ELASTIC SPLINES III: EXISTENCE OF STABLE NONLINEAR SPLINES

Albert Borbély & Michael J. Johnson

Abstract. Given points $P_1, P_2, \dots, P_n$ in the complex plane, a stable nonlinear spline is an interpolating curve, of arbitrary length, whose bending energy is minimal among all nearby interpolating curves. We show that if the chord angles of a restricted elastic spline f, at interior nodes, are less than $\frac{\pi}{2}$ in magnitude, then f is a stable nonlinear spline. As a consequence, existence of stable nonlinear splines is now proved for the case when the stencil angles $\psi_j := \arg \frac{P_{j+1}-P_j}{P_j-P_{j-1}}$ satisfy $|\psi_j| < \Psi$ for $j = 2, 3, \dots, n-1$, where Ψ ($\approx 37°$) is defined in our previous article. As in our previous articles, the optimal s-curves $c_1(\alpha, \beta)$ play an important role and here we show that, when $|\alpha|, |\beta| < \frac{\pi}{2}$, they are also optimal among Hermite interpolating curves whose tangent directions are never orthogonal to the chord.

1. Introduction

Let $P_1, P_2, \dots, P_n$, with $P_j \neq P_{j+1}$, be a list of points in the complex plane, and consider the problem of drawing a fair curve that passes (i.e., interpolates) sequentially through the given points. Historically, draftsman have drawn such a curve using a spline, nowadays called a draftsman's spline, which is a flexible straight-edge that can be bent so that it interpolates the given points. The **bending energy** of a curve f is defined as

$$\|f\|^2 := \frac{1}{4} \int_0^S \kappa(s)^2 \, ds,$$

where κ denotes signed curvature and s arclength. The naive model of the draftsman's spline is that it assumes the shape of an interpolating curve whose bending energy is minimal. Whether or not the length of the spline has been prescribed or constrained in some manner is a significant detail. In this article, we are concerned only with the case when length has not been prescribed or constrained–the spline is free to assume whatever length it pleases in pursuit of minimal bending energy. Unfortunately, interpolating curves with minimal bending energy never exist, except when the points lie sequentially along a line. This was first observed at General Motors (see [1] and [2]) by Birkhoff, de Boor, Burchard and Thomas, and they proposed seeking instead an interpolating curve whose bending energy is *locally minimal*; that is, minimal among all *nearby* interpolating curves. (When length is prescribed or constrained, existence of interpolating curves with minimal bending energy is proved in [14] and [7].) At General Motors, such interpolating curves were called *nonlinear splines* and they observed that the **pieces** (connecting one interpolation point

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to the next) of a nonlinear spline would be segments of rectangular elastica. **Rectangular elastica** (a.k.a simple elastica [9] or free elastica [5]), first described by James Bernoulli (1694), refers to a curve whose signed curvature κ satisfies the differential equation $2\frac{d^2\kappa}{ds^2} + \kappa^3 = 0$, where s denotes arclength (see [11] for a detailed account). The meaning of the term *nonlinear spline* has evolved over the years, so, to avoid confusion, we will call them *stable nonlinear splines*. Thus, an interpolating curve F is called a **stable nonlinear spline** if there exists $\varepsilon > 0$ such that $\|F\|^2 \leq \|G\|^2$ for all interpolating curves G whose *distance* from F is less than ε. How one defines the distance between interpolating curves depends on one's setup, but in our case (Def. 6.1) this distance is defined to be the maximal Hausdorff distance between corresponding pieces. Brunnett [5] also uses (one-sided) Hausdorff distance, while Jerome [14], Fisher & Jerome [7], Golomb & Jerome [10], Golomb [9], and Linnér [17, 18, 19] employ various Sobolev formulations of distance.

Lee and Forsythe [16] (see also [7] and [10]) have shown that in addition to having pieces that are segments of rectangular elastica, stable nonlinear splines are curvature continuous with zero curvature at P_1 and P_n. This result resonates with the result for cubic splines that states that if a smooth function $s : [a, b] \to \mathbb{R}$ minimizes $\int_a^b [s''(x)]^2\, dx$, subject to given interpolation conditions $s(x_i) = y_i$ ($a = x_1 < x_2 < \cdots < x_n = b$), then s is a C^2 piecewise cubic polynomial satisfying $s(x_i) = y_i$ ($i = 1, 2, \ldots, m$) and $s''(a) = s''(b) = 0$. For cubic splines, the converse is also true but that is not the case for nonlinear splines. In the language of Golomb and Jerome [10], an interpolating curve that satisfies the necessary conditions of Lee and Forsythe is called an **extremal interpolant**. Although every stable nonlinear spline is an extremal interpolant, the converse is false. Moreover, there exist points $P_1, P_2, \ldots, P_n$ which have extremal interpolants, but for which no stable nonlinear spline exists. An example of this (first put forth in [2]) are the four points $1 + i0$, $2 + i0$, $0 + i2$, $0 + i$. Golomb [9] has shown that there exists an extremal interpolant for these points, but a stable nonlinear spline does not exist.

Our list of interpolation points $P_1, P_2, \ldots, P_n$ is called a *configuration* and it is allowed to impose *clamps* at P_1 and/or P_n. This means that the direction of the interpolating curve at P_1 and/or P_n is prescribed. The configuration is called **free** if no clamps are imposed and is called **clamped** if both clamps are imposed. When only one clamp is imposed, the configuration is called **free-clamped** or **clamped-free**. Configurations are assumed to be free, unless otherwise specified. A special configuration, called a **ray configuration** in [10] and [9], occurs when the interpolation points lie sequentially along a line. Configurations can also be designated as **closed**, in which case the interpolating curve is required to be a closed curve, where the n-th piece runs from P_n back to P_1 ($P_1 \neq P_n$) with C^1 continuity across the node P_1. For example, if the interpolation points are the n vertices of a regular n-gon that is inscribed in a circle C, then C would be an interpolating curve for this closed configuration.

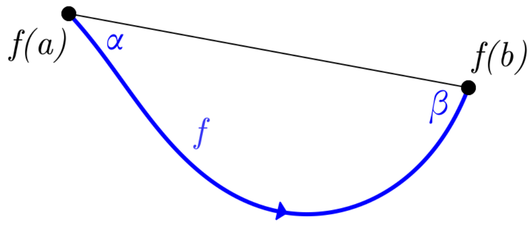
The term *nonlinear spline* means different things to different people (see [16] and [19], eg.), but in the present context it generally refers to an interpolating curve that is inclusively between an extremal interpolant and a stable nonlinear spline. Computation of nonlinear splines, often focusing on clamped 2-point configurations, is discussed in [5, 6, 8, 12, 17]. Existence (and enumeration) of extremal interpolants for regular configurations (eg., the closed configuration formed by the vertices of a regular n-gon) and for free configurations that are close to a ray configuration is proven in [10], while proofs of existence (and enumeration) of nonlinear splines for 2-point configurations that are free, clamped, or free-clamped can be found in [18, 19]. The importance of 2-point configurations can be explained as follows. If F, written piecewise as $F = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{n-1}$, is a stable nonlinear spline for the configuration $P_1, P_2, \ldots, P_n$, then each piece f_j is a stable nonlinear spline for the clamped 2-point configuration that it determines, and f_1 is also a stable nonlinear spline for its free-clamped configuration.

Jerome [13] and Fisher & Jerome [7] give a sufficient condition for the existence of a stable nonlinear spline. Unfortunately, this sufficient condition has not lead to any existence proofs for stable nonlinear splines. In his technical report, Golomb [9] builds rather impressive machinery for deciding whether a given extremal interpolant is stable. The upshot is that if one has a configuration in hand for which an extremal interpolant can be constructed, then it might be possible to apply Golomb's machinery to decide whether it is stable. Several examples are worked out in [9], where the existence of the extremal interpolant is first proved in [10]:

1. For every ray configuration $Q_1, Q_2, \dots, Q_n$, there exists $\varepsilon > 0$ such that if $|P_i - Q_i| < \varepsilon$, then there exists a stable nonlinear spline for the configuration $P_1, P_2, \dots, P_n$.
2. If $P_1, P_2, \dots, P_n$ are the corners of a regular n-gon ($n \geq 3$), then there exists a stable nonlinear spline for the closed configuration.
3. The existence or non-existence of stable nonlinear splines has been decided for a large swathe of clamped 2-point configurations (see comments after Remark 2.2 for more details).

Referring to item 1. above, Golomb writes, "This is probably the first general existence proof for locally minimizing interpolants which are not length-restricted." As we are not aware of any existence proofs (in the present context) of stable nonlinear splines other than these, we conclude that the present contribution is probably the second general existence proof for stable nonlinear splines.

In order to state our results, some preparation is needed. As in [3] and [4], a **curve** is a function $f : [a, b] \to \mathbb{C}$ whose first derivative f' is absolutely continuous and non-vanishing.



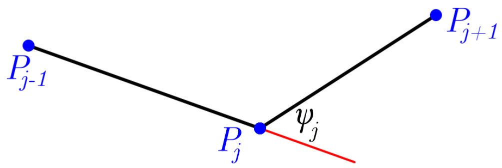


Fig. 1 the chord angles α and β **Fig. 2** the stencil angle ψ_j

When $f(a) \neq f(b)$, the **chord angles** of f are defined by

$$\alpha := \arg \frac{f'(a)}{f(b) - f(a)} \quad \text{and} \quad \beta := \arg \frac{f'(b)}{f(b) - f(a)} \qquad \text{(see Fig. 1)}, \tag{1.1}$$

where arg is defined with the standard range $(-\pi, \pi]$. For a configuration $P_1, P_2, \dots, P_n$, the **stencil angles**, $\psi_2, \psi_3, \dots, \psi_{n-1}$, are defined by

$$\psi_j := \arg \frac{P_{j+1} - P_j}{P_j - P_{j-1}} \qquad \text{(see Fig. 2)}.$$

An **s-curve** is a curve that first turns monotonically at most 180° in one direction (clockwise or counter-clockwise) and then turns monotonically at most 180° in the opposite direction. Let

$$\mathcal{A}(P_1, P_2, \dots, P_n)$$

denote the set of interpolating curves whose pieces (connecting one interpolation point to the next) are s-curves. We proved in [3] that $\mathcal{A}(P_1, P_2, \dots, P_n)$ contains a curve with minimal bending energy; such curves are called **elastic splines**. In order to improve the fairness of the obtained interpolating curves and their theoretical tractability, it was suggested in [15] that the chord angles of pieces be restricted to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Let

$$\mathcal{A}_{\pi/2}(P_1, P_2, \dots, P_n)$$

denote the set of interpolating curves whose pieces are s-curves with chord angles in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We proved in [4] that $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_n)$ contains a curve with minimal bending energy; such curves are called **restricted elastic splines**.

Definition. Let $F = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{n-1}$ be a restricted elastic spline, and let the chord angles of f_j be denoted (α_j, β_{j+1}). We say that F is **proper** if $\alpha_j, \beta_j \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $j = 2, 3, \ldots, n-1$; that is, if the chord angles at the interior nodes $P_2, P_3, \ldots, P_{n-1}$ belong to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

The main results in [4] are the following:
1. If a restricted elastic spline is proper, then it is curvature continuous.
2. Let Ψ ($\approx 37°$) be the angle defined in [4, eq. (8.1)]. If the stencil angles satisfy $|\psi_j| < \Psi$ ($j = 2, 3, \ldots, n-1$), then all restricted elastic splines through the points $P_1, P_2, \ldots, P_n$ are proper.
3. The angle Ψ is sharp.

The main result of the present contribution is the following.

Theorem 1.1. *Let F be a restricted elastic spline through given points $P_1, P_2, \ldots, P_n$. If F is proper, then F is a stable nonlinear spline.*

As an (almost) immediate corollary of Theorem 1.1 and item 2 above, we obtain the following.

Corollary 1.2. *Let Ψ ($\approx 37°$) be the angle defined in* [4, eq. (8.1)]. *If the stencil angles satisfy $|\psi_j| < \Psi$ ($j = 2, 3, \ldots, n-1$), then there exists a stable nonlinear spline through the points $P_1, P_2, \ldots, P_n$.*

To appreciate the novelty of Corollary 1.2, we mention that it is not a consequence of Golomb's results [9], even when $n = 3$ and Ψ is replaced by an arbitrarily small positive number.

An outline of the sequel is as follows. In Section 2, we review notation and results from [3] and [4]. Our proof of Theorem 1.1, given in Section 6, compares a proper restricted elastic spline $F = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{n-1}$ with a nearby interpolating curve $\widehat{F} = \widehat{f}_1 \sqcup \widehat{f}_2 \sqcup \cdots \sqcup \widehat{f}_{n-1}$. Our objective, of course, is to prove that $\|F\|^2 \le \|\widehat{F}\|^2$. This objective is achieved in several steps, the first of which is to show that each piece $\widehat{f}_j$ is *forward tracking* (see Def. 4.1). The machinery needed for this is developed in Section 5. Now that each piece $\widehat{f}_j$ is known to be forward tracking, we compare $\widehat{F}$ with a curve $G = g_1 \sqcup g_2 \sqcup \cdots \sqcup g_{n-1}$, in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_n)$, which has the same directions as $\widehat{F}$ at the interpolation nodes and whose bending energy is minimal among all such curves (of course $\|G\|^2 \ge \|F\|^2$). The purpose of Section 4 is to prove that $\|g_j\|^2 \le \|\widehat{f}_j\|^2$, on the grounds that both curves are forward tracking. A rather delicate element of this proof involves comparing $\widehat{f}_j$ with an s-curve constructed from the convex hull of the range of $\widehat{f}_j$, and Section 3 is dedicated to this task. In Section 6, in addition to proving Theorem 1.1 and Corollary 1.2, we make several concluding remarks relating to non-free configurations.

2. Review of Notation and Key Results

As mentioned above, a **curve** is a function $f : [a, b] \to \mathbb{C}$ whose first derivative f' is absolutely continuous. Every curve can be reparametrized by arclength and two curves f and g are deemed **equivalent**, written $f \equiv g$, if their arclength parametrizations are equal. Curves f and g are said to be **directly similar** if there exists a similarity transformation

$T(z) = c_1 z + c_2$ ($c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$) such that $f \equiv T \circ g$. A **unit tangent vector** is a pair $u = (u_1, u_2)$ consisting of a base point $u_1 \in \mathbb{C}$ and a direction $u_2 \in \mathbb{C}$, with $|u_2| = 1$. Let u and v be two unit tangent vectors with distinct base points. A curve $f : [a, b] \to \mathbb{C}$ is said to **connect** u to v if $u = (f(a), f'(a)/|f'(a)|)$ and $v = (f(b), f'(b)/|f'(b)|)$. The **chord angles** (α, β) determined by the pair (u, v) are those of any curve f that connects u to v (see 1.1); in terms of $u = (u_1, u_2)$ and $v = (v_1, v_2)$, they are

$$\alpha := \arg \frac{u_2}{v_1 - u_1} \quad \text{and} \quad \beta := \arg \frac{v_2}{v_1 - u_1}, \tag{2.1}$$

where (as mentioned above) arg is defined with the standard range $(-\pi, \pi]$. An **s-curve** is a curve that first turns monotonically at most 180° in one direction (clockwise or counter-clockwise) and then turns monotonically at most 180° in the opposite direction. The set of s-curves can be partitioned into the set of line segments, the set of **left c-curves** (nonlinear curves that turn monotonically at most 180° counter-clockwise), the set of **right c-curves** (nonlinear curves that turn monotonically at most 180° clockwise), the set of **left-right s-curves**, and the set of **right-left s-curves**. The set of all s-curves connecting u to v is denoted

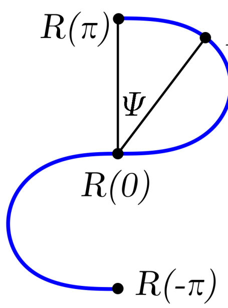
$$S(u, v),$$

and we mention that $S(u, v)$ is non-empty if and only if the the chord angles determined by (u, v) satisfy $|\alpha|, |\beta| < \pi$ and $|\alpha - \beta| \leq \pi$. We proved in [3] that when $S(u, v)$ is nonempty, it contains a curve with minimal bending energy. Optimal curves in $S(u, v)$ were further studied in [4] under the restriction $|\alpha|, |\beta| \leq \frac{\pi}{2}$; the following theorem and remark are proved in [4, Th. 5.4 and Cor. 6.1].

Theorem 2.1. *Let (α, β) be the chord angles determined by a pair of unit tangent vectors (u, v) with distinct base points. If $(\alpha, \beta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \backslash \{(-\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, -\frac{\pi}{2})\}$, then $S(u, v)$ contains a unique curve $c(u, v)$ (modulo equivalence) with minimal bending energy. Moreover, $c(u, v)$ is a segment of rectangular elastica.*

Remark 2.2. When (α, β) equals $(\frac{\pi}{2}, -\frac{\pi}{2})$ or $(-\frac{\pi}{2}, \frac{\pi}{2})$, the optimal curve in $S(u, v)$ fails to be unique. Nevertheless, among all optimal curves in $S(u, v)$, there is a unique one, denoted $c(u, v)$, that is a segment of rectangular elastica.

The family of curves $c(u, v)$ considered in Theorem 2.1 have been shown by Golomb [9] to be stable nonlinear splines for their clamped 2-point configurations. The curves $c(u, v)$ of Remark 2.2 are all directly similar to $U := c((0+i0, i), (1+i0, -i))$, which is a single arch of rectangular elastica. Golomb [9] has shown that U is an unstable extremal interpolant for the clamped configuration that it determines, but earlier Horn [12] had argued that U has minimal bending energy when compared to many other plausible curves that fit the same configuration. More recently, Linnér and Jerome [21] have proven that the bending energy of U is minimal among all graphs $(t \mapsto t + i f(t))$ that fit the configuration.



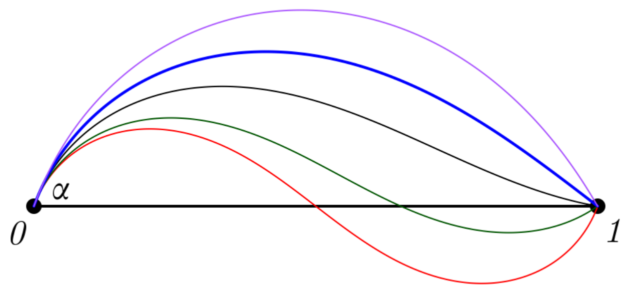


Fig. 3 parametrized rectangular elastica **Fig. 4** $c_1(\alpha, \beta)$ for $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Assume that the chord angles (α, β) determined by a pair of unit tangent vectors (u, v) (with distinct base points) satisfy $|\alpha|, |\beta| \leq \frac{\pi}{2}$. If $(\alpha, \beta) = (0, 0)$, $c(u, v)$ is simply the

line segment from u_1 to v_1; otherwise, $c(u,v)$ will be a segment of nonlinear rectangular elastica. Our preferred parametrization for the latter (see Fig. 3) is $R(t) := \sin t + i\xi(t)$, where ξ is defined by $\xi'(t) = \frac{\sin^2 t}{\sqrt{1+\sin^2 t}}$, $\xi(0) = 0$. If $(\alpha,\beta) \neq (0,0)$, then there exist $t_1 < t_2$ such that the optimal curve $c(u,v)$ is directly similar to the segment $R_{[t_1,t_2]}$ (see [4, Th. 5.4, Cor. 6.1]). The parameters (t_1,t_2) are unique in the sense described in [4, Th. 4.1], and more can be said:
1. If $c(u,v)$ is a right (resp. left) c-curve, then $-\pi \le t_1 < t_2 \le 0$ (resp. $0 \le t_1 < t_2 \le \pi$);
2. If $c(u,v)$ is a right-left (resp. left-right) s-curve, then $-\bar{t} \le t_1 < 0 < t_2 \le \bar{t}$ (resp. $\pi - \bar{t} \le t_1 < \pi < t_2 \le \pi + \bar{t}$), where $\bar{t}$ is defined in [4, Cor. 3.5] as the unique $\tau \in (0,\pi)$ for which the segment $R_{[-\tau,\tau]}$ has chord angles $(\frac{\pi}{2},\frac{\pi}{2})$. The critical angle Ψ mentioned above Theorem 1.1 and in Corollary 1.2, is shown in Fig. 3 and defined in [4, eq. (8.1)].

The bending energy of a curve f is invariant under translation and rotation and is inversely proportional to scale. More precisely, if $P, Q \in \mathbb{C}$ are constant, with $Q \neq 0$, then $\|P + Qf\|^2 = \frac{1}{|Q|}\|f\|^2$. Consequently, when studying $S(u,v)$, it suffices to focus on the case when $u = (0 + i0, e^{i\alpha})$ and $v = (1 + i0, e^{i\beta})$. In this case we write $S(u,v)$ and $c(u,v)$ simply as $S(\alpha,\beta)$ and $c_1(\alpha,\beta)$ and define

$$E_1(\alpha,\beta) := \|c_1(\alpha,\beta)\|^2, \quad |\alpha|,|\beta| \le \frac{\pi}{2}. \tag{2.2}$$

In the general case, when $u = (u_1,u_2)$, $v = (v_1,v_2)$ determine chord angles $(\alpha,\beta) \in [-\frac{\pi}{2},\frac{\pi}{2}]^2$, the bending energy of $c(u,v)$ is given by $\|c(u,v)\|^2 = \frac{1}{|u_1-v_1|}E_1(\alpha,\beta)$. The function $E_1 : [-\frac{\pi}{2},\frac{\pi}{2}]^2 \to [0,\infty)$ is continuous on its domain and is C^∞ on $[-\frac{\pi}{2},\frac{\pi}{2}]^2 \backslash \{(0,0),(-\frac{\pi}{2},\frac{\pi}{2}),(-\frac{\pi}{2},\frac{\pi}{2})\}$ (see [3, Th. 7.10] and [4, Cor. 7.4]). A fundamental identity is proved in [4, Th. 7.3] that relates partial derivatives of $E_1(\alpha,\beta)$ with the end curvatures of $c_1(\alpha,\beta)$. With $\kappa_a(c_1(\alpha,\beta))$ and $\kappa_b(c_1(\alpha,\beta))$ denoting the initial and terminal signed curvatures, respectively, it is proved that for all $(\alpha,\beta) \in [-\frac{\pi}{2},\frac{\pi}{2}]^2 \backslash \{(-\frac{\pi}{2},\frac{\pi}{2}),(-\frac{\pi}{2},\frac{\pi}{2})\}$,

$$-\kappa_a(c_1(\alpha,\beta)) = 2\frac{\partial E_1}{\partial\alpha}(\alpha,\beta) \text{ and } \kappa_b(c_1(\alpha,\beta)) = 2\frac{\partial E_1}{\partial\beta}(\alpha,\beta).$$

A question of both practical and theoretical importance is the following.
Let $\alpha \in [-\frac{\pi}{2},\frac{\pi}{2}]$ be fixed and consider all curves $c_1(\alpha,\beta)$ as β ranges over $[-\frac{\pi}{2},\frac{\pi}{2}]$ (see Fig. 4). Which of these has minimal bending energy?
In [4, Sec. 8], it is shown that $E_1(\alpha,\beta)$ (the bending energy of $c_1(\alpha,\beta)$) is uniquely minimized at $\beta = \beta^*(\alpha)$, where the function β^* is defined in [4, Def. 8.4]. The domain of β^* is an open interval containing $[-\frac{\pi}{2},\frac{\pi}{2}]$ and key properties of β^* are detailed in [4, Lem. 8.5]:

Lemma 2.3. *The function β^* is continuous, odd, and decreasing. Moreover, the following hold.*
(i) $|\beta^*(\alpha)| \le \frac{\pi}{2} - \Psi$ for all $\alpha \in [-\frac{\pi}{2},\frac{\pi}{2}]$.
(ii) On $[0,\frac{\pi}{2}]$, the function $\gamma \mapsto \Psi - \beta^*(\gamma)$ increases continuously from Ψ to $\frac{\pi}{2}$.
(iii) On $[0,\frac{\pi}{2}]$, the function $\gamma \mapsto \gamma + \beta^*(\gamma)$ increases continuously from 0 to Ψ.

In [4, Th. 8.6] it is proved that

$$\operatorname{sign}\left(\frac{\partial E_1}{\partial\beta}(\alpha,\beta)\right) = \operatorname{sign}(\beta - \beta^*(\alpha)) \tag{2.3}$$

holds for all $(\alpha,\beta) \in [-\frac{\pi}{2},\frac{\pi}{2}]^2 \backslash \{(0,0),(\frac{\pi}{2},-\frac{\pi}{2}),(-\frac{\pi}{2},\frac{\pi}{2})\}$. From (2.3), we deduce that the function $\beta \mapsto E_1(\alpha,\beta)$ is decreasing on $[-\frac{\pi}{2},\beta^*(\alpha)]$ and is increasing on $[\beta^*(\alpha),\frac{\pi}{2}]$; hence, $E_1(\alpha,\beta)$ is uniquely minimized at $\beta = \beta^*(\alpha)$.

3. The Convex Hull of a Curve

One of the primary challenges faced when proving Theorem 1.1 is that of comparing the bending energy of an optimal curve $c(u,v)$ ((u,v) being a pair of unit tangent vectors) with the bending energy of some generic curve F that also connects u to v. Although we cannot in general claim that $\|c(u,v)\|^2 \leq \|F\|^2$, we will prove this when F has the property, defined in the next section, of being *forward tracking*. A key ingredient in this is to produce an s-curve $g \in S(u,v)$ such that $\|g\|^2 \leq \|F\|^2$. It turns out that g can be assembled from the boundary of the convex hull of F. The theorem below lays the foundation for this.

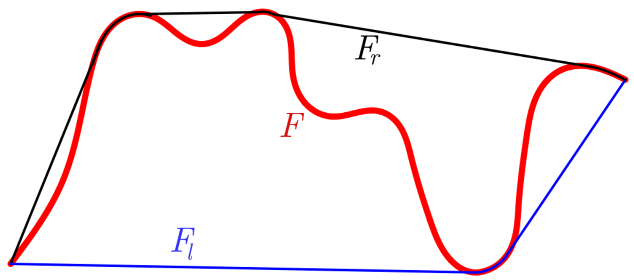


Fig. 5 The convex hull of F is bounded by F_r and F_l.

Theorem 3.1. *Let $F : [0,S] \to \mathbb{C}$ be a curve that satisfies*

$$-\frac{\pi}{2} < m := \min_{0\leq s\leq S} \arg F'(s) \text{ and } \frac{\pi}{2} > M := \max_{0\leq s\leq S} \arg F'(s). \tag{3.1}$$

Let $H(F)$ denote the convex hull of the range of F. Then $H(F)$ (see Fig. 5) is bounded above by a right c-curve $F_r : [0,S_r] \to \mathbb{C}$ and is bounded below by a left c-curve $F_l : [0,S_l] \to \mathbb{C}$. Moreover, the following hold:
(i) $F_r(0) = F(0) = F_l(0)$ and $F_r(S_r) = F(S) = F_l(S_l)$;
(ii) $\arg F'(0) \leq \arg F_r'(0) \leq M$ and $\arg F'(S) \geq \arg F_r'(S_r) \geq m$;
(iii) $\arg F'(0) \geq \arg F_l'(0) \geq m$ and $\arg F'(S) \leq \arg F_l'(S_l) \leq M$;
(iv) $\|F_r\|^2 \leq \|F\|^2$, with equality if and only if $F \equiv F_r$;
(v) $\|F_l\|^2 \leq \|F\|^2$, with equality if and only if $F \equiv F_l$.

Proof. It suffices to prove the assertions on F_r, since those on F_l can then be obtained by first reflecting F about the real axis. Set $a := \operatorname{Re} F(0)$ and $b := \operatorname{Re} F(S)$. Since $\operatorname{Re} F'(s) > 0$ for all s, the curve F can be parametrized as $x \mapsto x + if(x)$, $x \in [a,b]$, where f and f' are absolutely continuous. Note that if $F(s) = x + iy$, then $y = f(x)$ and $\arg F'(s) = \arctan f'(x)$. The upper boundary of $H(F)$ can be parametrized as $x \mapsto x + ig(x)$, $x \in [a,b]$, where the function $g : [a,b] \to \mathbb{R}$ is defined by

$$g(x) := \max\{\operatorname{Im} z : z \in H(F) \text{ and } \operatorname{Re} z = x\}.$$

Since $H(F)$ is convex, it follows that

(3.2) g is the smallest concave function with the property $g(x) \geq f(x)$ for all $x \in [a,b]$.

It is easy to construct a concave function $\widehat{g} : [a,b] \to \mathbb{R}$ that satisfies $\widehat{g}(a) = f(a)$, $\widehat{g}(b) = f(b)$, and $\widehat{g}(x) > f(x)$ for $a < x < b$. From this it follows that $g(a) = f(a)$ and $g(b) = f(b)$; hence (i).
Since g is concave, it follows that the left (resp. right) derivatives exist for all $x \in (a,b]$ (resp. $x \in [a,b)$); in particular, $g'(a)$ and $g'(b)$ exist. For $x \in (a,b)$, the following two

statements follow easily from (3.2) and the continuity of f'.
(A) If $g(x) = f(x)$, then $g'(x)$ exists and equals $f'(x)$.
(B) If $g(x) \neq f(x)$, then $g(x) > f(x)$ and there is a smallest interval $a_x < x < b_x$ such that $g(a_x) = f(a_x)$, $g(b_x) = f(b_x)$, and g is linear on $[a_x, b_x]$.
Note that if $g(x) \neq f(x)$, then it follows from these that $g'(x) = \frac{f(b_x)-f(a_x)}{b_x-a_x}$; consequently, $g'(x)$ exists for all $x \in [a,b]$ and g is continuous.
We can now prove (ii).
In order to prove that $\arg F'(0) \leq \arg F_r'(0) \leq M$, it suffices to show that $g'(a) \geq f'(a)$ and that there exists $c \in [a,b]$ such that $g'(a) = f'(c)$. Since $g(a) = f(a)$, it follows from (3.2) that $g'(a) \geq f'(a)$. If $g'(a) = f'(a)$, then the desired equality $g'(a) = f'(c)$ holds with $c = a$; so assume $g'(a) > f'(a)$. Then there exists $x \in (a,b)$ such that $g(t) > f(t)$ for all $t \in (a,x]$. The smallest interval $[a_x, b_x]$ described in property (B) will have $a_x = a$, and therefore

$$g'(a) = g'(x) = \frac{f(b_x) - f(a)}{b_x - a} = f'(c)$$

for some $c \in (a, b_x)$, by the mean value theorem. The proof that $\arg F'(S) \geq \arg F_r'(S_r) \geq m$ can be made in a similar manner, so this completes the proof of (ii).
At present we know that $g'(x)$ exists for all $x \in [a,b]$ and that g' is monotonically decreasing (since g is concave). It follows by Darboux's Theorem that the range of g' equals the interval (or singleton) $[g'(b), g'(a)]$. But since g' is monotonically decreasing, it now follows that g' is continuous. With (ii) in hand, in order to prove that F_r is a right c-curve, it suffices to show that g' is absolutely continuous on $[a,b]$. For this, since g' is continuous on $[a,b]$, it suffices to show that g' is absolutely continuous on the open interval (a,b).
Claim For all intervals $[c,d] \subset (a,b)$, there exists an interval $[\widehat{c}, \widehat{d}] \subset [c,d]$ such that $g'(c) - g'(d) \leq |f'(\widehat{c}) - f'(\widehat{d})|$.

proof: Let $[c,d] \subset (a,b)$. If $g'(c) - g'(d) \leq |f'(c) - f'(d)|$, then $[\widehat{c}, \widehat{d}] = [c,d]$ is as required. So assume $g'(c) - g'(d) > |f'(c) - f'(d)|$. If $g(c) = f(c)$, set $\widehat{c} = c$; otherwise (if $g(c) > f(c)$), let $[a_c, b_c]$ be the interval described in property (B) and set $\widehat{c} = b_c$. Similarly, if $g(d) = f(d)$, set $\widehat{d} = d$; otherwise, let $[a_d, b_d]$ be the interval described in property (B) and set $\widehat{d} = a_d$. Note that $g'(\widehat{c}) = g'(c) > g'(d) = g'(\widehat{d})$ and therefore $\widehat{c} < \widehat{d}$. Since $c \leq \widehat{c}$ and $\widehat{d} \leq d$ by construction, it follows that $[\widehat{c}, \widehat{d}] \subset [c,d]$. Note also that $g(\widehat{c}) = f(\widehat{c})$ and $g(\widehat{d}) = f(\widehat{d})$, and hence, by property (A), $g'(c) = g'(\widehat{c}) = f'(\widehat{c})$ and $g'(d) = g'(\widehat{d}) = f'(\widehat{d})$. We therefore have $g'(c) - g'(d) = |f'(\widehat{c}) - f'(\widehat{d})|$, which proves the claim.
Let $\varepsilon > 0$ be given. Since f' is absolutely continuous, there exists $\delta > 0$ such that if $a < c_1 < d_1 < c_2 < d_2 < \cdots < c_n < d_n < b$ satisfy $\sum_{j=1}^n (d_j - c_j) < \delta$, then $\sum_{j=1}^n |f'(c_j) - f'(d_j)| < \varepsilon$. We will show that the same δ works for g'. Assume $a < c_1 < d_1 < c_2 < d_2 < \cdots < c_n < d_n < b$ satisfy $\sum_{j=1}^n (d_j - c_j) < \delta$. For $j = 1, 2, \ldots, n$, let $[\widehat{c}_j, \widehat{d}_j] \subset [c_j, d_j]$ be as in the claim. Then

$$\sum_{j=1}^n |g'(c_j) - g'(d_j)| = \sum_{j=1}^n (g'(c_j) - g'(d_j)) \leq \sum_{j=1}^n |f'(\widehat{c}_j) - g'(\widehat{d}_j)| < \varepsilon,$$

since $\sum_{j=1}^n (\widehat{d}_j - \widehat{c}_j) \leq \sum_{j=1}^n (d_j - c_j) < \delta$. Therefore g' is absolutely continuous on (a,b) and this completes the proof that F_r is a right c-curve.
Lastly, we turn to (iv).
Let U denote the open interval (a,b) and set $V := \{x \in U : f''(x) \text{ and } g''(x) \text{ exist}\}$. Since both f' and g' are absolutely continuous, $U \backslash V$ has Lebesgue measure 0. Set $W := \{x \in V : g(x) = f(x)\}$. If $x \in V \backslash W$ we must have $g(x) > f(x)$ and hence, by property (B),

$g''(x) = 0$. On the other hand, if $x \in W$, then (by property (A)) $g'(x) = f'(x)$ and since $g \geq f$, it follows that $g''(x) \geq f''(x)$. Since g is concave, we have $g''(x) \leq 0$, and therefore $|g''(x)| = -g''(x) \leq -f''(x) \leq |f''(x)|$ for all $x \in W$. Thus,

$$\begin{aligned} \|F_r\|^2 &= \int_V \frac{|g''(x)|^2}{(1+g'(x)^2)^{5/2}}\, dx \\ &= \int_W \frac{|g''(x)|^2}{(1+g'(x)^2)^{5/2}}\, dx \quad \text{(since } g''=0 \text{ on } V\backslash W) \\ &= \int_W \frac{|g''(x)|^2}{(1+f'(x)^2)^{5/2}}\, dx \quad \text{(since } g'=f' \text{ on } W) \\ &\leq \int_W \frac{|f''(x)|^2}{(1+f'(x)^2)^{5/2}}\, dx \leq \|F\|^2. \end{aligned}$$

Now, suppose $\|F_r\|^2 = \|F\|^2$. Then the above holds with equalities and it follows that $\int_{V\backslash W} \frac{|f''(x)|^2}{(1+f'(x)^2)^{5/2}}\, dx = 0$; hence, $f'' = 0$ a.e. on $V\backslash W$. Moreover, on W we have $0 \geq g'' \geq f''$ and $\int_W \frac{|g''(x)|^2}{(1+f'(x)^2)^{5/2}}\, dx = \int_W \frac{|f''(x)|^2}{(1+f'(x)^2)^{5/2}}\, dx$. From these it follows that $f'' = g''$ a.e. on W. Consequently, $f'' = g''$ a.e. on V. Since $[a,b]\backslash V$ has measure 0 (and f', g' are absolutely continuous), there exists a constant C such that $g'(x) = f'(x) + C$ for all $x \in [a,b]$. But since $g(a) = f(a)$ and $g(b) = f(b)$, it follows that $f(x) = g(x)$ for all $x \in [a,b]$; hence, $F_r \equiv F$. □

4. Forward Tracking Curves and Optimality

Definition 4.1. Let $F : [a,b] \to \mathbb{C}$ be a curve. We say that F is **forward tracking** if $F(a) \neq F(b)$ and

$$-\frac{\pi}{2} < \arg \frac{F'(t)}{F(b)-F(a)} < \frac{\pi}{2} \quad \text{for all } t \in [a,b]. \tag{4.1}$$

Given a pair (u,v) of unit tangent vectors with distinct base points, let $\mathcal{F}(u,v)$ denote the set of all forward tracking curves that connect u to v. Given points $P_1, P_2, \ldots, P_n$, let $\mathcal{F}(P_1, P_2, \ldots, P_n)$ denote the set of all interpolating curves whose pieces are forward tracking.

Our purpose in this section is to prove the following theorem and corollary.

Theorem 4.2. *Let (u,v) be a pair of unit tangent vectors, with distinct base points, such that $\mathcal{F}(u,v)$ is non-empty. Then the curve $c(u,v)$ (defined in Section 2) is the unique curve in $\mathcal{F}(u,v)$ (modulo equivalence) with minimal bending energy.*

Corollary 4.3. *Given points $P_1, P_2, \ldots, P_n$, with $P_j \neq P_{j+1}$, let $F \in \mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_n)$ be a restricted elastic spline (i.e., a curve with minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_n)$). If $\mathcal{F}(P_1, P_2, \ldots, P_n)$ is non-empty (i.e., if none of the stencil angles equals π), then $\|F\|^2 \leq \|f\|^2$ for all $f \in \mathcal{F}(P_1, P_2, \ldots, P_n)$.*

The property of being forward tracking (applied to a curve) is invariant under translation, rotation and scaling, and therefore, as explained above (2.2), when proving Theorem 4.2 we can assume, without loss of generality, that $u = (0 + i0, e^{i\alpha})$ and $v = (1 + i0, e^{i\beta})$. With this in mind, we write $\mathcal{F}(\alpha,\beta)$ instead of $\mathcal{F}(u,v)$, which is consistent with the notation $S(\alpha,\beta)$ and $c_1(\alpha,\beta)$ in place of $S(u,v)$ and $c(u,v)$ already instituted in Section 2. One easily verifies that $\mathcal{F}(\alpha,\beta)$ is non-empty if and only if $(\alpha,\beta) \in (-\frac{\pi}{2}, \frac{\pi}{2})^2$.

We will first show that $c_1(\alpha,\beta)$ is a forward tracking curve when $(\alpha,\beta) \in (-\frac{\pi}{2}, \frac{\pi}{2})^2$. The line segment $c_1(0,0)$ is obviously forward tracking, so this case is excluded in the following.

Proposition 4.4. *Let $(\alpha,\beta)\in[-\frac{\pi}{2},\frac{\pi}{2}]^2\backslash\{(0,0)\}$, and assume $c_1(\alpha,\beta)$ is parametrized as $c_1(t;\alpha,\beta)$, $t\in[t_1,t_2]$. Then*

$$|\arg c_1'(t;\alpha,\beta)|<\max\{|\alpha|,|\beta|\}\ \textit{for all}\ t\in(t_1,t_2). \tag{4.2}$$

In particular, $c_1(\alpha,\beta)$ is forward tracking for all $(\alpha,\beta)\in(-\frac{\pi}{2},\frac{\pi}{2})^2\backslash\{(0,0)\}$.

Proof. We can assume, without loss of generality (see [4, Remark 5.5]), that $\alpha\geq|\beta|$ and it follows [3, Prop. 5.6] that $c_1(\alpha,\beta)$ is either a right c-curve or a right-left s-curve. Define $\theta(t):=\arg c_1'(t;\alpha,\beta)$, $t_1\leq t\leq t_2$, whereby $\theta(t_1)=\alpha$ and $\theta(t_2)=\beta$. If $c_1(\alpha,\beta)$ is a right c-curve, then θ decreases from α to β and inequality (4.2) follows easily.
Assuming now that $c_1(\alpha,\beta)$ is a right-left s-curve, there exists $t_0\in(t_1,t_2)$ such that θ is decreasing on $[t_1,t_0]$ and increasing on $[t_0,t_1]$. Since $\alpha\geq|\beta|$, in order to prove (4.2) it suffices to show that $|\theta(t_0)|<\alpha$. With $Q:=c_1(t_0;\alpha,\beta)$ denoting the inflection point, let ℓ denote the inflection line (the tangent line through Q) and let R denote the point of intersection between ℓ and the line segment $[0,1]$.

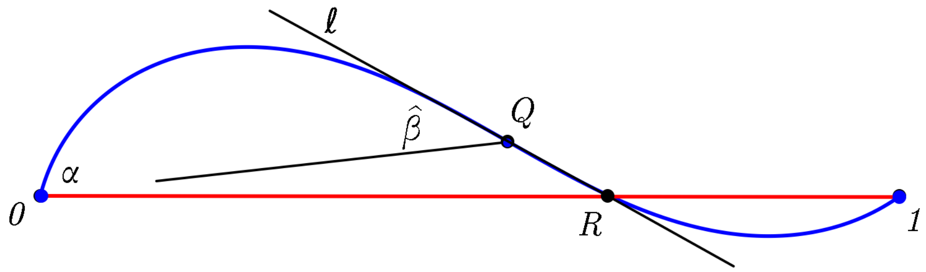


Fig. 6 The case when $c_1(\alpha,\beta)$ is a right-left s-curve and $\mathrm{Im}\,Q\geq 0$.
Case 1: $\mathrm{Im}\,Q\geq 0$ (see Fig. 6)
Let $(\widehat{\alpha},\widehat{\beta})$ be the chord angles of the segment $v(t):=c_1(t;\alpha,\beta)$, $t_1\leq t\leq t_0$. It follows from [3, Lemma 6.3] that $\widehat{\alpha}>|\widehat{\beta}|$, and since $\mathrm{Im}\,Q\geq 0$, we have $|\widehat{\beta}|\geq|\angle QR0|=|\theta(t_0)|$. Therefore $\alpha\geq\widehat{\alpha}>|\widehat{\beta}|\geq|\theta(t_0)|$.
Case 2: $\mathrm{Im}\,Q<0$
The same argument given for Case 1 shows that $|\beta|>|\theta(t_0)|$ and hence $\alpha\geq|\beta|>|\theta(t_0)|$. $\square$

Our proof of Theorem 4.2 hinges on the monotonicity of $E_1(\alpha,\beta)$ as described in the following.

Proposition 4.5. *Assume $\frac{\pi}{2}\geq\widehat{\alpha}\geq\alpha\geq|\beta|$. If $\frac{\pi}{2}\geq\widehat{\beta}\geq\beta\geq\beta^*(\alpha)$ or $-\frac{\pi}{2}\leq\widehat{\beta}\leq\beta\leq\beta^*(\alpha)$, then $E_1(\widehat{\alpha},\widehat{\beta})\geq E_1(\alpha,\beta)$, with equality if and only if $(\widehat{\alpha},\widehat{\beta})=(\alpha,\beta)$.*

Our proof of the proposition employs the following lemma.

Lemma 4.6. *For $\alpha,\beta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, the following hold.*
(i) If $\frac{\pi}{2}\geq\widehat{\beta}>\beta\geq\beta^*(\alpha)$, then $E_1(\alpha,\widehat{\beta})>E_1(\alpha,\beta)$.
(ii) If $-\frac{\pi}{2}\leq\widehat{\beta}<\beta\leq\beta^*(\alpha)$, then $E_1(\alpha,\widehat{\beta})>E_1(\alpha,\beta)$.
(iii) If $\frac{\pi}{2}\geq\widehat{\alpha}>\alpha\geq\beta^*(\beta)$, then $E_1(\widehat{\alpha},\beta)>E_1(\alpha,\beta)$.
(iv) If $\frac{\pi}{2}\geq\widehat{\alpha}>\alpha\geq|\beta|$, then $E_1(\widehat{\alpha},\beta)>E_1(\alpha,\beta)$.
(v) If $\frac{\pi}{2}\geq\widehat{\alpha}>\alpha\geq 0$, then $E_1(\widehat{\alpha},\beta^*(\widehat{\alpha}))>E_1(\alpha,\beta^*(\alpha))$.

Proof. We first consider (i) and (ii). Let $\alpha\in[-\frac{\pi}{2},\frac{\pi}{2}]$ be fixed. The function $\beta\mapsto E_1(\alpha,\beta)$ is continuous on $[\beta^*(\alpha),\frac{\pi}{2}]$ and C^∞ on $(\beta^*(\alpha),\frac{\pi}{2})$. It follows from (2.3) that $\frac{\partial E_1}{\partial\beta}(\alpha,\beta)>0$ on $(\beta^*(\alpha),\frac{\pi}{2})$. Therefore $\beta\mapsto E_1(\alpha,\beta)$ is increasing on $[\beta^*(\alpha),\frac{\pi}{2}]$, which proves (i). One proves (ii) by showing, in a similar fashion, that $\beta\mapsto E_1(\alpha,\beta)$ is decreasing on $[-\frac{\pi}{2},\beta^*(\alpha)]$.

Item (iii) is an immediate consequence of (i) and the symmetry $E_1(\alpha,\beta) = E_1(\beta,\alpha)$. It follows from Lemma 2.3 that $|\beta^*(\gamma)| < |\gamma|$ for all $\gamma \in [-\frac{\pi}{2}, \frac{\pi}{2}]\backslash\{0\}$, and consequently (iv) is an immediate consequence of (iii).

We turn now to (v), where our task is to show that the continuous function $g(\gamma) := E_1(\gamma, \beta^*(\gamma))$ is increasing on $[0, \frac{\pi}{2}]$. We first show that it is increasing on $[|\beta^*(a)|, a]$ for all $a \in (0, \frac{\pi}{2}]$. Suppose $a \geq \widehat{\alpha} > \alpha \geq |\beta^*(a)|$. Then $g(\widehat{\alpha}) = E_1(\widehat{\alpha}, \beta^*(\widehat{\alpha})) > E_1(\alpha, \beta^*(\widehat{\alpha}))$ by (iv) since $\alpha \geq |\beta^*(a)| \geq |\beta^*(\widehat{\alpha})|$ (by Lemma 2.3). And by (ii), we have $E_1(\alpha, \beta^*(\widehat{\alpha})) > E_1(\alpha, \beta^*(\alpha)) = g(\alpha)$, since $-\frac{\pi}{2} \leq \beta^*(\widehat{\alpha}) < \beta^*(\alpha)$. Therefore $g(\widehat{\alpha}) > g(\alpha)$, and hence g is increasing on $[|\beta^*(a)|, a]$ as claimed. In particular, g is increasing on $[\beta^*(\frac{\pi}{2}), \frac{\pi}{2}] = [\frac{\pi}{2} - \Psi, \frac{\pi}{2}]$. Define $\gamma_0 \in [0, \frac{\pi}{2} - \Psi]$ by

$$\gamma_0 := \inf\{a \in [0, \frac{\pi}{2} - \Psi] : g \text{ is increasing on } [a, \frac{\pi}{2}]\}.$$

If $\gamma_0 > 0$, then it follows that g is increasing on $(\gamma_0, \frac{\pi}{2}]$. But since g is continuous, g must be increasing on the closed interval $[\gamma_0, \frac{\pi}{2}]$. By the claim, g is increasing on $[|\beta^*(\gamma_0)|, \gamma_0]$, and hence g is increasing on the union $[|\beta^*(\gamma_0)|, \frac{\pi}{2}]$–a contradiction (since $|\beta^*(\gamma_0)| < \gamma_0$). Therefore $\gamma_0 = 0$ and it follows that g is increasing on $[0, \frac{\pi}{2}]$. $\square$

Proof of Proposition 4.5. If $\widehat{\alpha} = \alpha$ or $\widehat{\beta} = \beta$, then the desired result is already proved in Lemma 4.6 (i),(ii),(iv). So assume $\frac{\pi}{2} \geq \widehat{\alpha} > \alpha \geq |\beta|$ and $\widehat{\beta} \neq \beta$, and note that, by Lemma 2.3, $\beta^*(\widehat{\alpha}) < \beta^*(\alpha) \leq 0$.

Another easy case is when $\frac{\pi}{2} \geq \widehat{\beta} > \beta \geq \beta^*(\alpha)$. By Lemma 4.6 (i), we have $E_1(\widehat{\alpha}, \widehat{\beta}) > E_1(\widehat{\alpha}, \beta)$, while invoking Lemma 4.6 (iv) yields $E_1(\widehat{\alpha}, \beta) > E_1(\alpha, \beta)$. Therefore $E_1(\widehat{\alpha}, \widehat{\beta}) > E_1(\alpha, \beta)$.

We now consider the remaining case $-\frac{\pi}{2} \leq \widehat{\beta} < \beta \leq \beta^*(\alpha)$. Although $\beta^*(\widehat{\alpha}) < \beta^*(\alpha)$, we cannot say where $\beta^*(\widehat{\alpha})$ lies relative to $\widehat{\beta}$ and β. Consequently, we must branch into three cases.

Case A: $\beta \leq \beta^*(\widehat{\alpha}) < \beta^*(\alpha)$

Then $E_1(\widehat{\alpha}, \widehat{\beta}) > E_1(\widehat{\alpha}, \beta) > E_1(\alpha, \beta)$ by Lemma 4.6 (ii) and (iv).

Case B: $\widehat{\beta} \leq \beta^*(\widehat{\alpha}) < \beta$

Then $E_1(\widehat{\alpha}, \widehat{\beta}) \geq E_1(\widehat{\alpha}, \beta^*(\widehat{\alpha}))$, by Lemma 4.6 (ii). There exists $\alpha_1 \in [\alpha, \widehat{\alpha})$ such that $\beta^*(\alpha_1) = \beta$, and so, by Lemma 4.6 (v), we have $E_1(\widehat{\alpha}, \beta^*(\widehat{\alpha})) > E_1(\alpha_1, \beta^*(\alpha_1))$. Finally, $E_1(\alpha_1, \beta^*(\alpha_1)) = E_1(\alpha_1, \beta) \geq E_1(\alpha, \beta)$, by Lemma 4.6 (iv). Therefore, $E_1(\widehat{\alpha}, \widehat{\beta}) > E_1(\alpha, \beta)$.

Case C: $\beta^*(\widehat{\alpha}) < \widehat{\beta}$

There exists $\alpha_2 \in (\alpha, \widehat{\alpha})$ such that $\beta^*(\alpha_2) = \widehat{\beta}$. By Lemma 2.3, $\alpha_2 > |\beta^*(\alpha_2)| = |\widehat{\beta}|$, and hence $E_1(\widehat{\alpha}, \widehat{\beta}) > E_1(\alpha_2, \widehat{\beta})$ by Lemma 4.6 (iv). Since $\beta^*(\alpha_2) = \widehat{\beta}$, we are now in Case B whereby $E_1(\alpha_2, \widehat{\beta}) > E_1(\alpha, \beta)$. Therefore, $E_1(\widehat{\alpha}, \widehat{\beta}) > E_1(\alpha, \beta)$. $\square$

We can now prove the main results of this section.

Proof of Theorem 4.2. . We can assume, without loss of generality, that $\frac{\pi}{2} > \alpha \geq |\beta|$ and $\alpha > 0$. Let $F : [0, S] \to \mathbb{C}$ be a curve in $\mathcal{F}(\alpha, \beta)$ that satisfies $\|F\|^2 \leq E_1(\alpha, \beta)$. We will show that $F \equiv c_1(\alpha, \beta)$. Let m, M be as defined in (3.1) and note that (3.1) holds (i.e., $-\frac{\pi}{2} < m \leq M < \frac{\pi}{2}$) since F is a forward tracking curve from $0 + i0$ to $1 + i0$. Let F_r be as in Theorem 3.1.

Case 1: $\beta \leq \beta^*(\alpha)$

With $\widehat{\alpha} := \arg F_r'(0)$ and $\widehat{\beta} := \arg F_r'(S_r)$, we have from Theorem 3.1 (ii) that $M \geq \widehat{\alpha} \geq \alpha$ and $m \leq \widehat{\beta} \leq \beta$. It follows from Proposition 4.5 that $E_1(\widehat{\alpha}, \widehat{\beta}) \geq E_1(\alpha, \beta)$. Since $F_r \in$

$S(\widehat{\alpha}, \widehat{\beta})$, we conclude that

$$\|F\|^2 \geq \|F_r\|^2 \geq E_1(\widehat{\alpha}, \widehat{\beta}) \geq E_1(\alpha, \beta) \geq \|F\|^2,$$

and it follows that $F \equiv F_r \equiv c_1(\alpha, \beta)$. This settles Case 1.
Case 2: $\beta > \beta^*(\alpha)$ and $m = \beta$
Set $\widehat{\alpha} = \arg F_r'(0)$. By Theorem 3.1 (ii), we have $M \geq \widehat{\alpha} \geq \alpha$ and $\beta = \arg F_r'(S_r) = m$. Thus $F_r \in S(\widehat{\alpha}, \beta)$, and therefore, by Proposition 4.5,

$$\|F\|^2 \geq \|F_r\|^2 \geq E_1(\widehat{\alpha}, \beta) \geq E_1(\alpha, \beta) \geq \|F\|^2,$$

and it follows that $F \equiv F_r \equiv c_1(\alpha, \beta)$. This settles Case 2.
Case 3: $\beta > \beta^*(\alpha)$ and $m < \beta$

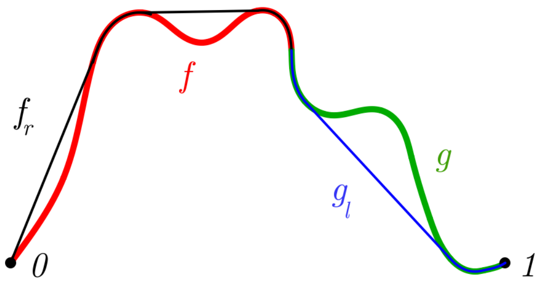


Fig. 7 The curve $F = f \sqcup g$ shown with $f_r \sqcup g_l$.
Let $s_1 \in (0, S)$ be such that $\arg F'(s_1) = m$. Writing $F = f \sqcup g$ (see Fig. 7), where $f := F_{[0,s_1]}$ and $g := F_{[s_1,S]}$, we observe that

$$\arg f'(s_1) = \min_{0 \leq s \leq s_1} \arg f'(s) = m = \min_{s_1 \leq s \leq S} \arg g'(s) = \arg g'(s_1).$$

Since f and g are sub-curves of F, it follows that (3.1) holds for both f and g. Let $f_r : [0, a] \to \mathbb{C}$ and $g_l[a, b] \to \mathbb{C}$ be as described in Theorem 3.1, and set $\widehat{\alpha} := \arg f_r'(0)$, $\widehat{\beta} := \arg g_l'(b)$. It follows from Theorem 3.1 that $M \geq \widehat{\alpha} \geq \alpha$, $\arg f_r'(a) = m$, $\beta \leq \widehat{\beta} \leq M$, and $\arg g_l'(a) = m$. Since $\arg f_r'(a) = \arg g_l'(a)$, the composite curve $f_r \sqcup g_l$ belongs to $S(\widehat{\alpha}, \widehat{\beta})$, and therefore, by Proposition 4.5,

$$\|F\|^2 = \|f\|^2 + \|g\|^2 \geq \|f_r\|^2 + \|g_l\|^2 = \|f_r \sqcup g_l\|^2 \geq E_1(\widehat{\alpha}, \widehat{\beta}) \geq E_1(\alpha, \beta) \geq \|F\|^2.$$

Noting that equality holds throughout the above, it follows that $F \equiv f_r \sqcup g_l \equiv c_1(\alpha, \beta)$. $\square$

Proof of Corollary 4.3. Assume $f \in \mathcal{F}(P_1, P_2, \ldots, P_n)$. We can write f piecewise as $f = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{n-1}$ where f_j connects $u_j = (P_j, d_j)$ to $u_{j+1} = (P_{j+1}, d_{j+1})$. Let $\widehat{F} \in \mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_n)$ be defined piecewise by $\widehat{F} := c(u_1, u_2) \sqcup c(u_2, u_3) \sqcup \cdots \sqcup c(u_{n-1}, u_n)$. By Theorem 4.2, $\|c(u_j, u_{j+1})\|^2 \leq \|f_j\|^2$, $j = 1, 2, \ldots, n$, and hence $\|\widehat{F}\|^2 \leq \|f\|^2$. But since F has minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_n)$, we have $\|F\|^2 \leq \|\widehat{F}\|^2$. Therefore $\|F\|^2 \leq \|f\|^2$. $\square$

5. Contagion of forward tracking curves

In this section we show that the property of being forward tracking is contagious, for if a curve f, with limited bending energy, gets too close to a forward tracking C^2 curve g, then f will also be forward tracking. The precise statement of this is given in the theorem below, where the distance from $f : [0, L] \to \mathbb{C}$ to $g : [0, S] \to \mathbb{C}$ is defined as the Hausdorff distance:

$$\text{dist}(f, g) := \max\{d(f, g), d(g, f)\}, \text{ with } d(f, g) := \max_{t\in[0,L]} \min_{s\in[0,S]} |f(t) - g(s)|. \tag{5.1}$$

Theorem 5.1. *Let $g : [0, S] \to \mathbb{C}$ be a C^2 unit speed curve that is forward tracking. For all $M > 0$ there exists $\varepsilon > 0$ such that if $f : [0, L] \to \mathbb{C}$ is a unit speed curve satisfying*

(i) $f(0) = g(0)$, $f(L) = g(S)$, (ii) $\|f\|^2 < M$, and (iii) dist$(f, g) < \varepsilon$,

then f is forward tracking.

Our proof Theorem 5.1 employs the following two lemmas, the first of which shows that if a sufficiently long curve is initially directed toward a line, then it must possess considerable bending energy in order to avoid intersecting the line.

Lemma 5.2. *Given $\theta_0 \in (0, \frac{\pi}{2}]$ and $\varepsilon > 0$, set $b_0 := \dfrac{9\varepsilon}{\theta_0}$. If $f : [0, b_0] \to \mathbb{C}$ is a unit speed curve satisfying*

(i) $f'(0) = e^{i\theta_0}$ and

(ii) $\operatorname{Im} f(s) - \operatorname{Im} f(0) \le 3\varepsilon$ for all $s \in [0, b_0]$,

then $\|f\|^2 \ge \dfrac{\theta_0^3}{108\varepsilon}$.

Proof. Assume that f satisfies (i) and (ii), and let $\kappa(s)$ denote its signed curvature and $\theta(s) := \theta_0 + \int_0^s \kappa(t)\, dt$ its direction angle. We can assume, without loss of generality, that $\frac{\pi}{2} \ge \theta(s) \ge 0$, since otherwise we can construct a curve $\widehat{f}$ (eg., $\widehat{\kappa}(s) := -|\kappa(s)|$ while $\widehat{\theta}(s) > 0$), with $\|\widehat{f}\|^2 \le \|f\|^2$, that satisfies $\frac{\pi}{2} \ge \widehat{\theta}(s) \ge 0$ while maintaining (i) and (ii). Since $\frac{1}{2}\theta(s) \le \sin\theta(s)$, we have

$$\int_0^{b_0} \frac{1}{2}\theta(s)\, ds \le \int_0^{b_0} \sin\theta(s)\, ds = \operatorname{Im} f(b_0) - \operatorname{Im} f(0) \le 3\varepsilon,$$

and hence $\int_0^{b_0} [\theta_0 + \int_0^s \kappa(t)\, dt]\, ds \le 6\varepsilon$. This can be written equivalently as

$$3\varepsilon = b_0\theta_0 - 6\varepsilon \le \int_0^{b_0} (b_0 - s)(-\kappa(s))\, ds.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$9\varepsilon^2 \le \left(\int_0^{b_0} (b_0 - s)(-\kappa(s))\, ds\right)^2 \le \int_0^{b_0} (b_0 - s)^2\, ds \int_0^{b_0} (\kappa(s))^2\, ds = \frac{b_0^3}{3}(4\|f\|^2),$$

and therefore $\|f\|^2 \ge \dfrac{27\varepsilon^2}{4b_0^3} = \dfrac{\theta_0^3}{108\varepsilon}$. $\square$

This lemma generalizes easily to the following.

Lemma 5.3. *Given* $\theta_0 \in (0, \frac{\pi}{2}]$ *and* $\varepsilon > 0$*, set* $b_0 := \dfrac{9\varepsilon}{\theta_0}$*. If* $f : [0, b_0] \to \mathbb{C}$ *is a unit speed curve satisfying*

(i) $\theta_0 \le |\arg f'(0)| \le \pi - \theta_0$ *and*

(ii) $|\mathrm{Im}\, f(s) - \mathrm{Im}\, f(0)| \le 3\varepsilon$ *for all* $s \in [0, b_0]$*,*

then $\|f\|^2 \ge \dfrac{\theta_0^3}{108\varepsilon}$.

Proof. Assume f satisfies (i) and (ii) and set $\theta_1 := \arg f'(0)$. We can assume without loss of generality that $\theta_0 \le \theta_1 \le \frac{\pi}{2}$ since otherwise we can replace f with one of $\overline{f}$ (complex conjugate), $-f$, or $-\overline{f}$, which leaves bending energy unchanged and preserves (ii). Set $b_1 := \frac{9\varepsilon}{\theta_1}$. Since $b_1 \le b_0$, it follows from Lemma 5.2 that

$$\|f\|^2 \ge \|f_{[0,b_1]}\|^2 \ge \frac{\theta_1^3}{108\varepsilon} \ge \frac{\theta_0^3}{108\varepsilon}.$$

□

Proof of Theorem 5.1. Let $B := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk and C its boundary, whereby the open disk of radius r and center c can be written as $c + rB$. With $P := g(0)$ and $Q := g(S)$, set

$$\theta_{max} := \max_{0 \le s \le S} |\arg \frac{g'(s)}{Q-P}|, \quad \theta_0 := \frac{1}{3}(\frac{\pi}{2} - \theta_{max}),$$
$$K := 1 + \max_{0 \le s \le S} |g''(s)|, \text{ and } \delta := \frac{1}{2K}.$$

Since g is forward tracking, we have $\theta_{max} < \frac{\pi}{2}$, which implies $\theta_0 > 0$. We extend g, in a C^1 fashion, by tangent rays:

$$g(s) := g(0) + sg'(0) \text{ for } s < 0, \qquad g(s) := g(S) + (s - S)g'(S) \text{ for } s > S,$$

and denote the range of g by $G := g(\mathbb{R})$. Set

$$\varepsilon := \min\{\frac{1}{81}\delta\theta_0^2, \frac{1}{18}L\theta_0, \frac{1}{108M}\theta_0^3\} \quad \text{and} \quad b_0 := \frac{9\varepsilon}{\theta_0},$$

and note that $b_0 \le \frac{L}{2}$ is an immediate consequence of $\varepsilon \le \frac{1}{18}L\theta_0$.
Assume f satisfies (i),(ii), and (iii). Since $\theta_{max} + \theta_0 < \frac{\pi}{2}$, in order to prove that f is forward tracking, it suffices to show that

$$|\arg \frac{f'(t)}{Q-P}| < \theta_{max} + \theta_0 \text{ for all } 0 \le t \le L. \tag{5.2}$$

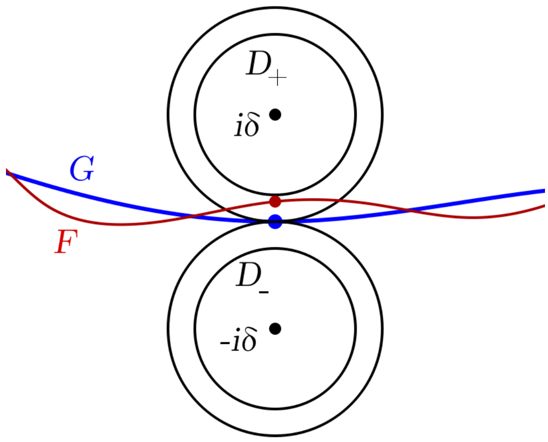
Suppose not. Since $f(0) = P$ and $f(L) = Q$, it follows by Cauchy's mean value theorem that there exists $t_2 \in (0, L)$ such that $\arg \frac{f'(t_2)}{Q-P} = 0$. Since we assume (5.2) does not hold, it follows, by the intermediate value theorem, that there exists $t_0 \in [0, L]$ such that $|\arg \frac{f'(t_0)}{Q-P}| = \theta_{max} + \theta_0$. Let $s_0 \in \mathbb{R}$ be such that $g(s_0)$ is a closest point of G to $f(t_0)$. By applying a translation and rotation, if necessary, we can assume without loss of generality that $g(s_0) = 0 + i0$ and $g'(s_0) = 1 + i0$. Note that it follows that $f(t_0)$ lies (on the imaginary axis) in the interval $[-i\varepsilon, i\varepsilon]$.

Recall that for nonzero $w, z \in \mathbb{C}$, $|\arg w| - |\arg z| \le |\arg \frac{w}{z}| \le |\arg w| + |\arg z|$. Writing $\arg f'(t_0) = \arg\left(\frac{f'(t_0)/(Q-P)}{g'(s_0)/(Q-P)}\right)$, we apply this to obtain

$$\theta_0 = (\theta_{max} + \theta_0) - \theta_{max} \le |\arg f'(t_0)| \le (\theta_{max} + \theta_0) + \theta_{max} < \pi - \theta_0. \tag{5.3}$$

Since K is greater than the curvature $|g''(s)|$, for all s, the definition of δ ensures that the range of g satisfies

$$G \cap (i\delta + \delta B) = \emptyset = G \cap (-i\delta + \delta B) \quad \text{(see Fig. 8).}$$



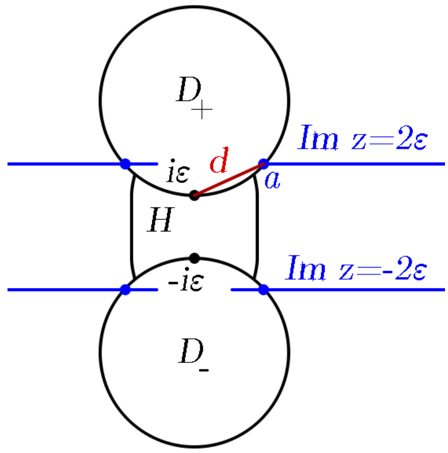


Fig. 8 The ranges $G := g(\mathbb{R})$ and $F = f([0, L])$. **Fig. 9** The set H.

Since $\operatorname{dist}(f, g) < \varepsilon$, it follows that the range of f, denoted $F = f([0, L])$, satisfies

$$F \cap (D_+ \cup D_-) = \emptyset, \text{ where } D_\pm := \pm i\delta + (\delta - \varepsilon)B \quad \text{(see Fig. 8).} \tag{5.4}$$

Either $t_0 \in [0, L/2]$ or $t_0 \in (L/2, L]$.

Case 1: $t_0 \in [0, L/2]$

We claim that

$$|\mathrm{Im}\, f(t) - \mathrm{Im}\, f(t_0)| \le 3\varepsilon \text{ for all } t \in [t_0, t_0 + b_0]. \tag{5.5}$$

Since $f(t_0) \in [-i\varepsilon, i\varepsilon]$ and $f_{[t_0, t_0+b_0]}$ has length b_0, it follows from (5.4) that

$$f([t_0, t_0 + b_0)) \subset H := ([-i\varepsilon, i\varepsilon] + b_0 B)\backslash(D_+ \cup D_-) \quad \text{(see Fig. 9).}$$

One easily verifies that the point $a := \sqrt{2\varepsilon\delta - 3\varepsilon^2} + i2\varepsilon$ lies on the circle (bounding D_+) $i\delta + (\delta - \varepsilon)C$ and that the distance from $i\varepsilon$ to a is $d := \sqrt{2\varepsilon\delta - 2\varepsilon^2}$. In order to prove (5.5), it suffices (by symmetry) to show that $d \ge b_0$.

The bound $\varepsilon \le \frac{1}{81}\delta\theta_0^2$ ensures that $\varepsilon \le \frac{1}{2}\delta$, and consequently $d^2 \ge 2\varepsilon\delta - \varepsilon\delta = \varepsilon\delta \ge \varepsilon\frac{81\varepsilon}{\theta_0^2} = b_0^2$; hence (5.5) as claimed.

It now follows from Lemma 5.3 that $\|f_{[t_0, t_0+b_0]}\|^2 \ge \frac{1}{108}\frac{\theta_0^3}{\varepsilon} \ge M$, which contradicts assumption (ii).

Case 2: $t_0 \in (\frac{L}{2}, L]$

Defining $\widehat{f}(t) := f(2t_0 - t)$, $t \in [t_0, t_0 + b_0]$, the proof of Case 1 above shows that $\|f_{[t_0-b_0, t_0]}\|^2 = \|\widehat{f}\|^2 \ge M$, which again contradicts assumption (ii). $\square$

6. Proofs of the Main Results and Concluding Remarks

In this section, we prove Theorem 1.1 and Corollary 1.2.

Definition 6.1. Let $P_1, P_2, \cdots, P_n$ be a list of points in $\mathbb{C}$ with $P_j \neq P_{j+1}$. Any interpolating curve F can be written piecewise as $F = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{n-1}$, where the j-th piece f_j connects P_j to P_{j+1}. We define the **distance** from F to another interpolating curve $\widehat{F} = \widehat{f}_1 \sqcup \widehat{f}_2 \sqcup \cdots \sqcup \widehat{f}_{n-1}$ by

$$\mathrm{Dist}(F, \widehat{F}) := \max_{1 \leq j < n} \mathrm{dist}(f_j, \widehat{f}_j),$$

where $\mathrm{dist}(f_j, \widehat{f}_j)$ is the Hausdorff distance defined in (5.1).

Proof of Theorem 1.1. Assume $F = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{n-1}$ is a proper restricted elastic spline through points $P_1, P_2, \cdots, P_n$, with $P_j \neq P_{j+1}$, and set $M := \|F\|^2 + 1$. As usual, let the chord angles of f_j be denoted (α_j, β_{j+1}). Since F is proper, we have $\alpha_j, \beta_j \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $j = 2, 3, \ldots, n-1$. That α_1 and β_n also belong to $(-\frac{\pi}{2}, \frac{\pi}{2})$ can be seen as follows. As explained at the end of Section 2, we necessarily have $\alpha_1 = \beta^*(\beta_2)$ and $\beta_n = \beta^*(\alpha_{n-1})$, since otherwise the bending energy of F would not be minimal in $\mathcal{A}_{\pi/2}(P_1, P_2, \ldots, P_n)$. By Lemma 2.3 (i), $|\alpha_1|, |\beta_n| \leq \frac{\pi}{2} - \Psi$, and hence $\alpha_1, \beta_n \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, all the chord angles belong to $(-\frac{\pi}{2}, \frac{\pi}{2})$. Let $u_1, u_2, \ldots, u_n$ be the unit tangent vectors determined by F, whereby $f_j \in S(u_j, u_{j+1})$. Since F has minimal bending energy in $\mathcal{A}_{\pi/2}(P_1, P_2, \cdots, P_n)$, it follows that f_j has minimal bending energy in $S(u_j, u_{j+1})$ and therefore, by Theorem 2.1, f_j is equivalent to $c(u_j, u_{j+1})$ (a segment of rectangular elastica). By Proposition 4.4, $c(u_j, u_{j+1})$ is forward tracking and therefore, by Theorem 5.1, there exists $\varepsilon_j > 0$ such that if a curve f, connecting P_j to P_{j+1}, satisfies $\|f\|^2 < M$ and $\mathrm{dist}(f, f_j) < \varepsilon_j$, then f is forward tracking. Set $\varepsilon := \min_{1 \leq j < n} \varepsilon_j$.

In order to prove that F is a stable nonlinear spline, we will show that if $\widehat{F} = \widehat{f}_1 \sqcup \widehat{f}_2 \sqcup \cdots \sqcup \widehat{f}_{n-1}$ is an interpolating curve with $\mathrm{Dist}(\widehat{F}, F) < \varepsilon$, then $\|\widehat{F}\|^2 \geq \|F\|^2$. Let $\widehat{F}$ be an interpolating curve with $\mathrm{Dist}(\widehat{F}, F) < \varepsilon$ and suppose, to the contrary, that $\|\widehat{F}\|^2 < \|F\|^2$. Since $\widehat{f}_j$ connects P_j to P_{j+1} and satisfies $\|\widehat{f}_j\|^2 \leq \|\widehat{F}\|^2 < M$ and $\mathrm{dist}(\widehat{f}_j, f_j) \leq \mathrm{Dist}(\widehat{F}, F) < \varepsilon$, it follows that $\widehat{f}_j$ is forward tracking $(j = 1, 2, \ldots, n-1)$. Therefore $\widehat{F} \in \mathcal{F}(P_1, P_2, \cdots, P_n)$. By Corollary 4.3, $\|\widehat{F}\|^2 \geq \|F\|^2$, which is a contradiction. $\square$

Proof of Corollary 1.2. Assume that the points $P_1, P_2, \ldots, P_n$ yield stencil angles satisfying $|\psi_j| < \Psi$, for $j = 2, 3, \ldots, n-1$. By [4, Prop. 1.1], there exists a restricted elastic spline F through $P_1, P_2, \ldots, P_n$, and by [4, Cor. 1.3] F is proper. Therefore, by Theorem 1.1, F is a stable nonlinear spline. $\square$

Remark 6.2. Although written and proved specifically for the free configuration, Theorem 1.1 and Corollary 1.2 remain true for the clamped-free, free-clamped, and clamped configurations provided that prescribed chord angles α_1 and/or β_n belong to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Remark 6.3. For the closed configuration through $P_1, P_2, \ldots, P_n$ (and back to P_1), a closed restricted elastic spline $F = f_1 \sqcup f_2 \sqcup \cdots \sqcup f_n$ has the additional piece f_n, with chord angles (α_n, β_1), and we also have two additional stencil angles ψ_1 and ψ_n at P_1 and P_n. Theorem 1.1 and Corollary 1.2 remain true with the following modifications:
(A) F is deemed **proper** if $\alpha_j, \beta_j \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $j = 1, 2, \ldots, n$, and
(B) the hypothesis of Corollary 1.2 includes the assumption $|\psi_1|, |\psi_n| < \Psi$.

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Department of Mathematics, Faculty of Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait

E-mail address: borbely.albert@gmail.com, yohnson1963@hotmail.com