Gramian analysis of affine bases and affine frames

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Abstract. Shift-invariance "fiberization techniques" are applied for the study of the synthesis and analysis operators of affine (wavelet) systems. In this approach, one has first to circumvent the fact that affine systems are *not* shift-invariant. The results obtained include characterizations of the Bessel property, the Riesz basis property and the frame property of such sets in terms of the behaviour of simpler operators. Various estimates of the lower and upper frame (Riesz) bounds are included, too. Complete details will appear in [7].

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§1 The synthesis and analysis operators

The setup is as follows: given a countable subset $X \subset L_2 := L_2(\mathbb{R}^d)$, we would like to consider X as a "basis" for either the entire L_2-space, or, in case X is not fundamental in L_2, for the closed subspace $H \subset L_2$ "spanned" by X. The goal is to use X for either decomposing functions in H or for reconstructing such functions. The reconstruction is done with the aid of the *synthesis operator*

$$T_X : \ell_2(X) \to L_2(\mathbb{R}^d) : c \mapsto \sum_{x \in X} c(x)x. \tag{1}$$

The underlying assumption here is that T_X is bounded, *i.e.*, that it is bounded on the finitely supported elements of $\ell_2(X)$ and is extended to the entire domain by continuity. In that event, X is said to be a *Bessel system*. Otherwise, T_X is only densely defined (say, on the finitely supported sequences). The adjoint of T_X^* of T_X is the *analysis operator*

$$T_X^* : L_2 \to \ell_2(X) : f \mapsto (\langle f, x \rangle)_{x \in X},$$

that can be used in the decomposition process.

The possible properties of X which are of interest to us here are listed in the following definition.

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Definition 2. Let X be a Bessel system. We say that X is
(a) a *basis* if T_X is 1-1.
(b) a *frame* if $\operatorname{ran} T_X$ is closed.
(c) a *Riesz basis* if it is a basis and a frame.
(d) *fundamental* if the finite span of X is dense in L_2.

Whenever X is a frame, the restriction of $T := T_X$ to the orthogonal complement (in $\ell_2(X)$) of $\ker T$ is bounded below, hence invertible. This partial-inverse of T is denoted here by T^{-1}. The *frame bounds* are then defined as the numbers $\|T\|^2$ (upper frame bound) and $\|T^{-1}\|^{-2}$ (lower frame bound), and are referred to as the *Riesz bounds*, if X is a basis. Quite clearly, the frame bounds can be equivalently defined with the aid of the analysis operator (simply replace T by T^*). However one should note that it is usually easier to handle inverses than partial-inverses, and it is thus desired to study the operator that is known to be 1-1; consequently, the study of a basis X is best done with the aid of T, and the study of a fundamental X is best done with T^*. Pseudo-inverses cannot be avoided if X is neither a basis nor fundamental, *e.g.*, a non-fundamental frame.

§2 Shift-invariant systems

In [4-6,8], we analyse the analysis and synthesis operators under the assumption that X is *shift-invariant*, *i.e.*, of the form

$$X = E(\Phi) := \{E^\alpha \phi : \ \phi \in \Phi, \ \alpha \in \mathbf{Z}^d\}, \tag{3}$$

with Φ a small (but not necessarily finite) set of "generators", and with E^α the *shift operator*

$$E^\alpha : f \mapsto f(\cdot - \alpha).$$

That shift-invariance assumption allowed us to decompose the Fourier transform analogues $\widehat{T}$ and $\widehat{T}^*$ of these operators into a collection of constant-coefficient matrices, which we call *pre-Gramians*, and to relate basic properties of the fibers to corresponding properties of $\widehat{T}$ and $\widehat{T}^*$, thereby to properties of the original operators T and T^*. A typical fiber $J(\omega)$, $\omega \in \mathbb{R}^d$, is a matrix whose columns are indexed by Φ, its rows indexed by $2\pi\mathbf{Z}^d$, and its (ϕ, α)-entry being

$$J(\omega)(\phi, \alpha) = \widehat{\phi}(\omega + \alpha).$$

Each fiber $J(\omega)$ is considered as a (possibly only densely defined) operator from $\ell_2(\Phi)$ to $\ell_2(2\pi\mathbf{Z}^d)$. In the same manner, one may view the matrix adjoint $J^*(\omega)$ of $J(\omega)$ as an operator from $\ell_2(2\pi\mathbf{Z}^d)$ to $\ell_2(\Phi)$. In agreement with previous definitions, we say that $J(\omega)$ is *partially invertible* whenever it is bounded and has a closed range. We further denote then the partial inverse by $J(\omega)^{-1}$. Similar definitions and notations apply to $J^*(\omega)$. The following theorem summarizes some of the key observations in [4,8]:

Theorem 4. *Let*

$$\begin{array}{llllllll} \mathcal{J} & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & \omega & \mapsto & \|J(\omega)\| & , \\ \mathcal{J}^* & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & \omega & \mapsto & \|J^*(\omega)\| & , \\ \mathcal{J}^- & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & \omega & \mapsto & \|J(\omega)^{-1}\| & , \\ \mathcal{J}^{*-} & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & \omega & \mapsto & \|J^*(\omega)^{-1}\| & . \end{array}$$

Then, the following is true.

(a) X is a Bessel system iff $\mathcal{J} \in L_\infty$ iff $\mathcal{J}^* \in L_\infty$. Furthermore, $\|T\| = \|T^*\| = \|\mathcal{J}\|_{L_\infty} = \|\mathcal{J}^*\|_{L_\infty}$.

(b) Assume X is a Bessel system. Then X is a frame iff $\mathcal{J}^- \in L_\infty$ iff $\mathcal{J}^{*-} \in L_\infty$. Furthermore, $\|T^{-1}\| = \|T^{*-1}\| = \|\mathcal{J}^-\|_{L_\infty} = \|\mathcal{J}^{*-}\|_{L_\infty}$.

(c) Assume X is a Bessel system. Then X is a basis if and only if $J(\omega)$ is 1-1 for a.e. ω, and X is fundamental if and only if $J^*(\omega)$ is 1-1 for a.e. ω.

§3 Affine systems

The notion of an *affine system* corresponds to the case when the "basis" set X is obtained by shifting and dilating each of the functions in a finite $\Psi \subset L_2$. Here *dilation* is meant as any integer power of the unitary operator

$$D : L_2 \to L_2 : f \mapsto |\det A|^{-1/2} f(A^{*-1}\cdot), \tag{5}$$

with A a $d \times d$ invertible matrix whose 2-norm is < 1, and whose inverse is an integer matrix. Precisely, the set X is defined as

$$X = \{D^k E^\alpha \psi : \ \psi \in \Psi,\ \alpha \in \mathbf{Z}^d,\ k \in \mathbf{Z}\}. \tag{6}$$

One notes that this set X is *not* shift-invariant: for a negative integer k, the shifts of $D^k\psi$ that belong to X are taken from the proper sublattice $A^{*k}\mathbf{Z}^d$ of $\mathbf{Z}^d$. This means that our analysis of shift-invariant systems from [4,8] cannot be applied directly to the present situation.

We circumvent this difficulty with the aid of the *truncated affine systems* X_k, $k \in \mathbf{Z}$, defined as follows:

$$X_k = \{D^{k'} E^\alpha \psi : \ \psi \in \Psi,\ \alpha \in \mathbf{Z}^d,\ k' \ge -k\}. \tag{7}$$

In what follows, we use the notation T, T^* for the operators associated with the full affine system X (viz., T_X and T_X^*) and T_k, T_k^* for the operators associated with the truncated system X_k.

We start the analysis by observing that T_k and $T_{k'}$ are *unitarily equivalent* (regardless of the values of k, k'), and thus T_0 is unitarily equivalent to each T_k. Since certain properties of the operator T (such as boundedness and invertibility) are fully captured by analysing the limiting properties of $(T_k)_k$ as $k \to \infty$, we easily obtain the following:

Proposition 8. *Let X be an affine system, and let X_0 be its truncated version. Then:*
(a) X is a Bessel system iff X_0 is a Bessel system. Moreover, $\|T\| = \|T_0\|$.
(b) X is Riesz basis iff X_0 is a Riesz basis. Moreover, whenever X and X_0 are Riesz bases, we have $\|T^{-1}\| = \|T_0^{-1}\|$.
(c) X is a frame if (but not only if!) X_0 is a frame. Moreover, $\|T^{-1}\| \le \|T_0^{-1}\|$.

Since the truncated system X_0 *is* shift-invariant, the proposition allows us to study some of the properties of X by applying our shift-invariance methods to the system X_0. However, as the last statement of the proposition indicates, not *all* properties of X can be completely recovered by this approach. For example, X may fail to be a basis while X_0 can still be so. This further difficulty can also be circumvented; but, before we discuss these further details, we list two consequences of Proposition 8.

Corollary 9. *Let X_0 be a truncated affine system. Then:*
(a) X_0 can never be a fundamental Riesz basis.
(b) X_0 is a tight frame (fundamental or not) if and only if the system $X_1 \backslash X_0$ is a tight frame which is orthogonal to X_0.

We recall that a *tight frame* is a frame whose frame bounds coincide. In (b), the "only if" assertion is the interesting one. The proof of (a) is actually trivial: if X_0 is a fundamental Riesz basis, its superset X cannot be a basis, in particular, cannot be a Riesz basis. By Proposition 8, X_0 is not a Riesz basis, either.

It seems instructive to present some heuristics before we proceed with the general discussion. A typical Fourier transform $\widehat{\psi}$ of ψ in the generating set Ψ of the affine system X has a high order zero at both the origin and ∞. The dilations of Ψ that can be found in X_0 have their Fourier transform attracted towards ∞. This entails that such system may be useful for decomposing functions whose Fourier transform vanishes on a sufficiently large neighborhood of the origin.

Theorem 10. *Let H_Ω be the space of all L_2-functions whose Fourier transform vanishes on Ω. Given any operator $S : L_2$, let S_Ω be the restriction of that operator to H_Ω. Consider, for a Bessel system X, the following two conditions:*
(a) $T^*_{0,\Omega}$ is bounded below, for some large neighborhood Ω of the origin.
(b) X is a fundamental frame.
Then (a)$\Longrightarrow$ (b), and $\|T^{*-1}\| \le \|T^{*}_{0,\Omega}{}^{-1}\|$. If, further, for some $\rho > d/2$, and every $\psi \in \Psi$,

$$|\widehat{\psi}| = O(|\cdot|^{-\rho}), \text{ as } \rho \to \infty, \tag{11}$$

then (b)$\Longrightarrow$(a) as well.

It is worth mentioning that the implication (b)$\Longrightarrow$ (a) in the above theorem holds under conditions that are weaker than the ones assumed there. Slightly weaker decay assumptions on each $\widehat{\psi}$ always suffice, and, significantly weaker assumptions may do as well, if we know that the functions in Ψ decay in some "nice" way. For further details see [7].

§4 Gramian analysis

The results of the last section allow us to apply our shift-invariant methods for the analysis of Bessel systems, Riesz bases, and fundamental frames (but not with respect to non-fundamental frames). The relevant functions are the functions $\mathcal{J}$, $\mathcal{J}^*$, $\mathcal{J}^-$, and $\mathcal{J}^{*-}$, *calculated with respect to the shift-invariant set* X_0. Further, the norms of $\mathcal{J}^*$ and $\mathcal{J}^{*-}$ should be calculated on an arbitrarily small neighborhood of ∞. These calculations can be done with the aid of the two non-negative matrices $G := J^*J$, $\widetilde{G} := JJ^*$, with J the pre-Gramian of X_0. We refer to G as the *Gramian* matrix, to $\widetilde{G}$ as the *dual Gramian* matrix, and to entire analysis as *Gramian analysis*. Though we do not provide computational details, we do mention that the set Φ whose shifts $E(\Phi)$ generate X_0 can be chosen as $\Phi := \{E^\gamma D^k\psi :\ \psi \in \Psi,\ k \geq 0,\ \gamma \in\ ,_k\}$, with $\ ,_k$ the quotient group $(A^{*k}\mathbf{Z}^d)/\mathbf{Z}^d$.

Three different types of results can be derived with aid of the presented approach. First, and foremost, by computing accurately the entries of G and $\widetilde{G}$ we can get complete characterizations of the Bessel, Riesz and fundamental frame properties of the affine X. Second, standard techniques can be applied for estimating from above the ℓ_2-norm of the fibers $G(\omega)$ and $\widetilde{G}(\omega)$ and this leads to sufficient conditions for X to be Bessel, as well as to upper bounds on $\|T\|$. Finally, by using straightforward diagonal dominance estimates with respect to $G(\omega)$ (respectively, $\widetilde{G}(\omega)$) we obtain sufficient conditions for X to be a Riesz basis (fundamental frame, respectively), together with upper bound estimates on $\|T^{-1}\|$. These estimates are intimately related to Daubechies' estimates in [2,3].

From the discussion concerning the pre-Gramian J and from our description of the present shift-invariant generators Φ, one can deduce that the dual Gramian matrix $\widetilde{G}(\omega)$ is indexed by $2\pi\mathbf{Z}^d \times 2\pi\mathbf{Z}^d$, and having entries of the form

$$\widetilde{G}(\omega)(\alpha,\beta) = \sum_{\psi\in\Psi}\sum_{k=0}^{k(\alpha,\beta)} \widehat{\psi}(A^k(\omega+\alpha))\overline{\widehat{\psi}}(A^k(\omega+\beta)),$$

with $k(\alpha,\beta)$ the maximal integer that satisfies $\alpha-\beta \in A^{-k}\mathbf{Z}^d$. The results of [4] allow us to restrict attention to $\omega \in C := [-\pi \mathinner{\ldotp\ldotp} \pi]^d$, and Theorem 10 allows us to remove from $\widetilde{G}$ any finite set of rows, together with their corresponding columns. We thus remove from $\widetilde{G}$ all rows and columns whose index α lies in some ball of radius r centered at the origin.

Theorem 12. *Let X be an affine system. For $r \geq 0$, let $\widetilde{G}(r,\omega)$ be the constant coefficient matrix indexed by $Z \times Z$ with $Z := \{\alpha \in 2\pi\mathbf{Z}^d : |\alpha| \geq r\}$, and whose (α,β)-entry is*

$$\sum_{\psi\in\Psi}\sum_{k=0}^{k(\alpha,\beta)} \widehat{\psi}(A^k(\omega+\alpha))\overline{\widehat{\psi}}(A^k(\omega+\beta)).$$

Consider $\widetilde{G}(r,\omega)$ as a map from $\ell_2(Z)$ to itself, with norm $\mathcal{G}^*(\omega)$ and norm of inverse $\mathcal{G}^{*-}(\omega)$.

(a) X is a Bessel system iff $\widetilde{G}(0,\omega)$ is bounded for a.e. ω, and the then-well-defined $\mathcal{G}^*$ is in L_∞. Furthermore, $\|T\|^2 = \|\mathcal{G}^*\|_{L_\infty}$.

(b) If, for some $r \geq 0$, the matrix $\widetilde{G}(r,\omega)$ is boundedly invertible for a.e. $\omega \in C := [-\pi \mathinner{..} \pi]^d$, and if $\mathcal{G}^{*-}$ is essentially bounded on C, then X is a fundamental frame and $\|T^{-1}\|^2 \leq \|\mathcal{G}^{*-}\|_{L_\infty(C)}$.

(c) If each $\psi \in \Psi$ satisfies condition (11), then the essential boundedness of $\mathcal{G}^{*-}$ for some r is necessary for X to be a fundamental frame.

The role of the essential infimum and essential supremum of the expression $\sum_{\psi\in\Psi,\,k\in\mathbf{Z}}|\widehat{\psi}(A^k\cdot)|^2$ as crude estimators for the frame bounds (cf. [1,2,3]) is easily revealed now from the fact that the diagonal entry $\widetilde{G}(r,\omega)(\alpha,\alpha)$ is the sum $\sum_{\psi\in\Psi}\sum_{k\geq 0}|\widehat{\psi}(A^k(\cdot+\alpha))|^2$, and the fact that $|\alpha|$ can be chosen arbitrarily large. The above theorem also leads to "oversampling" results that refine corresponding ones presented in [2,3].

Finer estimates for the frame bounds are derived from Theorem 12 as follows. For estimating the upper frame bound from *below* one can use the ℓ_2-norm of any row of any $\widetilde{G}(0,\omega)$. The upper frame bound can be estimated from *above* by finding the supremum of the ℓ_1-norms of the rows of all fiber matrices $\widetilde{G}(0,\omega)$, $\omega \in C$. The lower frame bound can be estimated from below in case the fibers $\widetilde{G}(r,\omega)$ are *diagonally dominant* for sufficiently large r. We collect some of these resulted estimates below.

Corollary 13. *Let X be an affine system.*

(a) If the function $F := \sum_{\psi\in\Psi,\,\alpha\in 2\pi\mathbf{Z}^d,\,k\geq 0}|\widehat{\psi}(A^k\cdot)\widehat{\psi}(A^k\cdot+\alpha)|$ is essentially bounded, then X is a Bessel system whose upper frame bound is no larger than $\|F\|_{L_\infty}$.

(b) If X is Bessel, and if, with $F_1 := 2\sum_{\psi\in\Psi,\,k\geq 0}|\widehat{\psi}(A^k\cdot)|^2 - F$, $\inf F_1$ is essentially positive on some neighborhood V of ∞, then X is a fundamental frame with lower frame bound no smaller than $\operatorname{ess\,inf} F_1$.

(c) If each $\widehat{\psi}$ is non-negative a.e., and X is a Bessel system, then the function

$$F_2 := \Big(\sum_{\psi\in\Psi,\,k\in\mathbf{Z}}|\widehat{\psi}(A^k\cdot)|^2\Big)^2 + \sum_{\psi\in\Psi,\,\alpha\in 2\pi\mathbf{Z}^d\backslash 0,\,k\in\mathbf{Z}}|\widehat{\psi}(A^k\cdot)\widehat{\psi}(A^k\cdot+\alpha)|^2$$

is essentially bounded. Furthermore, $\|T\|^2 \geq \|F_2\|_{L_\infty}^{1/2}$; thus, if X is a frame, $\|F_2\|_{L_\infty}$ bounds the upper frame bound from below.

The results concerning the Gramian matrix G are analogous, with the following notable differences. First, the parameter ω need not be restricted to any neighborhood of ∞, and no a-priori assumption on Ψ should be made. Second, the approach applies to the study of the Riesz basis property, not to frames. Third (and most obviously), the entries of the Gramian are quite different from these of the dual Gramian. In fact, we applied some unitary transformations in order to arrive at the entries listed below. As before, we state two results: one is a characterization of the Riesz basis property and the other contains estimates on the Riesz bounds.

Theorem 14. *Let X be an affine system generated by Ψ. For each $k \geq 0$, let $\tilde{\Gamma}_k$ be the quotient group $2\pi(\mathbf{Z}^d/(A^{-k}\mathbf{Z}^d))$. For each $\omega \in \mathbb{R}^d$, let $G(\omega)$ be a Hermitian matrix whose rows and columns are indexed by $(\psi, k, \gamma) \in \Psi \times \mathbf{Z}_+ \times \tilde{\Gamma}_k$, and whose $((\psi, k, \gamma), (\psi', k', \gamma'))$-entry is, say, for $k' \leq k$,*

$$\delta_{\gamma,\gamma'} \sum_{\alpha \in 2\pi\mathbf{Z}^d} \widehat{\psi}(\omega_k + \alpha)\overline{\widehat{\psi'}}(A^{k'-k}(\omega_k + \alpha)),$$

with $\omega_k := A^k(\omega + \gamma)$, and with $\delta_{\gamma,\gamma'} = 0$ whenever the cosets represented by γ, γ' have an empty intersection. Then:

(a) X is a Bessel system iff the map $G(\omega)$ is bounded a.e., and the function $\mathcal{G} : \omega \to \|G(\omega)\|$ is essentially bounded. Moreover, $\|T\|^2 = \|\mathcal{G}\|_{L_\infty}$.

(b) Assume that X is Bessel. Then X is a Riesz basis if and only if $G(\omega)$ is boundedly invertible for a.e. ω, and the map $\mathcal{G}^- : \omega \mapsto \|G(\omega)^{-1}\|$ is essentially bounded. Moreover, $\|T^{-1}\|^2 = \|\mathcal{G}^-\|_{L_\infty}$.

Corollary 15. *Let X be an affine system generated by Ψ.*

(a) Given $\psi \in \Psi$, define

$$R_\psi := \sum_{\psi' \in \Psi,\, \alpha \in 2\pi\mathbf{Z}^d,\, k \in \mathbf{Z}} |\widehat{\psi}(\cdot + \alpha)\widehat{\psi}'(A^k(\cdot + \alpha))|.$$

Then X is a Bessel system if $R_\psi \in L_\infty$, for every $\psi \in \Psi$. Moreover, the upper Riesz bound cannot exceed $\max_{\psi \in \Psi} \|R_\psi\|_{L_\infty}$.

(b) Assume that X is a Bessel system, and define

$$R_{\psi,1} = 2 \sum_{\alpha \in 2\pi\mathbf{Z}^d} |\widehat{\psi}(\cdot + \alpha)|^2 - R_\psi.$$

If the essential infimum of each $R_{\psi,1}$ is positive, then X is a Riesz basis, and the lower Riesz bound is no smaller than the minimum among the aforementioned infima.

It follows, in particular, that X is a Bessel system if the functions $R_E := \max_{\psi\in\Psi}\sum_{\alpha\in 2\pi\mathbf{Z}^d}|\widehat{\psi}(\cdot+\alpha)|$, and $R_D := \sum_{\psi\in\Psi, k\in\mathbf{Z}}|\widehat{\psi}(A^k\cdot)|$ are in L_∞; more precisely, Theorem 15 implies the crude (yet simple) estimate $\|T\|^2 \le \|R_E\|_{L_\infty}\|R_D\|_{L_\infty}$. That latter estimate improves Theorem 2 of [1].

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