L_p-error bounds for Hermite interpolation and the associated Wirtinger inequalities

Shayne Waldron*

Department of Mathematics
University of Wisconsin-Madison, Madison, WI 53706, U.S.A.
e-mail: waldron@math.wisc.edu

Abstract:

The B-spline representation for divided differences is used, for the first time, to provide L_p-bounds for the error in Hermite interpolation, and its derivatives, thereby simplifying and improving the results to be found in the extensive literature on the problem. These bounds are equivalent to certain Wirtinger inequalities (cf. [FMP91:p66]).

The major result is the inequality

$$|f(x) - H_\Theta f(x)| \leq \frac{n^{1/q}}{n!} \frac{|\omega_\Theta(x)|}{(\operatorname{diam}\{x, \Theta\})^{1/q}} \|D^n f\|_q,$$

where $H_\Theta f$ is the Hermite interpolant to f at the multiset of n points Θ,

$$\omega_\Theta(x) := \prod_{\theta \in \Theta} (x - \theta),$$

and $\operatorname{diam}\{x, \Theta\}$ is the diameter of $\{x, \Theta\}$. This inequality significantly improves upon 'Beesack's inequality' (cf. [Be62]), on which almost all the bounds given over the last 30 years have been based.

AMS (MOS) Subject Classifications: primary 26D10, 41A05, 41A80; secondary 41A10, 41A44

Key Words: Hermite interpolation, B-spline, Green's function, Beesack's inequality, Wirtinger inequality

* Supported by the Chebyshev professorship of Carl de Boor.

L_p-error bounds for Hermite interpolation and the associated Wirtinger inequalities

Shayne Waldron (waldron@math.wisc.edu)

1. Introduction

While working on the error in multivariate polynomial interpolation, I became aware of the suprisingly extensive literature on the error in Hermite interpolation by *univariate* polynomials. Two problems seem to have been the particular focus of these many papers. One is the bounding of the L_p-norm of the error in terms of the L_q-norm of the appropriate derivative. The other is the obviously related so-called 'Wirtinger problem' of bounding the L_p-norm of a function known to vanish to given orders at certain points in a given interval.

In hindsight, the existence of these many papers may have come about because the various workers in this area failed to *combine* the following well-known facts.

(i) The error formula for Hermite interpolation in terms of a divided difference which vanishes on the interpolating polynomial space.
(ii) The representation of a divided difference of order n as integration of the n-th derivative against an appropriate B-spline.
(iii) The L_p-bounds for B-splines.
(iv) Rolle's theorem, with multiplicities.

Apparently, see, e.g., the recent monograph of Agarwal and Wong [AW93], B-splines are not known to the many authors in this area.

Thus, it is the purpose of this paper to show how, in proper combination, these well-known facts imply error bounds for Hermite interpolation which, except for one or two very specific cases, imply the many different error bounds now in the literature.

In Section 2, we establish notation, state the problems of interest, and discuss some properties of the constants which we are interested in estimating.

In Section 3, we establish the main result, that:

$$|f(x) - H_\Theta f(x)| \le \frac{n^{1/q}}{n!} \frac{|\omega_\Theta(x)|}{\operatorname{diam}\{x,\Theta\}^{1/q}} \|D^n f\|_{L_q(\operatorname{conv}\{x,\Theta\})}, \tag{1.1}$$

where $H_\Theta f$ is the Hermite interpolant to f at the multiset Θ of n points, $\omega_\Theta(x) := \prod_{\theta\in\Theta}(x-\theta)$, and **diam** (resp. **conv**) is the diameter (resp. convex hull) of a (multi)set of points. This is seen to be the appropriate replacement for an inequality, due to Beesack [Be62], that has been used extensively over the last 30 years.

In Section 4, we use the inequality (1.1) to obtain L_p-error bounds for Hermite interpolation. There is a discussion of the relevant literature, including 'Wirtinger inequalities'.

In Section 5, Rolle's theorem (with multiplicities) is used to obtain bounds for the derivatives of the error.

In Section 6, we indicate how to compute G_Θ, the Green's function which occurs in the integral error formula for Hermite interrpolation, using MATLAB$^{\text{TM}}$. There is a short discussion of extensions of the results of this paper to *Birkhoff interpolation*. We end with a simple all-purpose estimate for the error in Hermite interpolation.

2. Statement of the problem

To simplify the presentation of our results, we find it convenient to use a certain amount of default notation. We reserve n for a positive integer, and $1 \le p, q \le \infty$. Our functions will be defined on the closed interval $[a, b]$, $b > a$. Thus $\|\cdot\|_p := \|\cdot\|_{L_p[a,b]}$, and $W_p^{(n)} := W_p^{(n)}[a, b]$, the **Sobolev** space of functions f with $D^{n-1}f$ absolutely continuous on $[a, b]$ and $D^n f \in L_p := L_p[a, b]$. We use Θ for a finite multiset of points in $[a, b]$ with cardinality $\#\Theta = n$, and $\omega_\Theta := \prod_{\theta\in\Theta}(\cdot - \theta)$.

The **Hermite interpolant** to $f \in W_1^{(n)}$ at the multiset Θ is the unique polynomial $H_\Theta f \in \Pi_{<n}$, of degree $< n$, which satisfies

$$D^j H_\Theta f(\theta) = D^j f(\theta), \quad \forall\, \theta \in \Theta, \quad j = 0, \ldots, \#\theta - 1,$$

where $\#\theta$ is the *multiplicity* of θ in Θ. Important special cases are **Lagrange interpolation** (where the points in Θ are distinct), and **Taylor interpolation** (where the points in Θ all coincide).

Recall the following fact, see, e.g., [AW93:p74,(2.3.9)]. There exists a piecewise polynomial function $G_\Theta : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x) - H_\Theta f(x) = \int G_\Theta(x, \cdot) D^n f, \quad \forall\, x \in \mathbb{R}, \quad \forall\, f \in W_1^{(n)}(\operatorname{conv}\{x, \Theta\}).$$

More precisely, if m is the number of distinct points in Θ, then G_Θ, restricted to each of the $2(m+1)$ components of the partition of $\mathbb{R}^2$ obtained by removing the m lines $\mathbb{R} \times \Theta$ and the diagonal $\mathbb{R}(1, 1)$, is a polynomial of degree $2n - 2$.

By applying $1 - H_\Theta$ to both sides of the Taylor identity

$$f = \sum_{j<n} \frac{D^j f(a)}{j!}(\cdot - a)^j + \int_a^b \frac{D^n f(t)}{(n-1)!}(\cdot - t)_+^{n-1}\, dt$$

we obtain

$$G_\Theta(\cdot, t) = (1 - H_\Theta)\frac{(\cdot - t)_+^{n-1}}{(n-1)!}$$

on $[a, b]$, where $(\cdot)_+^{n-1}$ is the *truncated power* function. In particular $G_\Theta(\cdot, t)$ is $C^{(n-1)}$ on $[a, b]$ except at t where $D^{(n-1,0)}G_\Theta(\cdot, t)$ jumps by 1. Thus, for $0 \le j \le n - 1$, there exists a smallest constant C such that

$$\|D^j(f - H_\Theta f)\|_p \le C(b - a)^{n-j+\frac{1}{p}-\frac{1}{q}}\|D^n f\|_q, \quad \forall f \in W_q^{(n)}, \tag{2.1}$$

with the determination of C being equivalent to finding the norm of the compact linear map $A : L_q \to L_p$, given by

$$(2.2) \qquad Af(x) := \int_a^b D^{(j,0)} G_\Theta(x,\cdot) f.$$

The exponent of $(b-a)$ in (2.1) is chosen so that C depends only on n, p, q, j, and something I choose to call the **position of Θ in** $[a,b]$, for short,

$$\text{pos}\Theta := \text{pos}(\Theta, [a,b]).$$

This is by definition, the collection of all pairs $(\Theta', [a', b'])$ for which there exists an invertible map A, with $A\Theta = \Theta'$ and $A[a,b] = [a', b']$. At the risk of having excessive notation, we prefer to show all dependencies, and write $C = C^{(j)}_{n,p,q}(\text{pos}\Theta)$, with the j omitted if it is 0.

In many cases, we will allow posΘ to be described, e.g.,

$$\begin{aligned} &\text{equally spaced} := \text{pos}(\{0,1,\ldots,n-1\}, [0, n-1]),\\ &\text{all at one endpoint} := \text{pos}(\{0,\ldots,0\}, [0,1]),\\ &\text{Chebyshev} := \text{pos}(\text{zeros of the Chebyshev polynomial } T_n, [-1,1]). \end{aligned}$$

The computation of $\|A\|$ can be recast into many equivalent forms. In Th. 10.1 of [Br72], for some cases where Θ consists solely of endpoints, it is shown to be equivalent to finding the largest eigenvalue of a related differential system. For more examples, including the norm of maps related to the adjoint of A, see Waldron [Wa94$_1$].

In some very special cases, where Θ lies in $\{a,b\}$, and $p, q = \infty$, $C^{(j)}_{n,p,q}(\text{pos}\Theta)$ has been determined by computing $\|A\|$ (which is taken on for a constant function). Results along these lines can be found in Tumara [Tu41], Birkhoff and Priver [BP67], and [AW93]. There is strong circumstantial evidence that each of these is a special case of the following.

(2.3) Conjecture. *Let Θ^* consist of one endpoint m times and the other $n-m$ times, where $2m \le n$. If Θ contains each of the endpoints at least m times, then*

$$C^{(j)}_{n,\infty,\infty}(\text{pos}\Theta) \le C^{(j)}_{n,\infty,\infty}(\text{pos}\Theta^*), \quad 0 \le j \le n-1,$$

with equality iff Θ consists of one endpoint m times and the other $n-m$ times.

There has been no attempt here to prove (or disprove) this conjecture. A major step towards this would be a close examination of the proof of Tumara's often quoted result of [Tu41], of which the author has yet to procure a copy.

For $q \ne \infty$, or $j > 0$, the computation of $\|A\|$ (and hence the determination of $C^{(j)}_{n,p,q}(\text{pos}\Theta)$) is in general very difficult. Some aspects of this difficulty are discussed in [BP67]. To get a feel for what is involved, consider the simplest case, namely of interpolation at one point. Suppose $b-a=1$, and $\Theta = \{\theta\}$. Then $C_{1,p,q}(\text{pos}\Theta) = \|A\|$, where

$$A : L_q \to L_p, \quad Af(x) := \int_\theta^x f.$$

In this case the author is unable to compute $\|A\|$ when $q \neq \infty$. The reader is urged to try this computation before seeking other exact values of $C_{n,p,q}(\mathrm{pos}\Theta)$, $q \neq \infty$.

On a more positive note, if one is prepared to give a little on the exact determination of $C_{n,p,q}(\mathrm{pos}\Theta)$, then reasonable estimates for it are possible, as we will see in Section 3.

We end this section with some useful properties of the constants $C^{(j)}_{n,p,q}(\mathrm{pos}\Theta)$.

(2.4) Properties. *Let* $0 \le j \le n-1$.

(a) *The function* $p \mapsto C^{(j)}_{n,p,q}(\mathrm{pos}\Theta)$ *is continuous and strictly increasing.*

(b) *The function* $q \mapsto C^{(j)}_{n,p,q}(\mathrm{pos}\Theta)$ *is continuous and strictly decreasing.*

(c) *The map* $[a,b]^n \to \mathbb{R}$ *given by* $\Theta = (\theta_1, \ldots, \theta_n) \mapsto C^{(j)}_{n,p,q}(\mathrm{pos}\Theta)$ *is continuous.*

Additional statements about the (continuous) dependence of G_Θ on its various parameters can be found in Gustafson [Gu76].

Since $[a,b]^n \to \mathbb{R} : \Theta \mapsto C_{n,p,q}(\mathrm{pos}\Theta)$ is a continuous map on a compact set, it attains its infimum and supremum. The corresponding set of positions will be denoted by **best** and **worst** respectively. We conjecture that there is a unique best position, and that the corresponding Θ consists of distinct points inside (a,b), and that the worst positions correspond to Taylor interpolation at an endpoint. These conjectures are supported by some special cases investigated by the author in [Wa94$_2$].

3. The main result

In this section we use the B-spline theory to prove

$$|f(x) - H_\Theta f(x)| \le \frac{n^{1/q}}{n!} \frac{|\omega_\Theta(x)|}{(\mathrm{diam}\{x,\Theta\})^{1/q}} \|D^n f\|_{L_q(\mathrm{conv}\{x,\Theta\})}, \tag{3.1}$$

which we refer to as the **basic estimate**. The function $x \mapsto \mathrm{diam}\{x,\Theta\}$ is nonnegative, continuous, and piecewise linear with break points at the endpoints of the interval $\mathrm{conv}\,\Theta$. It is zero only if $\mathrm{conv}\{x,\Theta\} = \{x\}$, in which case $\omega_\Theta(x) = 0$, and the quotient (3.1) is understood to be 0.

B-splines

For Θ with $\#\Theta = k+1$, $\mathrm{diam}\,\Theta > 0$, the **B-spline with knots** Θ is the function

$$M(\cdot|\Theta) : \mathbb{R} \to \mathbb{R} : t \mapsto M(t|\Theta) := k[\Theta](\cdot - t)_+^{k-1}, \tag{3.2}$$

where $[\Theta]$ is the *divided difference* at Θ. It suffices, for our purposes, to let $M(\cdot|\Theta) := 0$ when $\mathrm{diam}\,\Theta = 0$.

Recall the following, see, e.g., [DL93:p137].

(3.3) B-spline properties. *If* $\#\Theta = k+1$, *and* $\operatorname{diam}\Theta > 0$, *then*
(a) $M(\cdot|\Theta) > 0$ *on the interior of its support,* $\operatorname{supp} M(\cdot|\Theta) = \operatorname{conv}\Theta$.
(b) $0 \le M(\cdot|\Theta) \le k/\operatorname{diam}\Theta$.
(c) $\int M(\cdot|\Theta) = 1$.
(d) *The B-spline represents the divided difference of* f *at* Θ, *i.e.*

$$[\Theta]f = \frac{1}{k!}\int D^k f M(\cdot|\Theta), \quad \forall f \in W_1^{(k)}(\operatorname{conv}\Theta).$$

Property (d) allows the error formula for Hermite interpolation,

$$f(x) - H_\Theta f(x) = \omega_\Theta(x)[x,\Theta]f,$$

to be written in terms of B-splines.

(3.4) B-spline form of the error.

$$f(x) - H_\Theta f(x) = \frac{w_\Theta(x)}{n!}\int M(\cdot|x,\Theta)D^n f, \quad \forall f \in W_1^{(n)}(\operatorname{conv}\{x,\Theta\}).$$

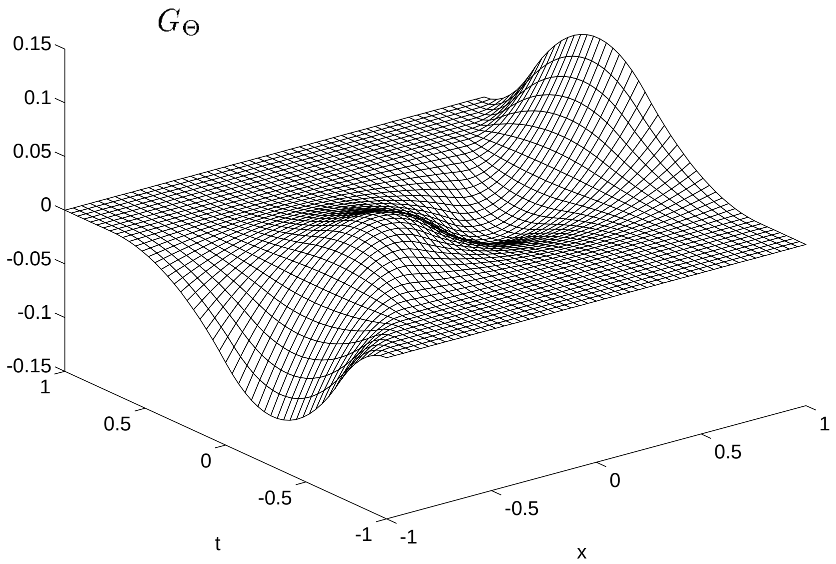
When it is not necessary to know the exact form of the kernel in this formula. we follow the standard practice and denote it by

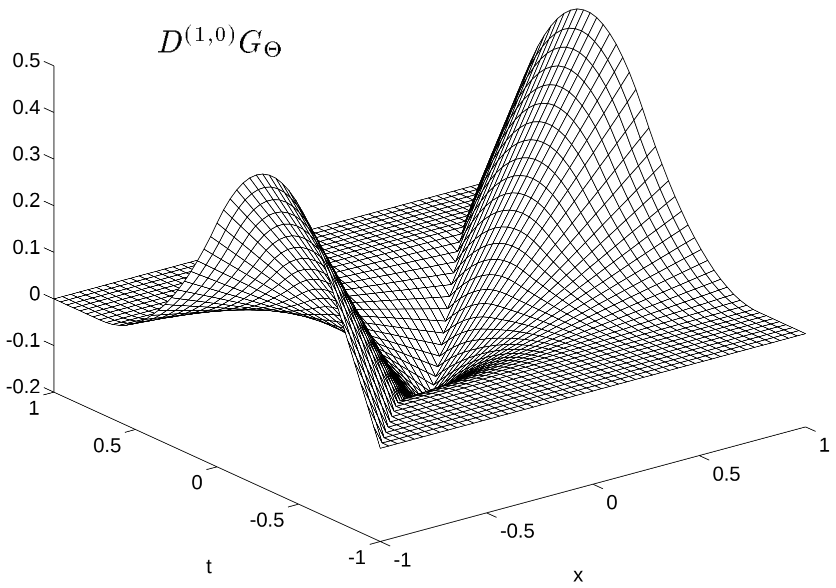
$$G_\Theta(x,t) := \frac{\omega_\Theta(x)}{n!}M(t|x,\Theta). \tag{3.5}$$

The choice of the letter G here is apt since G_Θ is the *Green's function* of the boundary value problem $D^n f = g$, with Hermite multipoint conditions given by $H_\Theta f = 0$; i.e., the solution of this problem can be written

$$f(x) = \int G_\Theta(x,\cdot)\,g.$$

Here are the graphs of G_Θ, $D^{(1,0)}G_\Theta$, $D^{(2,0)}G_\Theta$ over $[-1,1]^2$ for $\Theta = \{-1/\sqrt{2}, 0, 1/\sqrt{2}\}$ the roots of the cubic Chebyshev polynomial.


G_Θ
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
1
0.5
0
-0.5
-1
t
-1
-0.5
0
0.5
1
x


$D^{(1,0)}G_\Theta$
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
1
0.5
0
-0.5
-1
t
-1
-0.5
0
0.5
1
x

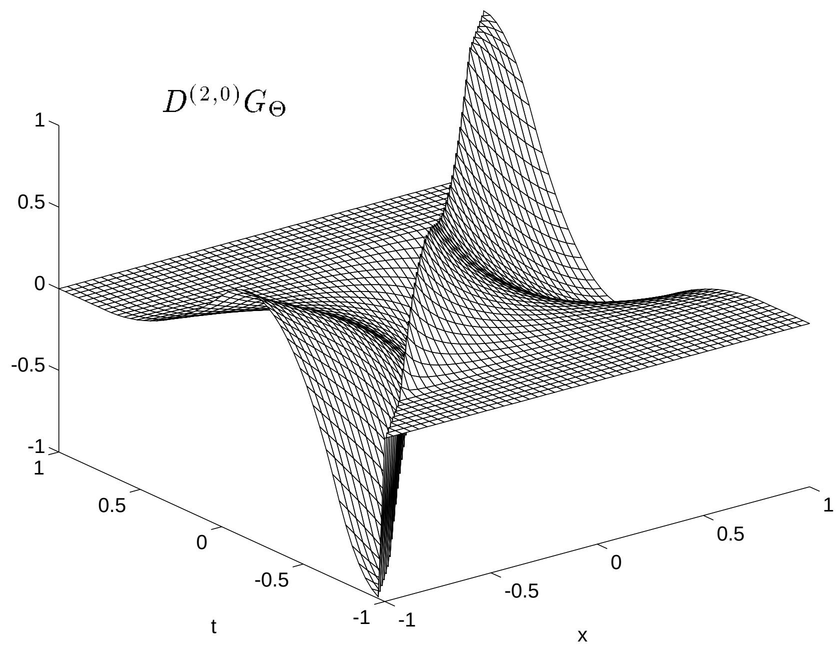


Fig 3.1 Graphs of G_Θ, $D^{(1,0)}G_\Theta$, $D^{(2,0)}G_\Theta$ over $[-1,1]^2$ for $\Theta = \{-1/\sqrt{2}, 0, 1/\sqrt{2}\}$ the roots of the cubic Chebyshev polynomial

A less well-known property of B-splines, which follows from (a), (b), and (c) of (3.3), is the following.

(3.6) B-spline L_p-estimate ([B73]). *If* $\operatorname{diam}\Theta > 0$, *then*

$$\|M(\cdot|\Theta)\|_{L_p(\mathbb{R})} \le \left(\frac{\#\Theta - 1}{\operatorname{diam}\Theta}\right)^{1-1/p},$$

with equality iff $p = 1$ *or* $\#\Theta = 2$.

This completes the list of B-spline properties needed to prove (3.1).

The proof of the basic estimate

(3.7) Basic estimate. *We have the pointwise estimate*

$$|f(x) - H_\Theta f(x)| \le \frac{n^{1/q}}{n!} \frac{|\omega_\Theta(x)|}{(\operatorname{diam}\{x,\Theta\})^{1/q}} \|D^n f\|_{L_q(\operatorname{conv}\{x,\Theta\})}, \quad \forall f \in W_q^n(\operatorname{conv}\{x,\Theta\}).$$

Proof. By applying Hölder's inequality to the B-spline form (3.4) of the error, then using the B-spline L_p-estimate (3.6), we get

$$\begin{aligned} |f(x) - H_\Theta f(x)| &\le \frac{|\omega_\Theta(x)|}{n!} \|M(\cdot|x,\Theta)\|_{L_{q^*}(\operatorname{conv}\{x,\Theta\})} \|D^n f\|_{L_q(\operatorname{conv}\{x,\Theta\})} \\ &\le \frac{|\omega_\Theta(x)|}{n!} \left(\frac{n}{\operatorname{diam}\{x,\Theta\}}\right)^{1/q} \|D^n f\|_{L_q(\operatorname{conv}\{x,\Theta\})}. \end{aligned}$$

$\square$

We observe that equality in (3.7) for $D^n f \neq 0$ can occur only if $q = \infty$, or $n = 1$, in which case it does so for polynomials of (exact) degree n.

For applications not directly related to Hermite interpolation, such as the analysis of boundary value problems (with Hermite multipoint conditions), the reader might prefer:

(3.8) Basic estimate in terms of G_Θ.

$$\|G_\Theta(x,\cdot)\|_{L_{q^*}(\mathbb{R})} = \frac{1}{n!}|\omega_\Theta(x)|\|M(\cdot|x,\Theta)\|_{L_{q^*}(\mathbb{R})} \leq \frac{n^{1/q}}{n!}\frac{|\omega_\Theta(x)|}{(\operatorname{diam}\{x,\Theta\})^{1/q}},$$

with equality iff $q = \infty$ or $\#\Theta = 1$.

Some history

An exhaustive search of the literature shows that (3.5) is not known to those working on error estimates for Hermite interpolation. For example, see the recent, elaborate, representation for G_Θ given in [AW92].

Instead, the main tool used in the literature has been the following.

(3.9) Beesack's inequality ([Be62]). *If $a, b \in \Theta$, then*

$$|G_\Theta(x,t)| \leq \frac{1}{(n-1)!}\frac{1}{b-a}|\omega_\Theta(x)|, \quad \forall\, a \leq x, t \leq b.$$

Beesack's inequality is considered difficult to prove, with alternative 'simpler' proofs given by Nehari [Ne64], and Gustafson [Gu76], amongst others. With the benefit of (3.5), and the B-spline L_∞-estimate, we can immediately offer the strengthening

$$(3.10) \qquad |G_\Theta(x,t)| \leq \frac{1}{(n-1)!}\frac{1}{\operatorname{diam}\{x,\Theta\}}|\omega_\Theta(x)| \leq \frac{1}{(n-1)!}|\omega_\Theta(x)|^{1-\frac{1}{n}}, \quad \forall x, t \in \mathbb{R}.$$

This strengthening (3.10) is a very special case of the basic estimate (3.7) (equivalently (3.8)). Thus, the basic estimate (3.7) should be considered the natural replacement for Beesack's inequality.

One major advantage of the basic estimate (3.7) over Beesack's inequality is that it allows the points in Θ to coalesce; a fact which we exploit, in Section 5, to derive bounds for the derivative of the error in Hermite interpolation.

In additional support of the rightful place of B-splines in Hermite error estimation, we mention some other properties of G_Θ which their use makes transparent.

With $\operatorname{conv}\Theta =: [a,b]$, by property (3.3) (a),

$$G_\Theta(x,t)/\omega_\Theta(x) = M(x|t,\Theta)/n! > 0, \quad \forall\, x \in [a,b] \setminus \Theta, \quad \forall\, a \leq t \leq b;$$

see Levin [Le63], Pokornyi [Po68], Coppel [Co71:p108], and Das and Vatsala [DV73].

By property (3.3) (c),

$$\|G_\Theta(x,\cdot)\|_1 = \frac{|\omega_\Theta(x)|}{n!}\|M(\cdot|x,\Theta)\|_1 = \frac{|\omega_\Theta(x)|}{n!};$$

see [DV73], [Gu76], and most likely earlier in the Russian literature.

4. L_p-error bounds and the corresponding inequalities of Wirtinger type

In this Section, the basic estimate (3.7) is used to provide bounds for $C_{n,p,q}(\mathrm{pos}\Theta)$ in terms of $\|\omega_\Theta\|_p$. These bounds are exact for $q=\infty$, and off by a factor of at most $n^{1/q}$ if $a,b\in\Theta$.

Given the appropriate knowledge of $\|\omega_\Theta\|_p$, the bounds of this section give a unified description of all the estimates for $C_{n,p,q}(\mathrm{pos}\Theta)$ in the literature. Some of these estimates are given in terms of inequalities of the so called 'Wirtinger type', which we discuss.

To facilitate the reader's own computations, we choose not to normalise $b-a=1$.

(4.1) Theorem.

$$\frac{\|\omega_\Theta\|_p}{n!}(b-a)^{-(n+\frac{1}{p})} \le C_{n,p,q}(\mathrm{pos}\Theta) \le \frac{n^{\frac{1}{q}}}{n!}\|\frac{\omega_\Theta}{(\mathrm{diam}\{\cdot,\Theta\})^{\frac{1}{q}}}\|_p(b-a)^{-(n+\frac{1}{p}-\frac{1}{q})}.$$

Note that $\|\omega_\Theta/(\mathrm{diam}\{\cdot,\Theta\})^{\frac{1}{q}}\|_p \le \|\omega_\Theta^{1-\frac{1}{nq}}\|_p$, *and that*

$$C_{n,p,q}(\mathrm{pos}\Theta) \le \frac{n^{\frac{1}{q}}}{n!}.$$

The following special cases are of particular interest.

(a) *If* $q=\infty$, *then*

$$C_{n,p,\infty}(\mathrm{pos}\Theta) = \frac{\|\omega_\Theta\|_p}{n!}(b-a)^{-(n+\frac{1}{p})}.$$

(b) *If* $a,b\in\Theta$, *then*

$$C_{n,p,q}(\mathrm{pos}\Theta) \in \frac{\|\omega_\Theta\|_p}{n!}(b-a)^{-(n+\frac{1}{p})}\left[1,n^{\frac{1}{q}}\right].$$

Proof. Taking $\|\cdot\|_p$ of the basic estimate (3.7) shows

$$C_{n,p,q}(\mathrm{pos}\Theta) \le \frac{n^{\frac{1}{q}}}{n!}\|\frac{\omega_\Theta}{(\mathrm{diam}\{\cdot,\Theta\})^{\frac{1}{q}}}\|_p(b-a)^{-(n+\frac{1}{p}-\frac{1}{q})} \le \frac{n^{\frac{1}{q}}}{n!}.$$

If $f=\omega_\Theta$, then $H_\Theta f=0$, and $D^n f = n!$, therefore

$$C_{n,p,q}(\mathrm{pos}\Theta) \ge \frac{\|f-H_\Theta f\|_p}{\|D^n f\|_q}(b-a)^{-(n+\frac{1}{p}-\frac{1}{q})} = \frac{\|\omega_\Theta\|_p}{n!}(b-a)^{-(n+\frac{1}{p})}.$$

Finally, for each $\theta\in\Theta$, $|x-\theta|\le \mathrm{diam}\{x,\Theta\}$, hence $|\omega_\Theta(x)|^{\frac{1}{n}} \le \mathrm{diam}\{x,\Theta\}$, giving

$$\|\omega_\Theta/(\mathrm{diam}\{\cdot,\Theta\})^{\frac{1}{q}}\|_p \le \|\omega_\Theta^{1-\frac{1}{nq}}\|_p. \qquad \square$$

In addition to the special case $p,q=\infty$, which is well known, only the following instance of Theorem (4.1) is known.

(4.2) Result ([Ag91$_1$]). *If $a, b \in \Theta$, then*

$$C_{n,p,q}(\mathrm{pos}\Theta) \le n^{\frac{1}{q}} \frac{\|\omega_\Theta\|_p}{n!}(b-a)^{-(n+\frac{1}{p})}.$$

This is proved there using Beesack's inequality.

Wirtinger type inequalities

In Fink, Mitrinović, and Pěcarić [FMP91:p66] an inequality of the form

$$\|f\|_p \le C(b-a)^{n+\frac{1}{p}-\frac{1}{q}} \|D^n f\|_q, \tag{4.3}$$

for all f in some set $\mathcal{F} \subset W_q^{(n)} \setminus \Pi_{<n}$ is said to be of **Wirtinger type**. The origin of this name is the following result of Wirtinger first appearing in [Bl16:p105], see also [HLP52:p184].

(4.4) Wirtinger's inequality ([Bl16:p105]). *For all 2π-periodic f with $\int_0^{2\pi} f = 0$*

$$\|f\|_{L_2[0,2\pi]} \le \|Df\|_{L_2[0,2\pi]},$$

with equality iff $f \in \mathrm{span}\{\cos, \sin\}$.

If $\mathcal{F}$ of (4.3) is taken to be the set of those $f \in W_q^{(n)}$ with n zeros at Θ (i.e., with $H_\Theta f = 0$), then we get the Wirtinger inequality

$$\|f\|_p = \|f - H_\Theta f\|_p \le C(b-a)^{n+\frac{1}{p}-\frac{1}{q}} \|D^n f\|_q, \quad \forall f \in \ker H_\Theta \cap W_q^{(n)}, \tag{4.5}$$

where the smallest possible C is precisely $C_{n,p,q}(\mathrm{pos}\Theta)$.

Another related class of Wirtinger inequalities was studied by Brink in [Br72]. Its description makes use of the following definition.

(4.6) Definition. *Let $\mathcal{A}_n(i_1, i_2)$ be the set of those Θ containing one endpoint at least i_1 times and the other at least i_2 times, where $i_1 + i_2 \le n$.*

Notice that $\mathcal{A}_n(i_1, i_2)$ is symmetric in i_1, i_2, and that $\mathcal{A}_n(0,0)$ consists of all Θ. In terms of this notation, Brink considered Wirtinger type inequalities where $\mathcal{F}$ consists of all f with n zeros at some $\Theta \in \mathcal{A}_n(i_1, i_2)$. For this choice of $\mathcal{F}$, the best constant in (4.3) is

$$C(n, p, q, i_1, i_2) = \max_{\Theta \in \mathcal{A}_n(i_1, i_2)} C_{n,p,q}(\mathrm{pos}\Theta). \tag{4.7}$$

In view of Theorem (4.1), good estimates for this constant require knowledge of the size of $\|\omega_\Theta\|_p$. This question is considered in [Wa94$_2$].

The minimiser of $\Theta \mapsto \|\omega_\Theta\|_p$ has close connections with best polynomial approximation, and Gauss quadrature.

We require the following result concerning the maximiser of $\Theta \mapsto \|\omega_\Theta\|_p$.

(4.8) Result ([Wa94$_2$]). *Let $m := \min\{i_1, i_2\}$, and $0^0 := 1$. Then*

$$\max_{\Theta\in\mathcal{A}_n(i_1,i_2)} \|\omega_\Theta\|_p = (b-a)^{n+\frac{1}{p}} \begin{cases} B(pm+1, p(n-m)+1)^{\frac{1}{p}}, & 1 \le p < \infty \\ m^m(n-m)^{n-m}/n^n, & p = \infty, \end{cases}$$

with the maximum achieved iff $\Theta \in \mathcal{A}_n(m, n-m)$.

Here B is the **beta function**

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall x, y > 0.$$

Note that B satisfies $0 < B(x,y) \le \min\{1, 1/\max\{x,y\}\}$, $\forall x, y > 0$.

This result allows the following estimate.

(4.9) Corollary. *Let $\Theta \in \mathcal{A}_n(i_1, i_2)$, $m := \min\{i_1, i_2\}$, and $0^0 := 1$. Then:*

(a) *If $m = 0$, then*

$$C_{n,p,q}(\text{pos}\Theta) \le \frac{n^{\frac{1}{q}}}{n!} \begin{cases} 1/(pn - p/q + 1)^{\frac{1}{p}}, & 1 \le p < \infty \\ 1, & p = \infty \end{cases}$$

with equality when $q = \infty$ and $\Theta \in \mathcal{A}_n(0, n)$.

(b) *If $m > 0$, or $q = \infty$, then*

$$C_{n,p,q}(\text{pos}\Theta) \le \frac{n^{\frac{1}{q}}}{n!} \begin{cases} B(pm+1, p(n-m)+1)^{\frac{1}{p}}, & 1 \le p < \infty \\ m^m(n-m)^{n-m}/n^n, & p = \infty \end{cases}$$

with equality when $q = \infty$ and $\Theta \in \mathcal{A}_n(m, n-m)$.

Proof. Without loss of generality, we may assume that $b - a = 1$. For (a), by Theorem (4.1),

$$C_{n,p,q}(\text{pos}\Theta) \le \frac{n^{\frac{1}{q}}}{n}\|\omega_\Theta\|_p \le \frac{n^{\frac{1}{q}}}{n!}\|\omega_\Theta^{1-\frac{1}{nq}}\|_p = \frac{n^{\frac{1}{q}}}{n!}\left(\|\omega_\Theta\|_{p(1-\frac{1}{nq})}\right)^{1-\frac{1}{nq}},$$

to which we apply Result (4.8). Similarly, for (b)

$$C_{n,p,q}(\text{pos}\Theta) \le \frac{n^{\frac{1}{q}}}{n!}\|\omega_\Theta\|_p,$$

which is estimated by Result (4.8). □

For $m = 0$, only the result with $p, q = \infty$ is known, see, e.g., [AW93:p105,Th.2.4.3]. For $m > 0$, the best known result is:

(4.10) Result ([Ag91$_1$]). *Let $\Theta \in \mathcal{A}_n(i_1, i_2)$, with $m := \min\{i_1, i_2\} > 0$. Then*

$$C_{n,p,q}(\mathrm{pos}\Theta) \le \frac{n^{\frac{1}{q}}}{n!} \begin{cases} (2B_{1/2}(pm+1, p(n-m)+1))^{\frac{1}{p}}, & 1 \le p < \infty \\ m^m (n-m)^{n-m}/n^n, & p = \infty. \end{cases}$$

Here $B_{1/2}$ is the **incomplete beta function**

$$B_{1/2}(x,y) := \int_0^{1/2} t^{x-1}(1-t)^{y-1}\, dt, \quad \forall x, y > 0,$$

which satisfies $B(x,y) \le 2B_{1/2}(x,y)$, $\forall\, 1 \le x \le y$, with strict inequality unless $x = y$.

Thus, Corollary (4.9) gives better bounds than Result (4.10) if $1 \le p < \infty$, $m \ne n-m$, and the same bounds otherwise.

In view of (4.7), Corollary (4.9) provides an estimate of Brink's constants (4.7).

(4.11) Estimate for Brink's constants. *Let $m := \min\{i_1, i_2\}$, where $i_1 + i_2 \le n$.*
(a) *If $m = 0$, then*

$$C(n,p,q,i_1,i_2) \le \frac{n^{\frac{1}{q}}}{n!} \begin{cases} 1/(pn - p/q + 1)^{\frac{1}{p}}, & 1 \le p < \infty \\ 1, & p = \infty, \end{cases}$$

with equality if $q = \infty$.
(b) *If $m > 0$, then*

$$C(n,p,q,i_1,i_2) \le \frac{n^{\frac{1}{q}}}{n!} \begin{cases} B(pm+1, p(n-m)+1)^{\frac{1}{p}}, & 1 \le p < \infty \\ (m^m (n-m)^{n-m}/n^n), & p = \infty, \end{cases}$$

which is within a factor of $n^{1/q}$ of being sharp.

This estimate reproduces the results of Brink [Br72], which are for the case when $p, q = \infty$, and, otherwise, provides bounds where none were previously known.

5. L_p-bounds for the derivative of the error in Hermite interpolation

With the exception of [Tu41], [BP67], and [AW93:Th.2.4.13], all existing estimates for $C^{(j)}_{n,p,q}(\mathrm{pos}\Theta)$, $j > 0$ are obtained, not by attacking $D^{(j,0)}G_\Theta$ directly, but by using bounds for $C_{n,p,q}(\mathrm{pos}\Theta)$ together with Rolle's Theorem. Due to the erratic behaviour of the known bounds for $C_{n,p,q}(\mathrm{pos}\Theta)$ as a function of Θ (especially as the points in Θ coalesce), this argument was limited to cases when $p, q = \infty$ and Θ contained the endpoints $\{a, b\}$ with high multiplicities.

In contrast, the bounds for $C_{n,p,q}(\mathrm{pos}\Theta)$ given in Theorem (4.1) depend continuously on Θ. This allows us, in this section, to perform the 'Rolle's Theorem argument' for any p, q, and Θ.

Let $\Theta^{(j)}$ be the multiset obtained from Θ by reducing the multiplicity of each point by j; e.g., $\{0,0,0,1\}^{(2)} = \{0\}$.

(5.1) Rolle's Theorem (with multiplicities). *Let $0 \le j \le n-1$. If $f \in W_q^{(n)}$ has (at least) n zeros Θ, then $D^j f$ has (at least) $n-j$ zeros, which include $\Theta^{(j)}$. In addition $D^j f$ has (at least) $n - j - \#\Theta^{(j)}$ zeros in $(\operatorname{conv}\Theta) \setminus \Theta$.*

Proof. The proof is by induction on n. The inductive step follows from the facts: that if θ is a zero of f with multiplicity m, then θ is a zero of Df with multiplicity $m-1$; and the special case $\Theta = \{a, b\}$ which is the classical Rolle's Theorem. □

Next the 'Rolle's Theorem argument'.

(5.2) Theorem. *If $0 \le j \le n-1$, then*

$$\frac{\|D^j \omega_\Theta\|_p}{n!}(b-a)^{-(n-j+\frac{1}{p})} \le C^{(j)}_{n,p,q}(\mathrm{pos}\Theta) \le \max_{\substack{\#\Theta'=n-j \\ \Theta^{(j)} \subset \Theta'}} C_{n-j,p,q}(\mathrm{pos}\Theta'),$$

with equality in both bounds for $q = \infty$ and $\Theta \in \mathcal{A}_n(0, n)$.

Proof. Let $f \in W_q^{(n)}$. By Rolle's Theorem (5.1), $D^j(f - H_\Theta f) \in W_q^{(n-j)}$ has $n-j$ zeros $\Theta' \supset \Theta^{(j)}$. Therefore, by the Wirtinger inequality (4.5),

$$\|D^j(f - H_\Theta f)\|_p \le C_{n-j,p,q}(\mathrm{pos}\Theta')(b-a)^{n-j+\frac{1}{p}-\frac{1}{q}}\|D^{n-j}D^j(f - H_\Theta f)\|_q.$$

Since $D^n H_\Theta f = 0$, $\|D^n(f - H_\Theta f)\|_q = \|D^n f\|_q$, and we conclude

$$C^{(j)}_{n,p,q}(\mathrm{pos}\Theta) \le \max_{\substack{\#\Theta'=n-j \\ \Theta^{(j)} \subset \Theta'}} C_{n-j,p,q}(\mathrm{pos}\Theta').$$

The lower bound follows as in Theorem (4.1). □

In the exceptional cases [Tu41], [BP67], and [AW93:Th.2.4.13], there is equality in the lower bound of Theorem (5.2). This phenomenon is equivalent to there being equality in

$$\|D^j(f - H_\Theta f)\|_p \le C^{(j)}_{n,p,q}(\mathrm{pos}\Theta)\,(b-a)^{n-j+\frac{1}{p}-\frac{1}{q}}\|D^n f\|_q$$

for all $f \in \Pi_n$. Further aspects of this phenomenon, which is deserving of closer scrutiny, can be found in [Wa94$_1$].

Here is a typical example of Theorem (5.2). If $\Theta \in \mathcal{A}_{2m}(m, m)$, and $0 \le j \le m$, then

$$C^{(j)}_{2m,\infty,\infty}(\mathrm{pos}\Theta) \le \frac{m^m(m-j)^{m-j}}{(2m-j)!(2m-j)^{2m-j}}.$$

This estimate improves upon the earlier result of Ciarlet, Schultz, and Varga [CSV67], and is proved in [Ag91$_2$:p774]. The following Corollary encompasses this, and many other variations on the 'Rolle's theorem argument' found in the literature.

(5.3) Corollary. *Let $\Theta \in \mathcal{A}_n(i_1, i_2)$, with $m := \min\{i_1, i_2\}$.*
(a) *If $0 \le j \le m-1$, then*

$$C^{(j)}_{n,p,q}(\mathrm{pos}\Theta) \le \frac{(n-j)^{\frac{1}{q}}}{(n-j)!} \begin{cases} B(p(m-j)+1, p(n-m)+1)^{\frac{1}{p}}, & 1 \le p < \infty \\ (m-j)^{m-j}(n-m)^{n-m}/(n-j)^{n-j}, & p = \infty. \end{cases}$$

(b) *If $m \le j \le n-1$, then*

$$C^{(j)}_{n,p,q}(\mathrm{pos}\Theta) \le \frac{(n-j)^{\frac{1}{q}}}{(n-j)!} \begin{cases} 1/(p(n-j)-p/q+1)^{\frac{1}{p}}, & 1 \le p < \infty \\ 1, & p = \infty \end{cases}$$

with equality when $q = \infty$ and $\Theta \in \mathcal{A}_n(0, n)$.

Proof. Apply the bounds of Corollary (4.9) to Theorem (5.2). □

Though sharp for the extreme case $q = \infty$, $\Theta \in \mathcal{A}_n(0, n)$, the exceptional cases previously discussed indicate that, in the general case, these bounds can be significantly improved.

6. Final remarks

Computing $D^{(j,0)}G_\Theta$

To obtain the figures of Section 2, $D^{(j,0)}G_\Theta$ was computed by using the SPLINE TOOLBOX for use with MATLAB™ ([B90]), then plotted using `mesh`.

The SPLINE TOOLBOX uses the 'partion of unity' normalisation, i.e.

$$B(\cdot|\Theta) := \frac{\operatorname{diam}\Theta}{\#\Theta - 1} M(\cdot|\Theta).$$

Thus, for example, with `n` $:= n$, `x` $:= x$, `t` $:= t$, `gTh` $:= \Theta$ (a $1 \times n$ matrix), `d` $:= \operatorname{diam}\{x, \Theta\}$, and `G` $:= G_\Theta(x,t)$ we have the MATLAB instructions

```
d=max([x gTh])-min([x gTh])
G=prod(x*ones(size(gTh))-gTh)/gamma(n)*fnval(spmak(sort([x,gTh]),1),t)/d
```

To compute $D^{(j,0)}G_\Theta$, for $0 \le j \le n-1$, we observe, by the formula for differentiating a B-spline with respect to one of its knots (see [B77:Ex.4]), that

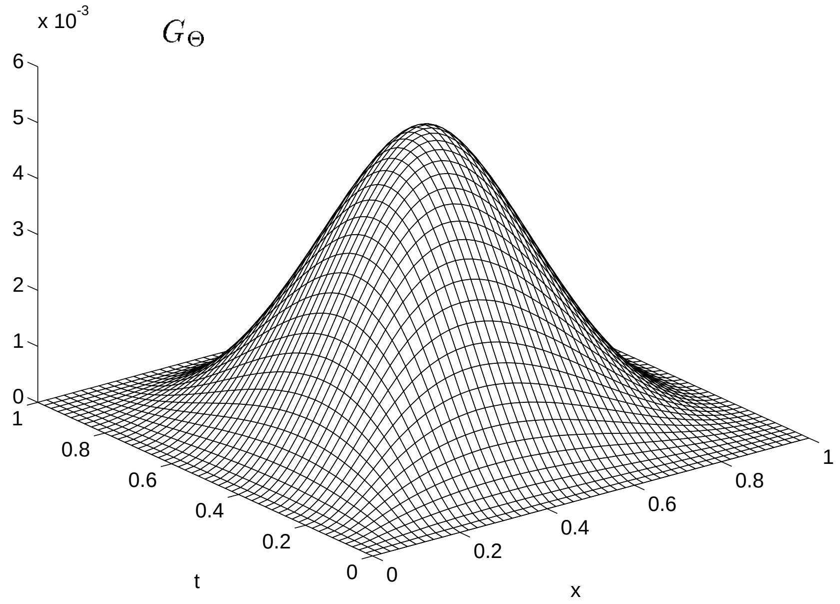
$$\begin{aligned} D^{(j,0)}G_\Theta(x,t) &= \sum_{i=0}^{j} \frac{(-1)^i j!}{(n+i)!(j-i)!} D^{j-i}\omega_\Theta(x)\, D^i M(t|\underbrace{x,\ldots,x}_{i+1 \text{ times}}, \Theta) \\ &= \frac{1}{\operatorname{diam}\{x,\Theta\}} \sum_{i=0}^{j} \frac{(-1)^i j!}{(n+i-1)!(j-i)!} D^{j-i}\omega_\Theta(x)\, D^i B(t|\underbrace{x,\ldots,x}_{i+1 \text{ times}}, \Theta). \end{aligned}$$

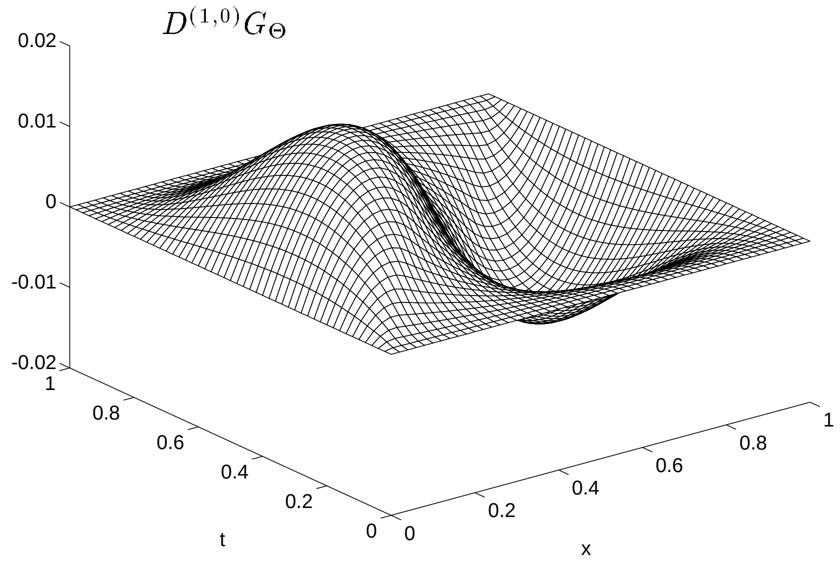
With $\mathtt{i} := i$, we then have the MATLAB instruction

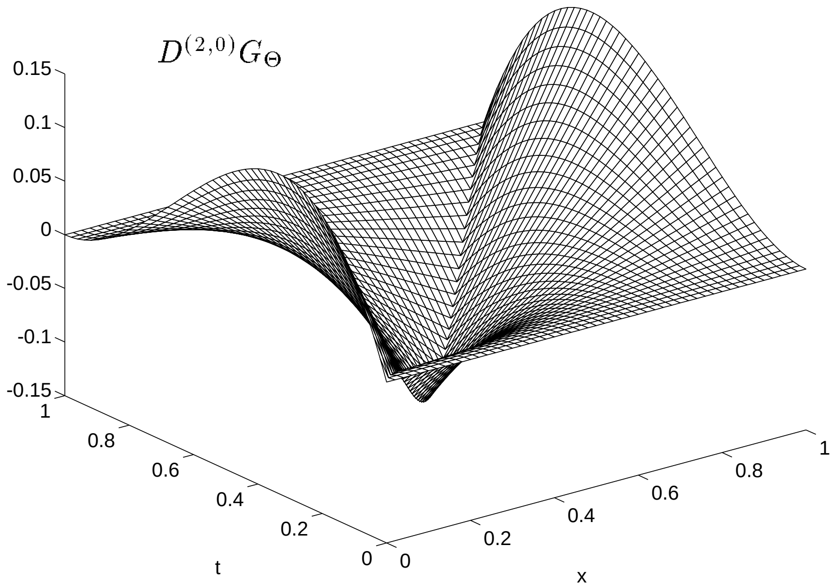
$$D^i B(t|\underbrace{x,\ldots,x}_{i+1 \text{ times}},\Theta) = \texttt{fnval(fnder(spmak(sort([x*ones(1,i+1),gTh]),1),i),t)}$$

For complete details see the technical report [Wa94$_1$].

Here are the `mesh` plots obtained for $\Theta = \{0, 0, 1, 1\}$.







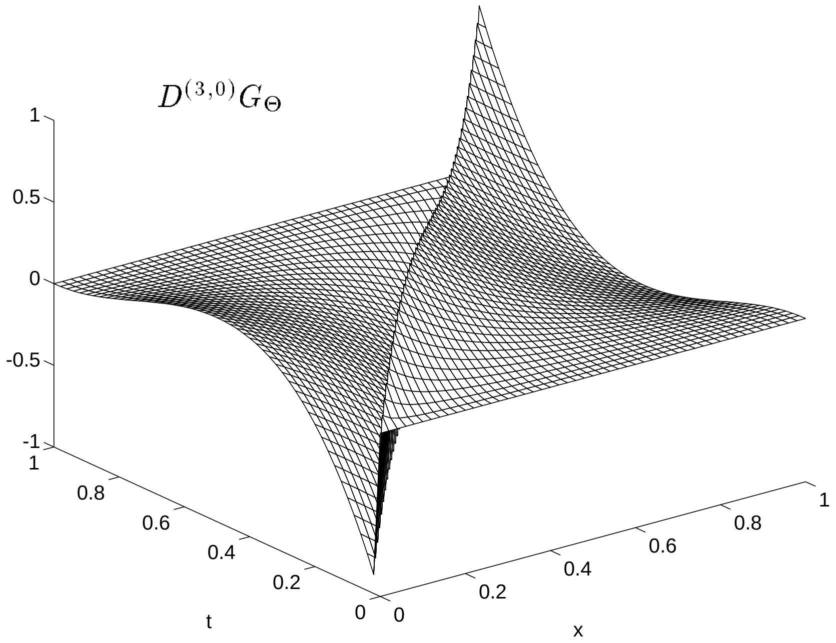


Fig 6.1 Graphs of G_Θ, $D^{(1,0)}G_\Theta$, $D^{(2,0)}G_\Theta$, $D^{(3,0)}G_\Theta$ over $[0,1]^2$ for $\Theta = \{0,0,1,1\}$

Note the jump discontinuity of 1 in $D^{(3,0)}G_\Theta$ along the line $x = t$.

Extensions

We indicate how the results of this paper can be extended to *Birkhoff interpolation*. See, e.g., Lorentz, Jetter, and Riemenschneider [LJR83].

Let E be a *regular* $m \times n$ *interpolation matrix*, $X = \{x_1, \cdots, x_m\}$ a set of m points which satisfies $x_1 < x_2 < \ldots < x_m$, and denote the corresponding *Birkhoff interpolant* to f at (E, X) by $Hf := H_{E,X} f \in \Pi_{<n}$.

If $x \notin X$, then let $(E^*, \{x, X\})$ denote the Birkhoff interpolation scheme obtained by adding to (E, X) the extra interpolation condition that $f(x)$ be matched. Let $\lambda_x(f)$ be the coefficient of $(\cdot)^n$ in the interpolant (from Π_n) to f at $(E^*, \{x, X\})$. In this notation

$$f(x) - Hf(x) = \bigl((\cdot)^n - H(\cdot)^n\bigr)\, \lambda_x(f).$$

The monic polynomial $(\cdot)^n - H(\cdot)^n \in \Pi_n$ plays the role of ω_Θ in the Hermite interpolation case, and $\lambda_x(f)$ the role of $[x, \Theta]f$. By Peano's theorem

$$\lambda_x(f) = \frac{1}{n!} \int M_{E,x,X}\, D^n f,$$

where $M_{E,x,X}$ is, in this case, a nonnegative, piecewise polynomial function supported on $\mathrm{conv}\{x, X\}$. Many of the properties of $M_{E,x,X}$ (which plays the role of $M(\cdot|x, \Theta)$) can be found in Bojanov, Hakopian, and Sahakian [BHS93], where it is denoted by $B((E^*, \{x, X\}); \cdot)$ and called a 'B-spline of degree $n-1$ with knots at $(E^*, \{x, X\})$', see also Uluchev [Ul89].

The only difficulty encountered with $B((E^*, \{x, X\}); \cdot)$ is finding the appropriate analog of the B-spline L_p-estimate (3.6), which is beyond the scope of this paper. Once this is done, it should be possible to give a unified treatment of the many error bounds for special cases of Birkhoff interpolation, most notably, *Lidstone interpolation* (see [AW93:Ch.1]), *Abel-Gontscharoff* (see [AW93:Ch.3]), and the 'miscellaneous interpolations' of Chapter 4 of [AW93].

For some simple examples of L_∞-estimates for Birkhoff interpolation see [Ho93].

Tumura's result?

Of the 'exceptional results' mentioned in Section 5, by far the most outstanding is Tumura's.

(6.1) Tumara's Result ([Tu41]). *Assume $n \geq 2$. If $\Theta \in \mathcal{A}_n(1,1)$, then for $1 \leq j \leq n-1$*

$$C^j_{n,\infty,\infty}(\mathrm{pos}\Theta) \leq \frac{j}{n(n-j)!},$$

with equality iff $\Theta \in \mathcal{A}_n(1, n-1)$. For the case of equality $C^j_{n,\infty,\infty}(\mathrm{pos}\Theta)$ is equal to the lower bound in Theorem (5.2).

This result, which is often quoted and used (cf. [Ag83]), was apparently mentioned in Hukuhara [Hu63]. The author has been entirely unsuccessful in locating [Tu41]. It is curious, given its significance, that none of those quoting [Tu41] whom the author was able to contact had ever seen a proof of Tumura's result.

Thus, the first step in dealing systematically with the few exceptional results mentioned in Section 5 (all of which are cases of Conjecture (2.3)) must be to locate a proof of

Tumura's result, if indeed one does exist. The author would be most grateful to anybody able to supply one.

A simple estimate

Finally, for those not worried about best constants, here is an all-purpose estimate.

(6.2) Proposition. *For all p, q, Θ, if $j = 0, \ldots, n-1$, then*

$$\|D^j(f - H_\Theta f)\|_p \le \frac{1}{(n-j-1)!}(b-a)^{n-j+\frac{1}{p}-\frac{1}{q}}\|D^n f\|_q, \quad \forall f \in W_q^{(n)}.$$

References

[Ag83] R. P. Agarwal (1983): *Some inequalities for a function having n zeros.* In: General inequalities 3(E.F. Beckenbach and W. Walter, eds). International series of numerical mathematics, vol.64, Birkhäuser Verlag, pp. 371–378.

[Ag91$_1$] R. P. Agarwal (1991): *Better error estimates in polynomial interpolation.* J. Math. Anal. Appl., **161**:241–257.

[Ag91$_2$] R. P. Agarwal (1991): *Sharp Hermite interpolation error bounds for derivatives.* Nonlinear Analysis, **17(8)**:773–786.

[AW92] R. P. Agarwal and P. J. Y. Wong (1992): *A new representation for the error function for the Hermite interpolation and sharper pointwise and uniform error bounds for the derivatives.* Nonlinear Analysis, **19(8)**:769–786.

[AW93] R. P. Agarwal and P. J. Y. Wong (1993): Error inequalities in polynomial interpolation and their applications. Kluwer Academic Publishers.

[Be62] P. R. Beesack (1962): *On the Green's function of an N-point boundary value problem.* Pacific J. Math., **12**:801–812.

[BP67] G. Birkhoff and A. Priver (1967): *Hermite interpolation errors for derivatives.* J. Math. Phys., **46**:440–447.

[Bl16] W. Blaschke (1916): Kreis und Kugel. Leipzig.

[BHS93] B. D. Bojanov, H. A. Hakopian and A. A. Sahakian (1993): Spline functions and multivariate polynomial interpolations. Kluwer Academic Publishers.

[B73] C. de Boor (1973): *The quasi-interpolant as a tool in elememtary spline theory.* In: Approximation Theory(G. G. Lorentz *et al.*, eds). Academic Press (New York), pp. 269–276.

[B77] C. de Boor (1977): *Package for calculating with B-splines.* SIAM J. Numer. Anal., **14(3)**:441–472.

[B90] C. de Boor (1990): Spline Toolbox for use with matlab. The MathWorks Inc.

[Br72] J. Brink (1972): *Inequalities involving $\|f\|_p$ and $\|f^{(n)}\|_q$ for f with n zeros.* Pacific J. Math., **42**:289–311.

[CSV67] P. G. Ciarlet, M. H. Schultz and R. S. Varga (1967): *Numerical methods of high-order accuracy for nonlinear boundary value problems.* Numer. Math., **9**:394–430.

[Co71] W. A. Coppel (1971): Disconjugacy. Lecture Notes in Mathematics, Springer-Verlag.

[DV73] K. M. Das and A. S. Vatsala (1973): *On Green's function of an n-point boundary value problem.* Trans. Amer. Math. Soc., **182**:469–480.

[DL93] R. A. DeVore and G. G. Lorentz (1993): Constructive approximation. Springer-Verlag.

[FMP91] A. M. Fink, D. S. Mitrinović and J. E. Pečarić (1991): Inequalities involving functions and their integrals and derivatives. Kluwer Academic Publishers.

[Gu76] G. B. Gustafson (1976): *A Green's function convergence principle, with applications to computation and norm estimates.* Rocky Mountain J. Math., **6(3)**:457–492.

[HLP52] G. H. Hardy, J. E. Littlewood and G. Pólya (1952): Inequalities. 2nd ed., Cambridge.

[Ho93] G. Howell (1994): *Generalisations of the Cauchy remainder.* In: Open Problems in Approximation Theory(xxx, ed). Science Culture Technology Publ. (Singapore), pp. 91–98.

[Hu63] M. Hukuhara (1963): *On the zeros of solutions of linear ordinary differential equations.* Súgaku, **15**:108-109.

[Le63] A. Ju. Levin (1963): *Some problems bearing on the oscillation of solutions of linear differential equations.* Soviet Math. Dokl., **4**:121-124.

[LJR83] G. G. Lorentz, K. Jetter and S. D. Riemenschneider (1983): Birkhoff interpolation. Addison-Wesley.

[Ne64] Z. Nehari (1964): *On an inequality of P. R. Beesack.* Pacific J. Math., **14**:261–263.

[Po68] Yu. V. Pokornyi (1968): *Some estimates of the Green's function of a multipoint boundary value problem.* Math. Notes, **4**:810–814.

[Tu41] M. Tumura (1941): *Kôkai zyôbibunhôteisiki ni tuite.* Kansû Hôteisiki, **30**:20–35.

[Ul89] R. K. Uluchev (1989): *B-splines with Birkhoff knots, applications in the approximations and shape preserving interpolation.* Math. Balkanica, **3**:225–239.

[Wa94$_1$] S. Waldron (1994): *L_p-error bounds for Hermite interpolation.* CMS Tech. Rep. #94-13, Center for Math. Sciences, U. Wisconsin-Madison (available by anonymous ftp from `stolp.cs.wisc.edu`).

[Wa94$_2$] S. Waldron (1994): *Extremising the L_p-norm of a monic polynomial with roots in a given interval and Hermite interpolation.* submitted for publication.