

ELASTIC SPLINES I: EXISTENCE

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ABSTRACT. Given interpolation points P_1, P_2, \dots, P_m in the plane, it is known that there does not exist an interpolating curve with minimal bending energy, unless the given points lie sequentially along a line. We say that an interpolating curve is *admissible* if each piece, connecting two consecutive points P_i and P_{i+1} , is an *s-curve*, where an *s-curve* is a planar curve which first turns monotonically at most 180° in one direction and then turns monotonically at most 180° in the opposite direction. Our main result is that among all admissible interpolating curves there exists a curve with minimal bending energy. We also prove, in a very constructive manner, the existence of an *s-curve*, with minimal bending energy, that connects two given unit tangent vectors.

1. Introduction

Given a sequence of points P_1, P_2, \dots, P_m in \mathbb{R}^2 with $P_i \neq P_{i+1}$, a curve $F : [a, b] \rightarrow \mathbb{R}^2$ is called an **interpolating curve** if there exist times $a = t_1 < t_2 < \dots < t_m = b$ such that $F(t_i) = P_i$. In the special case when the interpolation points can be written as $P_i = (x_i, y_i)$, with $x_1 < x_2 < \dots < x_m$, an interpolating curve can be constructed as the graph of a smooth function $g : [x_1, x_m] \rightarrow \mathbb{R}$, provided g satisfies the interpolation conditions $g(x_i) = y_i$. It is well known that if g is the natural cubic spline, then g minimizes the functional $\int_a^b (g''(x))^2 dx$ among all smooth functions which satisfy the interpolation conditions. This functional is often viewed as a simple approximation of the curve's bending energy $\int_0^L \kappa(s)^2 ds$, where s denotes arclength and κ denotes signed curvature, and it is natural to ask what would happen if one tried to minimize the bending energy among all smooth interpolating curves. Unfortunately, such optimal curves do not exist except in the trivial case when the interpolation points lie sequentially along a line. Apparently, this was first observed by Birkhoff and de Boor [1], along with Birkhoff, Burchard and Thomas

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[2]. This lack of existence can be understood as a consequence of the effect that scaling has on bending energy: the bending energy of a curve scaled by a factor q equals $\frac{1}{q}$ times the original bending energy. As a result, it is possible to construct smooth interpolating curves with arbitrarily small bending energy. For example, let $p \gg 1$ and consider the circles c_1, c_2, \dots, c_m which meet tangentially at the point $(p, 0)$; specifically, let c_i be the circle which begins and ends at $(p, 0)$, has center on the x -axis, and passes through the point P_i . Then the subcurve of $c = c_1 \cup c_2 \cup \dots \cup c_m$, starting at P_1 and ending at P_m , is an interpolating curve whose bending energy tends to 0 as $p \rightarrow \infty$ (see section 2.6 of [8] for other constructions).

Subsequent attention was directed towards interpolating curves whose bending energy is locally minimal (i.e. minimal among all nearby interpolating curves). It was reported in [2], and mentioned in [1], that if an interpolating curve F has a locally minimal bending energy, then each segment of F , connecting two consecutive interpolation points, will be a segment of ‘rectangular elastica’, meaning a planar curve whose signed curvature κ satisfies the differential equation $2\frac{d^2\kappa}{ds^2} + \kappa^3 = 0$. (Rectangular elastica was first described by James Bernoulli (1694) and is one of the nine types of elastica identified by Euler (1750), see [5].) Using a variational calculus and physical reasoning, Lee and Forsythe [7] (see also [3]) have confirmed that each segment of F is indeed a segment of rectangular elastica, and have moreover shown that the signed curvature of F is continuous throughout the curve and vanishes at the endpoints. Unfortunately, interpolating curves with locally minimal bending energy do not necessarily exist, and this constitutes a significant deficiency in the theoretical foundation of this interpolation method.

Rather than seeking an interpolating curve with a locally minimal bending energy, an alternate approach is to define a restricted class of ‘admissible’ interpolating curves and then seek a curve with minimal bending energy in the restricted class. Birkhoff proposed a restriction on *length* and conjectured that among all smooth interpolating curves of length at most L_0 , L_0 being a prescribed upper bound, there exists a curve with minimal bending energy. This conjecture was eventually proved by Jerome [6] (see [4] for a more comprehensive treatment and also [8] where ‘pinning’ and ‘clamping’ at interpolation nodes are treated).

Rather than a restriction on length, we propose a restriction on *shape*. The motivation for our restriction comes from the fact that if a smooth interpolating curve F has a locally minimal bending energy, then it can be shown that each segment of F , connecting two consecutive interpolation points, is what here is called an *s-curve*. In brief, an s-curve is a curve which first turns monotonically in one direction (either counter-clockwise or clockwise) at most 180° and then turns monotonically in the opposite direction at most 180° . An interpolating curve F is deemed *admissible* if each piece of F , connecting two consecutive interpolation points P_i and P_{i+1} , is an s-curve. The family of all admissible interpolating curves is denoted $\mathcal{A}(P_1, P_2, \dots, P_n)$, and we emphasize that our definition of $\mathcal{A}(P_1, P_2, \dots, P_n)$ includes no restrictions or constraints on length. Our main result is the following.

Theorem 1.1. *Given any sequence of points P_1, P_2, \dots, P_m in \mathbb{R}^2 with $P_i \neq P_{i+1}$, the family of admissible interpolating curves $\mathcal{A}(P_1, P_2, \dots, P_m)$ contains a curve with minimal bending energy.*

An essential sub-problem which arises in the proof of Theorem 1.1 is that of proving the existence of an s-curve, with minimal bending energy, which connects two given unit tangent vectors. In addition to facilitating our proof of Theorem 1.1, we anticipate that this sub-problem sits at the core of any numerical algorithm for solving the general problem, and with this in mind, we present a thorough analysis of the sub-problem along with a constructive solution. We mention that in [4] and [8], the gradient vector field approach is employed, but this approach would not apply to $\mathcal{A}(P_1, P_2, \dots, P_m)$ since it is not an open manifold.

An outline of the sequel is as follows. In section 2, we explain our notation and develop some basic formulae and properties of rectangular elastica. A curve which turns monotonically at most 180° in one direction is called a *c-curve* and in section 3, we show the existence of an optimal c-curve connecting a unit tangent vector to a line as well as connecting two unit tangent vectors. Incidentally, line segments are (degenerate) c-curves and c-curves are (degenerate) s-curves. In section 4, the uniqueness, or lack thereof, of the optimal c-curves found in section 3 is treated. The important sub-problem mentioned above, namely the existence of an optimal s-curve connecting two unit tangent vectors, is primarily solved in section 5, except that one particular case (where the optimal s-curve turns out to be a unique c-curve) is treated in section 6. Finally, in section 7, we prove Theorem 1.1.

2. Notation

We simplify our notation by using the complex plane \mathbb{C} in place of \mathbb{R}^2 . A **curve** is a differentiable function $f : [a, b] \rightarrow \mathbb{C}$ whose derivative f' is absolutely continuous and non-zero. The **length** of f is $\text{len}(f) = \int_a^b |f'(t)| dt$. With $L = \text{len}(f)$, let the variables $t \in [a, b]$ and $s \in [0, L]$ be related by $s = \int_a^t |f'(\tau)| d\tau$ and define $F : [0, L] \rightarrow \mathbb{C}$ by $F(s) = f(t)$. It can be shown that F is a curve (i.e. F' is absolutely continuous) satisfying $|F'| = 1$. The curve F is called the **unit speed curve described by f** and is denoted $[f]$. Two curves f and g are said to be **equivalent**, written $f \equiv g$, if $[f] = [g]$. Since F' is absolutely continuous and $|F'| = 1$, it follows that there exists an absolutely continuous function $\theta : [0, L] \rightarrow \mathbb{R}$, unique modulo an additive constant in $2\pi\mathbb{Z}$, such that $F' = e^{i\theta}$. We refer to θ at the **direction angle** of F , while the derivative of θ , denoted κ , is called the **signed curvature** of F . Since θ is absolutely continuous, it follows that κ is Lebesgue integrable (see [9, pp. 108–112]). The **turning angle** of f , denoted $\Delta(f)$, is defined by $\Delta(f) = \Delta(F) := \int_0^L \kappa(s) ds$. Note that the magnitude of the turning angle is bounded by the L_1 -norm of κ . If $\kappa \geq 0$ (resp. $\kappa \leq 0$) almost everywhere in $[0, L]$, then $|\Delta(F)| = \|\kappa\|_{L_1}$ and f is called a **left-curve** (resp. **right-curve**). A **c-curve** is a left-curve or a right-curve whose turning angle has magnitude at most π . A **u-turn** is a c-curve whose turning angle has magnitude π .

Given the signed curvature κ of F and its initial position and direction, we can recover F as follows:

Step 1. Define $\theta(s) = \theta_0 + \int_0^s \kappa(r) dr$, $s \in [0, L]$, where $\theta_0 = \arg(F'(0))$.

Step 2. $F(s) = F(0) + \int_0^s e^{i\theta(r)} dr$.

(Here, \arg is defined with the standard range $(-\pi, \pi]$.) This reconstruction can be used

to decide when two curves are close to each other. To see this, suppose F_1 is a unit speed curve having the same length and initial position and direction as F . It follows from step 1, that $|\theta(r) - \theta_1(r)| \leq \|\kappa - \kappa_1\|_{L_1}$ and then using the Lipschitz continuity of the function $r \mapsto e^{ir}$ in step 2, we obtain

$$(2.1) \quad |F(s) - F_1(s)| \leq \int_0^s \left| e^{i\theta(r)} - e^{i\theta_1(r)} \right| dr \leq \int_0^s |\theta(r) - \theta_1(r)| dt \leq s \|\kappa - \kappa_1\|_{L_1}.$$

Whereas the L_1 -norm of κ is necessarily finite, the L_2 -norm may or may not be finite. When it is finite, we say that f has finite bending energy, where the **bending energy** of f , denoted $\|f\|^2$, is essentially the square of the L_2 -norm of κ :

$$\|f\|^2 = \|F\|^2 := \frac{1}{4} \int_0^L |\kappa(s)|^2 ds.$$

The constant $\frac{1}{4}$ has been inserted for later convenience.

A **unit tangent vector** is an ordered pair of complex numbers $u = (u_1, u_2) \in \mathbb{C}^2$ such that $|u_2| = 1$ and can be visualized (see Fig. 5.1) as the directed line segment, of unit length, having base-point u_1 and direction u_2 . For any $t \in [a, b]$, the **unit tangent vector to f** at t , denoted $\vec{f}(t)$, has base-point $f(t)$ and direction $f'(t)/|f'(t)|$, whereby

$$\vec{f}(t) := (f(t), f'(t)/|f'(t)|).$$

The unit tangent vectors $u = \vec{f}(a)$ and $v = \vec{f}(b)$ are called, respectively, the **initial** and **terminal** unit tangent vectors of f (see Fig. 5.2), and we say that f **connects** u to v . We also say that f **connects** u to λ if λ is the line through $f(b)$ which is parallel to $f'(b)$ (see Fig. 5.3 where f_r connects u to λ). If g is a curve whose initial unit tangent vector equals the terminal unit tangent vector of f , then $[f]$ can be extended by $[g]$ obtaining a unit speed curve, denoted $f \sqcup g$, whose initial and terminal unit tangent vectors equal those of f and g , respectively, and whose bending energy satisfies $\|f \sqcup g\|^2 = \|f\|^2 + \|g\|^2$.

A **similarity transformation** is a mapping $T : \mathbb{C} \rightarrow \mathbb{C}$ of the form $T(z) = c_1 z + c_2$ or $T(z) = c_1 \bar{z} + c_2$, where c_1, c_2 are complex constants, $c_1 \neq 0$. The first form preserves the orientation (left or right) of a curve while the second form reverses it. The dilation factor is $q = |c_1|$, and the effect on a curve f is, as expected, $\text{len}(T \circ f) = q \text{len}(f)$, $\|T \circ f\|^2 = \frac{1}{q} \|f\|^2$ and $|\Delta(T \circ f)| = |\Delta(f)|$. If a curve g is equivalent to $T \circ f$, then we say that g is **similar** to f ; in case $q = 1$, T is called a **congruency transformation** and we say that g is **congruent** to f . Furthermore, we say that g is **directly similar** (or congruent) to f if T is orientation preserving.

The curves constructed in this article are formed by line segments (denoted $[A, B]$) and various segments of rectangular elastica. For the latter, we employ the parameterization

$$R(t) := \sin t + i\xi(t),$$

where $\xi(t)$ is defined by $\frac{d\xi}{dt} = \frac{\sin^2 t}{\sqrt{1 + \sin^2 t}}$, $\xi(0) = 0$ (see Figure 6.1a). This parameterization is derived simply by substituting $a = 1$ and $x = \sin t$ into James Bernoulli's equation

$dy = x^2 dx / \sqrt{a^4 - x^4}$, and we find it more suitable to the work at hand than the usual unit speed formulation involving Jacobi's elliptic functions. Since $\frac{d\xi}{dt}$ is even and π -periodic, it follows that ξ is odd and satisfies $\xi(t + \pi) = d + \xi(t)$, where

$$d := \xi(\pi).$$

Since the sine function is odd and 2π -periodic, we conclude that $R(t)$ is odd and satisfies $R(t + 2\pi) = i2d + R(t)$. We use the notation $R_{[a,b]}$ to denote the sub-curve $R(t)$, $t \in [a, b]$, and any curve which is similar to $R_{[a,b]}$ is called a **segment of rectangular elastica**. For later reference, we mention the following.

$$\begin{aligned} |R'(t)| &= \frac{1}{\sqrt{1 + \sin^2 t}}, & \frac{R'(t)}{|R'(t)|} &= \cos t \sqrt{1 + \sin^2 t} + i \sin^2 t, & \kappa(t) &= 2 \sin t, \\ \Delta(R_{[0,b]}) &= \int_0^b \kappa(t) |R'(t)| dt = 2 \cos^{-1} \left[\frac{\cos b}{\sqrt{2}} \right] - \frac{\pi}{2}, \\ \|R_{[a,b]}\|^2 &= \frac{1}{4} \int_a^b \kappa(t)^2 |R'(t)| dt = \xi(b) - \xi(a). \end{aligned}$$

For $t_0 \in (0, \pi]$, the segment $R_{[0,t_0]}$ plays an important role in the sequel. In the following lemma, we establish a connection between the turning angle of $R_{[0,t_0]}$ and the value of $\xi(t_0)$.

Lemma 2.1. *Let $t_0 \in (0, \pi]$ and put $\theta_0 = \Delta(R_{[0,t_0]})$. Then $\xi(t_0) = \frac{1}{2} \int_0^{\theta_0} \sqrt{\sin \tau} d\tau$.*

Proof. Fix $t_0 \in (0, \pi]$ and put $\theta = \Delta(R_{[0,t]}) = 2 \cos^{-1} \left[\frac{\cos t}{\sqrt{2}} \right] - \frac{\pi}{2}$, $t \in [0, t_0]$. Then $\frac{d\theta}{dt} = \kappa(t) |R'(t)|$, and since $e^{i\theta} = R'(t) / |R'(t)|$, we have $\sin \theta = \text{Im } R'(t) / |R'(t)| = \sin^2 t$, which implies $\sqrt{\sin \theta} = \sin t$. Hence,

$$\xi(t_0) = \int_0^{t_0} \frac{\sin^2 t}{\sqrt{1 + \sin^2 t}} dt = \frac{1}{2} \int_0^{t_0} \sin(t) \kappa(t) |R'(t)| dt = \frac{1}{2} \int_0^{\theta_0} \sqrt{\sin \tau} d\tau.$$

□

3. Existence of optimal c-curves

Given a unit tangent vector u and a line λ , let $C_l(u, \lambda)$ denote the set of left c-curves which connect u to λ . In this section, we consider the problem of finding a curve in $C_l(u, \lambda)$ which has minimal bending energy. We first consider $C_l(u_0, \lambda_d)$, where $u_0 = \vec{R}(0)$ and $\lambda_d = \{z \in \mathbb{C} : \text{Im } z = d\}$. We will show that $R_{[0,\pi]}$ has minimal bending energy in $C_l(u_0, \lambda_d)$. Note that $\|R_{[0,\pi]}\|^2 = \xi(\pi) = d$ and by Lemma 2.1, we have $d = \xi(\pi) = \frac{1}{2} \int_0^\pi \sqrt{\sin \tau} d\tau$.

Let $f \in C_l(u_0, \lambda_d)$, put $L = \text{len}(f)$, and let $F = [f]$ denote the unit speed curve described by f . Let θ and κ be the direction angle and signed curvature of F , respectively, and note that $\int_0^L \kappa(s) ds = \pi$ since the turning angle in $C_l(u, \lambda_d)$ is π . Furthermore, since F originates at 0 and terminates on λ_d , we have $d = \text{Im } F(L) = \int_0^L \sin \theta ds$.

Lemma 3.1. *If κ is continuous and positive, then $\|F\|^2 \geq d$.*

Proof. We adopt the viewpoint that $\theta \in [0, \pi]$ and $s \in [0, L]$ are variables related by $\theta = \int_0^s \kappa(r) dr$. The assumptions on κ ensure that θ and s are increasing C^1 functions of one another. Noting that $\frac{d\theta}{ds} = \kappa(s)$, we observe that $\|F\|^2 = \frac{1}{4} \int_0^L \kappa(s)^2 ds = \frac{1}{4} \int_0^L \kappa(s) \frac{d\theta}{ds} ds = \frac{1}{4} \int_0^\pi \kappa(s) d\theta$. Similarly, since $\frac{ds}{d\theta} = 1/\kappa(s)$, we have $\int_0^\pi \frac{\sin \theta}{\kappa(s)} d\theta = \int_0^\pi \sin \theta \frac{ds}{d\theta} d\theta = \int_0^L \sin \theta ds = d$. Now,

$$d^2 = \left[\frac{1}{2} \int_0^\pi \sqrt{\sin \theta} d\theta \right]^2 = \left[\int_0^\pi \frac{\sqrt{\sin \theta}}{\sqrt{\kappa(s)}} \frac{\sqrt{\kappa(s)}}{2} d\theta \right]^2 \leq \left(\int_0^\pi \frac{\sin \theta}{\kappa(s)} d\theta \right) \left(\frac{1}{4} \int_0^\pi \kappa(s) d\theta \right),$$

by the Cauchy-Schwarz inequality. Hence $d^2 \geq d\|F\|^2$, and therefore $\|F\|^2 \geq d$. \square

Returning now to the general case, suppose, by way of contradiction, that $\|F\|^2 < d$. Then κ is a nonnegative square integrable function satisfying $\int_0^L \kappa(s) ds = \pi$. It follows that for every $\varepsilon > 0$, there exists a positive continuous function $\kappa_\varepsilon : [0, L] \rightarrow [0, \infty)$ such that $\int_0^L \kappa_\varepsilon(s) ds = \pi$ and $\|\kappa - \kappa_\varepsilon\|_{L_2} < \varepsilon$. Let F_ε be the unit speed curve having signed curvature κ_ε and initial unit tangent vector u_0 . Since F_ε has turning angle π , it follows that the terminal unit tangent vector of F_ε is parallel to λ_d , but there is no guarantee that the terminal point $z_\varepsilon = F_\varepsilon(L)$ lies on λ_d . We repair this by multiplying F_ε with the positive scalar $c_\varepsilon = d/\text{Im } z_\varepsilon$ obtaining the curve $c_\varepsilon F_\varepsilon \in C_l(u_0, \lambda_d)$ with $\|c_\varepsilon F_\varepsilon\|^2 = \frac{1}{c_\varepsilon} \|F_\varepsilon\|^2$. Since $\|\kappa - \kappa_\varepsilon\|_{L_2} \rightarrow 0$, it follows that $\|F_\varepsilon\|^2 \rightarrow \|F\|^2$ as $\varepsilon \rightarrow 0$. And since the L_1 -norm of $\kappa - \kappa_\varepsilon$ is bounded by a constant multiple of its L_2 -norm, it follows from (2.1) that $c_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$. Hence $\|c_\varepsilon F_\varepsilon\|^2 < d$ when $\varepsilon > 0$ is sufficiently small, contradicting Lemma 3.1. Therefore, $\|F\|^2 \geq d$. We have thus proved the following.

Theorem 3.2. *Let $u_0 = \vec{R}(0)$ and $\lambda_d = \{z \in \mathbb{C} : \text{Im } z = d\}$. Then $R_{[0, \pi]}$ has minimal bending energy in $C_l(u_0, \lambda_d)$.*

Corollary 3.3. *Let $t_1 \in (0, \pi)$ and put $u_1 = \vec{R}(t_1)$. Then $R_{[t_1, \pi]}$ has minimal bending energy in $C_l(u_1, \lambda_d)$.*

Proof. If $f \in C_l(u_1, \lambda_d)$ has bending energy less than $R_{[t_1, \pi]}$, then $R_{[0, t_1]} \sqcup f$ will be a curve in $C_l(u_0, \lambda_d)$ with bending energy less than d , contradicting Theorem 3.2. \square

Corollary 3.4. *Let $0 \leq t_1 < t_2 \leq \pi$ and put $u_1 = \vec{R}(t_1)$, $u_2 = \vec{R}(t_2)$. Then $R_{[t_1, t_2]}$ has minimal bending energy in $C_l(u_1, u_2)$.*

Proof. If $f \in C_l(u_1, u_2)$ has bending energy less than $R_{[t_1, t_2]}$, then $R_{[0, t_1]} \sqcup f \sqcup R_{[t_2, \pi]}$ will be a curve in $C_l(u_0, \lambda_d)$ with bending energy less than d , contradicting Theorem 3.2. \square

Definition 3.5. Let λ be a line and u a unit tangent vector whose base-point lies off of λ , and assume $C_l(u, \lambda)$ is nonempty. Let $\delta \in (0, \pi]$ be the common turning angle in $C_l(u, \lambda)$ and let $t_1 \in [0, \pi)$ be such that $\Delta(R_{[t_1, \pi]}) = \delta$. There exists a unique similarity transformation T such that $T \circ R_{[t_1, \pi]}$ belongs to $C_l(u, \lambda)$. We define $l(u, \lambda) = T \circ R_{[t_1, \pi]}$. In other words, $l(u, \lambda)$ is the unique curve in $C_l(u, \lambda)$ which is similar to $R_{[t_1, \pi]}$.

Theorem 3.6. *Let λ be a line and u a unit tangent vector whose base-point lies off of λ , and assume $C_l(u, \lambda)$ is nonempty. Then $l(u, \lambda)$ has minimal bending energy in $C_l(u, \lambda)$. Moreover, if $\delta \in (0, \pi]$ denotes the common turning angle in $C_l(u, \lambda)$ and p denotes the orthogonal distance from the base-point of u to λ , then*

$$\|l(u, \lambda)\|^2 = \frac{1}{p} \left[\frac{1}{2} \int_0^\delta \sqrt{\sin \tau} \, d\tau \right]^2.$$

Proof. Let t_1 and T be as in Definition 3.5 and u_1 and λ_d as in Corollary 3.3. Note that T maps $C_l(u_1, \lambda_d)$ onto $C_l(u, \lambda)$ and a curve f has minimal bending energy in $C_l(u_1, \lambda_d)$ if and only if $T(f)$ has minimal bending energy in $C_l(u, \lambda)$. It therefore follows from Corollary 3.3 that $l(u, \lambda) = T \circ R_{[t_1, \pi]}$ has minimal bending energy in $C_l(u, \lambda)$. In order to compute the bending energy of $l(u, \lambda)$, recall that $\|l(u, \lambda)\|^2 = \frac{1}{q} \|R_{[t_1, \pi]}\|^2$, where q is the dilation factor in T . Since the orthogonal distance from the base-point of u_1 to λ_d is $d - \xi(t_1)$, it follows that $q = p/(d - \xi(t_1))$. And since $\|R_{[t_1, \pi]}\|^2 = d - \xi(t_1)$, we have $\|l(u, \lambda)\|^2 = \frac{1}{p}(d - \xi(t_1))^2$. By Lemma 2.1, $\xi(t_1) = \int_0^{\pi-\delta} \sqrt{\sin \tau} \, d\tau$ and hence $d - \xi(t_1) = \frac{1}{2} \int_{\pi-\delta}^\pi \sqrt{\sin \tau} \, d\tau = \frac{1}{2} \int_0^\delta \sqrt{\sin \tau} \, d\tau$. \square

Remark 3.7. The definitions and results for right c-curves are analogous to those for left c-curves. In brief, we denote the set of right c-curves connecting u to λ by $C_r(u, \lambda)$, and $r(u, \lambda)$ is defined the same as $l(u, \lambda)$ except that δ denotes the magnitude of the common turning angle in $C_r(u, \lambda)$ (right curves have a negative turning angle). Theorem 3.6 then holds with $C_r(u, \lambda)$ and $r(u, \lambda)$ in place of $C_l(u, \lambda)$ and $l(u, \lambda)$, respectively.

4. Uniqueness of optimal c-curves

Having settled the question of existence of an optimal curve in $C_l(u, \lambda)$ we now address uniqueness. As with existence we start with $C_l(u_0, \lambda_d)$, where u_0 and λ_d are as in Theorem 3.2.

Theorem 4.1. *For $i = 1, 2$, let $F_i : [0, L_i] \rightarrow \mathbb{C}$ be a unit speed curve in $C_l(u_0, \lambda_d)$ such that $\|F_i\|^2 = d$ and assume that F_i does not begin or end with a line segment. Then $F_1 = F_2$.*

Our proof of this employs the following technical result, which is left as a simple exercise in differential calculus.

Lemma 4.2. *Let $\nu_1, \nu_2 > 0$ and define $H : (0, 1) \rightarrow (0, \infty)$ by $H(\mu) = \frac{\nu_1^2}{\mu} + \frac{\nu_2^2}{1-\mu}$. Then H has a unique minimum at $\mu_0 = \nu_1/(\nu_1 + \nu_2)$, where $H(\mu_0) = (\nu_1 + \nu_2)^2$.*

Proof of Theorem 4.1. Let θ_i and κ_i be the direction angle and signed curvature of F_i , respectively. Since F_i does not begin or end with a line segment, we have $0 < \theta_i(s) < \pi$ for all $s \in (0, L_i)$, and it follows that F_i can be reparameterized as $t \mapsto g_i(t) + it$, $t \in [0, d]$, where g_i is continuous on $[0, d]$ and continuously differentiable on $(0, d)$. Fix $\gamma \in (0, \pi)$

and let $t \in (0, \pi)$ be such that $\Delta(R_{[0,t]}) = \gamma$. Let $s_i \in (0, L_i)$ be such that $\theta_i(s_i) = \gamma$, and put $v_i = \vec{F}_i(s_i)$ and $t_i = \text{Im } F_i(s_i)$. We claim that $t_1 = \xi(t) = t_2$. Noting that the turning angle in $C_l(v_i, \lambda_d)$ is $\pi - \gamma$ and the orthogonal distance from the base-point of v_i to λ_d is $d - t_i$, and since $F_{i[s_i, L_i]}$ belongs to $C_l(v_i, \lambda_d)$, we obtain from Theorem 3.6 and Lemma 2.1 that $\|F_{i[s_i, L_i]}\|^2 \geq \frac{1}{d-t_i}(d - \xi(t))^2$. By a similar argument (using right c-curves) we obtain $\|F_{i[0, s_i]}\|^2 \geq \frac{1}{t_i}\xi(t)^2$. Therefore,

$$d = \|F_i\|^2 = \|F_{i[0, s_i]}\|^2 + \|F_{i[s_i, L_i]}\|^2 \geq \frac{1}{t_i}\xi(t)^2 + \frac{1}{d-t_i}(d - \xi(t))^2.$$

With $\nu_1 = \xi(t)$, $\nu_2 = d - \xi(t)$, $\mu = t_i/d$, and with $H(\mu)$ as in Lemma 4.2, we can express the above inequality as $d \geq \frac{1}{d}H(\mu)$, or equivalently, $d^2 \geq H(\mu)$. By Lemma 4.2, H has a unique minimum at $\mu_0 = \xi(t)/d$ where $H(\mu_0) = d^2$. But since $d^2 \geq H(\mu)$, it must be the case that $\mu = \mu_0$; therefore $t_i = \xi(t)$ as claimed. In terms of the functions g_1 and g_2 , we have proved that if $g'_1(t_1) = \cot \gamma = g'_2(t_2)$, then $t_1 = \xi(t) = t_2$. Since, for $i = 1, 2$, g'_i is continuous and decreasing on $(0, d)$, with range $(-\infty, \infty)$, we conclude that $g'_1 = g'_2$ on $(0, d)$. Since $g_1(0) = 0 = g_2(0)$, we have $g_1 = g_2$ on $[0, d]$. From this we conclude that F_1 and F_2 are equivalent, but since both are unit speed curves, they must be equal. \square

As an immediate corollary, we have the following.

Corollary 4.3. *If $f \in C_l(u_0, \lambda_d)$ has minimal bending energy, then f contains a subcurve which is equivalent to $c + R_{[0, \pi]}$ for some real constant $c \geq 0$.*

Imitating the proof of Corollary 3.3 and Theorem 3.6, one easily obtains the following.

Corollary 4.4. *Let λ be a line and u a unit tangent vector whose base-point lies off of λ , and assume $C_l(u, \lambda)$ is nonempty. Let $\delta \in (0, \pi]$ denote the common turning angle in $C_l(u, \lambda)$ and let $f \in C_l(u, \lambda)$ have minimal bending energy.*

(i) *If $\delta = \pi$, then f contains a subcurve which is congruent to $l(u, \lambda)$.*

(ii) *If $\delta < \pi$, then either $f \equiv l(u, \lambda)$ or $f \equiv l(u, \lambda) \sqcup [A, B]$ for some line segment $[A, B]$.*

We have seen in Corollary 3.4 that $R_{[t_1, t_2]}$ has minimal bending energy in $C_l(u_1, u_2)$. Using the same technique as above, one easily obtains the following.

Theorem 4.5. *Let $0 \leq t_1 < t_2 \leq \pi$ and let $u_1 = \vec{R}(t_1)$, $u_2 = \vec{R}(t_2)$ denote the initial and terminal unit tangent vectors of the curve $R_{[t_1, t_2]}$, respectively. Let f be a curve with minimal bending energy in $C_l(u_1, u_2)$. If f is not equivalent to $R_{[t_1, t_2]}$, then $[t_1, t_2] = [0, \pi]$ and a sub-curve of f is congruent to $R_{[0, \pi]}$ (i.e. f is equivalent to $[0, c] \sqcup (c + R_{[0, \pi]}) \sqcup [c + id, id]$ for some real constant $c > 0$).*

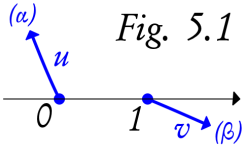
5. Optimal s-curves, part I

An **s-curve** is either a c-curve (considered a degenerate s-curve) or a curve of the form $f = f_1 \sqcup f_2$, where f_1 and f_2 are c-curves which turn in opposite directions. Let u and v be two unit tangent vectors and let $S(u, v)$ denote the set of all s-curves which connect u to v . In this section and the next, we will prove the following.

Theorem 5.1. *Let u and v be two unit tangent vectors with distinct base-points. If $S(u, v)$ is nonempty, then there exists a curve in $S(u, v)$ with minimal bending energy.*

In addition to proving existence, our proof of Theorem 5.1 will actually describe all optimal curves in $S(u, v)$. Expecting that the numerical problem of finding an optimal curve in $S(u, v)$ lies at the heart of future algorithms, we have structured our proof so that it easily translates into a numerical algorithm.

To begin, let u and v be two unit tangent vectors with distinct base-points. By applying a similarity transformation, if necessary, and possibly a direction reversal (i.e. $S(-v, -u)$ in place of $S(u, v)$), we can assume without loss of generality that $u = (0, e^{i\alpha})$ and $v = (1, e^{i\beta})$, where $\alpha \in [0, \pi]$ and $|\beta| \leq \alpha$ (see Figure 5.1, where (α) indicates the direction angle of u).



We leave it to the reader, as a worthwhile exercise, to verify that in this situation $S(u, v)$ is nonempty if and only if $\alpha < \pi$ and $\beta \geq \alpha - \pi$. With that in mind, we proceed assuming that $\alpha \in [0, \pi)$, $|\beta| \leq \alpha$ and $\beta \geq \alpha - \pi$.

If $\alpha = 0$, then $\beta = 0$ as well and the line segment $[0, 1]$ is the unique curve (modulo equivalence) in $S(u, v)$ having minimal bending energy. Having dispensed with the trivial case, we assume henceforth that $\alpha > 0$.

Our proof of existence will show that there exists an optimal curve in $S(u, v)$ having one of the following two forms.

Definition 5.2. A curve f is of

- (i) **first form** if there exist $-\pi < t_0 < t < \pi$ such that f is directly similar to $R_{[t_0, t]}$,
- (ii) **second form** if there exists $c \geq 0$ and $t \in [0, \pi]$ such that f is directly similar to

$$R_{[-\pi, 0]} \sqcup [0, c] \sqcup (c + R_{[0, t]}).$$

Note that curves of first form do not contain u-turns, while curves of second form do. While studying a generic right-left s-curve $f \in S(u, v)$, the following quantities will gradually take on significance, but for easy reference we gather and define them here. The minimum direction angle $\gamma = \min \arg(f')$ is illustrated in Fig. 5.3 for a non-degenerate right-left s-curve f , while $\gamma = \beta$ if f is a right c-curve. The set of all possible angles γ is denoted Γ .

Definition 5.3. For γ in $\Gamma := [\alpha - \pi, \beta] \cap (-\infty, 0)$, we define the following:

$$\begin{aligned} y_1 &:= y_1(\gamma) := \frac{1}{2} \int_0^{\alpha-\gamma} \sqrt{\sin \tau} \, d\tau && \text{(bending energy of } R_{[0,t_1]}) \\ y_2 &:= y_2(\gamma) := \frac{1}{2} \int_0^{\beta-\gamma} \sqrt{\sin \tau} \, d\tau && \text{(bending energy of } R_{[0,t_2]}) \\ G(\gamma) &:= \frac{1}{-\sin \gamma} (y_1 + y_2)^2 && \text{(lower bound on } \|f\|^2) \\ \sigma(\gamma) &:= \cos \gamma + \frac{\sin \gamma}{y_1 + y_2} (\sqrt{\sin(\alpha - \gamma)} + \sqrt{\sin(\beta - \gamma)}) && \text{(signed distance)} \\ q(\gamma) &:= \frac{-\sin \gamma}{y_1 + y_2} && \text{(dilation factor)} \end{aligned}$$

Note that, by Lemma 2.1 (see Fig. 2.1), y_1 and y_2 can also be expressed as $y_1 = \xi(t_1) = \|R_{[0,t_1]}^2\|$ and $y_2 = \xi(t_2) = \|R_{[0,t_2]}^2\|$, where $t_1, t_2 \in [0, \pi]$ are determined by $\Delta(R_{[0,t_1]}) = \alpha - \gamma$ and $\Delta(R_{[0,t_2]}) = \beta - \gamma$.

We mention further that $G(\gamma)$ (see Theorem 5.6) is a lower bound on the bending energy of our generic curve f , and $\sigma(\gamma)$ is a signed distance, which is illustrated in Fig. 5.4. Regarding $q(\gamma)$, we mention that the curves $r(u, \lambda)$ and $l(\lambda, v)$, shown in Fig. 5.4, are similar to $R_{[0,t_1]}$ and $R_{[0,t_2]}$, respectively, with common dilation factor $q(\gamma)$. The crucial identity relating $G(\gamma)$, $\sigma(\gamma)$ and $q(\gamma)$ is given in Lemma 5.11.

Our constructive proof that $S(u, v)$ contains an optimal curve is broken into three cases which depend on α and β . To help the reader track these cases, we give here a short description of each case and where in this section or the next it is treated.

Summary 5.4. We assume $\alpha \in (0, \pi)$, $|\beta| \leq \alpha$ and $\beta \geq \alpha - \pi$.

Case A: $\beta = \alpha - \pi$.

This case is treated just below and results in an optimal curve of second form.

Case B: $\beta \geq 0$ or $(\alpha - \pi < \beta < 0$ and $\sigma(\beta) \geq 0)$.

It is shown in Lemma 5.11 that the function G has a minimum value G_{min} , and in Corollary 5.12 (vi) it is shown that G_{min} equals the minimum bending energy in $S(u, v)$. Each $\gamma \in \Gamma$, where G is minimized, gives rise to an optimal curve in $S(u, v)$, but the form of the optimal curve depends on whether or not γ is the left endpoint of Γ . If G is minimized at the left endpoint $\gamma = \alpha - \pi$, then it is shown in Corollary 5.12 (iii), that the curve $f_{\alpha-\pi}$, which is of second form, is an optimal curve in $S(u, v)$. If G is minimized at any other point $\gamma > \alpha - \pi$, then it is shown in Corollary 5.12 (iv), (v) that the curve f_γ , which is of first form, is an optimal curve in $S(u, v)$.

Case C: $\alpha - \pi < \beta < 0$ and $\sigma(\beta) \leq 0$.

In Theorem 6.2, it is shown that the unique curve (modulo equivalence) in $S(u, v)$ having minimal bending energy is a c-curve of first form.

Remark. The reader may note that Case B and Case C have some overlap (namely, when $\alpha - \pi < \beta < 0$ and $\sigma(\beta) = 0$); this overlap is intentional and serves as a bridge from Case B to Case C. The construction under Case C is the better because it yields a unique optimal curve.

Proof of Theorem 5.1 for Case A. Assume $\beta = \alpha - \pi$. Then $\alpha \geq \pi/2$ and $S(u, v) = C_r(u, v)$ (since the common turning angle in $S(u, v)$ is $-\pi$). Let λ be the line through 1 which is parallel to v (and also parallel to u) and let P_1 be the terminal point of $r(u, \lambda)$ (see Fig. 5.2). Since $\alpha \geq \pi/2$, it follows that P_1 lies on or above the real axis and therefore $f := r(u, \lambda) \sqcup [P_1, 1]$ belongs to $C_r(u, v)$. Since $S(u, v) = C_r(u, v) \subset C_r(u, \lambda)$, it follows from Theorem 3.6 and Remark 3.7 that f has minimal bending energy in $S(u, v)$. Furthermore, one easily deduces from Corollary 4.4 (i) and Remark 3.7 that f is unique modulo equivalence and elongation of u-turns (see remark below). Note that f is of second form with $t = 0$.

Regarding the quantities defined in Definition 5.3, the set Γ reduces to the singleton $\Gamma = \{\beta\}$ and since the orthogonal distance from 0 to λ is $|P_1 - 0| = -\sin \beta$, it follows from Theorem 3.6 that $\|f\|^2 = d^2/(-\sin \beta) = G(\beta)$. It is easy to see that the distance from P_1 to 1 equals $\cos \beta$, and hence $|P_1 - 1| = \cos \beta = \sigma(\beta)$. Lastly, $q(\beta) = (-\sin \beta)/d$ corresponds to the dilation factor from $R_{[-\pi, 0]}$ to the similar curve $r(u, \lambda)$. \square

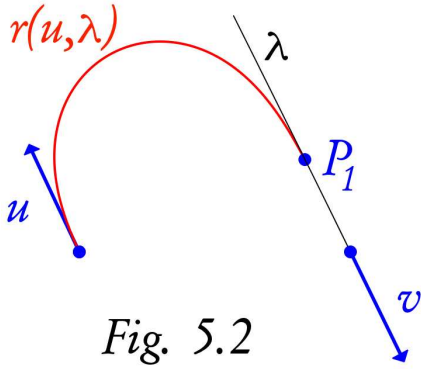


Fig. 5.2

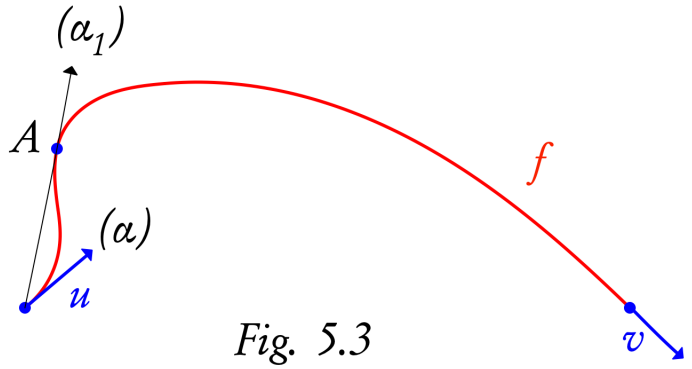


Fig. 5.3

Remark 5.5. If a curve $f \in S(u, v)$ contains a u-turn (eg. the curve $r(u, \lambda) \sqcup [P_1, 1]$ above), then it is always possible to *elongate* the u-turn by inserting a pair of congruent line segments before and after the u-turn. Although longer, the resulting curve still belongs to $S(u, v)$ and has the same bending energy as f .

Having settled the trivial case ($\alpha = 0$) and Case A, we proceed assuming that

$$(5.1) \quad \alpha \in (0, \pi), \quad |\beta| \leq \alpha, \quad \beta > \alpha - \pi.$$

Our analysis employs an initial partitioning $S(u, v) = S'_{lr}(u, v) \cup S_{rl}(u, v)$, where $S'_{lr}(u, v)$ (which is nonempty if and only if $\beta < 0$) denotes the set of all non-degenerate left-right s-curves in $S(u, v)$ and $S_{rl}(u, v)$ denotes the set of all right-left s-curves in $S(u, v)$.

Proposition 5.6. *If $\beta < 0$, then $\|f\|^2 > G(\beta)$ for all $f \in S'_{lr}(u, v)$.*

Proof. Let $f : [a, b] \rightarrow \mathbb{C}$ be a non-degenerate left-right s-curve in $S(u, v)$. Set $\alpha_1 = \min_{t \in (a, b)} \arg f(t)$ and let $A = f(s_1)$ be a point where this minimum is attained. It can be shown that $\arg f'(s_1) = \alpha_1 > \alpha$ and that $f_{[s_1, b]}$ is a right c-curve. Let u_1 be the unit tangent vector $u_1 = (0, e^{i\alpha_1})$ and let λ denote the line through 1 which is parallel to v .

Then $g = [0, A] \cup f_{[s_1, b]}$ belongs to $C_r(u_1, \lambda)$, and we have

$$\begin{aligned} \|f\|^2 &> \|f_{[s_1, b]}\|^2 = \|g\|^2 \geq \|r(u_1, \lambda)\|^2 = \frac{1}{-\sin \beta} \left(\frac{1}{2} \int_0^{\alpha_1 - \beta} \sqrt{\sin \tau} \, d\tau \right)^2 \\ &> \frac{1}{-\sin \beta} \left(\frac{1}{2} \int_0^{\alpha - \beta} \sqrt{\sin \tau} \, d\tau \right)^2 = \|r(u, \lambda)\|^2 = G(\beta). \end{aligned}$$

□

Although it is not yet apparent, it will eventually be clear that Proposition 5.6 is all we need to rule out curves in $S'_{lr}(u, v)$. We now turn our attention to $S_{rl}(u, v)$ and define subsets $s_\gamma^*(u, v) \subset s_\gamma(u, v) \subset S_{rl}(u, v)$. For $\gamma \in \Gamma$, let $s_\gamma(u, v)$ be the set of all curves $f \in S_{rl}(u, v)$ whose minimal direction angle, $\min \arg(f')$, equals γ . One easily verifies that $S_{rl}(u, v)$ partitions as $S_{rl}(u, v) = \bigcup_{\gamma \in \Gamma} s_\gamma(u, v)$.

If $\beta < 0$, then $s_\beta(u, v)$ simplifies to $s_\beta(u, v) = C_r(u, v)$ and we also define $s_\beta^*(u, v) = C_r(u, v)$. The definition of $s_\gamma^*(u, v)$ is much more involved when $\gamma < \beta$: Let $\gamma \in \Gamma$ with $\gamma < \beta$, and let $f \in s_\gamma(u, v)$. Then f , being a non-degenerate right-left s-curve, has a well defined inflection line λ with direction angle γ (see Figure 5.4).

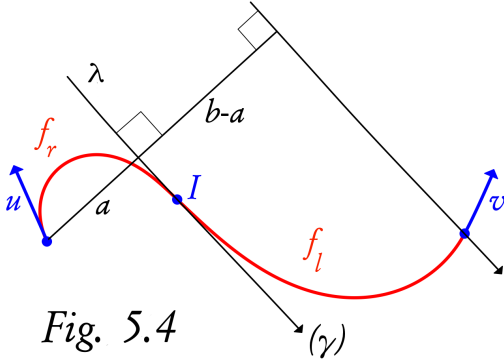


Fig. 5.4

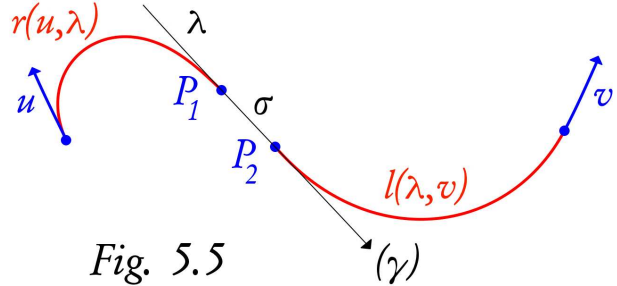


Fig. 5.5

Let $a(f)$ denote the orthogonal distance from 0 to λ . Since λ necessarily passes between 0 and 1, it follows that $a(f)$ belongs to the interval $(0, b)$, where $b = -\sin \gamma$ denotes the orthogonal distance from 0 to the line through 1 and parallel to λ . We define $s_\gamma^*(u, v)$ to be the set of all curves $f \in s_\gamma(u, v)$ for which $a(f) = -\sin \gamma \frac{y_1}{y_1 + y_2}$.

Theorem 5.7. *Assume (5.1). For $\gamma \in \Gamma$ and $f \in s_\gamma(u, v)$, the following hold.*

- (i) $\|f\|^2 \geq G(\gamma)$.
- (ii) If $\|f\|^2 = G(\gamma)$, then $f \in s_\gamma^*(u, v)$.

Proof. We first consider the simpler case when $\beta < 0$ and $\gamma = \beta$. Since $s_\beta^*(u, v) = s_\beta(u, v) = C_r(u, v)$ we need only prove (i). Let λ be the line through 1 which is parallel to v . Then $C_r(u, v) \subset C_r(u, \lambda)$ and it follows that $\|f\|^2 \geq \|r(u, \lambda)\|^2$, since $r(u, \lambda)$ has minimal bending energy in $C_r(u, \lambda)$. Noting that the orthogonal distance from 0 to λ is $-\sin \beta$ and the common turning angle in $C_r(u, \lambda)$ has magnitude $\alpha - \beta$, it follows from Theorem 3.6 and Remark 3.7 that $\|r(u, \lambda)\|^2 = G(\beta)$; hence (i).

Now assume that $\gamma \in \Gamma$ with $\gamma < \beta$ and let $f \in s_\gamma(u, v)$. Let the inflection line λ and the distances $a(f)$ and $b = -\sin \gamma$ be as described above (see Fig. 5.4), and let us write

$f = f_r \sqcup f_l$, where f_r terminates (and f_l originates) at an inflection point I of f . Since $f_r \in C_r(u, \lambda)$ and $f_l \in C_l(\lambda, v)$, it follows that

$$(5.2) \quad \|f\|^2 = \|f_r\|^2 + \|f_l\|^2 \geq \|r(u, \lambda)\|^2 + \|l(\lambda, v)\|^2 = \frac{y_1^2}{a(f)} + \frac{y_2^2}{b - a(f)}.$$

Let H be the function defined in Lemma 4.2 with $\nu_1 = y_1$, $\nu_2 = y_2$ and $\mu = \frac{a(f)}{b} \in (0, 1)$. Then (5.2) can be expressed as $\|f\|^2 \geq \frac{1}{b}H(\mu)$, and it follows from Lemma 4.2 that $H(\mu) \geq (y_1 + y_2)^2$, with equality if and only if $\mu = y_1/(y_1 + y_2)$; that is, if and only if $a(f) = -\sin \gamma \frac{y_1}{y_1 + y_2}$. We therefore conclude that $\|f\|^2 > G(\gamma)$ if $f \notin s_\gamma^*(u, v)$ and $\|f\|^2 \geq G(\gamma)$ if $f \in s_\gamma^*(u, v)$, which proves both (i) and (ii). \square

For $\gamma \in \Gamma$ with $\gamma < \beta$, let λ_γ denote the common inflection line in $s_\gamma^*(u, v)$, and let $a_\gamma = -\sin \gamma \frac{y_1}{y_1 + y_2}$ denote the orthogonal distance from 0 to λ_γ ; furthermore (see Fig. 5.5), let P_1 and P_2 denote the terminal and initial points of $r(u, \lambda_\gamma)$ and $l(\lambda_\gamma, v)$, respectively. If $\beta < 0$ and $\gamma = \beta$, let λ_β denote the line through 1 which is parallel to v and let a_γ and P_1 be as defined above (note that $a_\beta = -\sin \beta$ still equals the orthogonal distance from 0 to λ_β), but for convenience set $P_2 = 1$ (the base-point of v). We now show that the quantity $q(\gamma)$ corresponds to a common dilation parameter.

Proposition 5.8. *Assume (5.1) and let $\gamma \in \Gamma$. Then the following hold.*

- (i) *The curve $r(u, \lambda_\gamma)$ is directly congruent to $q(\gamma)R_{[-t_1, 0]}$.*
- (ii) *If $\gamma < \beta$, then $l(\lambda_\gamma, v)$ is directly congruent to $q(\gamma)R_{[0, t_2]}$.*

Proof. Let T_1 be the similarity transformation such that $r(u, \lambda_\gamma) = T_1 \circ R_{[-t_1, 0]}$ and if $\gamma < \beta$, let T_2 be such that $l(\lambda_\gamma, v) = T_2 \circ R_{[0, t_2]}$. Since the orthogonal distance from $R(-t_1)$ to the real axis is $\xi(t_1)$ and the orthogonal distance from 0 to λ_γ is a_γ , it follows that the dilation parameter of T_1 equals $\frac{a_\gamma}{\xi(t_1)} = \frac{1}{y_1} \frac{y_1}{y_1 + y_2} (-\sin \gamma) = q(\gamma)$. If $\gamma < \beta$, we see by similar reasoning, that the dilation parameter of T_2 equals $\frac{b - a_\gamma}{\xi(t_2)} = \frac{1}{y_2} (-\sin \gamma + y_1 \sin \gamma / (y_1 + y_2)) = q(\gamma)$, where $b = -\sin \gamma$ denotes the orthogonal distance from 0 to the line through 1 which is parallel to λ_γ . \square

The following result shows that the quantity $\sigma(\gamma)$ corresponds to the signed distance from P_1 to P_2 .

Proposition 5.9. *Assume 5.1 and let $\gamma \in \Gamma$. Then $\sigma(\gamma)$ equals the signed distance, in the direction $e^{i\gamma}$, from P_1 to P_2 .*

Proof. We consider first the case $\gamma < \beta$. Let h denote the signed distance in question, and put $B = P_2 - P_1 = he^{i\gamma}$ (see Figure 5.5). It follows from Proposition 5.8 that $f = r(u, \lambda_\gamma) \sqcup (l(\lambda_\gamma, v) - B)$ is directly congruent to $q(\gamma)R_{[-t_1, t_2]}$. Since the projected distance, in the direction e^{i0} , from $R(-t_1)$ to $R(t_2)$ equals $\sin t_1 + \sin t_2$, it follows that the projected distance, in the direction $e^{i\gamma}$, from the initial point to the terminal point of f equals $q(\gamma)(\sin t_1 + \sin t_2)$. Noting that the projected distance, in the direction $e^{i\gamma}$, from 0 to 1 equals $\cos \gamma$, we deduce that $q(\gamma)(\sin t_1 + \sin t_2) = \cos \gamma - h$. Solving for h and then employing the identity $\sin^2 \delta = \sin t$, when $\delta = \Delta(R_{[0, t]})$ and $t \in [0, \pi]$, yields the desired conclusion $h = \sigma(\gamma)$.

If $\beta < 0$ and $\gamma = \beta$, then $t_2 = 0$ and the above proof, with $f = r(u, \lambda_\beta)$, yields the same conclusion $h = \sigma(\gamma)$. \square

Remark 5.10. Two important consequences of Proposition 5.8 and Proposition 5.9 are:

1. Let $\gamma \in \Gamma$, with $\gamma > \alpha - \pi$. If $\sigma(\gamma) = 0$, then $f_\gamma := r(u, \lambda_\gamma) \sqcup l(\lambda_\gamma, v)$ has bending energy $G(\gamma)$ and is directly congruent to $q(\gamma)R_{[-t_1, t_2]}$. It follows from the latter that f_γ is of first form with $t_0 = -t_1$ and $t = t_2$.
2. If $\sigma(\alpha - \pi) \geq 0$, then $f_{\alpha - \pi} := r(u, \lambda_\gamma) \sqcup [P_1, P_2] \sqcup l(\lambda_\gamma, v)$ has bending energy $G(\alpha - \pi)$ and is of second form with $c = \sigma(\alpha - \pi)/q(\alpha - \pi)$ and $t = t_2$.

In the following result, we see that $\sigma(\gamma)$ appears as a factor in the derivative $G'(\gamma)$.

Lemma 5.11. *Assume (5.1). The function $G : \Gamma \rightarrow (0, \infty)$ is continuously differentiable, has a minimum value G_{min} , and satisfies $\frac{d}{d\gamma}G(\gamma) = \frac{1}{q(\gamma)^2}\sigma(\gamma)$ for all $\gamma \in \Gamma$.*

Proof. For $\gamma \in \Gamma$, we have

$$\begin{aligned} G'(\gamma) &= \frac{\cos \gamma}{\sin^2 \gamma} (y_1 + y_2)^2 - \frac{2}{\sin \gamma} (y_1 + y_2)(y_1'(\gamma) + y_2'(\gamma)) \\ &= \frac{1}{q(\gamma)^2} \left(\cos \gamma - \frac{2 \sin \gamma}{y_1 + y_2} \left(-\frac{1}{2} \sqrt{\sin(\alpha - \gamma)} - \frac{1}{2} \sqrt{\sin(\beta - \gamma)} \right) \right) = \frac{1}{q(\gamma)^2} \sigma(\gamma), \end{aligned}$$

and we note that both q and σ are continuous on Γ and q is positive. If $\beta < 0$, then $\Gamma = [\alpha - \pi, \beta]$ and it is clear that G has a minimum value. On the other hand, if $\beta \geq 0$, then $\Gamma = [\alpha - \pi, 0)$, but we note that $G(\gamma) \rightarrow \infty$ as $\gamma \rightarrow 0^-$; hence G has a minimum value. \square

In preparation for the main result of this section, we remind the reader that $S(u, v)$ has been partitioned as

$$(5.3) \quad S(u, v) = S'_{lr}(u, v) \cup \bigcup_{\gamma \in \Gamma} s_\gamma(u, v),$$

where $S'_{lr}(u, v)$ is nonempty only when $\beta < 0$.

Corollary 5.12. *Let (5.1) be in force, and in case $\beta < 0$, assume $\sigma(\beta) \geq 0$. The following hold.*

- (i) *If $\beta < 0$, then $\|f\|^2 > G_{min}$ for all $f \in S'_{lr}(u, v)$.*
- (ii) *If $\gamma \in \Gamma$ and $G(\gamma) > G_{min}$, then $\|f\|^2 > G_{min}$ for all $f \in s_\gamma(u, v)$.*
- (iii) *If $G(\alpha - \pi) = G_{min}$, then $\sigma(\alpha - \pi) \geq 0$ and the curve $f_{\alpha - \pi}$, defined in Remark 5.10, is the unique curve, modulo equivalence and elongation of u -turns, in $s_{\alpha - \pi}(u, v)$ with bending energy G_{min} .*
- (iv) *Let $\gamma \in \Gamma$ with $\alpha - \pi < \gamma < \beta$. If $G(\gamma) = G_{min}$, then $\sigma(\gamma) = 0$ and the curve f_γ , defined in Remark 5.10, is the unique curve (modulo equivalence) in $s_\gamma(u, v)$ with bending energy G_{min} .*
- (v) *If $\beta < 0$ and $G(\beta) = G_{min}$, then $\sigma(\beta) = 0$ and $f_\beta := r(u, \lambda_\beta)$, which is of first form with $t_0 = -t_1$ and $t = 0$, is the unique curve (modulo equivalence) in $s_\beta(u, v)$ with bending*

energy G_{min} .

(vi) The minimum bending energy in $S(u, v)$ is G_{min} .

Proof. Items (i) and (ii) are immediate consequences of Proposition 5.6 and Theorem 5.7 (i), respectively. For (iii), assume $G(\alpha - \pi) = G_{min}$. If $\sigma(\alpha - \pi) < 0$, then it follows from Theorem 5.11 that $G'(\alpha - \pi) < 0$, which contradicts the assumption that G attains its minimum at $\alpha - \pi$; therefore, $\sigma(\alpha - \pi) \geq 0$. It now follows from Remark 5.10 that $f_{\alpha-\pi}$ has bending energy G_{min} . Now, suppose $f \in s_{\alpha-\pi}(u, v)$ has bending energy $G(\alpha - \pi) = G_{min}$. By Theorem 5.7 (ii), f belongs to $s_{\alpha-\pi}^*(u, v)$ and writing $f = f_1 \cup f_2$, as in the discussion preceding (5.2), it follows from (5.2) that $\|f_1\|^2 = \|r(u, \lambda_{\alpha-\pi})\|^2$ and $\|f_2\|^2 = \|l(\lambda_{\alpha-\pi}, v)\|^2$. It can then be deduced from the results of section 4 that f is equivalent to $f_{\alpha-\pi}$ or can be obtained (equivalently) from $f_{\alpha-\pi}$ by elongation of u-turns. We have thus proved (iii). Turning next to (iv), assume $G(\gamma) = G_{min}$. Then $G'(\gamma) = 0$ (since γ is an interior point of Γ) and by Lemma 5.11, we have $\sigma(\gamma) = 0$. It now follows from Remark 5.10 that f_γ has bending energy G_{min} and the previous argument can be applied to show that if $f \in s_\gamma(u, v)$ has bending energy G_{min} , then f is equivalent to f_γ (elongation of u-turns is ruled out since curves in $s_\gamma(u, v)$ do not have u-turns). This proves (iv). For (v), assume $\beta < 0$ and $G(\beta) = G_{min}$. If $\sigma(\beta) > 0$, then $G'(\beta) > 0$, by Lemma 5.11, which contradicts the assumption that G is minimized at β . Therefore, $\sigma(\beta) = 0$ and it follows that $r(u, \lambda_\beta)$ belongs to $s_\beta(u, v) = C_r(u, v)$ (i.e. $P_1 = 1$). From Proposition 5.8 (i) we have that $r(u, \lambda_\beta)$ is directly congruent to $q(\beta)R_{[-t_1, 0]}$, and from this it is easy to verify that $\|r(u, \lambda_\beta)\|^2 = G(\beta)$. Thus $\|r(u, \lambda_\beta)\|^2 = G_{min}$, and we note that $r(u, \lambda_\beta)$ is of first form, with $t_0 = -t_1$ and $t = 0$. Since the turning angle in $r(u, \lambda_\beta)$ has magnitude less than π , it easily follows from Theorem 4.5 and Remark 3.7 that $r(u, \lambda_\beta)$ is the unique curve (modulo equivalence) in $s_\beta(u, v)$ with bending energy G_{min} and the proof of (v) is complete. We now prove (vi). It follows from (i), Theorem 5.7 and (5.3) that $\|f\|^2 \geq G_{min}$ for all $f \in S(u, v)$. Since the function G has a minimum, there exists $\gamma \in \Gamma$ such that $G(\gamma) = G_{min}$, and it then follows from items (iii), (iv) and (v) that f_γ is a curve in $S(u, v)$ with bending energy G_{min} . This proves (vi). \square

As explained in Summary 5.4, Case B of Theorem 5.1 is a consequence of Corollary 5.12.

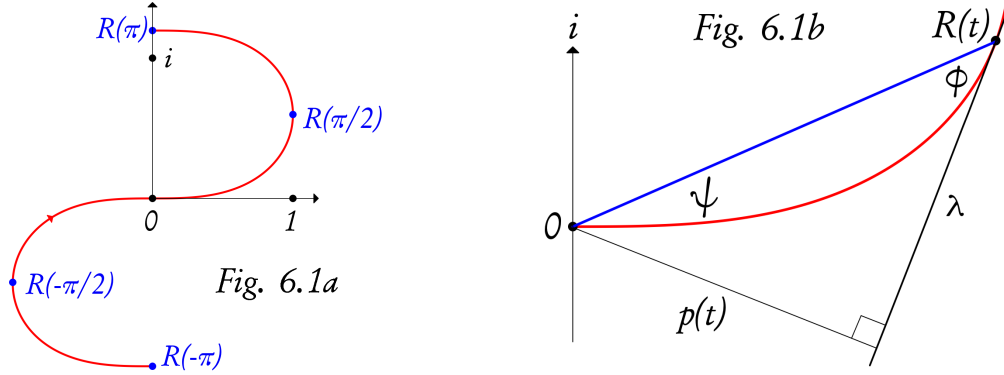
6. Optimal s-curves, part II

The purpose of this section is to prove the following two results, where we note that Case C (see Summary 5.4) of Theorem 5.1 follows from the latter.

Theorem 6.1. *Let $0 \leq t_1 < t_2 \leq \pi$ satisfy $t_2 - t_1 < \pi$, and let $u = \vec{R}(t_1)$ and $v = \vec{R}(t_2)$ be the initial and terminal unit tangent vectors, respectively, to the curve $R_{[t_1, t_2]}$. Then $R_{[t_1, t_2]}$ is the unique curve (modulo equivalence) in $S(u, v)$ with minimal bending energy.*

Theorem 6.2. *In the notation of section 5, let $\alpha \in (0, \pi)$ and $\beta < 0$ satisfy (5.1) and suppose $\sigma(\beta) \leq 0$. Then there exist $-\pi < t_1 < t_2 \leq 0$ such that $R_{[t_1, t_2]}$ is directly similar to a curve $f \in S(u, v)$. Moreover, the curve f is the unique curve (modulo equivalence) in $S(u, v)$ with minimal bending energy.*

For $t \in (0, \pi]$, let ψ and ϕ , as shown in Figure 6.1b, be the positive angles made by the chord $[0, R(t)]$ and the segment $R_{[0,t]}$. With λ denoting the tangent line to R at $R(t)$, let $p(t)$ denote the orthogonal distance from 0 to λ .



Lemma 6.3. For $t \in (0, \pi)$, $\phi(t) > \psi(t)$.

Proof. We will first show, by way of contradiction, that $\phi(t) \neq \psi(t)$. Assume $\phi(t) = \psi(t)$ for some $t \in (0, \pi)$. Let $T(z) = c_1 \bar{z} + c_2$ be the congruency transformation which interchanges $R(t)$ and 0, and set $W = T \circ R_{[t,0]}$, where $R_{[t,0]}$ denotes the reversal of $R_{[0,t]}$. Since $\phi(t) = \psi(t)$, it follows that W belongs to $C_l(\vec{R}(0), \vec{R}(t))$. But Theorem 4.5 asserts that $R_{[0,t]}$ is the unique curve (modulo equivalence) in $C_l(\vec{R}(0), \vec{R}(t))$ with minimal bending energy. Therefore, since W and $R_{[0,t]}$ have the same bending energy, they must be equivalent. However, they cannot be equivalent because W begins with a nonzero curvature, namely $2 \sin t$, while $R_{[0,t]}$ begins with curvature 0. This proves that $\phi(t) \neq \psi(t)$ for all $t \in (0, \pi)$. While $\phi(\pi)$ and $\psi(\pi)$ both equal $\pi/2$, a simple computation shows that their derivatives satisfy $-\phi'(\pi) = \psi'(\pi) = 1/d$, and it follows that $\phi(t) > \psi(t)$ for $t \in (0, \pi)$ sufficiently close to π . Since ϕ and ψ are continuous, we conclude that $\phi(t) > \psi(t)$ for all $t \in (0, \pi)$. \square

Lemma 6.4. For $t \in [0, \pi)$, $p(t)\xi(t) < (2d - \xi(t))^2$.

Proof. The orthogonal distance $p(t)$ can be formulated as the magnitude of the cross product $R(t) \times \frac{R'(t)}{|R'(t)|}$ which yields

$$p(t) = \det \begin{bmatrix} \sin t & \xi(t) \\ \cos t \sqrt{1 + \sin^2 t} & \sin^2 t \end{bmatrix} = \sin^3 t - \xi(t) \cos t \sqrt{1 + \sin^2 t}, \quad 0 \leq t \leq \pi.$$

We therefore have

$$\begin{aligned} p(t)\xi(t) - (2d - \xi(t))^2 &= (\sin^3 t + 4d)\xi(t) - 4d^2 - \left(1 + \cos t \sqrt{1 + \sin^2 t}\right) \xi(t)^2 \\ &\leq (\sin^3 t + 4d)\xi(t) - 4d^2 =: g(t), \end{aligned}$$

where the inequality holds since $-1 \leq \cos t \sqrt{1 + \sin^2 t} \leq 1$. We note that $g(\pi) = 0$ and $g'(t) = \sin^2 t \left(3\xi(t) \cos t + \frac{\sin^3 t + 4d}{\sqrt{2 - \cos t}} \right)$. It is clear that $g'(t) > 0$ for $t \in (0, \pi/2]$, and for

$t \in (\pi/2, \pi)$ (where $-\cos t > 0$), we have

$$g'(t) = (-3 \cos t) \sin^2 t \left(-\xi(t) + \frac{\sin^3 t + 4d}{-3 \cos t \sqrt{2 - \cos t}} \right) \geq (-3 \cos t) \sin^2 t \left(-\xi(t) + \frac{4d}{3} \right),$$

as $0 < -\cos t \sqrt{2 - \cos t} < 1$ on $(\pi/2, \pi)$. Since $0 \leq \xi(t) \leq d$, it follows that $g'(t) > 0$ for all $t \in (0, \pi)$ and hence g is increasing on $[0, \pi]$. For $t \in [0, \pi)$, we therefore have $p(t)\xi(t) - (2d - \xi(t))^2 \leq g(t) < g(\pi) = 0$, which completes the proof. \square

In the following, we again use the notation $S'_{lr}(u, v)$ (resp. $S'_{rl}(u, v)$) for the set of all non-degenerate left-right (resp. right-left) s-curves connecting u to v .

Lemma 6.5. *For $t \in (0, \pi)$, the following hold:*

(i) *If $f \in S'_{lr}(\vec{R}(0), \vec{R}(t))$, then $\|f\|^2 > \|R_{[0,t]}\|^2$.*

(ii) *If $f \in S'_{rl}(\vec{R}(0), \vec{R}(t))$ ends with a left u-turn, then $\|f\|^2 > \|R_{[0,t]}\|^2$.*

Proof. We will employ the notation and results of the previous section, so in order to minimize confusion, we will actually prove the following equivalent formulations:

(i') *If $f \in S'_{lr}(\vec{R}(-t), \vec{R}(0))$, then $\|f\|^2 > \|R_{[-t,0]}\|^2$.*

(ii') *If $f \in S'_{rl}(\vec{R}(-t), \vec{R}(0))$ begins with a right u-turn, then $\|f\|^2 > \|R_{[-t,0]}\|^2$.*

Let $T(z) = c_1 z + c_2$ be the similarity transformation determined by $T(\vec{R}(-t)) = 0$ and $T(0) = 1$, and note that T brings the configuration $(\vec{R}(-t), \vec{R}(0))$ to the canonical form (u, v) (see Figure 6.2), where $u = (0, e^{i\alpha})$, $v = (1, e^{i\beta})$ with $\alpha = \phi(t)$, $\beta = -\psi(t)$. Since $0 < \psi(t) < \phi(t) < \pi$, it follows that (5.1) holds. Noting that $r(u, \lambda_\beta) = T \circ R_{[-t,0]}$, we see that $\sigma(\beta) = 0$ and $G(\beta) = \|r(u, \lambda_\beta)\|^2$. For (i'), suppose $f \in S'_{lr}(\vec{R}(-t), \vec{R}(0))$. Then $T \circ f \in S'_{lr}(u, v)$, and it follows from Proposition 5.6 that $\|T \circ f\|^2 > \|r(u, \lambda_\beta)\|^2$. Since $r(u, \lambda_\beta) = T \circ R_{[-t,0]}$, we immediately obtain (i'). Now suppose $f \in S'_{rl}(\vec{R}(-t), \vec{R}(0))$ begins with a right u-turn. Then $T \circ f$ belongs to the set $s_{\alpha-\pi}(u, v)$ defined just above Theorem 5.7, and it follows from this theorem that $\|T \circ f\|^2 \geq G(\alpha - \pi)$. Since $G(\beta) = \|r(u, \lambda_\beta)\|^2$ and $r(u, \lambda_\beta) = T \circ R_{[-t,0]}$, in order to establish (ii'), it suffices to show that $G(\alpha - \pi) > G(\beta)$. From Definition 5.3, we have $G(\beta) = \xi(t)^2 / \sin \psi(t)$ and $G(\alpha - \pi) = (2d - \xi(t))^2 / \sin \phi(t)$. Referring to Figure 6.1b, we see that $\sin \psi(t) = \xi(t) / |R(t)|$ and $\sin \phi(t) = p(t) / |R(t)|$. Hence

$$G(\alpha - \pi) - G(\beta) = \frac{|R(t)|}{p(t)} ((2d - \xi(t))^2 - p(t)\xi(t)) > 0,$$

by the previous lemma, and this completes the proof of (ii'). \square

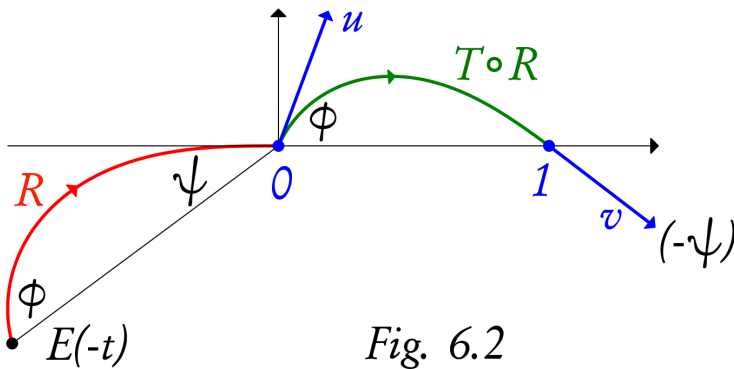


Fig. 6.2

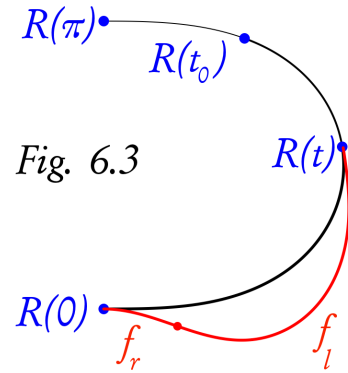


Fig. 6.3

Proposition 6.6. *Let $t \in (0, \pi)$. Then $R_{[0,t]}$ is the unique curve (modulo equivalence) in $S(\vec{R}(0), \vec{R}(t))$ having minimal bending energy.*

Proof. With Corollary 3.4, Theorem 4.5 and Lemma 6.5 in view, it suffices to show that $\|f\|^2 > \|R_{[0,t]}\|^2$ whenever $f \in S'_{rl}(\vec{R}(0), \vec{R}(t))$ does not end with a left u-turn. Let $f = f_r \sqcup f_l$ be as stated, where f_r is a right c-curve and f_l is a left c-curve (see Figure 6.3). Since f_r originates on $\vec{R}(0)$, it follows that there exists $t_0 \in (t, \pi)$ such that $f_l \sqcup R_{[t,t_0]}$ is a left u-turn. Thus $f \sqcup R_{[t,t_0]}$ belongs to $S'_{rl}(\vec{R}(0), \vec{R}(t_0))$ and ends with a left u-turn. By Lemma 6.5 (ii),

$$\|f\|^2 + \|R_{[t,t_0]}\|^2 = \|f \sqcup R_{[t,t_0]}\|^2 > \|R_{[0,t_0]}\|^2 = \|R_{[0,t]}\|^2 + \|R_{[t,t_0]}\|^2,$$

whence we obtain $\|f\|^2 > \|R_{[0,t]}\|^2$. \square

Remark 6.7. By symmetry, it follows from Proposition 6.6 that $R_{[t,\pi]}$ is the unique curve (modulo equivalence) in $S(\vec{R}(t), \vec{R}(\pi))$ having minimal bending energy.

In the context of the previous section, Proposition 6.6 asserts the following.

Corollary 6.8. *Let direction angles $\alpha \in (0, \pi)$, $\beta < 0$ satisfy (5.1) and suppose $\sigma(\beta) = 0$. Then $G(\gamma) > G(\beta)$ for all $\gamma \in [\alpha - \pi, \beta]$; that is, $G(\gamma)$ is uniquely minimized at $\gamma = \beta$.*

Proof of Theorem 6.1. The extreme cases $t_1 = 0$ and $t_2 = \pi$ have been settled in Proposition 6.6 and Remark 6.7, respectively, so assume $0 < t_1 < t_2 < \pi$. By symmetry, and with Corollary 3.4 and Theorem 4.5 in view, it suffices to show that $\|f\|^2 > \|R_{[t_1,t_2]}\|^2$ whenever f belongs to $S'_{rl}(\vec{R}(t_1), \vec{R}(t_2))$. Let f be as stated, and let $\gamma \in (-\pi, \pi)$ be the direction angle of f at an inflection point I .

Case 1: $\gamma \in [0, \pi)$.

Then $f \sqcup R_{[t_2,\pi]}$ belongs to $S'_{rl}(\vec{R}(t_1), \vec{R}(\pi))$, and it follows from Remark 6.7 that $\|f \sqcup R_{[t_2,\pi]}\|^2 > \|R_{[t_1,\pi]}\|^2$, which implies $\|f\|^2 > \|R_{[t_1,t_2]}\|^2$.

Case 2: $\gamma \in (-\pi, 0)$.

Since f begins at $R(t_1)$ with a direction angle in $(0, \pi)$, there exists a point B on f , between $R(t_1)$ and I , where f has direction angle 0 (see Figure 6.4). Let us write $f := f_1 \sqcup f_2$, where f_1 terminates (and f_2 originates) at B . Let λ be the (horizontal) tangent line to f at B , and set $g := l(\lambda, \vec{R}(t_2))$. Since g and $R_{[0,t_2]}$ are similar and terminate at the same unit tangent vector, it follows that g originates at the point of intersection A between λ and the line segment $[0, R(t_2)]$. Moreover, since g is at a smaller scale than $R_{[0,t_2]}$, we have $\|g\|^2 > \|R_{[0,t_2]}\|^2$. Now, it follows from Proposition 6.6 that $\|[A, B] \sqcup f_2\|^2 > \|g\|^2$, and therefore

$$\|f\|^2 > \|[A, B] \sqcup f_2\|^2 > \|g\|^2 > \|R_{[0,t_2]}\|^2 > \|R_{[t_1,t_2]}\|^2.$$

\square

Remark 6.9. By symmetry, Theorem 6.1 remains valid when $-\pi \leq t_1 < t_2 \leq 0$.

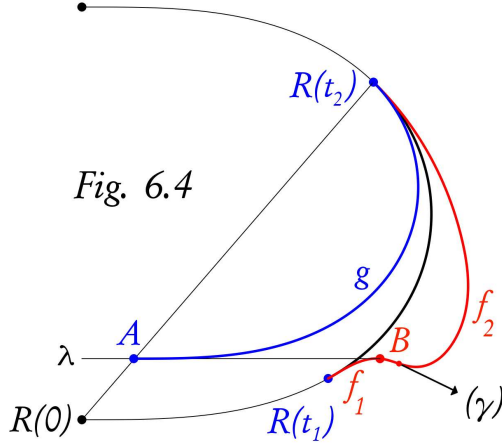


Fig. 6.4

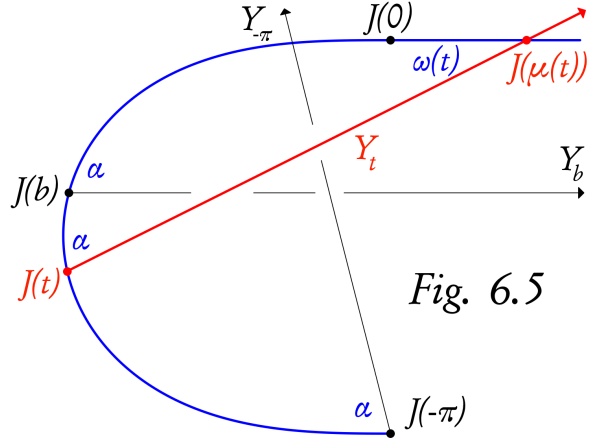


Fig. 6.5

Lemma 6.10. Define $J : [-\pi, \infty) \rightarrow \mathbb{C}$ by $J(t) := \begin{cases} R(t) & \text{if } t \in [-\pi, 0] \\ t & \text{if } t > 0 \end{cases}$. Given positive angles $\alpha \geq \delta > 0$, with $\alpha + \delta < \pi$, there exist $t_1 \in (-\pi, 0)$ and $t_2 > t_1$ such that the chord $[J(t_1), J(t_2)]$ intersects J with interior angles α and δ at $J(t_1)$ and $J(t_2)$, respectively.

Proof. We refer to Figure 6.5. For $t \in [-\pi, 0]$, let $\theta(t)$ denote the direction angle of $\vec{R}(t)$. As t ranges from $-\pi$ to 0 , $\theta(t)$ decreases continuously from π to 0 , and it follows that there exists $b \in (-\pi, 0)$ such that $\theta(b) = \alpha$. For $t \in [-\pi, b)$, let Y_t denote the ray emanating from $J(t)$ with direction angle $\theta(t) - \alpha$ and note that since the direction angle is positive, Y_t intersects J at a unique point $J(\mu(t))$, where $\mu(t) > t$. Let $\omega(t)$ denote the interior angle, at $J(\mu(t))$, made when the chord $[J(t), J(\mu(t))]$ intersects J (the interior angle at $J(t)$ equals α by construction). It is clear that $\omega(t)$ depends continuously on $t \in [-\pi, b)$ and tends to 0 as $t \rightarrow b^-$. We claim that $\omega(-\pi) > \delta$. If $\mu(-\pi) \geq 0$ (i.e. $J(\mu(-\pi))$ lies on $[0, \infty)$), then $\omega(-\pi) = \pi - \alpha$ and the claim follows immediately from the assumption that $\alpha + \delta < \pi$. On the other hand, if $\mu(-\pi) < 0$, then $\alpha = \psi(-\mu(-\pi))$ and $\omega(-\pi) = \phi(-\mu(-\pi))$; hence, by Lemma 6.3, $\omega(-\pi) > \alpha$ and now the claim follows from the assumption $\alpha \geq \delta$. By the intermediate value property of continuous functions, there exists $t_1 \in [-\pi, b)$ such that $\omega(t_1) = \delta$, and the lemma is proved with $t_2 = \mu(t_1)$. \square

Proof of Theorem 6.2. Put $\delta = -\beta > 0$ and note that the hypothesis of Lemma 6.10 follows from (5.1), and we obtain the conclusion of the lemma. We claim that $t_2 \leq 0$. To see this, assume to the contrary that $t_2 > 0$. Let $T(z) = c_1 z + c_2$ be the similarity transformation determined by $T(J(t_1)) = 0$ and $T(J(t_2)) = 1$. It follows that $T \circ R_{[t_1, 0]} = r(u, \lambda_\beta)$ and therefore, by Proposition 5.9, that $\sigma(\beta) = |c_1|(t_2 - 0) > 0$, which contradicts the assumption that $\sigma(\beta) \leq 0$. Therefore, $t_2 \leq 0$ and we conclude, from Theorem 6.1 and Remark 6.9, that $f = T \circ R_{[t_1, t_2]}$ is the unique curve (modulo equivalence) in $S(u, v)$ with minimal bending energy. \square

7. Proof of Theorem 1.1

For the convenience of the reader let us recall the main theorem of the paper stated in the introduction (but with \mathbb{C} in place of \mathbb{R}^2).

Theorem 1.1. *Given any sequence of points P_1, P_2, \dots, P_m in \mathbb{C} with $P_i \neq P_{i+1}$, the family of admissible interpolating curves $\mathcal{A}(P_1, P_2, \dots, P_m)$ contains a curve with minimal bending energy.*

The outline of the proof is as follows. First we show that the family $\mathcal{A}(P_1, P_2, \dots, P_m)$ is non-empty (Prop. 7.1). With M denoting the infimum of the bending energies of curves in $\mathcal{A}(P_1, P_2, \dots, P_m)$, it follows that there exists a sequence of curves c^1, c^2, c^3, \dots in $\mathcal{A}(P_1, P_2, \dots, P_m)$ such that $\|c^n\|^2 \rightarrow M$ as $n \rightarrow \infty$.

Let v_i^n denote the unit tangent vector to the curve c^n at the point P_i . Note that the vectors $v_i^1, v_i^2, v_i^3, \dots$ are all of unit length and have a common base point P_i . Appealing to the Heine-Borel theorem, and passing to a subsequence if necessary, we can assume without loss of generality that the sequence $v_i^1, v_i^2, v_i^3, \dots$ converges to a unit tangent vector v_i , for $i = 1, 2, \dots, m$. To dispel any possible confusion, we mention that the sequence of curves c^1, c^2, c^3, \dots need not converge in any sense—only the unit tangent vectors need converge.

Next we show that for each i , the pair v_i, v_{i+1} is s-feasible (Prop. 7.2), and therefore (by Theorem 5.1), there exists an s-curve s_i with minimal bending energy in the family $S(v_i, v_{i+1})$. Joining these pieces together, we construct our candidate $c = s_1 \sqcup s_2 \sqcup \dots \sqcup s_{m-1}$ which belongs to $\mathcal{A}(P_1, P_2, \dots, P_m)$.

The proof of Theorem 1.1 is then completed by showing that $\|c\|^2 = M$. The proof of this equality uses the fact (Theorem 7.10) that the minimal bending energy of curves in the family $S(u, v)$ depends continuously on the directions of u and v , and most of the work in the current section (Prop. 7.5 – Prop. 7.9) goes towards establishing this fact.

Proposition 7.1. *Under the hypothesis of Theorem 1.1, the family of admissible curves $\mathcal{A}(P_1, P_2, \dots, P_m)$ is nonempty.*

Proof. If we show that there exist unit tangent vectors $\{u_j\}$, with base-points $\{P_j\}$, such that $S(u_j, u_{j+1})$ is nonempty for $j = 1, 2, \dots, m-1$, then $f = f_1 \sqcup f_2 \sqcup \dots \sqcup f_{m-1}$, with $f_j \in S(u_j, u_{j+1})$, will be a curve in $\mathcal{A}(P_1, P_2, \dots, P_m)$. We will actually prove a slightly stronger result in that we will show that $\mathcal{A}(P_1, P_2, \dots, P_m, P_1)$ contains a periodic (closed) curve, where we have tacitly assumed (without loss of generality) that $P_m \neq P_1$. Let $\{P_j\}$ be extended periodically by $P_{j+m} = P_j$. For $j \in \mathbb{Z}$, let $w_j = (P_{j+1} - P_j)/|P_{j+1} - P_j|$ be the complex unit in the same direction as $P_{j+1} - P_j$ (see Figure 7.1). We then set $u_j = (P_j, z_j)$, where z_j , the direction of u_j , is defined as follows:

If $w_{j-1} + w_j$ is nonzero, then z_j is the complex unit in the same direction as $w_{j-1} + w_j$; otherwise, $z_j = e^{i\pi/2}w_j$. Since $\{u_j\}$ has inherited the periodicity of $\{P_j\}$ (namely $u_{j+m} = u_j$), in order to complete the proof, it suffices to show that $S(u_j, u_{j+1})$ is nonempty for all $j \in \mathbb{Z}$.

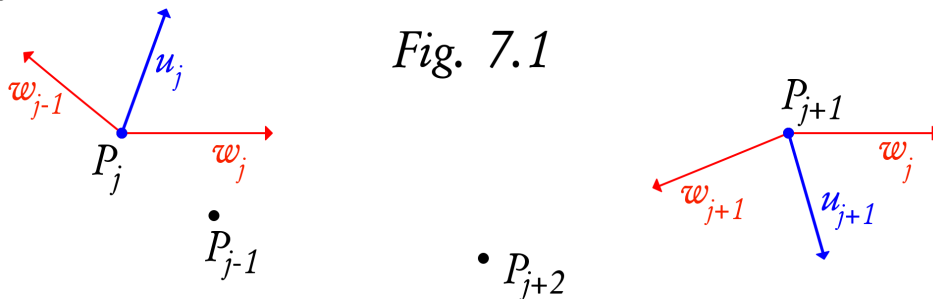


Fig. 7.1

Fix $j \in \mathbb{Z}$. By applying a rigid motion, if necessary, we can assume without loss of generality that $P_j = 0$ and that P_{j+1} lies on the positive real axis. That $S(u_j, u_{j+1})$ is nonempty can now be established simply by showing that both directions z_j and z_{j+1} have nonnegative real part. Note that $w_j = 1$. If $w_{j-1} \neq -1$, then z_j is the complex unit in the same direction as $w_{j-1} + 1$ and hence $\operatorname{Re} z_j \geq 0$; otherwise, $\operatorname{Re} z_j = \operatorname{Re} e^{i\pi/2} = 0$. By the same reasoning, one sees that $\operatorname{Re} z_{j+1} \geq 0$. \square

Let u, v be s-feasible ($S(u, v) \neq \emptyset$) unit tangent vectors. Denote by $E(u, v)$ the minimum of the bending energy in $S(u, v)$. By the previous sections (Theorem 5.1) this minimum is assumed by a curve in $S(u, v)$.

We will need the following proposition which shows that the limit of s-feasible vectors is also s-feasible assuming that the bending energy is bounded.

Proposition 7.2. *Let $P_u \neq P_v$ be different points of \mathbb{C} . Let u_n, v_n be s-feasible unit tangent vectors with base-points P_u and P_v respectively, such that $\lim u_n = u$ and $\lim v_n = v$. If $\{E(u_n, v_n)\}$ is bounded then u, v is also s-feasible.*

Proof. Assume $\{E(u_n, v_n)\}$ is bounded. Without loss of generality we can assume that $P_u = 0 \in \mathbb{C}$ and $P_v = 1 \in \mathbb{C}$. Let $\alpha, \alpha_n, \beta, \beta_n \in (-\pi, \pi]$ be the direction angles of u, u_n, v, v_n , respectively (see Figure 5.1). Since u_n, v_n are s-feasible configurations, we must have $\alpha_n, \beta_n \in (-\pi, \pi)$. Moreover, it is easy to see that if with $|\alpha_n| \rightarrow \pi$ or $|\beta_n| \rightarrow \pi$, then $E(u_n, v_n) \rightarrow \infty$; therefore $\alpha, \beta \in (-\pi, \pi)$.

If $\alpha = 0$ then u, v is s-feasible, regardless of $\beta \in (-\pi, \pi)$. So assume $\alpha \neq 0$. Without loss of generality, we may assume that $\alpha > 0$ and $\alpha_n > 0$ for all $n \in \mathbb{N}$. Similar to the exercise assigned to the reader at the beginning of section 5, we leave it to the reader to verify that u_n, v_n is s-feasible if and only if $\beta_n \in [\alpha_n - \pi, \pi)$. Since $\beta \in (-\pi, \pi)$, it follows that $\beta \in [\alpha - \pi, \pi)$, and therefore the limit configuration u, v is s-feasible. \square

Next, we show that the bending energy $E(u, v)$ is continuous in u and v . For this we will need some preparation.

Let $u = (0, e^{i\alpha})$ and $v = (1, e^{i\beta})$ be two unit tangent vectors with direction angles $\alpha, \beta \in (-\pi, \pi]$ and for comparison, let $\bar{u} = (0, e^{i\bar{\alpha}})$ and $\bar{v} = (1, e^{i\bar{\beta}})$ be unit tangent vectors with the same base-points as u and v , but possibly different directions. The **diameter** of a curve $f : [a, b] \rightarrow \mathbb{C}$ is defined by $\operatorname{diam}(f) = \max_{t, \tau} |f(t) - f(\tau)|$.

Lemma 7.3. *With the notations introduced above, let $D > 0$ and $\eta \in (0, \pi/4)$ be given and assume that $\alpha, -\beta \in (\eta, \pi - \eta)$ are such that there exists a curve $f : [a, b] \rightarrow \mathbb{C}$ in $C_r(u, v)$ with $\operatorname{diam}(f) \leq D$. Then, for every $\varepsilon > 0$ there is a $\delta_1 = \delta_1(\varepsilon, D, \eta) > 0$ (depending only on ε, D, η) such that if $|\bar{\alpha} - \alpha| < \delta_1$ and $C_r(\bar{u}, v)$ is nonempty, then there is a curve $c \in C_r(\bar{u}, v)$ such that $\|c\|^2 \leq \|f\|^2 + \varepsilon$ and $\operatorname{diam}(c) \leq 2D + 1$.*

Proof. We will describe, in two cases, how to modify the curve f near the base point of u to obtain a new curve c in $C_r(\bar{u}, v)$ satisfying $\|c\|^2 \leq \|f\|^2 + \varepsilon$.

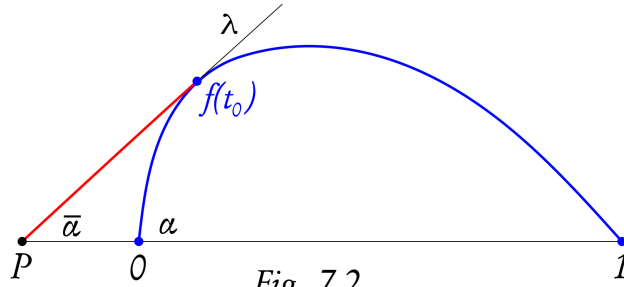


Fig. 7.2

Case 1: $0 < \bar{\alpha} < \alpha$.

Draw a line with direction angle $\bar{\alpha}$ through a distant point on the negative real axis, and then translate it horizontally towards f until it makes first contact, obtaining a line λ (see Figure 7.2). Let P denote the intersection point of λ with the negative real axis and let $f(t_0)$ be a point on λ . Set $c_1 = [P, f(t_0)] \sqcup f_{[t_0, b]}$ and note that c_1 is similar to a curve c in $C_r(\bar{u}, v)$ having bending energy $\|c\|^2 = (1 - P)\|c_1\|^2$, while $\|c_1\|^2 = \|f_{[t_0, b]}\|^2 \leq \|f\|^2$. It is easy to see that there is a $\delta_1 = \delta_1(\varepsilon, \eta, D) \in (0, \eta/2)$ such that if $\alpha - \delta_1 < \bar{\alpha} < \alpha$, then the distance from P to 0 is less than the minimum of 1 and $\frac{\varepsilon}{\|f\|^2}$; hence $\|c\|^2 \leq (1 - P)\|f\|^2 \leq \|f\|^2 + \varepsilon$ and $\text{diam}(c) \leq 2D + 1$.

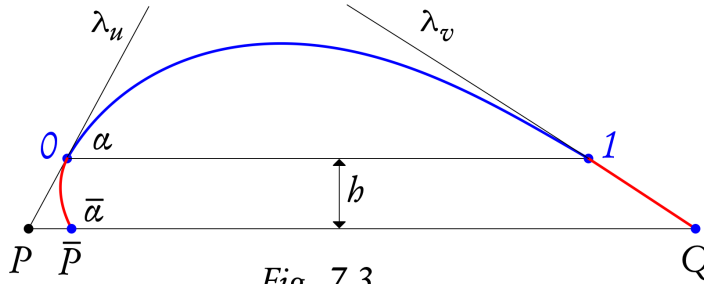


Fig. 7.3

Case 2: $\bar{\alpha} > \alpha$.

We obtain our curve c in two steps (see Figure 7.3). First, draw the tangent lines λ_u and λ_v to f at 0 and 1, respectively. For a given $h > 0$, let P and Q be the points on λ_u and λ_v , respectively, having $\text{Im } P = \text{Im } Q = -h$. Since, $\alpha, -\beta \in (\eta, \pi - \eta)$, there exists $h = h(\eta, \varepsilon) \in (0, 1/2)$ such that the resulting points P and Q will satisfy $|P| < 1/9$, $|Q - 1| < 1/9$ and $Q - P < \frac{\|f\|^2 + \varepsilon}{\|f\|^2 + \varepsilon/2}$. Let h, P and Q be thus fixed.

Second, replace the line segment $[P, 0]$ by an arc of a circle which emanates from a point \bar{P} on $[P, Q]$, with direction angle $\bar{\alpha}$, and terminates at 0 with direction angle α , as shown in Figure 7.3. If $\bar{\alpha}$ is sufficiently close to α , one can construct such an arc easily and we leave the details to the reader. With g denoting this circular arc, we define $c_1 = g \sqcup f \sqcup [1, Q]$, which is similar to a curve c in $C_r(\bar{u}, v)$, and we note that $\|c\|^2 = (Q - \bar{P})\|c_1\|^2$, while $\|c_1\|^2 = \|f\|^2 + \|g\|^2$. The radius of the circle containing g tends to ∞ as $\bar{\alpha}$ tends to α , and therefore $\|g\|^2$ can be made arbitrarily small by taking $\bar{\alpha}$ sufficiently close to α . Consequently, there exists $\delta_1 = \delta_1(\eta, \varepsilon) \in (0, \eta/2)$ such that if $\alpha < \bar{\alpha} < \alpha + \delta_1$, then $\|g\|^2 < \varepsilon/2$ and consequently we have $\text{diam}(c) < 2D + 1$ and $\|c\|^2 < (Q - P)(\|f\|^2 + \|g\|^2) \leq \|f\|^2 + \varepsilon$. \square

Remark 7.4. Under the same hypothesis as Lemma 7.3, we can prove in a similar manner that for every $\varepsilon > 0$ there is a $\delta_2 = \delta_2(\varepsilon, D, \eta) > 0$ (depending only on ε, D, η) such that if $|\bar{\beta} - \beta| < \delta_1$ and $C_r(u, \bar{v})$ is nonempty, then there is a curve $c \in C_r(u, \bar{v})$ such that $\|c\|^2 \leq \|f\|^2 + \varepsilon$ and $\text{diam}(c) \leq 2D + 1$.

Proposition 7.5. *Under the same hypothesis as Lemma 7.3, for every $\varepsilon > 0$ there is a $\delta_3 = \delta_3(\varepsilon, D, \eta) > 0$ (depending only on ε, D, η) such that if $|\bar{\alpha} - \alpha|, |\bar{\beta} - \beta| < \delta_3$ and $C_r(\bar{u}, \bar{v})$ is nonempty, then there is a curve $c \in C_r(\bar{u}, \bar{v})$ such that $\|c\|^2 \leq \|f\|^2 + \varepsilon$.*

Proof. Let $\varepsilon > 0$ be given, and let δ_3 be the minimum of $\eta/2, \delta_1(\varepsilon/2, D, \eta), \delta_2(\varepsilon/2, 2D + 1, \eta/2), \delta_2(\varepsilon/2, D, \eta), \delta_1(\varepsilon/2, 2D + 1, \eta/2)$. Assume that $\alpha, -\beta \in (\eta, \pi - \eta)$ are such that there exists a curve $f : [a, b] \rightarrow \mathbb{C}$ in $C_r(u, v)$ with $\text{diam}(f) \leq D$, and let $\bar{\alpha}, \bar{\beta}$ be such that $|\bar{\alpha} - \alpha|, |\bar{\beta} - \beta| < \delta$ and $C_r(\bar{u}, \bar{v})$ is nonempty. Although we cannot claim that both $C_r(\bar{u}, v)$ and $C_r(u, \bar{v})$ are nonempty (a counterexample can be easily found when f is a u-turn), it is easy to see that at least one of them is nonempty. We will address the case when $C_r(\bar{u}, v)$ is nonempty, as the other case is similar. Since $|\bar{\alpha} - \alpha| < \delta_3 \leq \delta_1(\varepsilon/2, D, \eta)$, we obtain from Lemma 7.3 that there exists a curve $c_1 \in C_r(\bar{u}, v)$ such that $\|c_1\|^2 \leq \|f\|^2 + \varepsilon/2$ and $\text{diam}(c_1) \leq 2D + 1$. Note that $\bar{\alpha}, -\beta \in (\eta/2, \pi/\eta/2)$ (since $\delta_3 \leq \eta/2$) and that c_1 is a curve in $C_r(\bar{u}, v)$ with $\text{diam}(c_1) \leq 2D + 1$. Since $C_r(\bar{u}, \bar{v})$ is nonempty and $|\bar{\beta} - \beta| < \delta_3 \leq \delta_2(\varepsilon/2, 2D + 1, \eta/2)$, it follows from Remark 7.4 that there exists a curve $c \in C_r(\bar{u}, \bar{v})$ such that $\|c\|^2 < \|c_1\|^2 + \varepsilon/2 < \|f\|^2 + \varepsilon$. \square

For given $\alpha, \beta \in (-\pi, \pi)$ let $u = (0, e^{i\alpha})$ and $v = (1, e^{i\beta})$ be the unit tangent vectors with base points 0 and 1 and direction angles α and β , respectively. It will be useful to introduce the following notations: $C_r(\alpha, \beta) := C_r(u, v)$, $S(\alpha, \beta) := S(u, v)$ and $E(\alpha, \beta) := E(u, v)$, which we will call the bending energy of the pair α, β . We say α, β are s-feasible if $S(u, v)$ is nonempty.

Proposition 7.6. *The bending energy $E(\alpha, \beta)$ tends to 0 as $\alpha, \beta \rightarrow 0$.*

Proof. Assume that $|\alpha|, |\beta| \leq \pi/3$, and let $f : [0, 1] \rightarrow \mathbb{C}$ be the curve given by $f(t) = t + ig(t)$, where g is the cubic polynomial $g(t) = t(\tan \alpha - (\tan \alpha + \tan \beta)t)(1 - t)$. Then f belongs to $S(\alpha, \beta)$ and it is easy to see that there exists a constant C such that $\|f\|^2 \leq \frac{C}{4} \int_0^1 |g''(t)|^2 dt$. A simple calculation shows that the latter quantity equals $C(\tan^2 \alpha + \tan \alpha \tan \beta + \tan^2 \beta)$ which tends to 0 as $\alpha, \beta \rightarrow 0$. \square

In what follows we will rely heavily on the results of sections 5 and 6, where it is assumed that u and v are in ‘canonical’ arrangement: $\alpha \geq |\beta|$. However, if we perturb α and β , the resultant pair, $\bar{\alpha}$ and $\bar{\beta}$, may no longer be in ‘canonical’ arrangement. The following two propositions will help deal with this situation.

Proposition 7.7.

- (i) $E(\alpha, \beta) = E(-\beta, -\alpha)$
- (ii) $E(\alpha, \beta) = E(-\alpha, -\beta)$
- (iii) $E(\alpha, \beta) = E(\beta, \alpha)$.

Proof. For any curve in $S(\alpha, \beta)$ if we reflect the curve across the $x = 1/2$ line and reverse its orientation, we obtain a curve in $S(-\beta, -\alpha)$ with the same bending energy. This means that there is a bijection between $S(\alpha, \beta)$ and $S(-\beta, -\alpha)$ which preserves the bending energy. This implies (i). Similarly, reflection across the x -axis gives a bending energy-preserving bijection between $S(\alpha, \beta)$ and $S(-\alpha, -\beta)$, which yields (ii). Combining (i) and (ii) we obtain (iii). \square

Proposition 7.8. *Let us assume that $\alpha \geq |\beta|$ but $0 < \bar{\alpha} < |\bar{\beta}|$.*

$$\text{Define } (\tilde{\alpha}, \tilde{\beta}) := \begin{cases} (\bar{\beta}, \bar{\alpha}) & \text{if } \bar{\beta} > 0 \\ (-\bar{\beta}, -\bar{\alpha}) & \text{if } \bar{\beta} < 0 \end{cases}.$$

If $|\alpha - \bar{\alpha}|, |\beta - \bar{\beta}| < \delta$, then $|\alpha - \tilde{\alpha}|, |\beta - \tilde{\beta}| < \delta$ and $\tilde{\alpha} \geq |\tilde{\beta}|$.

Proof. The proof is elementary and we will leave it to the reader. \square

Let us indicate how we will use the previous two propositions in the proof of the next one. Assume that $\alpha \geq |\beta|$, and $|\alpha - \bar{\alpha}|, |\beta - \bar{\beta}| < \delta$ with $\alpha > \delta > 0$. If $\bar{\alpha} < |\bar{\beta}|$, then we will replace $\bar{\alpha}, \bar{\beta}$ with a new pair $\tilde{\alpha}, \tilde{\beta}$ as in Proposition 7.8. Then the new pair will be in ‘canonical’ arrangement ($\tilde{\alpha} \geq |\tilde{\beta}|$), $|\alpha - \tilde{\alpha}|, |\beta - \tilde{\beta}| < \delta$ and from Proposition 7.7 we have $E(\alpha, \beta) = E(\tilde{\alpha}, \tilde{\beta})$.

Proposition 7.9. *With the notations introduced above let us assume that α, β are s -feasible with $\alpha \geq |\beta|$, and let $\eta \in (0, \pi/4)$. If $\alpha \in (\eta, \pi - \eta)$, then for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, \eta) > 0$ (depending only on ε, η) such that if $|\bar{\alpha} - \alpha|, |\bar{\beta} - \beta| < \delta$ and $\bar{\alpha}, \bar{\beta}$ are s -feasible, then $E(\bar{\alpha}, \bar{\beta}) < E(\alpha, \beta) + \varepsilon$.*

Before we start the proof of Proposition 7.9 let us recall some quantities defined in Definition 5.3. For $\gamma \in \Gamma = [\alpha - \pi, \beta] \cap (-\infty, 0)$ we have

$$y_1(\alpha, \gamma) = \frac{1}{2} \int_0^{\alpha-\gamma} \sqrt{\sin \tau} \, d\tau, \quad y_2(\beta, \gamma) = \frac{1}{2} \int_0^{\beta-\gamma} \sqrt{\sin \tau} \, d\tau,$$

$$G(\alpha, \beta, \gamma) = \frac{(y_1 + y_2)^2}{-\sin \gamma}, \quad G_{min}(\alpha, \beta) = \min\{G(\alpha, \beta, \gamma) : \gamma \in \Gamma\}.$$

Recall from Section 3 that $d = \xi(\pi) = \frac{1}{2} \int_0^\pi \sqrt{\sin \tau} \, d\tau$ and define the quantity γ_0 by $\gamma_0 = -\sin^{-1}((\sin \eta)(1 - \cos \eta)^2 / (16d^2))$. If $\alpha \in (\eta, \pi - \eta)$, from the formulas above one can verify immediately that

$$(7.1) \quad G(\alpha, \beta, \alpha - \pi) \leq \frac{4d^2}{\sin \eta} < G(\alpha, \beta, \gamma) \quad \text{if } \gamma_0 < \gamma < \beta.$$

This implies that if $G(\alpha, \beta, \gamma) = G_{min}(\alpha, \beta)$, then $\gamma \leq \beta^* = \min\{\beta, \gamma_0\}$.

It will be convenient to extend the domain of $G(\alpha, \beta, \gamma)$ to include any $\gamma \in [-\pi, 0]$ without changing the minimum $G_{min}(\alpha, \beta)$ or values γ where the minimum is assumed. We define the set K_η by $K_\eta = \{(\alpha, \beta) : \eta \leq \alpha \leq \pi - \eta, \quad |\beta| \leq \alpha, \quad \alpha - \pi \leq \beta\}$. For $(\alpha, \beta) \in K_\eta$, $\beta^* := \min\{\beta, \gamma_0\}$ and $\gamma \in [-\pi, 0)$ we set

$$\widehat{G}(\alpha, \beta, \gamma) = \begin{cases} G(\alpha, \beta, \beta^*) + \gamma - \beta^* & \text{if } \beta^* < \gamma \leq 0 \\ G(\alpha, \beta, \gamma) & \text{if } \alpha - \pi \leq \gamma \leq \beta^* \\ G(\alpha, \beta, \alpha - \pi) + \alpha - \pi - \gamma & \text{if } -\pi \leq \gamma < \alpha - \pi \end{cases}.$$

From the remark following inequality (7.1) and from the construction of \widehat{G} it is clear that $G_{min}(\alpha, \beta) = \widehat{G}_{min}(\alpha, \beta) = \min\{\widehat{G}(\alpha, \beta, \gamma) : \gamma \in [-\pi, 0]\}$. Moreover G and \widehat{G}

assume their minimum at the same points, that is $G_{min}(\alpha, \beta) = G(\alpha, \beta, \gamma)$ if and only if $\widehat{G}_{min}(\alpha, \beta) = \widehat{G}(\alpha, \beta, \gamma)$.

The quantity $\sigma(\gamma)$ will be interesting for us only in the case when $\gamma = \beta$. Therefore we have

$$\sigma(\alpha, \beta) = \cos \beta + \frac{\sin \beta}{y_1(\alpha, \beta)} \sqrt{\sin(\alpha - \beta)}, \quad \alpha - \pi \leq \beta < 0.$$

It is easy to see that $\sigma(\alpha, \beta) \rightarrow 1$ as $\beta \rightarrow 0$. Therefore we can extend the domain of $\sigma(\alpha, \beta)$ to the region $0 \leq \beta$ by setting

$$\sigma(\alpha, \beta) = 1, \quad \text{if} \quad \beta \geq 0.$$

We can summarize the results of sections 5 and 6 as follows:

If $\sigma(\alpha, \beta) \leq 0$, then we are in Case C of Summary 5.4 and there is a segment of rectangular elastica in $C_r(\alpha, \beta)$ which has minimal bending energy in $S(\alpha, \beta)$.

If $\sigma(\alpha, \beta) > 0$, then we are in Case A or B and there is an s-curve in $S(\alpha, \beta)$ with minimal bending energy $E(\alpha, \beta) = G_{min}(\alpha, \beta)$.

Proof of Proposition 7.9.

Case 1: $\sigma(\alpha, \beta) \leq 0$.

Then $\beta < 0$ and there is a segment of rectangular elastica $f \in C_r(\alpha, \beta)$ which has minimal bending energy in $S(\alpha, \beta)$. Before invoking Proposition 7.5, we remark that the diameter of f cannot exceed 10 since the ratio of length over breadth for any segment of rectangular elastica is bounded by 10. Since $\sigma(\alpha, \beta)$ is continuous and $\alpha \in (\eta, \pi - \eta)$, one can see that there is an $\eta_1 = \eta_1(\eta) \in (0, \eta)$ such that $|\beta| > \eta_1(\eta)$. By Proposition 7.5, there is a $\delta = \delta(\eta, \varepsilon) > 0$ (namely, $\delta_3(\varepsilon/2, 10, \eta_1)$ in the language of Prop. 7.5) such that if $|\bar{\alpha} - \alpha|, |\bar{\beta} - \beta| < \delta$, then

$$E(\bar{\alpha}, \bar{\beta}) < E(\alpha, \beta) + \varepsilon/2,$$

which completes the proof for Case 1.

Case 2: $\sigma(\alpha, \beta) > 0$.

Then $E(\alpha, \beta) = G_{min}(\alpha, \beta) = G(\alpha, \beta, \gamma)$, for some (not necessarily unique) $\gamma = \gamma(\alpha, \beta)$. From the remark following inequality (7.1) we have $\gamma(\alpha, \beta) \in [\alpha - \pi, \beta^*]$, where $\beta^* = \min\{\beta, \gamma_0\}$. From the definition of G and \widehat{G} one can see that $\widehat{G}(\alpha, \beta, \gamma)$ is continuous, hence uniformly continuous on the region $K_\eta \times [-\pi, 0]$. Therefore, there is a $\delta_0 = \delta_0(\eta, \gamma_0) > 0$ such that for $(\alpha, \beta), (\alpha', \beta') \in K_\eta$ we have

$$(7.2) \quad |\widehat{G}(\alpha, \beta, \gamma) - \widehat{G}(\alpha', \beta', \gamma')| < \frac{\varepsilon}{2}, \quad \text{whenever} \quad |\alpha - \alpha'|, |\beta - \beta'|, |\gamma - \gamma'| < \delta_0.$$

Let us assume that $|\bar{\alpha} - \alpha|, |\bar{\beta} - \beta| < \delta$, where $\delta \leq \min\{\delta_0, \frac{\eta}{4}\}$ is determined later. We can further assume, without loss of generality, that $\bar{\alpha} \geq |\bar{\beta}|$, since otherwise, we can replace $\bar{\alpha}, \bar{\beta}$ with $\tilde{\alpha}, \tilde{\beta}$, keeping in mind that $|\tilde{\alpha} - \alpha|, |\tilde{\beta} - \beta| < \delta$, by Proposition 7.8, and $E(\tilde{\alpha}, \tilde{\beta}) = E(\alpha, \beta)$, by Proposition 7.7.

Case 2a: $\sigma(\bar{\alpha}, \bar{\beta}) \geq 0$.

Let $\bar{\gamma} = \gamma(\bar{\alpha}, \bar{\beta})$ be an angle where $G(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ assumes its minimum. Since G and \widehat{G} assume their minimum at the same points (see the remarks following the definition of \widehat{G}) we have

$E(\bar{\alpha}, \bar{\beta}) = G_{\min}(\bar{\alpha}, \bar{\beta}) = G(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = \widehat{G}(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ and $E(\alpha, \beta) = G_{\min}(\alpha, \beta) = G(\alpha, \beta, \gamma) = \widehat{G}(\alpha, \beta, \gamma)$. Taking (7.2) into consideration, we then obtain

$$E(\bar{\alpha}, \bar{\beta}) = \widehat{G}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \leq \widehat{G}(\bar{\alpha}, \bar{\beta}, \gamma) < \widehat{G}(\alpha, \beta, \gamma) + \frac{\varepsilon}{2} = E(\alpha, \beta) + \frac{\varepsilon}{2},$$

which completes the proof for Case 2a (with $\delta = \min\{\delta_0, \frac{\eta}{4}\}$).

Case 2b: $\sigma(\bar{\alpha}, \bar{\beta}) < 0$.

Since $\delta \leq \eta/4$ we have $(\alpha, \beta), (\bar{\alpha}, \bar{\beta}) \in K_{\eta/2}$, which is a convex set. Therefore the line segment $[(\alpha, \beta), (\bar{\alpha}, \bar{\beta})]$ is also a subset of $K_{\eta/2}$. Since σ is a continuous function of α, β there is a pair $(\alpha_1, \beta_1) \in [(\alpha, \beta), (\bar{\alpha}, \bar{\beta})]$ with $\sigma(\alpha_1, \beta_1) = 0$. Applying the previous argument for α_1, β_1 instead of $\bar{\alpha}, \bar{\beta}$ we obtain $E(\alpha_1, \beta_1) < E(\alpha, \beta) + \frac{\varepsilon}{2}$.

Since $\sigma(\alpha_1, \beta_1) = 0$ there is a segment of rectangular elastica $f \in C_r(\alpha_1, \beta_1)$ with $\|f\|^2 = E(\alpha_1, \beta_1)$. Noting that $\alpha_1 > \frac{\eta}{2}$, we have, as in Case 1, $|\beta_1| > \eta_1(\eta/2)$. We can now apply Proposition 7.5 to obtain $E(\bar{\alpha}, \bar{\beta}) < E(\alpha_1, \beta_1) + \frac{\varepsilon}{2}$, provided $\delta \leq \delta_3(\varepsilon/2, 10, \eta_1(\eta/2))$ in the language of Proposition 7.5. Combining this with the previous inequality we obtain

$$E(\bar{\alpha}, \bar{\beta}) < E(\alpha, \beta) + \varepsilon,$$

which completes the proof for Case 2b. \square

With propositions 7.5–7.9 in hand, we can finally prove the following.

Theorem 7.10. *The bending energy $E(u, v)$ depends continuously on the directions of the unit tangent vectors u and v .*

Proof. Let u and v be s-feasible unit tangent vectors. As explained at the beginning of section 5, we can assume, without loss of generality, that $u = (0, e^{i\alpha})$ and $v = (1, e^{i\beta})$, where $\alpha, \beta \in (-\pi, \pi)$ satisfy $\alpha \geq |\beta|$. The case $\alpha = 0$ has been settled in Proposition 7.6, so assume $\alpha \in (0, \pi)$. Let $\eta \in (0, \pi/8)$ be such that $\alpha \in (2\eta, \pi - 2\eta)$.

Let $\varepsilon > 0$ and set $\delta = \min\{\eta, \delta_4\}$, where $\delta_4 = \delta(\varepsilon, \eta)$ is as described in Proposition 7.9. Let $\bar{\alpha}, \bar{\beta}$ be s-feasible with $|\bar{\alpha} - \alpha|, |\bar{\beta} - \beta| < \delta$. We will show, in two cases, that

$$(7.3) \quad |E(\bar{\alpha}, \bar{\beta}) - E(\alpha, \beta)| < \varepsilon.$$

Case 1: $\bar{\alpha} \geq |\bar{\beta}|$.

As written, Proposition 7.9 yields $E(\bar{\alpha}, \bar{\beta}) < E(\alpha, \beta) + \varepsilon$, but Proposition 7.9 can also be applied with (α, β) and $(\bar{\alpha}, \bar{\beta})$ interchanged, since $\bar{\alpha} \in (\eta, \pi - \eta)$ and $\bar{\alpha} \geq |\bar{\beta}|$. This yields $E(\alpha, \beta) < E(\bar{\alpha}, \bar{\beta}) + \varepsilon$, and we obtain (7.3).

Case 2: $\bar{\alpha} < |\bar{\beta}|$.

Let $\tilde{\alpha}, \tilde{\beta}$ be as defined in Proposition 7.8, whereby $|\tilde{\alpha} - \alpha|, |\tilde{\beta} - \beta| < \delta$ and $\tilde{\alpha} \geq |\tilde{\beta}|$. It follows from Case 1 that $|E(\tilde{\alpha}, \tilde{\beta}) - E(\alpha, \beta)| < \varepsilon$, but since $E(\tilde{\alpha}, \tilde{\beta}) = E(\bar{\alpha}, \bar{\beta})$ (by Proposition 7.7), we have (7.3). \square

Proof of Theorem 1.1. Most of the proof has been explained in the discussion following the theorem's statement at the beginning of this section. All that remains is to show that

our candidate $c = s_1 \cup s_2 \cup \dots \cup s_{m-1}$ has bending energy M (the infimum of bending energies in $\mathcal{A}(P_1, P_2, \dots, P_m)$).

For $n = 1, 2, 3, \dots$, let us write $c^n \in \mathcal{A}(P_1, \dots, P_m)$ as $c^n = f_1^n \sqcup f_2^n \sqcup \dots \sqcup f_m^n$, where f_j^n belongs to $S(v_j^n, v_{j+1}^n)$ for $j = 1, 2, \dots, m$. Note that

$$\|c^n\|^2 = \sum_{j=1}^{m-1} \|f_j^n\|^2 \geq \sum_{j=1}^{m-1} E(v_j^n, v_{j+1}^n) \quad \text{while} \quad \|c\|^2 = \sum_{j=1}^{m-1} \|s_j\|^2 = \sum_{j=1}^{m-1} E(v_j, v_{j+1})$$

(since s_j has minimal bending energy in $S(v_j, v_{j+1})$). For $j = 1, 2, \dots, m-1$, it follows from Theorem 7.10 that $E(v_j^n, v_{j+1}^n) \rightarrow E(v_j, v_{j+1})$ as $n \rightarrow \infty$, and therefore we obtain $\|c\|^2 \leq \lim_n \|c^n\|^2 = M$. Since c belongs to $\mathcal{A}(P_1, P_2, \dots, P_m)$, we conclude that $\|c\|^2 = M$. \square

Remark 7.11. We conjecture, and hope to show in a subsequent paper, that if each of the curves s_i is of form one (see Definition 5.2), then the resulting optimal curve s is twice continuously differentiable.

Remark 7.12. Let us denote by $\mathcal{A}_{\text{periodic}}(P_1, P_2, \dots, P_m, P_1)$ the set of periodic (closed) curves passing through the points $P_1, P_2, \dots, P_m, P_1$ such that they are s-curves between any two consecutive points. Notice that the proof of Theorem 1.1 works equally well for periodic admissible curves, provided that $\mathcal{A}_{\text{periodic}}(P_1, P_2, \dots, P_m, P_1)$ is nonempty. This is exactly what is shown in the proof of Proposition 7.1. Therefore we have the following extension of Theorem 1.1:

Theorem 7.13. *Given any sequence of points $P_1, P_2, \dots, P_m \in \mathbb{C}$ with $P_j \neq P_{j+1}$ and $P_m \neq P_1$, the family $\mathcal{A}_{\text{periodic}}(P_1, P_2, \dots, P_m, P_1)$ contains a curve with minimal bending energy.*

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