## Bivariate truncated powers:
## Complete intersection decompositions and the spline representation

*Amos Ron*[*], *Shengnan (Sarah) Wang*[†]

ABSTRACT

Cone polynomials, also known as volume polynomials and/or spline polynomials, are the polynomials that appear in the local structure of the truncated powers, hence in the local structure of any derived construction such as box splines, simplex splines, character formulæ and moment maps.

We provide a fresh look at bivariate cone polynomials. Two main principles underlie our approach here. The first is that understanding the truncated powers does not necessarily require us to analyse as a whole the ideal of differential operators that defines the cone polynomials. Instead, we create a host of much simpler ideals, and analyse each of them separately, with each of them making a contribution of a single cone polynomial. This concept is formalized under the notion of Complete Intersection Decomposition (CID) of ideals. The second observation is that the coefficients of the cone polynomial in a suitable monomial representation are piecewise-analytic. In a sense, one can say that not only cone polynomials underlie the structure of truncated powers, but truncated powers underlie the structure of cone polynomials, too. This surprising cycle must go beyond piecewise-polynomials: the coefficients of cone *polynomials* are rarely piecewise-*polynomial*: in general they are only piecewise-exponential.

**AMS (MOS) Subject Classifications:** 41A15, 41A63, 52C35, 13A02, 13C40, 13F20, 13N10, 13P25, 05B20, 05B35, 39A12, 47F05, 47L20, 52B40.

**Key Words and Phrases:** Multivariate polynomials, bivariate polynomials, splines, box splines, truncated powers, partition functions, Weyl's character formula, moment maps, discrete splines, polynomial ideals, complete intersection ideals, complete intersection decompositions, volume polynomials, zonotopal algebra.

**Authors' affiliation and address:**

[*] Computer Sciences Department
University of Wisconsin-Madison
1210 West Dayton Street
Madison WI 53706
`amos@cs.wisc.edu`

[†] Morgridge Institute for Research
Madison WI 53706
`swang@morgridge.org`
*and*
Department of Electrical and Computer Engineering
University of Illinois - Urbana Champaign
Urbana IL 61801

This work was supported by the National Science Foundation under Grant DMS-1419103.

## 1. Introduction

Truncated powers, also known as partition functions, are piecewise-analytic "cone functions". They provide the core building block for a variety of constructions in analysis, algebra and geometry. Simplex splines and box splines are the main such constructs in approximation theory. Schur functions and their various generalizations are good examples in algebra. Moment maps are a typical example in geometry. One of the objectives of this article is to recast truncated powers in a language that is coherent for these seemingly incompatible application domains. We review some of the efforts in this area later on in this introduction.

In our attempt to achieve such coherence, we noticed that we are, quite accidentally, realizing another important goal: to explore the duality between geometry and algebra in spline approximation. We will explain this point later, but let us list it right now as a second objective.

It is an ambitious program, and it will be based on the introduction of a few new techniques and methodologies that are off the beaten road of analysis, and may not be a common theme in algebra, either. We begin in this paper with a rather modest objective: understanding *bivariate* truncated powers. It will enable us to introduce some of the general concepts in a rather friendly environment, where the geometry is simple to understand and easy to explore. With the exception of the notion of "flow" that may be bypassed in 2D, the concepts we introduce and analyse cover the core aspects of the novel approach, and we do it at a level that is beyond our present reach in higher dimensions.

Details aside, two main principles underlie our approach here. The first is that understanding truncated powers does not necessarily require us to analyse the complicated ideal of differential operators that define the "cone polynomials". We may, instead, create a host of much simpler ideals, and analyse each of them separately, with each of them making a contribution of a single cone polynomial. This concept is formalized under the notion of Complete Intersection Decomposition (CID) of ideals. The second major observation is that the coefficients of the cone polynomial in a suitable monomial representation are piecewise-analytic. In a slightly oversimplified language, one can say that cone polynomials underlie the structure of truncated powers, while truncated powers underlie the structure of cone polynomials. An important reservation, however, is that this cycle must go beyond piecewise-polynomials: the coefficients of cone *polynomials* are rarely piecewise-*polynomial*: in general they are piecewise-exponential.

### 1.1. Review of main results

The geometry of truncated powers in 2D is of a hyperplane arrangement $\mathcal{H}$ of $k+1$ lines in $\mathbb{R}^2$ that go through the origin. Let $x_0, \ldots, x_k$ be the directions of these lines, indexed consecutively, say counter-clockwise. Each line $x_j$ appears with some positive multiplicity $\mu(j)$. We record the above arrangement in a matrix

$$X,$$

where each column $x_j$, $j \in [0:k]$, appears $\mu(j)$ times. The matrix is thus of order $2 \times N$, with $N := \sum_{j=0}^{k} \mu(j)$. Now, let us consider $x_0, \dots, x_k$ as *rays* only (each generating the ray $x_j \mathrm{I\!R}_+$), and add their opposite rays separately: $x_{k+j+1} := -x_j$, $j \in [0:k]$. Choose then any consecutive $k+1$ rays $(x_j, \dots, x_{j+k})$ from $(x_0, \dots, x_{2k+1})$. Every such selection defines a $2 \times N$ matrix

$$X_j$$

where each $x_i \in [j : j+k]$ appears $\mu(i)$ times, a cone

$$C_j := \mathrm{pos}(X_j) := X_j(\mathrm{I\!R}_+^N),$$

and a piecewise-polynomial function, $\mathrm{TP}_j$, known as *truncated power* or *partition function*, supported on $C_j$, and coinciding on each basic cone $c_i := \mathrm{pos}(x_i, x_{i+1})$, $i \in [j : j+k-1]$, with some polynomial $Q_{j,i}$, homogeneous of degree $N-2$. These $k$ "cone polynomials" form a basis for a $k$-dimensional polynomial subspace

$$\mathrm{soc}(\mathcal{D}(X)) := \mathrm{span}\{Q_{j,i} : i \in [j : j+k-1]\}$$

of the $(N-1)$-dimensional space

$$\Pi^0_{N-2},$$

of bivariate homogeneous polynomials of degree $N-2$. The cone polynomial $Q_{j,i}$ depends on the choice of *both* $j$ and $i$. However, the space $\mathrm{soc}(\mathcal{D}(X))$ depends only on $X$: the span of $(Q_{j,i} : i \in [j : j+k-1])$ is independent of $j$.

The goal is to understand the nature of these cone polynomials and their inter-relationships. A prevailing approach, that works well in 2D, is to focus on the cross-a-line cone polynomials

$$Q_{j,i} - Q_{j,i-1}, \quad Q_{j,j-1} := Q_{j,j+k} := 0, \quad i \in [j : j+k] :$$

They capture the increment as we cross the line $x_i$, and sum up to zero. Up to normalization, they are independent of $j$. We denote

$$Q_i := Q_{i,i},$$

and refer to it the the $x_i$-cone polynomial.

In our approach, we encode the geometry of $\mathcal{H}$ in the algebraic structure of a space $S(F(X), \gamma)$ of discrete *univariate* splines with a single knot, and provide a solution to the problem in terms of those splines. The approach is also connected to decompositions of polynomial ideals in terms of complete intersection ideals, but we skip this part for the time being.

Clearly, polynomials in the space $\Pi^0_{N-2}$, the ambient space of $\mathrm{soc}(\mathcal{D}(X))$, should be indexed linearly. With[1]

$$[\widetilde{t}^{\alpha}] := \widetilde{t}^{\alpha}/\alpha! := \frac{\widetilde{t}(1)^{\alpha(1)}\,\widetilde{t}(2)^{\alpha(2)}}{\alpha(1)!\alpha(2)!},$$

[1] We reserve the notation $t = (t(1), t(2))$ for the standard polynomial variables in $\mathrm{I\!R}^2$, i.e., those that are bi-orthogonal to the standard basis of $\mathrm{I\!R}^2$. Since we work with a different basis for $\mathrm{I\!R}^2$, our variables $\widetilde{t} = (\widetilde{t}(1), \widetilde{t}(2))$ are bi-orthogonal to our chosen basis.

the normalized monomial, and $\Pi_m^0$ any homogeneous polynomial space (of degree $m$ in 2D), we order the monomials in $\Pi_m^0$ by $\alpha(1)$, and then index them by any set $I \subset \mathbb{Z}$ of $m+1$ consecutive integers:

$$I \ni i \ \leftrightarrow\ [\widetilde{t}^{(\alpha_I(i), m-\alpha_I(i))}], \quad \alpha_I(i) := i - \min(I).$$

Given thus any $f$, defined at least on $I$, we obtain in this way a homogeneous polynomial of degree $m := \#I - 1$:

$$\iota_I(f) := \sum_{i \in I} f(i) [\widetilde{t}^{(\alpha_I(i), m-\alpha_I(i))}].$$

In particular, $\iota_I$ induces a linear bijection $\iota_I : \mathbb{R}^I \to \Pi_m^0$: In this language, our goal is to understand the space

$$S(X) := S_I(X) := \iota_I^{-1}\big(\mathrm{soc}(\mathcal{D}(X))\big)$$

as a subspace of $\mathbb{R}^I$, with $I$ any set of consecutive $N-1$ integers.

Skipping a few additional technical details, our assertion is that $S(X)$ is a space of *discrete splines with one knot*. More specifically, we start by partitioning the domain $I$ into three intervals

$$I = [I_- : \gamma : I_+]: \quad \max(I_-) = \min(\gamma) - 1 \quad \text{and} \quad \max(\gamma) + 1 = \min(I_+).$$

Then we introduce a finite dimensional $F(X) \subset \mathbb{R}^{\mathbb{Z}}$ with the following properties:

(1) $\dim F(X) = k - 1$, and $(f_i)_{i=1}^{k-1}$ is some specific basis for it.

(2) The sum $f_0 := \sum_{i=1}^{k-1} f_i$ vanishes on $\gamma$, but vanishes identically on neither of $I_\pm$.

“defsgam **Definition 1.1.** *With the details of $F(X)$ as above, we define the spline space $S(F(X), \gamma)$ as the collection of functions in $f \in \mathbb{R}^{\mathbb{Z}}$ that satisfy the two properties:*

*(1)* $f_{|[\min(\gamma):\infty)} \in F(X)$.

*(2)* $f_{|(-\infty:\max(\gamma)]} \in F(X)$.

So, $S(F(X), \gamma)$ is piecewise in $F(X)$ with a single knot $\gamma$: the two pieces are glued at $\gamma$. We denote the splines in $S(F(X), \gamma)$ by $(f, g)_\gamma$, i.e.,

$$(f, g)_\gamma := \begin{cases} g, & \text{on } [\min(\gamma) : \infty), \\ f, & \text{on } (-\infty : \max(\gamma)]. \end{cases}$$

We will see later that $\dim(S(F(X), \gamma)) = k$. A basis for $S(F(X), \gamma)$ can be obtained by appending to $(f_1, \ldots, f_{k-1})$ the spline $(0, f_0)_\gamma$, or any other function in $S(F(X), \gamma) \backslash F(X)$.

Now, we have omitted (for the time being) the requisite details on: (1) How the partitioning of $I$ into the triple interval set $I_-, \gamma, I_+$ is done, (2) What space $F(X)$ is selected, and (3) What is the basis for $F(X)$, $(f_i)_i$, that is selected here. As one should expect, these details depend on the directions of the lines in $\mathcal{H}$ and their multiplicities: in fact, for $i \in [1 : k-1]$, the function $f_i$ corresponds to the line $x_i$ in $\mathcal{H}$:

$$\iota_I(f_i) = Q_i, \quad i \in [1 : k-1].$$

The remaining two lines are used as our basis for $\mathbb{R}^2$. However, while the details of $S(F(X), \gamma)$ are tied to $\mathcal{H}$, the truncated power function is universal in terms of the spline space components:

“mainthm **Theorem 1.2.** *With* $\iota := \iota_I$ *considered as a map*

$$\iota : \mathbb{R}^{\mathbb{Z}} \to \Pi^0_{N-2}[\widetilde{t}],$$

*and with*

$$f_{j,i} := \sum_{m=j}^{i} f_m, \quad j \le i,$$

*we have:*

*(1) The cone polynomials* $(Q_{j,i} : j \in [1:k])$ *are the following:*

$$Q_{j,i} = \begin{cases} \iota(f_{j,i}), & j \le i < k, \\ \iota(-f_{i-k,j-1}), & k < i < j+k, \\ \iota((f_{j,k-1} - f_0, f_{j,k-1})_\gamma), & i = k. \end{cases}$$

*(2) The cone polynomials* $(Q_{0,i})_i$ *are the following:*

$$Q_{0,i} = \begin{cases} \iota((f_{1,i}, f_{1,i} - f_0)_\gamma), & 0 < i < k, \\ \iota((0, -f_0)_\gamma), & i = 0. \end{cases}$$

□

The rule of the theorem is very simple: whatever truncated power we access, crossing the line $x_i$, $0 < i < k$, while moving clockwise means that we subtract $f_i$ from the previous cone function:

$$Q_{j,i} - Q_{j,i-1} = \iota(f_i).$$

The only exception is when $i = 0, k$: when we cross $x_0$ clockwise we add $(0, f_0)_\gamma$:

$$Q_{j,1} - Q_{j,0} = -\iota((0, f_0)_\gamma).$$

When we cross $x_k$ clockwise we add $(f_0, 0)_\gamma$:

$$Q_{j,k} - Q_{j,k-1} = -\iota((f_0, 0)_\gamma).$$

So, the above identifies the $x_i$-cone polynomials in terms of their representation in the spline space, and shows that their definition is independent of the choice of the ambient cone $C_j$. The spline space clearly records the normalizations correctly since

$$\sum_{i=1}^{k-1} f_i - (f_0, 0)_\gamma - (0, f_0)_\gamma = f_0 - f_0 = 0.$$

Figure 1 provides an illustration when $k = 4$. As said, these rules are universal, and are independent of the geometry: the latter is encoded in the details of $S(F(X), \gamma)$ but not in the conversion of the discrete splines to the cone polynomials.

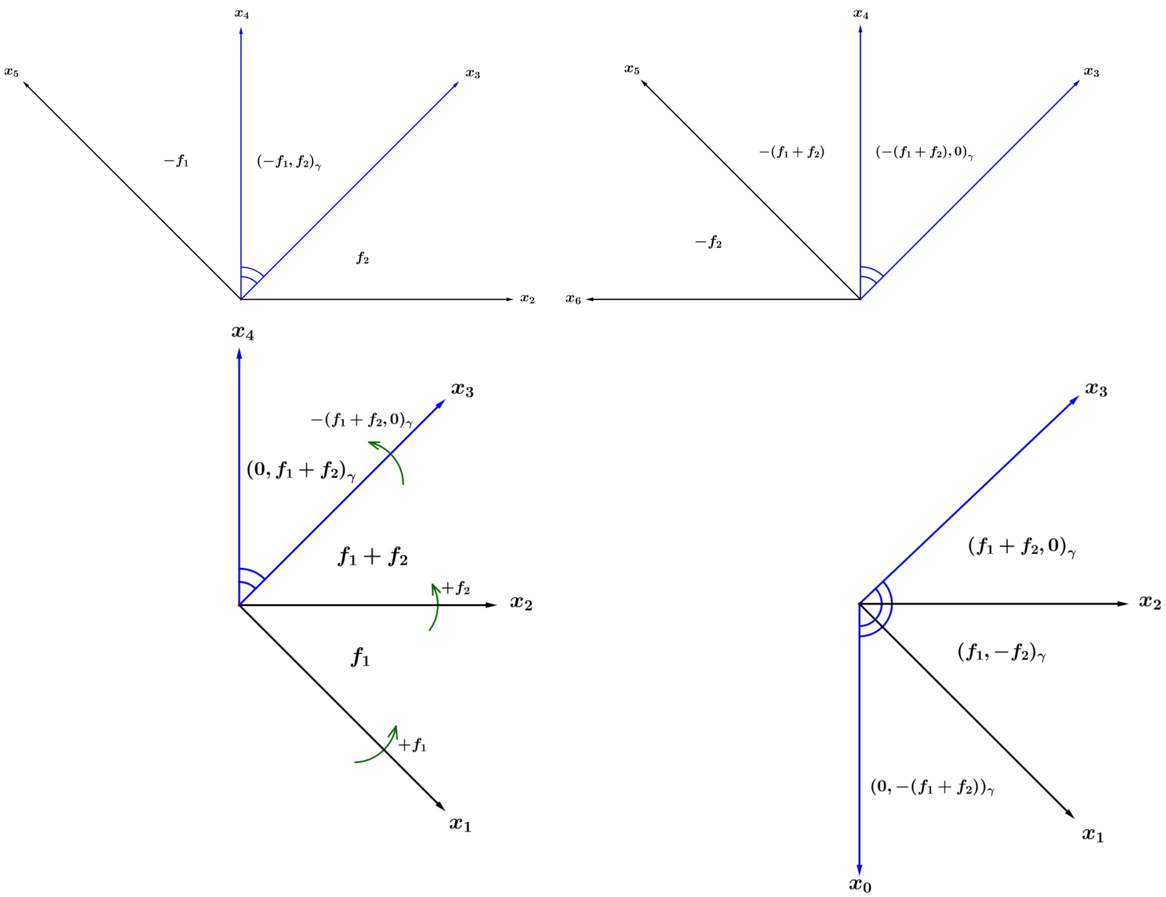


**Figure 1**: The four truncated powers that contain the ray $x_3$ in their support. The different cone functions are represented by the corresponding spline sequence.

### 1.2. The spline space $S(F(X), \gamma)$

Let us now describe how the spline space $S(F(X), \gamma)$ is constructed. As our reader may have observed, we select first two rays as our basis, or stabilizer, $B$. We selected in the above

$$B := (x_0, x_k).$$

We then define, for $i \in [1 : k-1]$,

$$\lambda_i := -(B^{-1}x_i)(2)/(B^{-1}x_i)(1),$$

and consider the univariate sequence space

$$F_i := \{\langle \lambda_i \rangle f\}, \quad f \in \pi_{<\mu(i)},$$

with $\pi_{<n}$ the univariate polynomials of degree $< n$, and with $\langle \lambda \rangle$ the exponential sequence

$$\langle \lambda \rangle : s \mapsto \lambda^s.$$

Now, with

$$N' := \sum_{i=1}^{k-1} \mu(i),$$

and with $\gamma$ any set of $N'-1$ consecutive integers, there is a unique sequence (up to normalization)

$$f_0 = \sum_{i=1}^{k-1} f_i, \quad f_i \in F_i,$$

that vanishes on $\gamma$. We define then

$$F(X) := \operatorname{span}\{f_i : i \in [1 : k-1]\}.$$

Next, we denote by

$$I_- \text{ (resp., } I_+)$$

the immediate $\mu(0)$ (resp., $\mu(k)$) integers to the left (resp., right) of $\gamma$, and define $I$ as the concatenation of $I_-, \gamma, I_+$. Then

$$\#I = \sum_{i=0}^{k} \mu(i) - 1 = N - 1 = \dim(\Pi^0_{N-2}).$$

The polynomial variables $\widetilde{t}(1), \widetilde{t}(2)$ in the monomial basis for

$$\Pi := \mathrm{IR}[\widetilde{t}(1), \widetilde{t}(2)]$$

are the rows of $B^{-1}$. The only real computation to be done, thus, is to compute the spectral functions $(f_i)_i$ above, which is the classical problem of finding the fundamental solution of a difference operator with constant coefficients (the difference operator annihilates $\sum_i F_i$, and the function $(0, f_0)_\gamma$ is a fundamental solution of it; we have avoided the 'bias' between using forward differencing vs. the backward one by not specifying 'where $\gamma$ is').

One may notice that $S(F(X), \gamma)$ is independent of the multiplicities $\mu(0)$ and $\mu(k)$: only when the interval $I$ is constructed, these multiplicities play a role. So, for example, the sequence $f_1$ is independent of $\mu(0), \mu(k)$, and hence "$\iota(f_1)$ is an element of $\operatorname{soc}(\mathcal{D}(X))$, whatever $\mu(0), \mu(k)$ are." However, without knowing $I$, we do not know what values of $f_1$ to collect, and what is the degree of the homogeneous polynomials that we represent. Here is a more detailed discussion.

**Discussion.** How to determine the cone polynomial $\iota(f_i)$ for different multiplicities of $\mu(0), \mu(k)$? The following recursion is easy to perform. Suppose that for some multiplicities $\mu(0), \mu(k)$ we already calculated that

$$\iota(f_1) = \sum_{s=s_0}^{s_1} f_1(s)[\widetilde{t}^{(s-s_0, s_1-s)}],$$

with $s_0, s_1$ some integers. Now,

(1) If we increase $\mu(0)$ by 1, the modified cone polynomial is

$$\iota(f_1) = \sum_{s=s_0-1}^{s_1} f_1(s)[\widetilde{t}^{(s+1-s_0,s_1-s)}].$$

(2) If we increase $\mu(k)$ by 1, the modified cone polynomial is

$$\iota(f_1) = \sum_{s=s_0}^{s_1+1} f_1(s)[\widetilde{t}^{(s-s_0,s_1+1-s)}].$$

So, we just take the previous coefficients, sample $f_1$ at one additional neighboring point, and align the new coefficients against the extended set of monomials.

**Discussion: geometry translated to algebra**. The process we described above can be fairly understood as "affinization": homogeneous polynomials in 2D are "just a finite interval of integer points in $\mathbb{R}^2$", so there should be some univariate theory that captures them, right? But such theory must encode not only the homogeneous polynomials, but also the geometric environment. So, the easy part is to replace $\Pi^0_{N-2}$ by $\mathbb{R}^I$. The more challenging part was to embed the geometry of the hyperplane arrangement $\mathcal{H}$ into $\mathbb{R}^I$: this is the spline space $S(F(X), \gamma)$. The exponent $\lambda_i$ records the direction of $x_i$ (viz., the polynomial $\iota(\langle\lambda_i\rangle)$ vanishes to the maximal $N-2$ degree on $x_i$) while the degree of the polynomials in $F_i$ (hence in $f_i$) records the multiplicity $\mu(i)$ (viz., the polynomial $\iota(\langle\lambda_i\rangle f)$ vanishes on $x_i$ only to order $N-2-\deg f$, while no polynomial in $\mathrm{soc}(\mathcal{D}(X))$ can vanish on $x_i$ to an order larger than $N-1-\mu(i)$).

### 1.3. Example: four-direction mesh

We illustrate the general discussion with a detailed analysis of the four-direction mesh. We first treat the case $(2,2,2,2)$ by choosing *some* basis. We then show how the choice of a better basis can simplify the computations, and treat the case $(m, l, m, l')$.

#### 1.3.1 The case $(2,2,2,2)$

**Four-direction mesh** means that we have four lines, i.e., $k = 3$. It means more, since there is a standard choice of these lines, viz., $x_1 = (1,-1)$, $x_2 = (1,0)$, $x_3 = (1,1)$, and $x_4 = (0,1)$. We assume in this example that $\mu(i) = 2$, $\forall i$. Thus, the cone polynomials are of degree 6:

$$\mathrm{soc}(\mathcal{D}(X)) \subset \Pi^0_6, \quad \dim \mathrm{soc}(\mathcal{D}(X)) = 3.$$

Our basis is

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

hence

$$B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

which means that our variables are $(\widetilde{t}(1), \widetilde{t}(2)) = (t(1), t(2) - t(1))$. Also, $\lambda_1 = 2$ and $\lambda_2 = 1$. $\gamma \subset \mathbb{Z}$ is any set of three consecutive integers. Choosing $\gamma := [-1 : 1]$, we have $I_- = [-3, -2]$, $I_+ = [2, 3]$, hence $I = [-3 : 3]$. One then computes that

$$f_1 : s \mapsto (s-3)2^s,$$

while

$$f_2 : s \mapsto s + 3.$$

(In order to verify that the above $f_1, f_2$ are correct, one only needs to observe that, for $i = 1, 2$, $f_i = \langle \lambda_i \rangle p_i$, with $p_i$ a linear polynomial, and to check that $f_0 := f_1 + f_2$ vanishes on $\gamma$.) A basis for $S(F(X), \gamma)$ is given by the splines

$$(0, f_0)_\gamma, (-f_0, 0)_\gamma, (-f_1, f_2)_\gamma,$$

which are three fundamental solutions of the underlying 4th-order difference operator, and, more importantly, are mapped by $\iota_I$ to the three cone polynomials in $c_3$. The spline $(0, f_0)_\gamma$ is supported on $I_+$ and assumes the values $(1, 6)$ there. It generates the cone polynomial

$$[\widetilde{t}^{(5,1)}] + 6[\widetilde{t}^{(6,0)}] = \frac{t(1)^5(t(2) - t(1))}{5!} + 6\frac{t(1)^6}{6!} = [t^{(5,1)}].$$

One computes that $\iota(f_2) = [t^{(1,5)}]$. The spline $(-f_0, 0)_\gamma$ is supported on $I_-$ and assumes the values $(3/4, 1/4)$ there. It corresponds to the cone polynomial

$$\frac{1}{4}(3[\widetilde{t}^{(0,6)}]+[\widetilde{t}^{(1,5)}]) = \frac{1}{4}(3\frac{(t(2) - t(1))^6}{6!}+\frac{t(1)(t(2) - t(1))^5}{5!}) = \frac{1}{8}(t(1)+t(2))\frac{(t(2) - t(1))^5}{5!}.$$

Finally, $\iota(f_1) = \iota(f_0 - f_2) = \iota((0, f_0)_\gamma) + \iota((f_0, 0)_\gamma) - \iota(f_2)$:

$$\iota(f_1) = \frac{1}{8}((t(1) - t(2))\frac{(t(2) + t(1))^5}{5!}.$$

### 1.3.2 The case $(m+1, l, m+1, l')$

We now choose the standard basis, $B = (x_2, x_4)$, as our basis, hence our variables are the standard $t = (t(1), t(2))$. Here, $\lambda_1 = 1$, while $\lambda_3 = -1$. We will use thus the notation $f_3$ instead of $f_2$. Assuming that $\mu(1) = \mu(3) = m + 1 > 0$, we select $\gamma := [-m : m]$, and define a polynomial $f \in \pi_m$ by

$$f(s) := \prod_{i=1}^{m}(s + m + 1 - 2i).$$

Then

$$f_1 = f, \quad f_3 = (-1)^{m+1}\langle -1 \rangle f, \quad f_0 = f_1 + f_3.$$

Since obviously $f_i \in F_i$, $i = 1, 3$, one only needs to check that $f_0$ vanishes on $\gamma$, for verification.

Now, denote $\mu(4) =: l'$. Then $\iota_I((0, f_0)_\gamma)$ (which is $Q_4$, corresponding to crossing $x_4 = e_2$) is obtained by evaluating $f_0$ at $[m+1 : m+l']$, and aligning those values against the *last* $l'$ monomials of $\Pi^0_{N-2}$. Note that $f_0$ vanishes on $m + 2\mathbb{Z}$, while on $m + 1 + 2\mathbb{Z}$ it assumes the values $2f$. In conclusion, the formal series

$$\sum_{i=0}^{\infty} 2f(m+1+2i)[t^{\alpha+(2i,-2i)}]$$

captures *all* the possible 'crossing-$e_2$' polynomials that exist for the case $\mu(1) = \mu(3) = m+1$. One needs to know what the initial $\alpha$ is, and then to truncate the series once negative powers of $t(2)$ appear. One calculates that $\alpha(1) = 2m + l + 1$. Since $f(m+1+2i) = 2^m m! \binom{m+i}{i}$, we conclude that the series is

$$2^{m+1} m! \sum_{i=0}^{\infty} \binom{m+i}{i} [t^{(2(m+i)+l+1, l'-1-2i)}].$$

If $l' = 1$ or $l' = 2$, there is only one summand:

$$\iota_I((0, f_0)_\gamma) = 2^{m+1} m! [t^{(2m+l+1, l'-1)}].$$

Thus, ignoring (the important) normalization, the crossing function of $e_2$ for $l' = 2$ is then the product of

(1) The linear polynomial the vanishes on $e_2$ (viz., $t(1)$) to the power $N - 3$.
(2) The linear polynomial in the direction of $e_2$, (viz., $t(2)$).

The conclusion required two conditions: $\mu(1) = \mu(3)$, and $l' = 2$. By symmetry, we can extend it to the case $\mu(2) = \mu(4)$, and $\mu(1) = 2$. In that case, with $B = (x_1, x_3)$,

"deftilt (1.3)
$$\widetilde{t}(1) = (t(1) - t(2))/2, \quad \widetilde{t}(2) = (t(1) + t(2))/2.$$

"propfourdir **Proposition 1.4.**

*(1) Assume that the 4-dir multiplicities are* $(m+1, l, m+1, 2)$. *Then the cone polynomial associated with crossing* $x_4 = e_2$ *is*

$$2^{m+1} m! [t^{(2m+l+1, 1)}].$$

*(2) Assume that the 4-dir multiplicities are* $(l, m+1, 2, m+1)$. *Then the cone polynomial associated with crossing* $x_3 = e_1 + e_3$ *is*

$$2^{m+1} m! [\widetilde{t}^{(2m+l+1, 1)}],$$

*with* $\widetilde{t}$ *as in (1.3).*

**Comments.** The observation clearly does not extend to $l > 2$: only in the cases $l = 1, 2$ the spline $(0, f_0)_\gamma$ has a 1-point support! Also, we allowed $f$ in the proposition to be normalized arbitrarily. However, the normalization of $f$ determines the correct normalization of the functions $f_1, f_2 \in S(F(X), \gamma)$, hence determines the correct normalization of all the other "cross-a-line" cone functions.

### 1.4. Literature

Truncated powers as a theme in univariate approximation theory has a rich history; cf. [1] and its references for details. Multivariate truncated powers underlie Weyl's character formulæ, [28], hence are almost one century old. These character formulæ involve the *discrete* version, and only special configurations (root systems). In approximation theory, multivariate truncated powers appear, perhaps, for the first time in [12], in the effort to provide a computational framework for *simplex splines*, [19]. The books [5],[20] contain, each, substantial expositions on splines, with the latter having detailed analysis of truncated powers. The connection between truncated powers and certain problems in algebra and combinatorics is studied in [30].

We forgo general review of multivariate truncated powers, since this article deals with the bivariate case only. We also forgo the extensive literature on piecewise-polynomials on arbitrary meshes, whose focus is elsewhere. Bivariate truncated powers are relevant to the study of bivariate splines on *regular* meshes; cf. [2],[4],[9],[10],[15],[22],[24],[25]. The complete structure of *discrete* bivariate truncated powers appears in [29], albeit only for simple multiplicities. The paper [26] contains explicit formulaæ for the same case studied here, but uses a different approach, resulting in a very different representation.

## 2. Bivariate truncated powers

### 2.1. Cone polynomials

Let

$$\{x_0, \ldots, x_{2k+1}\} \subset \mathrm{I\!R}^2 \backslash 0$$

be a set of vectors that satisfy $x_{i+k+1} = -x_i$, $i \in [0:k]$. Each vector $x_i$ is associated with positive multiplicity $\mu(i)$, where $\mu(i) = \mu(i+k+1)$, $i \in [0:k]$. Let

$$X_j, \quad j \in [0:k],$$

be a matrix whose columns are comprised of the vectors $(x_j, \ldots, x_{j+k})$, each appearing according to its multiplicity. The matrix is thus $2 \times N$, with $N := \sum_{i=0}^{k} \mu(i)$. In this subsection we assume that the vectors are ordered consecutively counter-clockwise. It then follows that each $X_j$ satisfies the acyclicity assumption

$$X_j(\mathrm{I\!R}_+^N) =: C_j \neq \mathrm{I\!R}^2.$$

“truncatedpower **Definition 2.1.** *The truncated power* $\mathrm{TP}_j$ *is defined by*

$$\mathrm{TP}_j(t) := \mathrm{TP}_{X_j}(t) := \mathrm{vol}(\mathrm{I\!R}_+^N \cap X_j^{-1} t), \quad t \in \mathrm{I\!R}^2,$$

*with the volume* 'vol' *being with respect to the Lebesgue measure in* $\mathrm{I\!R}^{N-2}$. *The exact way such measure is normalized on* $\ker X_j$ *is not important to us, so* $\mathrm{TP}_j$ *is defined here up to normalization by a non-zero constant.*

Now, each $\mathrm{TP}_j$ is a homogeneous piecewise-polynomial of degree $N-2$, with support

$$\mathrm{supp}\, \mathrm{TP}_j = C_j.$$

The cones $c_i := \mathrm{pos}(x_i, x_{i+1})$, $i = j, \ldots, j+k-1$, are the cones of polynomiality of $\mathrm{TP}_j$. There are $k$ such cones, thus, for each $\mathrm{TP}_j$.

In the discussion below, we let

$$X := X_0.$$

It is easily observed, however, that the definitions and the results do not depend on the choice of $X_j$. For $x \in \mathbb{R}^2 \backslash 0$, let

$$p_x : t \mapsto x \cdot t.$$

Set

$$P_i := p_{x_i}^{\mu(i)},$$

and

$$P := P_X := \prod_{i=0}^{k} P_i.$$

Each $\mathrm{TP}_j$ is a fundamental solution (up to normalization) of the $N$th-order differential operator

$$P_X(D).$$

It is easily observed that, with

$$\mathcal{D}(X) := \{q \in \Pi : (P/P_i)(D)q = 0, \quad i \in [0:k]\},$$

each $\mathrm{TP}_j$ is piecewise in $\mathcal{D}(X)$. We define

$$\mathrm{soc}(\mathcal{D}(X)) := \mathcal{D}(X) \cap \Pi^0_{N-2},$$

and refer to this space as the *socle* of $\mathcal{D}(X)$. Thus, each $\mathrm{TP}_j$ is actually piecewise in $\mathrm{soc}(\mathcal{D}(X))$. It is classically known that the different polynomials from the different cones of any *fixed* $X_j$ form a *basis* for $\mathrm{soc}(\mathcal{D}(X))$, [20]. In particular,

$$\dim(\mathrm{soc}(\mathcal{D}(X))) = k.$$

Our primary goal is to a build a basis for the above socle space. While $\mathcal{D}(X)$-spaces in higher dimension are very involved and building a useful basis for their socle is highly non-trivial, it is not hard to build in 2D *some* basis for $\mathrm{soc}(\mathcal{D}(X))$ (cf. [26], for example). So, it is really the nature of the construction and the details of the basis that matter here.

## 2.2. Complete intersection ideals

In the discussion of this subsection, we do not assume any more that the lines $(x_0, \ldots, x_k)$ are ordered in any way. They are any (pairwise different) lines with the positive multiplicities $(\mu(i))_i$.

The polynomials $P/P_i$, $i \in [0:k]$, generate a polynomial ideal that is denoted by

$$\mathcal{J}(X).$$

Thus, $\mathcal{D}(X)$ can be equivalently defined as the *kernel* of the ideal $\mathcal{J}(X)$:

$$\mathcal{D}(X) = \mathcal{J}(X)\perp := \{q \in \Pi : p(D)q = 0,\ p \in \mathcal{J}(X)\}.$$

The hope is then that some structural properties of the ideal $\mathcal{J}(X)$ will shed some light on the nature of the polynomials in $\mathcal{D}(X)$. In this article we realize this goal via a decomposition of $\mathcal{J}(X)$ with the aid of a very special type, and much simpler, ideal: complete intersection.

A complete intersection ideal in 2D is an ideal that has only two generators $\rho_1, \rho_2$ with only trivial relations, i.e., if $p\rho_1 + q\rho_2 = 0$, for some $p, q \in \Pi$, then $\rho_2|p$ and $\rho_1|q$. If $\rho_1$ and $\rho_2$ are products of homogeneous linear forms, it is equivalent to requiring that no linear form divides both $\rho_1$ and $\rho_2$. Thus, once we require that 0 is the only common zero of $\rho_1, \rho_2$, and require further that each generator $\rho_i$, $i = 1, 2$, is a product of linear forms, the ideal

$$J := \mathrm{Ideal}(\rho_1, \rho_2)$$

is complete intersection. In such a case, there is (up to normalization) a unique homogeneous polynomial $q_J \in J\perp$ of maximal degree

$$\deg(\rho_1) + \deg(\rho_2) - 2.$$

For completeness, we provide a proof of this assertion in §4, using the "zonotopality" of ideals of this type.

We deduce thus the following:

“socleone **Proposition 2.2.** *Let $(K, K')$ be a non-trivial partition of $[0:k]$. Define*

$$P_K := \prod_{i \in K} P_i,$$

*and*

$$\mathcal{J}_K := \mathrm{Ideal}(P_K, P_{K'}).$$

*Then:*

*(1) $\mathcal{J}(X) \subset \mathcal{J}_K$, hence $\mathcal{J}_K\perp \subset \mathcal{D}(X)$.*

*(2) $\mathcal{J}_K$ is complete intersection. The single (up to normalization) homogeneous polynomial $q_K$ of top degree in $\mathcal{J}_K\perp$ is of degree $N-2$, hence*

$$q_K \in \mathrm{soc}(\mathcal{D}(X)).$$

**Proof:** Given a generator $P_X/P_i$ of $\mathcal{J}(X)$, we assume without loss that $i \in K'$. Then $P_K$ divides $P_X/P_i$, hence $P_X/P_i \in \mathcal{J}_K$, and (1) follows, while (2) follows directly from the discussion that precedes this proposition. □

**Comments.** The ideal $\mathcal{J}(X)$ has $k+1$ generators, $P/P_i$, $i \in [0:k]$, each of degree $N-\mu(i)$. So the sum of the degrees of all the generators is $kN$. $\mathcal{J}_K$ has two generators, with degree sum $N$. Nonetheless, the kernel of this much larger ideal is not "too small": it does make a contribution to $\mathrm{soc}(\mathcal{D}(X))$. Since we have about $2^k$ polynomials of the form $q_K$, while $\dim \mathrm{soc}(\mathcal{D}(X)) = k$, it comes at little surprise that the polynomials $(q_K)_K$ span $\mathrm{soc}(\mathcal{D}(X))$.

Of particular interest for us is the singleton selection $K := \{i\}$, $i \in [0:k]$. We denote then

"defqi (2.3)
$$\mathcal{J}_i := \mathcal{J}_K, \quad q_i := q_K.$$

The requirement $P_i(D)q_i = 0$, coupled with the fact that $\deg P_i = \mu(i)$, while $\deg q_i = N-2$, entails that $q_i$ must vanish to degree $N-\mu(i)-1$ on the line $x_i$. In fact, $\mathcal{J}_i\perp$ contains *all* the polynomials in $\mathcal{D}(X)$ that vanish to that order on $x_i$.

A *Complete Intersection Decomposition (CID)* of $\mathcal{J}(X)$ is defined in our context as the writing

"cid (2.4)
$$\mathcal{J}(X) = \cap_{K\in\mathcal{K}} \mathcal{J}_K,$$

with the intersection running over $\mathcal{K} \subset 2^{[0:k]}$. The CID identity (2.4) is equivalent to

"cidspan (2.5)
$$\mathrm{span}\{q_K : K \in \mathcal{K}\} = \mathrm{soc}(\mathcal{D}(X)),$$

so the minimal number of components in (2.4) is $k$, and in that case $\{q_K : K \in \mathcal{K}\}$ is a basis for $\mathrm{soc}(\mathcal{D}(X))$. For all practical purposes, establishing (2.4) is done by verifying (2.5) and not the other way around.

### 2.3. The spline calculus

We choose now our variables $\widetilde{t} = (\widetilde{t}(1), \widetilde{t}(2))$ to be bi-orthogonal to $x_0, x_k$,[2] i.e., $p_{x_0}(D)(\widetilde{t}(1), \widetilde{t}(2)) = (1,0)$, and $p_{x_k}(D)(\widetilde{t}(1), \widetilde{t}(2)) = (0,1)$. Our goal is to find explicitly the coefficients of $q_K$ in the spline representation $S(F(X),\gamma)$ of

$$\Pi^0_{N-2}[\widetilde{t}].$$

First, let's recall our spline spaces. For each $x_i$, $i \in [1:k-1]$, we write $x_i = a(i)x_0 - b(i)x_k$, and define

$$\lambda_i := b(i)/a(i).$$

Recall from §1 the maps of the form $\iota_I$, with $I \subset \mathbb{Z}$ made of consecutive integers. Each such map maps $\mathbb{R}^{\mathbb{Z}}$ onto $\Pi^0_{\#I-1}$. Recall also the notation

$$F_i \subset \mathbb{R}^{\mathbb{Z}}$$

for the space of all sequences of the form $\langle\lambda_i\rangle f$, $f \in \pi_{<\mu(i)}$.

[2] Since the lines in $\mathcal{H}$ are not assumed to be ordered, $x_0$ and $x_k$ are actually any two lines in the arrangement.

“lemtwo **Lemma 2.6.** *Let $I$ be any subset of $\mathbb{Z}$ of consecutive integers. Then:*
*(1)* $p_{x_0}(D)\iota_I = \iota_{I'}$, *with* $I' = I\backslash\{\min(I)\}$.
*(2)* $p_{x_k}(D)\iota_I = \iota_{I'}$, *with* $I' = I\backslash\{\max(I)\}$.

**Proof:** $p_{x_0}(D)$ is partial differentiation with respect to $\widetilde{t}(1)$, hence annihilates the *first* monomial in our ordering of the monomial basis for any $\Pi^0_m$, $m := \#I - 1$. Since the monomials are normalized, $p_{x_0}(D)$ maps the other monomials in the monomial basis for $\Pi^0_m$ bijectively and in order-preserving onto the monomial basis of $\Pi^0_{m-1}$. The first claim then easily follows from these observations, while the second claim follows by a similar argument, since $p_{x_k}(D)$ is partial differentiation with respect to $\widetilde{t}(2)$. □

“lemone **Lemma 2.7.** *Let $i \in [1 : k-1]$. Let $\langle\lambda_i\rangle f \in F_i$. Let $I$ be any subset of $\mathbb{Z}$ of consecutive integers. Then $P_i(D)\iota_I(\langle\lambda_i\rangle f) = 0$.*

**Proof:** Up to normalization, $p_{x_i}(D)([\widetilde{t^\alpha}]) = [\widetilde{t^{(\alpha-e_1)}}] - \lambda_i[\widetilde{t^{(\alpha-e_2)}}]$, with obvious modifications if $\alpha(1)\alpha(2) = 0$. Thus

$$p_{x_i}(D)\sum_{j=0}^{m} c(j)[\widetilde{t^{(j,m-j)}}] = \sum_{j=1}^{m}(c(j) - \lambda_i c(j-1))[\widetilde{t^{(j-1,m-j)}}].$$

So, $\iota^{-1}p_{x_i}(D)\iota$ is degree reducing on $\langle\lambda_i\rangle f$: by removing one of the endpoints of $I$, one obtains a subset $I' \subset I$ such that

$$p_{x_i}(D)\iota_I(\langle\lambda_i\rangle f) = \iota_{I'}(\langle\lambda_i\rangle g),$$

with $\deg g < \deg f$. The claim now follows, since $\deg P_i = \mu(i)$, while $\deg f < \mu(i)$. □

In what follows, we select $\gamma \subset \mathbb{Z}$ to be any sequence of $N' - 1$ consecutive integers, with

$$N' := \sum_{i=1}^{k-1}\mu(i) = \sum_{i=1}^{k-1}\dim(F_i).$$

Then there exists a non-zero $f_0 \in \mathbb{R}^{\mathbb{Z}}$ of the form

$$f_0 = \sum_{i=1}^{k-1} f_i, \quad f_i \in F_i, \tag{2.8}$$

“deffzero

that vanishes on $\gamma$.

“lemthree **Lemma 2.9.** *If $f_0 \in \sum_{i=1}^{k-1} F_i$, and $f_0$ vanishes on a set of $N'$ consecutive integers, then $f_0 = 0$.*

**Proof:** Assume that $f_0 = \sum_{i=1}^{k-1} f_i$, $f_i \in F_i$, and $f_0$ vanishes on $[1 : N']$. For each $i$, we can find a difference operator $\nabla_i$ of order $N' - \mu(i)$ that annihilates each $F_j$, $j \neq i$, and is injective on $F_i$. Then $\nabla_i f_0 = \nabla_i f_i \in F_i$ is a sequence that vanishes at $\mu(i)$ points, which implies $\nabla_i f_i = 0$, hence that $f_i = 0$. □

"cortwo **Corollary 2.10.** *With $\gamma$ and $f_0$ as in (2.8), $f_0$ is unique up to normalization.*

**Proof:** If $f_0$ and $g_0$ vanish both on $\gamma$, and $f_0 \notin \mathrm{span}\{g_0\}$, then there exist $h_0 \in \mathrm{span}\{f_0, g_0\}\backslash 0$ which vanishes on a set of consecutive $N'$ integers. □

So, fixing $\gamma$ as above, we obtain from the discussion a uniquely defined

$$F(X) := \mathrm{span}\{f_i\}_{i=1}^{k-1},$$

and that completes the definition of our spline space

$$S(F(X), \gamma).$$

Moreover, let $I_+$ be the first $\mu(k)$ integers to the right of $\gamma$ and $I_-$ the first $\mu(0)$ integers to the left of $\gamma$. We concatenate these three sets into

$$I := [I_- : \gamma : I_+].$$

Since $\#I = N' - 1 + \mu(0) + \mu(k) = N - 1$, we can use $I$ to represent $\Pi^0_{N-2}$ via $\iota_I$.

"theoremone **Theorem 2.11.** *Let $K, K'$ be a non-trivial partition of $[0{:}k]$, $0 \in K'$. Let $\mathcal{J}_K$ be the corresponding complete intersection ideal, and let $q_K \in \Pi^0_{N-2} \cap \mathcal{J}_K\perp$ be the unique socle polynomial of $\mathcal{J}_K$. Then, with $\iota := \iota_I$:*
*(1) If $k \in K'$,*

$$q_K = \iota(f_K), \quad f_K := \sum_{i \in K} f_i.$$

*(2) If $k \in K$,*

$$q_K = \iota((f_K, -f_{K'})_\gamma), \quad f_K := \sum_{i \in K\backslash k} f_i, \quad f_{K'} := \sum_{i \in K'\backslash 0} f_i = f_0 - f_K.$$

**Proof:** (1) By Lemma 2.7, $P_i(D)\iota_I(f_i) = 0$, hence $P_K(D)\iota_I(f_K) = 0$, too. Now, by Lemma 2.6, $P_0(D)P_k(D)\iota_I(f_K) = \iota_\gamma(f_K)$. However, on $\gamma$, $f_0 = 0$, hence, $\iota_\gamma(f_K) = \iota_\gamma(f_K - f_0) = -\iota(f_{K'})$, with $f_{K'} := \sum_{i \in K'\backslash\{0,k\}} f_i$. Thus

$$P_{K'}(D)\iota_I(f_K) = -P_{K'\backslash\{0,k\}}(D)(\iota_\gamma(f_{K'})) = 0,$$

with the last equality following from another application of Lemma 2.7.

(2) By Lemma 2.6, $P_k(D)\iota_I((f_K, -f_{K'})_\gamma) = \iota_{I\backslash I_+}((f_K, -f_{K'})_\gamma)$. However, on $I\backslash I_+$ the spline $(f_K, -f_{K'})_\gamma$ coincides with $f_K$, hence

$$P_k(D)\iota_I((f_K, -f_{K'})_\gamma) = \iota_{I\backslash I_+}(f_K).$$

But then, by Lemma 2.7, $(P_K/P_k)(D)\iota_{I'}(f_K) = 0$, for any interval $I'$. This proves that $\iota_I((f_K, -f_{K'})_\gamma)$ is annihilated by $P_K(D)$, and an analogous argument shows that it is also annihilated by $P_{K'}(D)$. □

In fact, the statement of the theorem above is simpler to write in term of the "cross-a-line" polynomials $q_i = q_{K_i}$, i.e., those that correspond to $K_i = \{i\}$, $i \in [0:k]$.

"simcor **Corollary 2.12.**

"useful
$$q_K = \sum_{i \in K} q_i. \tag{2.13}$$

**Proof:** Theorem 2.11 provides explicit formulæ for each $q_K$ and in particular for $q_i$. One just verifies directly from these formulæ that (2.13) is valid. □

**Comment.** The ideal $\mathcal{J}_K$ is based on the partition $K, K'$, hence, with $K'$ the complement $[0:k]\backslash K$, the two ideals $\mathcal{J}_K$ and $\mathcal{J}_{K'}$ are the same, hence $q_K = q_{K'}$, essentially by definition. However, we claim that

$$q_K = \sum_{i \in K} q_i, \; q_{K'} = \sum_{i \in K'} q_i.$$

There is no contradiction here, since $\sum_{i=0}^{k} q_i = 0$, hence $q_K + q_{K'} = 0$, and our definitions of the polynomials $q_K$ are up to normalization. □

"uniquecross **Corollary 2.14.** *Fix $i \in [0:k]$.*

*(1) If $q \in \mathcal{D}(X)$ vanishes on $x_i$ to order $N - \mu(i)$, then $q = 0$.*

*(2) $q_i$ is, up to normalization, the only polynomial in $\mathcal{D}(X) \cap \Pi^0_{N-2}$ that vanishes on $x_i$ to order $N - \mu(i) - 1$.*

*(3) $\mathcal{J}_i\perp$ contains all the polynomials in $\mathcal{D}(X)$ that vanish to order $N - \mu(i) - 1$ on $x_i$. Each such polynomial is of the form $l(D)q_i$ for some $l$.*

**Proof:** For (3), assume $r \in \mathcal{D}(X)$. Then $(P_X/P_i)(D)r = 0$. Also, $\deg r \le N - 2$; so, if, in addition, $r$ vanishes on $x_i$ to degree $N - \mu(i) - 1$, then $r = r_1 r_2$, with $p_{x_i}(D) r_1 = 0$, and $\deg r_2 < \mu(i)$. Therefore, $P_i(D)r = 0$, too. Consequently, $r \in \mathcal{J}_i\perp$.

The fact that $q_i$ has the right order of zero on $x_i$ can be verified directly from Theorem 2.11 (e.g., if $i \in [1:k-1]$, from the fact that $q_i = \iota(f_i)$); alternatively, the smoothness properties of the truncated power $\mathrm{TP}_i$, [20], imply that there must be a polynomial in $\mathrm{soc}(\mathcal{D}(X))$ with such vanishing, and due to (3), that polynomial must be in $(\mathcal{J}_i\perp) \cap \mathrm{soc}(\mathcal{D}(X)) = \mathrm{span}\{q_i\}$.

Finally, by (3), the polynomial $q$ in (1) must be in $\mathcal{J}_i\perp$, hence must be of the form $l(D)q_i$, for some $l$. So, unless $l(D)q_i = 0$, it will not have on $x_i$ an order of zero larger than that of $q_i$. □

## 3. Main and other results

### 3.1. Complete intersection decompositions

Having found an explicit representation in the spline space $S(F(X), \gamma)$ for the socle polynomials $q_K$ of the ideals $\mathcal{J}_K$, it becomes straightforward to check whether and when $(q_K : K \in \mathcal{K})$ span $\operatorname{soc}(\mathcal{D}(X))$; after all, the spline space has a reasonably transparent structure. Writing

$$F(X) \ni f = (f, f)_\gamma,$$

we can write any $(f, g)_\gamma \in S(F(X), \gamma)$ as

$$f + (0, g - f)_\gamma.$$

Then $g - f$ vanishes on $\gamma$, hence, by Corollary 2.10,

“relation (3.1) $$(f, g)_\gamma = f + c\,(0, f_0)_\gamma.$$

So, a basis for $S(F(X), \gamma)$ is given by

“basisgam (3.2) $$f_1, \ldots, f_{k-1}, (0, f_0)_\gamma,$$

and one can use Theorem 2.11 together with (3.1) to derive a host of bases of the form $(q_K)_K$. Each such basis yields a CID of $\mathcal{J}(X)$. We list below two choices. In the first choice, we use $k$ (cross-a-line) $x_i$-polynomials. In the second basis, we assume that our basis $B$ of $\mathbb{R}^2$ is made of two rays $(x_0, x_1)$ that are consecutive in $\mathbb{R}^2$, and collect the cone polynomials in the fixed cone $c_0$ over all the truncated powers whose support include $c_0$.

“corcid **Corollary 3.3.** *Assume that the lines $(x_i)_{i=0}^k$ are ordered counter-clockwise.*

*(1) Each of the following is a minimal CID:*

*(1a) $\mathcal{J}(X) = \cap_{i \in [0:k] \setminus j} \mathcal{J}_i$, with $j \in [0 : k]$ arbitrary.*

*(1b) $\mathcal{J}(X) = \cap_{K \in \mathcal{K}} \mathcal{J}_K$, with*

$$\mathcal{K} := \{[1 : j] : j \in [1 : k]\}.$$

*(2) Each of the following is a basis for $\operatorname{soc}(\mathcal{D}(X))$.*

*(2a) Any $k$ of the following $k + 1$ polynomials*

$$\iota((0, f_0)_\gamma),\ \iota((f_0, 0)_\gamma),\quad \iota(f_i),\quad i \in [1 : k - 1].$$

*(2b)*

$$\iota((-\sum_{j=1}^{i-1} f_j, \sum_{j=i}^{k-1} f_j)_\gamma),\ i \in [1 : k].$$

**Proof:** For each basis listed in (2), since it is written via the bijection $\iota$ between $S(F(X), \gamma)$ and $\operatorname{soc}(\mathcal{D}(X))$, we just need to check that the corresponding sequences in $S(F(X), \gamma)$ form a basis for that latter space. This is straightforward to check, in view of the basis in (3.2). The statement in (1) is equivalent to the basis statement in (2).

The fact (stated before this corollary) that the second basis in (2) is made of cone polynomials from a fixed cone will be established later. □

Having found bases for $\mathrm{soc}(\mathcal{D}(X))$, we would like to compute their dual $\mathcal{P}$-polynomials. Recall that, given any multiset $X \subset \mathbb{R}^2\backslash 0$, and with

$$p_Y := \prod_{y \in Y}, \quad Y \subset X,$$

one defines

$$\mathcal{P}(X) := \mathrm{span}\{p_Y : Y \subset X,\ \mathrm{rank}(X\backslash Y) = 2\}.$$

It is known, [21],[20], that $\mathcal{P}(X)$ is isomorphic to $\mathcal{D}(X)'$ via the pairing

$$(p, q) := p(D)q(0).$$

Thus, the top degree homogeneous grade $\mathrm{soc}(\mathcal{P}(X))$ is dual to $\mathrm{soc}(\mathcal{D}(X))$. Hence, every basis for $\mathrm{soc}(\mathcal{D}(X))$ has a dual basis in $\mathrm{soc}(\mathcal{P}(X))$, which, in many cases, is quite explicit.

Note that $\mathrm{soc}(\mathcal{P}(X))$ is spanned by the polynomials

$$p_{X\backslash B}, \quad B \in \mathbb{B}(X),$$

with $\mathbb{B}(X)$ the bases of $X$, i.e., the multiset

$$\mathbb{B}(X) := \{B \subset X : B = \{x_i, x_j\},\ i \neq j\}.$$

There are $\binom{k+1}{2}$ polynomials of the form $p_{X\backslash B}$, while $\dim(\mathrm{soc}(\mathcal{P}(X)) = \dim(\mathrm{soc}(\mathcal{D}(X)) = k$, so we are dealing with a highly redundant spanning set.

“propdual **Proposition 3.4.** *The $\mathcal{P}$-dual basis for the polynomials $(q_1, \ldots, q_k)$ (2(a) in Corollary 3.3) are, up to normalization, the polynomials*

$$p_{X\backslash B_j},\ B_j := (x_0, x_j).$$

**Proof:** Fix $j \in [1:k]$, and let $i \in [1:k]\backslash j$. Since $P_i$ divides $p_{X\backslash B_j}$, it follows that $p_{X\backslash B_j} \in \mathcal{J}_i$, hence that

$$(p_{X\backslash B_j}, q_i) = 0.$$

It remains then to show that $(p_{X\backslash B_i}, q_i) \neq 0$. One may prove this part by factoring $q_i$ as in the proof of (3) in Corollary 2.14. Another way to establish it is as follows. Given $B = (x_i, x_j) \in \mathbb{B}(X)$, we easily verify that

$$p_{X\backslash B} \in \mathrm{span}\{p_{X\backslash B_i}, p_{X\backslash B_j}\}.$$

Therefore the $k$ polynomials $p_{X\backslash B_i}$, $i \in [1:k]$ span $\mathrm{soc}(\mathcal{P}(X))$ (hence for a basis for it). Since $\mathrm{soc}(\mathcal{P}(X))$ is isomorphic to $\mathrm{soc}(\mathcal{D}(X))$, $q_i$ cannot vanish identically on $\mathrm{soc}(\mathcal{P}(X))$, hence cannot vanish on $p_{X\backslash B_i}$. □

Other dual bases can be derived from the above. One just needs to identify the linear transformation from the $(q_i)_i$ above to the new basis, and to invert it on the dual basis.

### 3.2. Cone polynomials

While the discussion so far unraveled successfully the structure of $\mathrm{soc}(\mathcal{D}(X))$, we are yet to prove Theorem 1.2, i.e., to write down explicitly the cone polynomials $Q_{j,i}$. Let us assume now that the lines $(x_0, \ldots, x_k)$ are ordered, say, counter-clockwise.

Importantly, when accessing the cone $c_i \subset C_j$, we do *not* study the cone *geometrically* by crossing the lines $x_j, \ldots, x_i$ and trying to calculate the increment for each crossing. Instead, we access the problem *algebraically* by selecting a complete intersection ideal $\mathcal{J}_K$ that fits the pair $(C_j, c_i)$ and proving that the cone polynomial $Q_{j,i}$ coincides with the socle polynomial $q_K$ of $\mathcal{J}_K$. Once this is done, it only remains to find the correct normalization for $Q_{j,i}$, since $q_K$ is defined up to normalization.

Identifying the cone polynomials $Q_{j,i}$, i.e., the restriction of $\mathrm{TP}_j$ to $c_i$, is strikingly easy. We need to recall one property of truncated powers:

“resone **Result 3.5.** *Given a truncated power* $\mathrm{TP}_X$ *and* $x \in X$, *we have that* $p_x(D)\mathrm{TP}_X = \mathrm{TP}_{X\backslash x}$.

Then, given $j$ and $i$, we partition the lines $x_j, \ldots, x_{j+k} \in C_j$ into two sets, $K, K'$, as follows. Let

$$B := (x_i, x_{i+1}).$$

Then

$$K_{j,i} := \{m \in [j : j+k] : (B^{-1}x_m)(1) > 0\}.$$

Thus, $K_{j,i}$ records the indices of the rays that are found in *clockwise* direction from $c_i$. Since we assume here that the rays are ordered, then

$$K_{j,i} = [j : i].$$

Recall the socle polynomial $q_{K_{j,i}}$, which, by Corollary 2.12, is $q_{K_{j,i}} = \sum_{m=j}^{i} q_m$.

“thmcone **Theorem 3.6.** *The cone polynomial*

$$Q_{j,i} = \mathrm{TP}_{j|c_i}$$

*is, up to normalization, the socle polynomial* $q_{K_{j,i}}$.

The theorem is illustrated in Figure 2.

**Proof:** We collect from the matrix $X_j$ the lines $x_m$, $m \in K_{j,i}$ (each according to its multiplicity) in a matrix $X_{j,i}$. With that, $X'_{j,i} := X_j \backslash X_{j,i}$. Note that the cone $c_i$ has null intersection with either of

$$\mathrm{pos}(X_{j,i}),\ \mathrm{pos}(X'_{j,i}),$$

as we selected $K_{j,i}$ exactly in order to have this property. Now, by Result 3.5,

$$P_{K_{j,i}}(D)\mathrm{TP}_j = \mathrm{TP}_{X'_{j,i}}.$$

Since $\mathrm{supp}(\mathrm{TP}_{X'_{j,i}}) = \mathrm{pos}(X'_{j,i})$, we conclude that $\mathrm{TP}_{X'_{j,i}}$ vanishes a.e. on $c_i$, hence that

$$P_{K_{j,i}}(D)Q_{j,i} = 0.$$

An analogous argument shows that

$$P_{K'_{j,i}}(D)Q_{j,i} = 0.$$

Thus, $Q_{j,i} \in \mathcal{J}_{K_{j,i}}\perp$. Since $Q_{j,i} \in \Pi^0_{N-2}$, then, by Proposition 2.2,

$$Q_{j,i} \in \text{span}\{q_{K_{j,i}}\}.$$

□

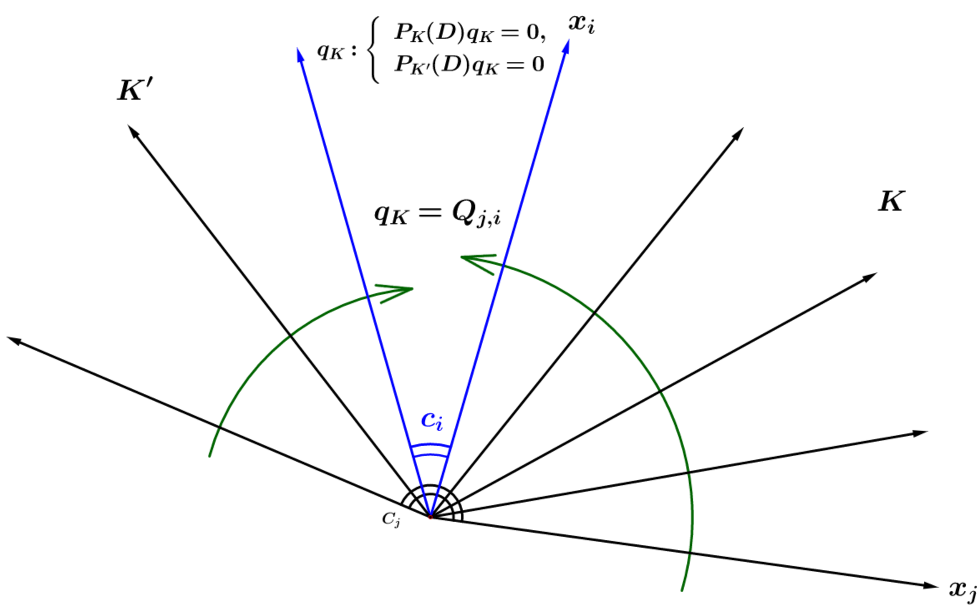


**Figure 2**: The partition $K, K'$ for the cone function $Q_{j,i}$.

**Proof of Theorem 1.2.** The smoothness properties of the truncated power $\text{TP}_{X_j}$ entails that $Q_{j,i} - Q_{j,i-1}$ vanishes on $x_i$ to order $N - \mu(i) - 1$. It follows then from Corollary 2.14 that

“temp (3.7) $$Q_{j,i} - Q_{j,i-1} \in \text{span}\{q_i\}.$$

By Theorem 3.6, when combined with Corollary 2.12, for some $a, b \in \mathbb{R}$,

(3.8) $$Q_{j,i} - Q_{j,i-1} = a \sum_{m=j}^{i} q_m - b \sum_{m=j}^{i-1} q_m = aq_i + (a-b) \sum_{m=j}^{i-1} q_m.$$

“tempa

Now, by Corollary 3.3, $(q_j, \ldots, q_i)$ are linearly independent (since $i < j + k$), thus (3.7) and (3.8) combined entail that $a - b = 0$.

We conclude thus that

$$Q_{j,i} = a \sum_{m=j}^{i} q_m,$$

with the constant $a$ independent of $i$, hence dependent only on the truncated power $\mathrm{TP}_{X_j}$, hence can be chosen as $a = 1$.

Now, the conclusion is identical to the claim of Theorem 1.2, only that there the $Q_{j,i}$ polynomials are written via the spline representation. □

## 4. More on $\mathcal{J}_K$

Let $\mathcal{J}$ be an ideal with two generators

“defJ
$$\mathcal{J} := \mathrm{Ideal}(\rho_1, \rho_2), \tag{4.1}$$

each of which a product of linear forms

“defJa
$$\rho_i = \prod_{y \in Y_i} p_y, \ \ i = 1, 2. \tag{4.2}$$

Assume further that $\rho_1, \rho_2$ do not have a common linear divisor. Equivalently, it means that every member of the multiset

$$\mathcal{B}_{\mathcal{J}} := Y_1 \times Y_2$$

is a basis for $\mathbb{R}^2$. Now, let $X := Y_1 \cup Y_2$. Then, with

$$\mathbb{B}(X)$$

the collection of all bases from $X$ (all $2 \times 2$ matrices with columns from $X$), one defines

$$L(X) := \{L \subset X : L \cap B \neq \emptyset, \ \forall B \in \mathbb{B}(X)\}.$$

Then it clearly follows that

$$\mathcal{J}(X) = \mathrm{Ideal}(p_L : L \in L(X)),$$

while it is well-known, [14],[5], that

$$\dim(\mathcal{D}(X)) = \#\mathbb{B}(X).$$

Now, let

$$\mathcal{B} \subset \mathbb{B}(X)$$

be an arbitrary subset of $\mathbb{B}(X)$; one then defines

$$L(X,\mathcal{B}) := \{L \subset X : L \cap B \neq \emptyset,\ \forall B \in \mathcal{B}\},$$

and

$$\mathcal{J}(X,\mathcal{B}) := \mathrm{Ideal}(p_L : L \in L(X,\mathcal{B})).$$

It follows from general arguments, [6], that, whatever the selected $\mathcal{B}$ is,

$$\dim(\mathcal{J}(X,\mathcal{B})\perp) \geq \#\mathcal{B}.$$

The selection $(X,\mathcal{B})$ is called *zonotopal* if we have the stronger result

$$\dim(\mathcal{J}(X,\mathcal{B})\perp) = \#\mathcal{B}.$$

This notion was studied extensively, [7],[13],[8]. Note that, whatever $\mathcal{B}$ we select, it is always true that

$$\mathcal{J}(X) \subset \mathcal{J}(X,\mathcal{B}),$$

hence that

$$\mathcal{J}(X,\mathcal{B})\perp \subset \mathcal{D}(X).$$

Note also that $\mathcal{J}$ fits the above discussion, i.e.,

$$\mathcal{J} = \mathcal{J}(X,\mathcal{B}_{\mathcal{J}}),$$

since $Y_1, Y_2$ are clearly the only two minimal sets in $L(X,\mathcal{B}_{\mathcal{J}})$. We claim that $\mathcal{J}$ is then zonotopal, but we claim actually more. To this end, we recall from [7] the notion of "placibility".

“placible **Definition 4.3.** *Let $\mathcal{B}$ be a subset of $\mathbb{B}(X)$, with $X$ a finite multiset in $\mathbb{R}^2\backslash 0$ of rank 2. A vector $y \in X$ is $\mathcal{B}$-placible, if, for every $B \in \mathcal{B}$, there exists $y' \in B$ such that $(y, y') \in \mathcal{B}$.* □

A placible vector induces a decomposition of $\mathcal{B}$ into

$$\mathcal{B} = \mathcal{B}_{\backslash} y \cup \mathcal{B}_{/} y,$$

where

$$\mathcal{B}_{\backslash} y := \{B \in \mathcal{B} : y \notin B\}, \quad \mathcal{B}_{/} y := \{B \in \mathcal{B} : y \in B\}.$$

The placibility of $y$ is non-trivial if both sets above are non-empty. We recall the following result, [7]:

“resbrs **Result 4.4.** *With $X$ and $\mathcal{B}$ as above, assume that $y \in X$ is non-trivially placible into $\mathcal{B}$. Consider the map*

$$M : \mathcal{J}(X, \mathcal{B})\perp \to \Pi, \ f \mapsto p_y(D)f.$$

*Then:*

$$\ker M = \mathcal{J}(X, \mathcal{B}_{/}y)\perp, \quad \operatorname{ran} M = \mathcal{J}(X, \mathcal{B}_{\backslash}y)\perp.$$

“socci **Proposition 4.5.** *Let $\mathcal{J}$ be the ideal in (4.1,4.2). Then:*

*(1) $\mathcal{J}$ is zonotopal, and hence*

$$\dim(\mathcal{J}\perp) = (\#Y_1)(\#Y_2).$$

*(2) Up to normalization, there exists a unique homogeneous polynomial $q \in \mathcal{J}\perp$ of maximal degree $\#X - 2$.*

**Proof:** We prove the two claims by induction on $\#X$. At the outset, note that, since $\mathcal{J} \supset \mathcal{J}(X)$, there are no polynomials in $\mathcal{J}\perp$ of degree $> \#X - 2$.

Now, the two claims here are trivial if both $Y_1$ and $Y_2$ are singletons. Otherwise, assume without loss that $\#Y_1 > 1$ and choose $y \in Y_1$. Then $y$ is non-trivially placible in $\mathcal{B} := \mathcal{B}_{\mathcal{J}}$. Now,

$$\mathcal{B}_{\backslash}y = (Y_1 \backslash y) \times Y_2,$$

and

$$\mathcal{B}_{/}y = (y) \times Y_2,$$

hence, with $M$ as in Result 4.4, our induction assumption applies to yield that

$$\dim(\mathcal{J}(X, \mathcal{B}_{\mathcal{J}})\perp) = \dim(\ker M) + \dim(\operatorname{ran} M) = \#\mathcal{B}_{/}y + \#\mathcal{B}_{\backslash}y = \#\mathcal{B}_{\mathcal{J}}.$$

This proves (1).

By induction, $\ker M$ has all its polynomials of degree $\leq 1 + \#Y_2 - 2 < \#X - 2$. On the other hand, the induction tells us that $\operatorname{ran} M$ has a unique polynomial (up to normalization) of highest degree $\#(Y_1 \backslash y) + \#Y_2 - 2 = \#X - 3$. Combined, these two observations imply (2). □

# References


[1] Carl de Boor, *A practical guide to splines*, Appl. Math. Sci., Springer, New York, 2001. MR 1900298

[2] Carl de Boor and Ronald DeVore, *Approximation by smooth multivariate splines*, Trans. Amer. Math. Soc. **276** (1983), no. 2, 775–788. MR 0688977

[3] Carl de Boor and Klaus Höllig, *B-splines from parallelepipeds*, J. d'Analyse Math. **42** (1982/83), 99–115. MR 0729403

[4] ______, *Bivariate box splines and smooth pp functions on a three direction mesh*, J. Comput. Appl. Math. **9** (1983), no. 1, 13–28. MR 0702228

[5] Carl de Boor, Klaus Höllig, and Sherman Riemenschneider, *Box splines*, Appl. Math. Sci., Springer, New York, 1993. MR 1243635

[6] Carl de Boor and Amos Ron, *On ideals of finite codimension and applications to box splines theory*, Journal of Mathematical Analysis and Its Applications **158** (1991), 168–193.

[7] Carl de Boor, Amos Ron, and Zuowei Shen, *On ascertaining inductively the dimension of the joint kernel of certain commuting linear operators*, Adv. in Appl. Math. **17** (1996), no. 3, 209–250. MR 1406403

[8] ______, *On ascertaining inductively the dimension of the joint kernel of certain commuting linear operators II*, Adv. in Math. **123** (1996), no. 2, 223–242. MR 1420485

[9] Charles K Chui and Ren-Hong Wang, *Bivariate cubic B-splines relative to cross-cut triangulations*, Chinese Ann. Math. Ser. **B4** (1983), no. 4, 509–523. MR MR0741749

[10] ______, *On smooth multivariate spline functions*, Math. Comp. **41** (1983), no. 163, 131–142. MR MR0701629

[11] Elaine Cohen, Tom Lyche, and Richard. F. Riesenfeld, *Cones and recurrence relations for simplex splines*, Constr. Approx. **3** (1987), no. 2, 131–141. MR MR0889550

[12] Wolfgang Dahmen, *Multivariate B-splines  recurrence relations and linear combinations of truncated powers*, Multivariate Approximation Theory, International Series of Numerical Mathematics, Springer, New York, 1979, pp. 64–82. MR MR0560664

[13] Wolfgang Dahmen, Andreas Dress, and Charles A. Micchelli, *On multivariate splines, matroids, and the Ext-functor*, Adv. in Appl. Math. **17** (1996), no. 3, 251–307. MR 1406404

[14] Wolfgang Dahmen and Charles Micchelli, *On the local linear independence of translates of a box spline*, Studia Mathematica **82** (1985), no. 3, 243–263.

[15] Wolfgang Dahmen and Charles A. Micchelli, *On the optimal approximation rates for criss-cross finite element spaces*, Journal of Comput. Appl. Math **10** (1984), no. 3, 255–273. MR MR0755803

[16] ______, *On the piecewise structure of discrete box splines*, Comput. Aided Geom. Design **3** (1986), no. 3, 185–191. MR MR0871112

[17] ______, *Algebraic properties of discrete box splines*, Constr. Approx. **3** (1987), no. 2, 209–221. MR MR0889556

[18] ———, *Local dimension of piecewise polynomial spaces, syzygies, and solutions of systems of partial differential equations*, Math. Nachr. **148** (1990), 117–136. MR MR1127336

[19] Wolfgang Dahmen, Charles A. Micchelli, and Hans-Peter Seidel, *Blossoming begets B-spline bases built better by B-patches*, Math. Comp. **59** (1992), no. 199, 97–115. MR MR1134724

[20] Corrado De Concini and Claudio Procesi, *Topics in hyperplane arrangements, polytopes and box-splines.*, Universitext, Springer, New York, 2011. MR MR2722776

[21] N. Dyn and A. Ron, *Local approximation by certain spaces of exponential polynomials, approximation order of exponential box splines and related interpolation problems*, Transactions of Amer. Math. Soc. **319** (1990), 381–404.

[22] Rong-Qing Jia, *Approximation order from certain spaces of smooth bivariate splines on a three-direction mesh*, Trans. Amer. Math. Soc. **295** (1986), no. 1, 199–212. MR MR0831196

[23] ———, *A Bernstein-type inequality associated with wavelet decomposition*, Constr. Approx. **9** (1993), no. 2-3, 299–318. MR MR1215774

[24] Edward Neuman, *A new formula for box splines on three-directional meshes*, Math. Comp. **62** (1994), no. 205, 227–229. MR MR1208840

[25] Ren-Hong Wang and Jin-Cai Chang, *Bivariate splines and golden section based on theory of elasticity*, J. Math. Res. Exposition **31** (2011), no. 1, 1–11. MR MR2791195

[26] Ren-Hong Wang, Mian Li, and Chun-Gang Zhu, *Multivariate splines and hyperplane arrangements*, J. Comput. Appl. Math. **236** (2011), no. 5, 775–781. MR MR2853502

[27] Ren-hong Wang and Zhi-qiang Xu, *Discrete truncated powers and lattice points in rational polytope*, J. Comput. Appl. Math. **159** (2003), no. 1, 149–159. MR MR2022325

[28] Hermann Weyl, *Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformation III.*, Mathematische Zeitschrift **24** (1926), 377–395.

[29] Zhiqiang Xu, *An explicit formulation for two dimensional vector partition functions*, Integer points in polyhedrageometry, number theory, representation theory, algebra, optimization, statistics, Contemp. Math., Amer. Math. Soc., Providence, RI, 2008, pp. 163–178. MR MR2405771

[30] ———, *Multivariate splines and polytopes*, J. Approx. Theory **163** (2011), no. 3, 377–387. MR MR2771249