Short Note

# A divided difference expansion of a divided difference

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running head: DIVIDED DIFFERENCE EXPANSION OF DIVIDED DIFFERENCE

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**Abstract** A divided difference expansion with remainder for a general divided difference is derived that contains Floater's recent derivative expansion as a special case.

**key words** divided difference, univariate, remainder, interpolation

It is the purpose of this note to record a divided difference expansion of a divided difference, as suggested by the intriguing derivative expansion of a divided difference recently derived by M. Floater (see [F]), and containing the latter as a special case.

With $[t_1, \ldots, t_n]$ the divided difference (functional) at the point sequence $(t_1, \ldots, t_n)$, here is the formula.

**Proposition.** *Let $t := (t_1, \ldots, t_n)$ and $s := (s_1, \ldots, s_m)$ be real sequences, with $n \le m$, and set*

$$\psi_{i,j} := \prod_{k=i}^{j-1} (\cdot - s_k), \quad i, j = 1, \ldots, m+1.$$

*Then,*

$$[t_1, \ldots, t_n] = \sum_{j=n}^{m} ([t_1, \ldots, t_n]\psi_{j-n+2,m+1})[s_{j-n+1}, \ldots, s_m] + R_m(t, s), \tag{1a}$$

*with*

$$R_m(t, s) = \sum_{i=1}^{n} (t_i - s_i)\left([t_i, \ldots, t_n]\psi_{i+1,m+1}\right)[t_1, \ldots, t_i, s_i, \ldots, s_m]. \tag{1b}$$

**Proof:** The proof is by induction on $m$, the case $m = n$ being the readily derivable identity

$$[t_1, \ldots, t_n] - [s_1, \ldots, s_n] = \sum_{i=1}^{n} (t_i - s_i)[t_1, \ldots, t_i, s_i, \ldots, s_n],$$

which occurred to me after reading Floater's account in [F] of an argument in [DL] that proves this identity for a constant sequence $t$, but which I eventually found already in Eberhard Hopf's 1926 dissertation [H].

Assuming (1b) to be correct for a given $m$, let $s_0$ be an arbitrary point in $\mathbb{R}$. Since, by Leibniz' formula, $[t_i, \ldots, t_n]((\cdot - s_i)f) = (t_i - s_i)[t_i, \ldots, t_n]f + [t_{i+1}, \ldots, t_n]f$, hence

$$(t_i - s_i)[t_i, \ldots, t_n]f = [t_i, \ldots, t_n](\cdot - s_i)f - [t_{i+1}, \ldots, t_n]f, \tag{2}$$

(1b) implies that

$$\begin{aligned}
R_{m+1}(t, (s_0, s)) &= R_m(t, s) - \left([t_1, \ldots, t_n]\psi_{1,m+1}\right)[s_0, \ldots, s_m] \\
&= \sum_{i=1}^{n} ([t_i, \ldots, t_n]\psi_{i,m+1} - [t_{i+1}, \ldots, t_n]\psi_{i+1,m+1})[t_1, \ldots, t_i, s_i, \ldots, s_m] \\
&\qquad - \left([t_1, \ldots, t_n]\psi_{1,m+1}\right)[s_0, \ldots, s_m] \\
&= \sum_{i=1}^{n} \left([t_i, \ldots, t_n]\psi_{i,m+1}\right)\left([t_1, \ldots, t_i, s_i, \ldots, s_m] - [t_1, \ldots, t_{i-1}, s_{i-1}, \ldots, s_m]\right) \\
&= \sum_{i=1}^{n} \left([t_i, \ldots, t_n]\psi_{i,m+1}\right)(t_i - s_{i-1})[t_1, \ldots, t_i, s_{i-1}, \ldots, s_m]
\end{aligned}$$

and moving the factors $(t_i - s_{i-1})$ to the left and renaming $(s_0, \ldots, s_m)$ to $(s_1, \ldots, s_{m+1})$ finishes the proof. □

Floater's formula is the special case when $s_i = x$ for all $i$, hence $\psi_{i+1,m+1} = (\cdot - x)^{m-i}$, while, as Floater kindly pointed out to me, the Dokken/Lyche formula (see [DL]) for the derivatives of the error in Hermite interpolation is the special case when $t$ is constant. More than that, Floater also pointed out that, with $p := m - n$, (1a-b) can also be written

$$[t_1, \dots, t_n] = [t_1, \dots, t_n] \sum_{j=n}^{m} \psi_{1,j}[s_1, \dots, s_j] + \sum_{i=1}^{n} (t_i - s_{i+p})([t_1, \dots, t_i]\psi_{1,i+p})[s_1, \dots, s_{i+p}, t_i, \dots, t_n]. \tag{3}$$

Indeed, reversing the order of the entries of both $t$ and $s$ converts (1a-b) into (3).

Floater [F] also proves, for the case of constant $s$ and using properties of the elementary symmetric functions, that, for odd $m - n$,

$$R_m(t, s)f = ([t_1, \dots, t_n]\psi_{1,m+1})\ D^m f(\xi)/m! \tag{4}$$

for some $\xi$ in the interval containing both $t$ and $s$ (and assuming that $f$ is sufficiently smooth). Such a result can also be proved in our more general context, using elementary properties of the divided difference, as follows.

By (2) and induction,

$$\begin{aligned} [t_1, \dots, t_n]\psi_{1,m+1} &= (t_1 - s_1)[t_1, \dots, t_n]\psi_{2,m+1} + [t_2, \dots, t_n]\psi_{2,m+1} \\ &= \cdots \\ &= \sum_{i=1}^{n} (t_i - s_i)[t_i, \dots, t_n]\psi_{i+1,m+1}. \end{aligned} \tag{5}$$

Since this is the sum of the coefficients in (1b), (4) follows provided one can show that these coefficients are all of the same sign. This is indeed possible for the case $m - n$ odd, under some assumption on $s$. The simplest such assumption is that the smallest interval containing $s$ contains no $t_j$ in its interior (certainly satisfied when $s$ is constant).

Since (as already used) $[t_1, \dots, t_n]f = D^{n-1}f(\xi)/(n-1)!$ for some $\xi$ in the smallest interval containing all the $t_j$, it is clear that $[t_1, \dots, t_n]\psi_{2,m+1}$ is positive in case all the $t_j$ are to the right of all the $s_i$. Also, when $m - n = \deg D^{n-1}\psi_{2,m+1}$ is odd, then $[t_1, \dots, t_n]\psi_{2,m+1}$ is negative in case all the $t_j$ are to the left of all the $s_i$. Hence, in both cases, $(t_1 - s_1)[t_1, \dots, t_n]\psi_{2,m+1}$ is nonnegative. Otherwise, there are $t_j$ both to the left and to the right of $s_1$, hence, after exchanging $t_1$ with some more suitable $t_j$ if necessary, $(t_1 - s_1)[t_1, \dots, t_n]\psi_{2,m+1}$ is nonnegative in this case, too. Thus, by (5) and induction, there is a reordering of $t$ so that all the coefficients in (1b) are nonnegative, and (4) follows.

The simple example $s = (0, 0, 3)$, $t = (2, 2)$, for which $[t_1, \dots, t_n]\psi_{1,m+1} = 0$, shows that (4) does not hold in general.

## References


[DL] T. Dokken and T. Lyche, A divided difference formula for the error in Hermite interpolation, *BIT* **19** (1979), 540–541.

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