-0.3930555556	0.0625	0.0625	
-3		-0.5409722222	-0.1722222222			
-4		-0.1621527778	-0.0430555556			
-5		-0.0215277778				
-6		-0.0053819444				

Table 3. The coefficients of the six filters (two low-pass ones and four high-pass ones) of the example in §6. The three reconstruction filters are listed first, followed by the three decomposition filters. Note that all the coefficients are actually rational.

Figure 7 depicts the graphs of the four wavelets constructed in this way, while Table 3 records the non-zero coefficients of the underlying six filters (courtesy of Steven Parker).

APPENDIX: Ad-hoc constructions of tight spline framelets with shorter filters. We construct here some more tight wavelet frames by OEP from several low-order B-spline functions. The ad-hoc constructions given here typically yield tight framelets whose mother wavelets have shorter support than the results of the general construction in §3.2. The computation in the following examples was done with the help of two computer algebra systems, `Maple` and `Singular` [GPS], and the graphs are produced by `Matlab`.

Example A.1. Let $\tau_0(\omega) = (1 + e^{-i\omega})^3/8$; then the refinable function ϕ is the quadratic B-spline, whose MRA provides approximation order 3. We choose $\Theta(\omega) = (3 - \cos(\omega))/2$, and find that $1 - \Theta|\widehat{\phi}|^2 = O(|\cdot|^4)$. This implies that every OEP construction that is based on this Θ and ϕ yields a wavelet system with 2 vanishing moments, and with approximation order $\min\{3, 4\} = 3$ (cf. Theorem 2.8). One possible construction of the mother wavelets is as follows:

$$\begin{aligned}
\tau_1(\omega) &= -\frac{\sqrt{2}}{24}(1 - e^{-i\omega})^3,\\
\tau_2(\omega) &:= -\frac{1}{24}(1 - e^{-i\omega})^3(1 + 6e^{-i\omega} + e^{-i2\omega}),\\
\tau_3(\omega) &= -\frac{\sqrt{13}}{48}(1 - e^{-i\omega})^2(1 + 5e^{-i\omega} + 5e^{-i2\omega} + e^{-i3\omega}).
\end{aligned}\tag{A.1}$$

Then the (symmetric!) filters are of sizes $4, 6, 6$. For the sake of comparison, note that among the $(6, 5, 5)$ filters of the type I construction of pseudo-splines of type $(4, 1)$, one is not symmetric; that system does have approximation order 4 (as compared to only 3 here). The corresponding ψ_1, ψ_2, ψ_3 are shown in Figure 8. Another choice is the following: Let $\Theta(\omega) = \bigl(219 - 112\cos(\omega) + 13\cos(2\omega)\bigr)/120$. Set

$$\begin{aligned}
\tau_1(\omega) &= t_1(1 - e^{-i\omega})^3\bigl[(5776 + 8t_0)(1 + 6e^{-i\omega}) + 4849e^{-i2\omega}\bigr],\\
\tau_2(\omega) &= t_2(1 - e^{-i\omega})^3\Bigl[(73233 + 60t_0)(1 + 6e^{-i\omega}) + (957098 + 700t_0)e^{-i2\omega}\\
&\qquad + 616278e^{-i3\omega} + 102713e^{-i4\omega}\Bigr].
\end{aligned}\tag{A.2}$$

where $t_0 = \sqrt{458247}$, $t_1 = \frac{\sqrt{154244433994641 - 226211192304t_0}}{284121413784}$ and $t_2 = \frac{\sqrt{37714995 - 30900t_0}}{15234392160}$. Then $\{\psi_1, \psi_2\}$ generates a tight framelet and has vanishing moments of order 3, as well as approximation order 3. The filters are 6-tap and 8-tap, hence are much shorter than the type III $(4, 1)$ pseudo-spline wavelets (whose filters are 6-tap and 14-tap. The approximation order of the systems there is 4, however, and the wavelets there are a notch smoother). The graphs of the corresponding ψ_1, ψ_2 are given in Figure 9.

The exact (but more complex) expressions of the wavelet filters in radicals can be obtained for the following examples as well; for simplicity, however, we will present them in decimal notation.

Example A.2. Take $\tau_0(\omega) = (1 + e^{-i\omega})^4/16$; then the refinable function ϕ is the cubic B-spline. Choosing

$$\Theta(\omega) = 2452/945 - 1657/840\cos(\omega) + 44/105\cos(2\omega) - 311/7560\cos(3\omega).$$

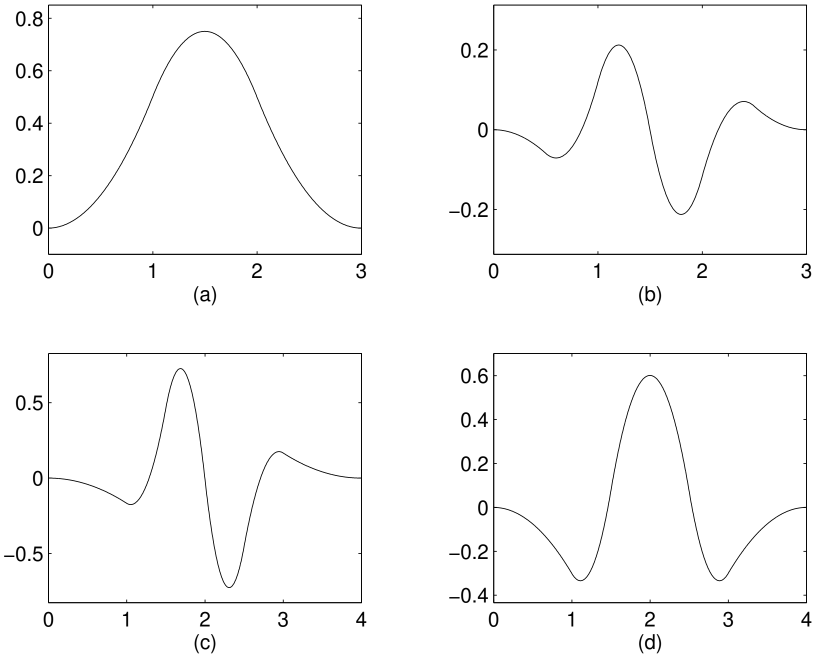

Figure 8. (b), (c) and (d) are the graphs of the symmetric wavelet functions ψ_1, ψ_2 and ψ_3 corresponding to (A.1) in Example A.1. The tight framelet provides approximation order 3 and has two vanishing moments. The filters are of size $(4, 6, 6)$.

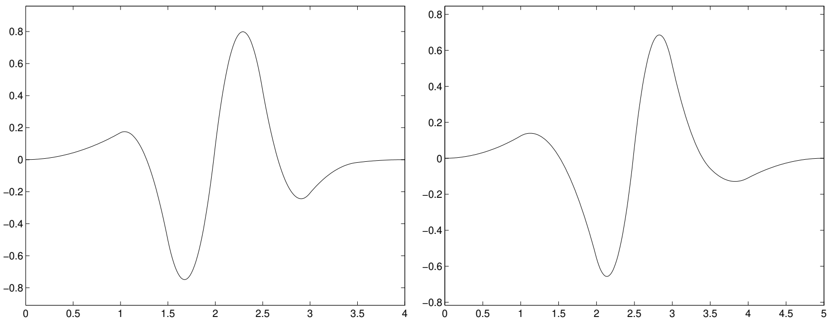

Figure 9. The graphs of the mother wavelets corresponding to (A.2) in Example A.1 The system provides approximation order 3 and has 3 vanishing moments. The filters are of lengths 6 and 8.

we define

$$\begin{aligned}
\tau_1(\omega) = (1-e^{-i\omega})^4 \Big[& 0.004648178373 + 0.037185426987e^{-i\omega} + 0.231579575890e^{-i2\omega} \\
& + 0.077492027449e^{-i3\omega} + 0.009686503431e^{-i4\omega} \Big], \\
\tau_2(\omega) = (1-e^{-i\omega})^4 \Big[& 0.00815406597 + 0.065232527739e^{-i\omega} + 0.221444746610e^{-i2\omega} \\
& + 0.401674890361e^{-i3\omega} + 0.257134715206e^{-i4\omega} \\
& + 0.078828706252e^{-i5\omega} + 0.009853588281e^{-i6\omega} \Big].
\end{aligned}$$

Then $\{\psi_1, \psi_2\}$ generates a tight framelet with vanishing moments of order 4, hence with approximation order 4. The filter are 9- and 11-tap. The functions ψ_1, ψ_2 are shown in Figure 10.

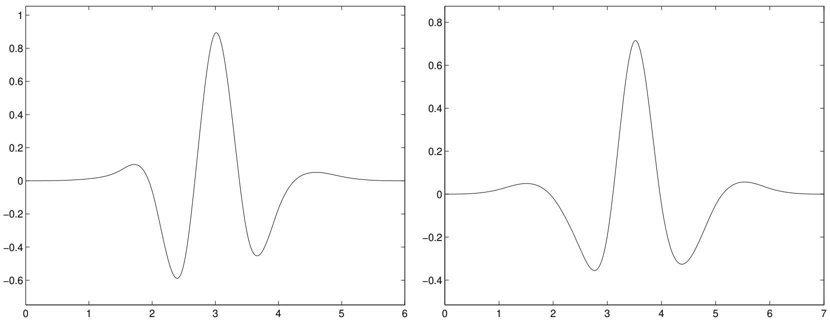

Figure 10. The graphs of the wavelet functions ψ_1 and ψ_2 in Example A.2. $\{\psi_1, \psi_2\}$ generates a wavelet tight frame in $L_2(\mathbb{R})$ and has vanishing moments of order 4.

Example A.3.

Let $\tau_0(\omega) = (1 + e^{-i\omega})^5/32$. Then ϕ is the B-spline function of order 5. Let

$$\Theta(\omega) = \Big[3274 - 2853\cos(\omega) + 654\cos(2\omega) - 67\cos(3\omega)\Big]/1008.$$

Define

$$\tau_1(\omega) = t_1(1 - e^{-i\omega})^5 \Big[1 + 10e^{-i\omega} + c_1 e^{-i2\omega} + 10e^{-i3\omega} + e^{-i4\omega}\Big],$$
$$\tau_2(\omega) = t_2(1 - e^{-i\omega})^5 \Big[1 + 10e^{-i\omega} + c_2 e^{-i2\omega} + (10c_2 - 330)e^{-i3\omega} + c_2 e^{-i4\omega} + 10e^{-i5\omega} + e^{-i6\omega}\Big]$$
$$\tau_3(\omega) = t_3(1 - e^{-i\omega})^4 \Big[1 + 9e^{-i\omega} + c_3 e^{-i2\omega} + (9c_3 - 240)(e^{-i3\omega} + e^{-i4\omega}) + c_3 e^{-i5\omega} + 9e^{-i6\omega} + e^{-i7\omega}\Big],$$

where $t_1 = 0.002079820445, t_2 = 0.002143933408, t_3 = 0.006087006866$ and

$$c_1 = 27.8020039303, \quad c_2 = 43.597827553, \quad c_3 = 34.9890169103.$$

Then we obtain tight framelet that has vanishing moments of order 4, hence provides approximation order 5. The three filters are of sizes $10, 12, 12$, which is longer than the $(8, 7, 7)$ filters of the type I construction of pseudo-spline of type $(5, 2)$, which also provide approximation order 5; the increase in length is the price to pay for having splines and 4 instead of 3 vanishing moments; moreover the wavelets in this example are symmetric. The scaling function ϕ and the three wavelets ψ_1, ψ_2, ψ_3 are shown in Figure 11.

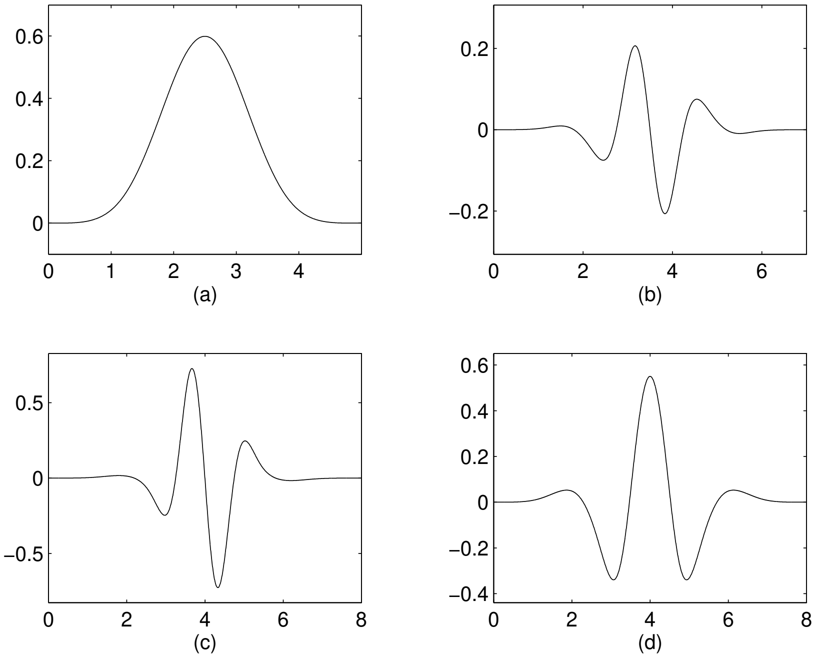

Figure 11. (b), (c) and (d) are the graphs of the symmetric mother wavelets derived from the B-spline function of order 5 in (a) in Example A.3. This tight framelet has vanishing moments of order 4, hence the approximation order is maximal, i.e., 5.

Another choice is the following:

$$\Theta(\omega) = \Big[927230 - 455536\cos(\omega) + 135068\cos(2\omega) - 24208\cos(3\omega) + 2021\cos(4\omega)\Big]/120960,$$

$$\begin{aligned}\tau_1(\omega) = (1-e^{-i\omega})^5 \Big[&0.025119887085 + 0.251198870848e^{-i\omega} + 0.262546371853e^{-i2\omega} \\ &+ 0.166262760002e^{-i3\omega} + 0.065011596958e^{-i4\omega} \\ &+ 0.014662218472e^{-i5\omega} + 0.001466221847e^{-i6\omega}\Big],\end{aligned}$$

$$\begin{aligned}\tau_2(\omega) = (1-e^{-i\omega})^5 \Big[&0.008881894968 + 0.088818949683e^{-i\omega} + 0.328950148428e^{-i2\omega} \\ &+ 0.358476144742e^{-i3\omega} + 0.250181103408e^{-i4\omega} + 0.123734867140e^{-i5\omega} \\ &+ 0.042684669937e^{-i6\omega} + 0.009185207037e^{-i7\omega} + 0.000918520704e^{-i8\omega}\Big].\end{aligned}$$

This time we obtain a tight framelet with 5 vanishing moments, hence with approximation order 5. The two wavelets are shown in Figure 12.

Example A.4. Take $\tau_0(\omega) = (1+e^{-i\omega})^6/64$. Then ϕ is the B-spline function of order 6.

Let

$$\begin{aligned}
\Theta(\omega) = \Big[&78020840 - 91378878\cos(\omega) + 33897504\cos(2\omega) - 8438339\cos(3\omega) \\
&+ 1298168\cos(4\omega) - 93695\cos(5\omega)\Big]/13305600, \\
\tau_1(\omega) = (1-e^{-i\omega})^6\Big[&0.002145656868 + 0.025747882416\,e^{-i\omega} + 0.119255331090\,e^{-i2\omega} \\
&+ 0.203748244582\,e^{-i3\omega} + 0.119255331090\,e^{-i4\omega} + 0.025747882416\,e^{-i5\omega} \\
&+ 0.002145656868e^{-i6\omega}\Big], \\
\tau_2(\omega) = (1-e^{-i\omega})^6\Big[&0.002080123603 + 0.02496148323\,e^{-i\omega} + 0.1259950758241\,e^{-i2\omega} \\
&+ 0.322110209123\,e^{-i3\omega} + 0.398690839006\,e^{-i4\omega} + 0.322110209123\,e^{-i5\omega} \\
&+ 0.125995075824\,e^{-i6\omega} + 0.024961483233\,e^{-i7\omega} + 0.002080123603e^{-i8\omega}\Big], \\
\tau_3(\omega) = (1-e^{-i\omega})^6\Big[&0.000927141464 + 0.011125697570\,e^{-i\omega} + 0.057997824965\,e^{-i2\omega} \\
&+ 0.165648982061\,e^{-i3\omega} + 0.266351327951\,e^{-i4\omega} + 0.249980354007\,e^{-i5\omega} \\
&+ 0.266351327951\,e^{-i6\omega} + 0.165648982061\,e^{-i7\omega} + 0.057997824965\,e^{-i8\omega} \\
&+ 0.011125697570\,e^{-i9\omega} + 0.000927141464e^{-i10\omega}\Big].
\end{aligned}$$

Here, we obtain a tight framelet with vanishing moments of order 6, and with symmetric mother wavelets, shown in Figure 13.

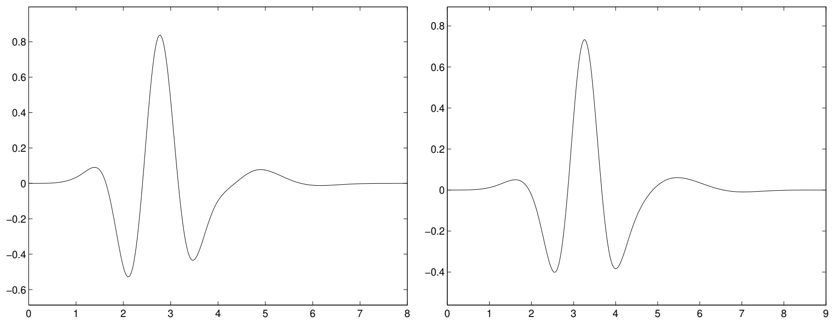

Figure 12. The graphs of the two (non-symmetric) mother wavelets in Example A.3. The tight framelet has approximation order 5 as well as vanishing moments of order 5. Note that the filters are 12- and 14-tap.

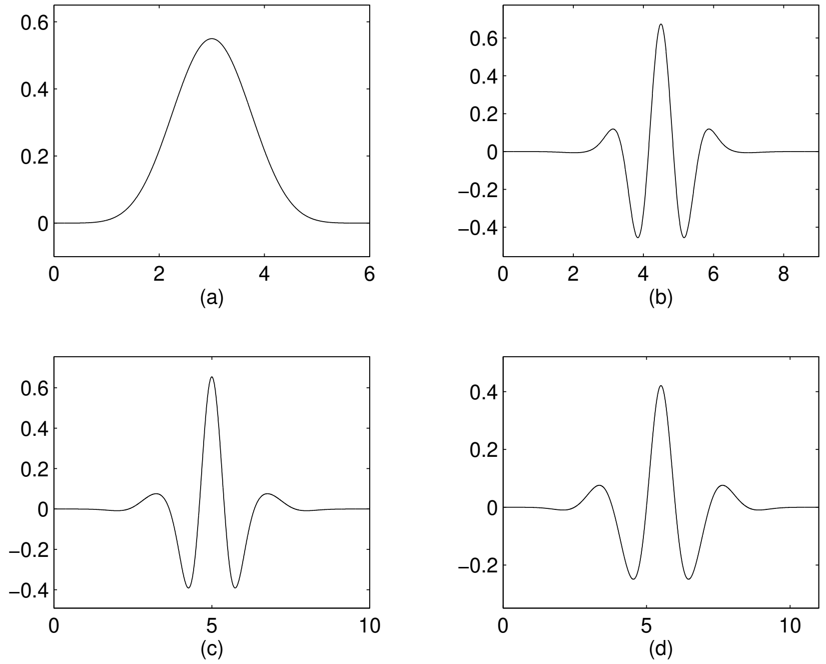

Figure 13. (b), (c) and (d) are the graphs of the symmetric mother framelets derived from the B-spline function of order 6 in (a) in Example A.4. The tight framelets has 6 vanishing moments, hence approximation order 6, as well.

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