# AN IMPROVED ORDER OF APPROXIMATION FOR THIN-PLATE SPLINE INTERPOLATION IN THE UNIT DISC

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Abstract. We show that the $L_p$-norm of the error in thin-plate spline interpolation in the unit disc decays like $O(h^{\gamma_p+1/2})$, where $\gamma_p := \min\{2, 1+2/p\}$, under the assumptions that the function to be approximated is $C^\infty$ and that the interpolation points contain the finite grid $\{hj : j \in \mathbb{Z}^2,\ |hj| < 1-h\}$.

## 1. Introduction

Let $H$ be the set of all continuous functions $f : \mathbb{R}^2 \to \mathbb{C}$ having square integrable second order derivatives, and let $|||\cdot|||$ be the semi-norm defined on $H$ by

$$|||f||| := \sqrt{\int_{\mathbb{R}^2} \left|\frac{\partial^2 f(x)}{\partial x_1^2}\right|^2 + 2\left|\frac{\partial^2 f(x)}{\partial x_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 f(x)}{\partial x_2^2}\right|^2 \, dx}.$$

Let $\Xi$ be any bounded set of non-collinear points in $\mathbb{R}^2$. Duchon [2] has shown that to each $f \in H$, there exists a unique $s \in H$ which minimizes $|||s|||$ subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. The function $s$ is called the *thin-plate spline interpolant to $f$ at $\Xi$* and will be denoted by $T_\Xi f$. When $\Xi$ contains only finitely many points, Duchon further characterized $T_\Xi f$ as the unique function in $S(\phi;\Xi)$ which interpolates $f$ at $\Xi$. Here $\phi : \mathbb{R}^2 \to \mathbb{R}$ is the radially symmetric function given by

$$\phi(x) := |x|^2 \log |x|, \qquad x \in \mathbb{R}^2,$$

and $S(\phi;\Xi)$ is the space of all functions $g$ of the form

$$g = \sum_{\xi\in\Xi} \lambda_\xi \phi(\cdot - \xi) + p,$$

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where $p \in \Pi_1 :=$ {polynomials of total degree $\leq 1$} and the $\lambda_\xi$'s satisfy

$$\sum_{\xi \in \Xi} \lambda_\xi q(\xi) = 0, \quad \forall q \in \Pi_1.$$

An important problem relating to thin-plate spline interpolation is that of determining the rate at which $T_\Xi f$ converges to $f$ as the points $\Xi$ become dense. Let us assume that $\Omega \subset \mathbb{R}^2$ is an open bounded domain over which the error will be measured. We assume that $\Xi \subset \overline{\Omega}$, and we define the 'density' of $\Xi$ in $\Omega$ to be the number

$$\delta := \delta(\Xi; \Omega) := \sup_{x \in \Omega} \min_{\xi \in \Xi} |x - \xi| .$$

Thin-plate spline interpolation in $\Omega$ is said to provide *$L_p$-approximation of order $\gamma$* if

$$\|f - T_\Xi f\|_{L_p(\Omega)} = O(\delta^\gamma)$$

for all sufficiently smooth functions $f$. Duchon [3] has shown that if $\Omega$ is connected, satisfies a uniform cone condition, and has a Lipschitz boundary, then thin-plate spline interpolation in $\Omega$ provides $L_p$-approximation of order at least

$$\gamma_p := \min\{2, 1 + 2/p\}$$

for $p \in [1 \, .. \, \infty]$. More precisely, it was shown that

(1.1)
$$\|f - T_\Xi f\|_{L_p(\Omega)} \leq \text{const}\, \delta^{\gamma_p} |||T_\Omega f - T_\Xi f|||, \quad \forall f \in H,\ p \in [1 \, .. \, \infty], \quad \text{and}$$

(1.2)
$$|||T_\Omega f - T_\Xi f||| \to 0 \text{ as } \delta \to 0.$$

Powell [7] (see also [10]) has obtained similar results for the case $p = \infty$ with less restrictive assumptions on the domain $\Omega$, and has even found the best const in (1.1) for some special cases. In the limiting case when the points $\Xi$ are taken as the infinite grid $h\mathbb{Z}^2$ and $\Omega$ is taken as all of $\mathbb{R}^2$, it was shown by Buhmann [1] that $\|f - T_\Xi f\|_{L_\infty(\mathbb{R}^2)} = O(h^4)$ as $h \to 0$ for all sufficiently smooth $f$. His approach employed techniques developed in the context of approximation from shift-invariant spaces; however, this shift-invariant space approach has yet to provide any results on the approximation order of thin-plate spline interpolation in bounded domains (as defined above). Recently, Johnson [6] has shown that one should not in general expect thin-plate spline interpolation to provide $L_p$-approximation of order greater than $2 + 1/p$. Precisely, it was shown that if $\Omega$ is the unit disk $B := \{x \in \mathbb{R}^2 : |x| < 1\}$, then there exists $f \in C^\infty(\mathbb{R}^2)$ such that $\|f - T_\Xi f\|_{L_p(B)} \neq o(\delta^{2+1/p})$. Note that the difference between this upper bound on the approximation order of $2 + 1/p$ and Duchon's lower bound of $\gamma_p$ is $\frac{1}{2} + \left|\frac{1}{p} - \frac{1}{2}\right|$.

The purpose of the present paper is to build on Duchon's work to obtain $L_p$-approximation of order $\gamma_p + 1/2$ in a special case. Our point of attack is the factor $|||T_\Omega f - T_\Xi f|||$ on the right hand side of (1.1) which, according to (1.2), decays to 0 as $\delta \to 0$. It seems plausible that if $f$ is sufficiently smooth, then this factor might decay to zero as some power of $\delta$. We can see immediately, that one should not in general expect this factor to decay faster than $O(\sqrt{\delta})$:

**Theorem 1.3.** *If $\Omega = B$, then there exists $f \in C^\infty(\mathbb{R}^2)$ such that*

$$|||T_\Omega f - T_\Xi f||| \neq o(\sqrt{\delta}) \text{ as } \delta \to 0.$$

*Proof.* According to [6], there exists $f \in C^\infty(\mathbb{R}^2)$ such that $\|f - T_\Xi f\|_{L_2(\Omega)} \neq o(\delta^{5/2})$. If $|||T_\Omega f - T_\Xi f||| = o(\sqrt{\delta})$, then it follows from (1.1) that $\|f - T_\Xi f\|_{L_2(\Omega)} = o(\delta^{5/2})$ which is a contradiction. $\square$

In order to investigate the decay of $|||T_\Omega f - T_\Xi f|||$ in the most favorable of circumstances, we make the following simplifying assumptions:
First, we assume that the function to be approximated, $f$, belongs to $C^\infty(\mathbb{R}^2)$. Second, we assume that our domain $\Omega$ is the open unit disc $B$. And last, we assume that our centres $\Xi$ satisfy

$$(1.4) \qquad \Xi_h \subset \Xi \subset \overline{B}, \qquad \text{where } \Xi_h := h\mathbb{Z}^2 \cap (1-h)B.$$

Note that $S(\phi; \Xi_h) \subset S(\phi; \Xi)$ and $\delta(\Xi; B) \leq \delta(\Xi_h; B) = O(h)$ as $h \to 0$.

Under these assumptions, we show that the factor $|||T_B f - T_\Xi f|||$ decays to 0 as $O(\sqrt{h})$. Precisely, we show the following

**Theorem 1.5.** *If $\Xi$ satisfies (1.4) and $f \in C^\infty(\mathbb{R}^2)$, then $|||T_B f - T_\Xi f||| = O(\sqrt{h})$ as $h \to 0$, and consequently*

$$\|f - T_\Xi f\|_{L_p(B)} = O(h^{\gamma_p + 1/2}) \text{ as } h \to 0,$$

*where $\gamma_p := \min\{2, 1 + 2/p\}$, $p \in [1\,..\,\infty]$.*

Note that, for $p = 2$, we obtain $L_2$-approximation of order $5/2$ (modulo assumption (1.4)) which matches Johnson's upper bound on the $L_2$-approximation order.

In the sequel we use standard multi-index notation: $D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}$. The Laplacian operator is denoted $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. For multi-indices $\alpha \in \{0, 1, 2, \dots\}^2$, we define $|\alpha| := \alpha_1 + \alpha_2$, while for $x \in \mathbb{R}^2$, we define $|x| := \sqrt{x_1^2 + x_2^2}$. For multi-indices $\alpha$, we employ the notation $()^\alpha$ to represent the monomial $x \mapsto x^\alpha$, $x \in \mathbb{R}^2$. The space of bivariate polynomials of total degree $\leq k$ can then be expressed as $\Pi_k := \operatorname{span}\{()^\alpha : |\alpha| \leq k\}$. For $x \in \mathbb{R}^2$, we define the complex exponential $e_x$ by $e_x(t) := e^{ix\cdot t}$, $t \in \mathbb{R}^2$. The Fourier transform of a function $f$ can then be expressed as $\widehat{f}(w) := \int_{\mathbb{R}^2} e_{-w}(x) f(x)\, dx$. The space of compactly supported $C^\infty$ functions is denoted $C_c^\infty(\mathbb{R}^2)$. If $\mu$ is a distribution and $g$ is a test function, then the application of $\mu$ to $g$ is denoted $\langle g, \mu \rangle$. Familiarity with tempered distributions is assumed throughout the sequel. Two important facts in this regard are first that the Fourier transform of $\phi$ can be identified on $\mathbb{R}^2 \backslash 0$ with $8\pi\,|\cdot|^{-4}$ (cf. [4]), and second that

$$|||f||| = (2\pi)^{-1} \left\| |\cdot|^2\, \widehat{f} \right\|_{L_2(\mathbb{R}^2)} = \|\Delta f\|_{L_2(\mathbb{R}^2)}$$

which is an application of the Plancherel Theorem [8; page 172].

## 2. A Preliminary Result

**Definition 2.1.** Let $\mathcal{F}$ be the collection of all functions of the form $\phi*\mu+p$, where $p \in \Pi_1$ and $\mu$ is a distribution of order at most 1 satisfying

$$\begin{aligned} &(i) \qquad \operatorname{supp}\mu \subset \overline{B}, \qquad \text{and} \\ &(ii) \qquad |\widehat{\mu}(w)| \le \text{const}\frac{|w|^2}{1+|w|^{3/2}}, \quad w \in \mathbb{R}^2. \end{aligned}$$

We point out that since $\phi$ and all its first order derivatives are continuous and exhibit only polynomial growth at $\infty$, and since $\mu$ is a compactly supported distribution of order at most 1, it follows that the function $\phi * \mu$, defined by

$$\phi * \mu(x) := \langle \phi(x-\cdot), \mu\rangle,$$

is continuous, has only polynomial growth at $\infty$, and satisfies $(\phi*\mu)\widehat{\ } = \widehat{\phi}\widehat{\mu}$. The purpose of this section is to prove the following

**Theorem 2.2.** *Let $\Xi$ satisfy (1.4). If $f \in \mathcal{F}$, then the following hold:*

$$\begin{aligned} &(i) \qquad f \in H, \\ &(ii) \qquad |||f - T_\Xi f||| = O(\sqrt{h}) \text{ as } h \to 0, \\ &(iii) \qquad T_B f = f, \\ &(iv) \qquad \|f - T_\Xi f\|_{L_p(B)} = O(h^{\gamma_p + 1/2}) \text{ as } h \to 0, \end{aligned}$$

*where* $\gamma_p := \min\{2, 1+2/p\}$, $p \in [1\,..\,\infty]$.

The following lemma is crucial to proving (i).

**Lemma 2.3.** *If $g \in C_c(\mathbb{R}^2)$ satisfies $|g(w)| \le \text{const}\,|w|^3$, then*

$$\langle g, \widehat{\phi}\rangle = 8\pi \int_{\mathbb{R}^2} |w|^{-4}\, g(w)\, dw.$$

*Proof.* Let $\sigma \in C_c^\infty(\mathbb{R}^2)$ satisfy $\sigma_{|B} = 1$, and define the tempered distribution $\widehat{\nu}$ according to

$$\langle g, \widehat{\nu}\rangle := 8\pi \int_{\mathbb{R}^2} |w|^{-4} \left( g(w) - \sum_{|\alpha|\le 2} \frac{D^\alpha g(0)}{\alpha!} \sigma(w) w^\alpha \right) dw, \quad g \in C_c^\infty(\mathbb{R}^2).$$

(Note that $\left| g(w) - \sum_{|\alpha|\le 2} \frac{D^\alpha g(0)}{\alpha!}\sigma(w) w^\alpha \right| \le \text{const}(g)\,|w|^3$ and hence the above integrand is absolutely integrable.) Since $\widehat{\nu} = \widehat{\phi}$ on $\mathbb{R}^2\backslash 0$, it follows that $\phi = \nu + p$ for some polynomial $p$. In order to show that $p \in \Pi_2$, we will estimate the growth of $|\nu(x)|$ for large $|x|$. Assume $|x| \geq 1$, and put $k_x(w) := e_x(w) - \sum_{|\alpha|\leq 2} \frac{D^\alpha e_x(0)}{\alpha!}\sigma(w)w^\alpha$, $w \in \mathbb{R}^2$. Since $\widehat{\nu}$ can be identified with an integrable function on $\mathbb{R}^2\backslash B$, it follows that

$$\nu(x) = (2\pi)^{-2}\langle e_x, \widehat{\nu}\rangle = (2\pi)^{-2}8\pi \int_{\mathbb{R}^2} |w|^{-4}\, k_x(w)\, dw.$$

Since $|e_x| = 1$ and $\max_{|\alpha|\leq 2} \|D^\alpha e_x(0)\| \leq \text{const}\,|x|^2$, we have the crude estimate $|k_x(w)| \leq \text{const}\; |x|^2$, $\forall w \in \mathbb{R}^2$. Noting that $D^\alpha k_x(0) = 0$, $\forall\, |\alpha| \leq 2$, it follows from Taylor's theorem that for $w \in B$,

$$|k_x(w)| \leq \text{const}\,|w|^3 \max_{|\alpha|=3} \|D^\alpha k_x\|_{L_\infty(B)} \leq \text{const}\,|w|^3\,|x|^3\,.$$

Employing these two estimates on $\mathbb{R}^2\backslash\,|x|^{-1/3}\,B$ and $|x|^{-1/3}\,B$, respectively, we obtain

$$\begin{aligned}
|\nu(x)| &\leq \text{const} \int_{|w|\geq x^{-1/3}} |w|^{-4}\,|k_x(w)|\;dw + \text{const} \int_{|w|<x^{-1/3}} |w|^{-4}\,|k_x(w)|\;dw \\
&\leq \text{const}\;|x|^2 \int_{|w|\geq x^{-1/3}} |w|^{-4}\;dw + \text{const}\,|x|^3 \int_{|w|<x^{-1/3}} |w|^{-4}\,|w|^3\;dw \\
&\leq \text{const}\;|x|^2 \int_{|x|^{-1/3}}^{\infty} r^{-4} r\,dr + \text{const}\;|x|^3 \int_0^{|x|^{-1/3}} r^{-1} r\,dr = \text{const}\;|x|^{8/3}\,.
\end{aligned}$$

Since $|\phi(x)|$ is also bounded by const $|x|^{8/3}$ for $|x| \geq 1$, it follows that $|p(x)| = |\phi(x) - \nu(x)| \leq$ const $|x|^{8/3}$ for $|x| \geq 1$. Hence, $p \in \Pi_2$. Therefore, there exists constants $a_\alpha$, $|\alpha| \leq 2$ such that

$$\langle g, \widehat{\phi}\rangle = \langle g, \nu\rangle + \sum_{|\alpha|\leq 2} a_\alpha D^\alpha g(0), \quad \forall g \in C_c^\infty(\mathbb{R}^2). \tag{2.3}$$

Now if $g \in C_c^\infty(\mathbb{R}^2)$ satisfies $|g(w)| \leq \text{const}\,|w|^3$, then $D^\alpha g(0) = 0$, $\forall\,|\alpha| \leq 2$ and consequently (2.3) reduces to $\langle g, \widehat{\phi}\rangle = 8\pi \int_{\mathbb{R}^2} |w|^{-4}\, g(w)\, dw$. $\square$

*Proof of Theorem 2.2 (i).* Let $f \in \mathcal{F}$, and let $\mu$ and $p$ be as in Definition 2.1. In order to show that $f \in H$, we must show that $D^\alpha f \in L_2\ \forall\,|\alpha| = 2$. Assume $|\alpha| = 2$. Then $(D^\alpha f)\widehat{\ } = -()^\alpha \widehat{\phi}\widehat{\mu}$. Now if $g \in C_c^\infty(\mathbb{R}^2)$, then $g_1 := -()^\alpha \widehat{\mu} g \in C_c^\infty(\mathbb{R}^2)$ and $|g_1(w)| \leq \text{const}\,|w|^4$, and so it follows by Lemma 2.3 that

$$\begin{aligned}
\langle g, (D^\alpha f)\widehat{\ }\rangle &= \langle g_1, \widehat{\phi}\rangle = 8\pi \int_{\mathbb{R}^2} |w|^{-4}\, g_1(w)\, dw \\
&= -8\pi \int_{\mathbb{R}^2} |w|^{-4}\, w^\alpha \widehat{\mu}(w) g(w)\, dw.
\end{aligned}$$

It follows from condition (ii) of Definition 2.1 that

$$\left|-8\pi\,|w|^{-4}\,w^{\alpha}\widehat{\mu}(w)\right| \le \text{const}\,(1+|w|)^{-3/2}.$$

Hence, $-8\pi\,|\cdot|^{-4}\,()^{\alpha}\widehat{\mu} \in L_2(\mathbb{R}^2)$, and it now follows from the Plancherel Theorem [8; page 172] that $D^{\alpha} f \in L_2(\mathbb{R}^2)$. $\square$

In order to get a handle on the quantity $|||f - T_{\Xi} f|||$, we make use of the fact [2] that

$$|||g - T_{\Xi} g||| = \min\{|||g - s||| : s \in S(\phi;\Xi)\}, \quad \forall g \in H.$$

The upshot is that rather than being forced to estimate $|||f - T_{\Xi} f|||$ directly, we may instead estimate $|||f - s_h|||$ where $s_h \in S(\phi;\Xi_h) \subset S(\phi;\Xi)$ can be chosen at our convenience. Given the form of $f \in \mathcal{F}$, namely $f = \phi * \mu + p$, a natural way to construct $s_h$ would be to first convolve $\mu$ with some function $\psi(\cdot/h)$ and then put $s_h = \sum_{j\in\mathbb{Z}^2}(\psi(\cdot/h) * \mu)(hj)\phi(\cdot - hj)$. The only problem with this attempt is that the coefficients $(\psi(\cdot/h) * \mu)(hj)$ will not in general vanish when $hj$ is outside $(1-h)B$, and hence we cannot expect $s_h$ to belong to $S(\phi;\Xi_h)$. This problem can be overcome by convolving $\psi(\cdot/h)$ not with $\mu$, but rather with $\mu((1+r_0h)\cdot)$. With $\psi$ and $r_0$ chosen appropriately, it will follow that $(\psi(\cdot/h) * \mu)(hj) = 0$ whenever $hj$ is outside $(1-h)B$.

Let $\psi := \eta * \sigma$, where $\eta \in C_c(\mathbb{R}^2)$ and $\sigma \in C_c^{\infty}(\mathbb{R}^2)$ are some functions satisfying

$$\sup_{j\in\mathbb{Z}^2} |\delta_{0,j} - \widehat{\eta}(w - 2\pi j)| \le \text{const}\,|w|^2, \quad w \in \mathbb{R}^2 \tag{2.4}$$

$$|1 - \widehat{\sigma}(w)| \le \text{const}\,|w|^2, \quad w \in \mathbb{R}^2. \tag{2.5}$$

For example, one could choose $\eta(x) = \chi_{[-1..1]^2}(x)(1-|x_1|)(1-|x_2|)$ and $\sigma(x) = c^{-1}\chi_B(x)\exp(-1/(1-|x|^2))$ with $c = \int_B \exp(-1/(1-|x|^2))\,dx$. Let $r_0 > 2$ be such that $\operatorname{supp}\psi \subset (\frac{r_0}{2} - 1)B$.

**Lemma 2.6.** *Let $f \in \mathcal{F}$, and let $\mu$ be as in Definition 2.1. If $\mu_h$ and $s_h$ are given by*

$$\mu_h := ((1+r_0h)^2\psi(\cdot/h)) * (\mu((1+r_0h)\cdot)),$$
$$s_h := \sum_{j\in\mathbb{Z}^2} \mu_h(hj)\phi(\cdot - hj),$$

*then $s_h \in S(\phi;\Xi_h)$ whenever $0 < h < r_0^{-1}$.*

*Proof.* It is a straightforward exercise to verify that the choice of $r_0$ ensures that $\operatorname{supp}\mu_h \subset (1-h)B$. Hence it remains only to show that $\sum_{\xi\in\Xi_h}\mu_h(\xi)q(\xi/h) = 0$ for all $q \in \Pi_1$. For that note that $\sum_{\xi\in\Xi_h}\mu_h(\xi)q(\xi/h) = \sum_{j\in\mathbb{Z}^2}\mu_h(hj)q(j)$. If we put $g(x) := \mu_h(hx)q(x)$, then we obtain from Poisson's summation formula (cf. [9], Chapter 7) that $\sum_{j\in\mathbb{Z}^2} g(j) = \sum_{j\in\mathbb{Z}^2} \widehat{g}(2\pi j)$. Now $\widehat{\mu}_h = h^2\widehat{\psi}(h\cdot)\widehat{\mu}(\cdot/(1+r_0h))$; hence, if $q = \sum_{|\alpha|\le 1} i^{-|\alpha|}a_\alpha()^\alpha$, then

$$\widehat{g} = \sum_{|\alpha|\le 1} a_\alpha D^\alpha(h^{-2}\widehat{\mu}_h(\cdot/h)) = \sum_{|\alpha|\le 1} a_\alpha D^\alpha(\widehat{\eta}\widehat{\sigma}\widehat{\mu}(\cdot/(h+r_0h^2))).$$

Condition (2.4) ensures that $D^\alpha(\widehat{\eta}\widehat{\sigma}\widehat{\mu}(\cdot/(h+r_0h^2))) = 0$ at $2\pi j$ whenever $j\in\mathbb{Z}^2\backslash 0$ and $|\alpha|\le 1$. On the other hand, Definition 2.1 (ii) ensures that $D^\alpha(\widehat{\eta}\widehat{\sigma}\widehat{\mu}(\cdot/(h+r_0h^2))) = 0$ at 0 for all $|\alpha|\le 1$. Hence,

$$\sum_{\xi\in\Xi_h} \mu_h(\xi)q(\xi/h) = \sum_{j\in\mathbb{Z}^2} \widehat{g}(2\pi j) = 0.$$

$\square$

The effect of convolving $\widehat{\psi}(\cdot/h)$ with the $(1+r_0h)$-dilate of $\mu$ rather than with $\mu$ itself is that $s_h$ is best compared not to $f$, but rather to the $(1+r_0)$-dilate of $f$. For this, we define

$$f_h := (1+r_0h)^{-2}f((1+r_0h)\cdot), \tag{2.7}$$

and use the triangle inequality to write

$$|||f - s_h||| \le |||f - f_h||| + |||f_h - s_h|||.$$

We consider each of these terms separately in the following two lemmata.

**Lemma 2.8.** *Let $f\in\mathcal{F}$. If $f_h$ is as defined in (2.7), then*

$$|||f - f_h||| = O(\sqrt{h}) \text{ as } h\to 0.$$

*proof.* Let $\mu$ be as in Definition 2.1, and note that

$$\begin{aligned} |||f - f_h||| &= (2\pi)^{-1}\left\| |\cdot|^2 \left(\widehat{f} - \widehat{f_h}\right)\right\|_{L_2(\mathbb{R}^2\backslash 0)} \\ &= (2\pi)^{-1}\left\| |\cdot|^2 \left[\widehat{\phi}\widehat{\mu} - (1+r_0h)^{-4}\widehat{\phi}(\cdot/(1+r_0h))\widehat{\mu}(\cdot/(1+r_0h))\right]\right\|_{L_2(\mathbb{R}^2\backslash 0)} \\ &= 4\left\| |\cdot|^{-2}\left(\widehat{\mu} - \widehat{\mu}(\cdot/(1+r_0h))\right)\right\|_{L_2}. \end{aligned}$$

Now,

$$\left\| |\cdot|^{-2}(\widehat{\mu} - \widehat{\mu}(\cdot/(1+r_0h))\right\|_{L_2}^2 = \int_{\mathbb{R}^2} |w|^{-4}\left|\widehat{\mu}(w) - \widehat{\mu}(w/(1+r_0h))\right|^2\, dw.$$

We estimate this integral by breaking $\mathbb{R}^2$ into the three pieces $B$, $h^{-1}B\backslash B$, and $\mathbb{R}^2\backslash h^{-1}B$. For the first piece, we note that since $\widehat{\mu}$ is entire it can be written as a power series

$$\widehat{\mu}(w) = \sum_\alpha c_\alpha w^\alpha,$$

and it follows from Definition 2.1 (ii) that $c_{(0,0)} = c_{(1,0)} = c_{(0,1)} = 0$. Thus

$$\widehat{\mu}(w) - \widehat{\mu}(w/(1+r_0h)) = \sum_{|\alpha|\geq 2} c_\alpha (w^\alpha - (w/(1+r_0h))^\alpha) = \sum_{|\alpha|\geq 2} c_\alpha w^\alpha (1-(1+r_0h)^{-|\alpha|}).$$

It now follows that

$$\begin{aligned}
&\left\| |\cdot|^{-2}(\widehat{\mu} - \widehat{\mu}(\cdot/(1+r_0h)) \right\|_{L_2(B)} = \left\| |\cdot|^{-2} \sum_{|\alpha|\geq 2} c_\alpha ()^\alpha (1-(1+r_0h)^{-|\alpha|}) \right\|_{L_2(B)} \\
&\leq \sum_{|\alpha|\geq 2} |c_\alpha| (1-(1+r_0h)^{-|\alpha|}) \left\| |\cdot|^{-2} ()^\alpha \right\|_{L_2(B)} \\
&\leq r_0 \sqrt{\pi} h \sum_{|\alpha|\geq 2} |c_\alpha| |\alpha| \leq \text{const}\, h,
\end{aligned}$$

where we have used the estimate $1-(1+r_0h)^{-|\alpha|} \leq r_0h|\alpha|$ and the fact that $\left\| |\cdot|^{-2}()^\alpha \right\|_{L_2(B)} \leq \sqrt{\pi}$.

For the second piece, we note that $\widehat{\mu}(w) - \widehat{\mu}(w/(1+r_0h)) = (1-(1+r_0h)^{-1}) w \cdot (\nabla \widehat{\mu}(\xi))$ for some $\xi$ between $w/(1+r_0h)$ and $w$. Since $\mu$ is compactly supported, it follows from Definition 2.1 (ii) that $|\nabla \widehat{\mu}(w)| \leq \text{const}\,(1+\sqrt{|w|})$. Consequently, $|\widehat{\mu}(w) - \widehat{\mu}(w/(1+r_0h))| \leq \text{const}\, h(1+|w|^{3/2})$. Hence,

$$\begin{aligned}
&\int_{h^{-1}B\setminus B} |w|^{-4} \left| \widehat{\mu}(w) - \widehat{\mu}(w/(1+r_0h)) \right|^2 \, dw \leq \text{const} \int_{h^{-1}B\setminus B} |w|^{-4} h^2 |w|^3 \, dw \\
&= \text{const}\, h^2 \int_1^{h^{-1}} r^{-4} r^3 r \, dr = \text{const}\, h^2 (h^{-1} - 1) \leq \text{const}\, h.
\end{aligned}$$

For the third piece we use the bound $|\widehat{\mu}(w) - \widehat{\mu}(w/(1+r_0h))| \leq \text{const}\,(1+\sqrt{|w|})$ (a consequence of Definition 2.1 (ii)) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus h^{-1}B} |w|^{-4} \left| \widehat{\mu}(w) - \widehat{\mu}(w/(1+r_0h)) \right|^2 \, dw \leq \text{const} \int_{\mathbb{R}^2\setminus h^{-1}B} |w|^{-4} |w| \, dw \\
&= \text{const} \int_{h^{-1}}^{\infty} r^{-3} r \, dr = \text{const}\, h.
\end{aligned}$$

□

**Lemma 2.9.** *Let $f \in \mathcal{F}$. If $f_h$ is as defined in (2.7) and $\mu_h$, $s_h$ are as defined in Lemma 2.6, then*

$$|||f_h - s_h||| = O(\sqrt{h}) \text{ as } h \to 0.$$

*Proof.* Assume $0 < h \leq r_0^{-1}$. In order to simplify the notation, we introduce

$$\widehat{\nu}_h := \widehat{\mu}(\cdot/(1+r_0h)),$$

and note that $\widehat{f_h} = \widehat{\phi}\widehat{\nu}_h$ on $\mathbb{R}^2\backslash 0$. On the other hand, for $w \in \mathbb{R}^2\backslash 0$, we have $\widehat{s}_h(w) = \sum_{j\in\mathbb{Z}^2} \widehat{\phi}(w)\mu_h(hj)e^{-ihj\cdot w}$. If we define $g(x) := \mu_h(hx)e^{-ihx\cdot w}$, $x \in \mathbb{R}^2$, then we obtain from Poisson's summation formula (cf. [9], Chapter 7) that $\sum_{j\in\mathbb{Z}^2} g(j) = \sum_{j\in\mathbb{Z}^2} \widehat{g}(2\pi j)$. Hence,

$$\begin{aligned}\widehat{s}_h(w) &= \widehat{\phi}(w) \sum_{j\in\mathbb{Z}^2} g(j) = \widehat{\phi}(w) \sum_{j\in\mathbb{Z}^2} \widehat{g}(2\pi j) \\ &= \widehat{\phi}(w) \sum_{j\in\mathbb{Z}^2} h^{-2}\widehat{\mu}_h(w + 2\pi j/h) \\ &= \widehat{\phi}(w) \sum_{j\in\mathbb{Z}^2} \widehat{\psi}(hw + 2\pi j)\widehat{\nu}_h(w + 2\pi j/h).\end{aligned}$$

Hence,

$$\begin{aligned}&|||f_h - s_h||| = (2\pi)^{-1} \left\| |\cdot|^2 \left(\widehat{f}_h - \widehat{s}_h\right)\right\|_{L_2(\mathbb{R}^2\backslash 0)} \\ &= 4 \left\| |\cdot|^{-2} \left[\widehat{\nu}_h - \sum_{j\in\mathbb{Z}^2} \widehat{\psi}(h\cdot + 2\pi j)\widehat{\nu}_h(\cdot + 2\pi j/h)\right]\right\|_{L_2}\end{aligned}$$

(2.10)

$$\leq 4\left\| |\cdot|^{-2}\, \widehat{\nu}_h(1 - \widehat{\psi}(h\cdot))\right\|_{L_2} + 4\left\| |\cdot|^{-2} \sum_{j\in\mathbb{Z}^2\backslash 0} \widehat{\psi}(h\cdot + 2\pi j)\widehat{\nu}_h(\cdot + 2\pi j/h)\right\|_{L_2}.$$

We consider first the term $\left\| |\cdot|^{-2}\, \widehat{\nu}_h(1 - \widehat{\psi}(h\cdot))\right\|_{L_2}$. Since $1 \leq 1 + r_0 h \leq 2$, it follows from Definition 2.1 (ii) that

$$|\widehat{\nu}_h(w)|^2 \leq \text{const}\frac{|w|^4}{1 + |w|^3}, \quad w \in \mathbb{R}^2. \tag{2.11}$$

From (2.4) and (2.5) we obtain

$$\left|1 - \widehat{\psi}(w)\right|^2 \leq \text{const}\frac{|w|^4}{1 + |w|^4}, \quad w \in \mathbb{R}^2.$$

Consequently,

$$\begin{aligned}&\left\| |\cdot|^{-2}\, \widehat{\nu}_h(1 - \widehat{\psi}(h\cdot))\right\|_{L_2}^2 \leq \text{const} \int_{\mathbb{R}^2} |w|^{-4}\, \frac{|w|^4}{1 + |w|^3}\frac{|hw|^4}{1 + |hw|^4}\, dw \\ &= \text{const} \int_0^\infty \frac{h^4 r^4}{(1 + r^3)(1 + h^4 r^4)} r\, dr \\ &\leq \text{const} \left(\int_0^1 h^4\, dr + \int_1^{1/h} \frac{h^4 r^5}{r^3}\, dr + \int_{1/h}^\infty \frac{r}{1 + r^3}\, dr\right)\end{aligned}$$

(2.12)

$$\leq \text{const} \left(h^4 + h^4 \int_1^{1/h} r^2\, dr + \int_{1/h}^\infty r^{-2}\, dr\right) \leq \text{const}\, h.$$

Let $C := [-\frac{1}{2} \,..\, \frac{1}{2})^2$. Employing the partition of $\mathbb{R}^2$, $\mathbb{R}^2 = \cup_{k\in\mathbb{Z}^2} 2\pi h^{-1}(k+C)$, we expand the square of the second term at the bottom of (2.10) as

$$\begin{aligned}
&\left\| |\cdot|^{-2} \sum_{j\in\mathbb{Z}^2\backslash 0} \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|_{L_2}^2 \\
&= \sum_{k\in\mathbb{Z}^2} \left\| |\cdot|^{-2} \sum_{j\in\mathbb{Z}^2\backslash 0} \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|_{L_2(2\pi h^{-1}(k+C))}^2 .
\end{aligned} \tag{2.13}$$

For $j \in \mathbb{Z}^2\backslash 0$ and $k \in \mathbb{Z}^2\backslash\{-j\}$, we have

$$\begin{aligned}
&\left\| |\cdot|^{-2}\, \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|_{L_2(2\pi h^{-1}(k+C))}^2 \\
&= \int_{2\pi h^{-1}(k+C)} |w|^{-4} \left|\widehat{\nu}_h(w + 2\pi j/h)\right|^2 \left|\widehat{\psi}(hw + 2\pi j)\right|^2 \, dw \\
&= \int_{2\pi h^{-1}C} \left|2\pi h^{-1}k + w\right|^{-4} \left|\widehat{\nu}_h(w + 2\pi(k+j)/h)\right|^2 \left|\widehat{\psi}(hw + 2\pi(k+j))\right|^2 \, dw \\
&\le \text{const} \int_{2\pi h^{-1}C} \left|2\pi h^{-1}k + w\right|^{-4} \left|w + 2\pi(k+j)/h\right| \left|\widehat{\psi}(hw + 2\pi(k+j))\right|^2 \, dw \\
&\le \text{const} \int_{2\pi h^{-1}C} \left|2\pi h^{-1}k + w\right|^{-4} \left|w + 2\pi(k+j)/h\right| \|\widehat{\sigma}\|_{L_\infty(2\pi(j+k+C))}^2 |hw|^4 \, dw, \qquad \text{by (2.4)}, \\
&\le \text{const}\, h^4 \left|h^{-1}(k+j)\right| \|\widehat{\sigma}\|_{L_\infty(2\pi(j+k+C))}^2 \int_{2\pi h^{-1}C} \frac{|w|^4}{\left|2\pi h^{-1}k + w\right|^4}\, dw \\
&\le \text{const}\, h(1+|k|)^{-4} \, |k+j| \, \|\widehat{\sigma}\|_{L_\infty(2\pi(j+k+C))}^2 .
\end{aligned}$$

For $j \in \mathbb{Z}^2\backslash 0$ and $k = -j$, we have

$$\begin{aligned}
&\left\| |\cdot|^{-2}\, \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \right\|_{L_2(2\pi h^{-1}(k+C))}^2 \\
&= \int_{2\pi h^{-1}(k+C)} |w|^{-4} \left|\widehat{\nu}_h(w + 2\pi j/h)\right|^2 \left|\widehat{\psi}(hw + 2\pi j)\right|^2 \, dw \\
&= \int_{2\pi h^{-1}C} \left|2\pi h^{-1}k + w\right|^{-4} \left|\widehat{\nu}_h(w)\right|^2 \left|\widehat{\psi}(hw)\right|^2 \, dw \\
&\le \text{const} \int_{2\pi h^{-1}C} \left|2\pi h^{-1}k + w\right|^{-4} \frac{|w|^4}{1+|w|^3}\, dw, \quad \text{by (2.11)}, \\
&\le \text{const}\, h^4 \int_{2\pi h^{-1}C} \frac{|w|}{|2\pi k + hw|^4}\, dw \le \text{const}\, h\, |k|^{-4} .
\end{aligned}$$

Since $\sigma \in C_c^\infty(\mathbb{R}^2)$, it follows that $\sum_{j\in\mathbb{Z}^2} \sqrt{|j|}\,\|\widehat{\sigma}\|_{L_\infty(2\pi(j+C))} < \infty$. Hence,

$$\begin{aligned}
&\Big\| |\cdot|^{-2} \sum_{j\in\mathbb{Z}^2\setminus 0} \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \Big\|_{L_2(2\pi h^{-1}(k+C))} \\
&\le \sum_{j\in\mathbb{Z}^2\setminus 0} \Big\| |\cdot|^{-2}\, \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \Big\|_{L_2(2\pi h^{-1}(k+C))} \\
&\le \mathrm{const}\sqrt{h}(1+|k|)^{-2} + \mathrm{const}\sqrt{h}(1+|k|)^{-2} \sum_{j\in\mathbb{Z}^2\setminus\{0,-k\}} \sqrt{|k+j|}\,\|\widehat{\sigma}\|_{L_\infty(2\pi(k+j+C))} \\
&\le \mathrm{const}\sqrt{h}(1+|k|)^{-2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^2} \Big\| |\cdot|^{-2} \sum_{j\in\mathbb{Z}^2\setminus 0} \widehat{\nu}_h(\cdot + 2\pi j/h)\widehat{\psi}(h\cdot + 2\pi j) \Big\|^2_{L_2(2\pi h^{-1}(k+C))} \\
&\le \mathrm{const}\, h \sum_{k\in\mathbb{Z}^2} (1+|k|)^{-4} \le \mathrm{const}\, h. \qquad (2.14)
\end{aligned}$$

And so with (2.14), (2.13), (2.12), and (2.10) in view, the lemma is proved. □

With Lemma 2.8 and Lemma 2.9 in hand we can prove the intended result.

*Proof of Theorem 2.2 (ii)–(iv).* Let $f \in \mathcal{F}$. Assume that $0 < h < r_0^{-1}$, and let $s_h$ be as in Lemma 2.6 and $f_h$ as defined in (2.7). Then

$$\begin{aligned}
|||f - T_\Xi f||| &= \min_{s\in S(\phi;\Xi)} |||f - s||| \\
&\le |||f - s_h|||, \text{ by Lemma 2.6,} \\
&\le |||f - f_h||| + |||f_h - s_h||| = O(\sqrt{h})
\end{aligned}$$

by Lemma 2.8 and Lemma 2.9. Hence (ii). Since $|||T_B f - T_\Xi f|||$ also converges to 0 (by (1.2)), it follows that $|||T_B f - f||| = 0$. Since both $f$ and $T_B f$ belong to $H$ and $f_{|B} = (T_B f)_{|B}$, it must be the case that $f = T_B f$. Hence (iii). Employing (1.1) we obtain

$$\begin{aligned}
\|f - T_\Xi f\|_{L_p(B)} &\le \mathrm{const}\, h^{\gamma_p} |||T_B f - T_\Xi f||| \\
&= \mathrm{const}\, h^{\gamma_p} |||f - T_\Xi f||| = O(h^{\gamma_p + 1/2})
\end{aligned}$$

which proves (iv). □

## 3. Proof of Theorem 1.5

With Theorem 2.2 in view, in order to prove Theorem 1.5, it suffices to prove the following

**Theorem 3.1.** *For all $f \in C^\infty(\mathbb{R}^2)$, there exists $\widetilde{f} \in \mathcal{F}$ such that $f_{|B} = \widetilde{f}_{|B}$.*

Recall that the form of $\widetilde{f} \in \mathcal{F}$ is $\widetilde{f} = \phi * \mu + p$ where $p \in \Pi_1$ and the distribution $\mu$ satisfies the conditions of Definition 2.1, the most restrictive of which is $|\widehat{\mu}(w)| = O(\sqrt{|w|})$ for large $|w|$. The following two lemmas display a class of distributions which satisfy this condition.

Let $y(\theta)$ denote the point $(\cos\theta, \sin\theta) \in \partial B$, and note that $y(\theta)$ is the outward unit normal to $\partial B$ at $y(\theta)$.

**Lemma 3.2.** *If $\nu$ is the distribution given by*

$$\langle g, \nu\rangle := \int_{-\pi}^{\pi} D_{y(\theta)} g(y(\theta))\, d\theta,$$

*then $|\widehat{\nu}(w)| \le \text{const}\,(1 + \sqrt{|w|})$, $w \in \mathbb{R}^2$.*

*Proof.* Since $\nu$ is compactly supported, its Fourier transform is entire, and so it suffices to show that $|\widehat{\nu}(w)| \le \text{const}\sqrt{|w|}$, $|w| > 2\pi$. Since $\nu$ is radially symmetric, so is its Fourier transform, and hence it suffices to consider only $w = (0, t)$, $t > 2\pi$, wherein

$$\begin{aligned}
\widehat{\nu}(w) = \langle e_{-w}, \nu\rangle &= \int_{-\pi}^{\pi} D_{y(\theta)} e_{-w}(y(\theta))\, d\theta \\
&= \int_{-\pi}^{\pi} -it\sin\theta\, e^{-it\sin\theta}\, d\theta \\
&= \int_{-\pi}^{\pi} -t\sin\theta\, \sin(t\sin\theta)\, d\theta, \qquad \text{since } -it\sin\theta\, \cos(t\sin\theta) \text{ is odd in } \theta, \\
&= -4\int_0^{\pi/2} t\sin\theta\, \sin(t\sin\theta)\, d\theta.
\end{aligned}$$

Employing the change of variables $x = t\sin\theta$, we arrive at $\widehat{\nu}(w) = -4\displaystyle\int_0^t \sin x \frac{x}{\sqrt{t^2 - x^2}}\, dx$. Let $N$ be the largest integer for which $N\pi \le t$, and define $A_n := \displaystyle\int_{n\pi}^{(n+1)\pi} \sin x \frac{x}{\sqrt{t^2 - x^2}}\, dx$, for $n = 0, 1, \ldots, N-1$. Then

$$\widehat{\nu}(w) = -4\sum_{n=0}^{N-1} A_n - 4\int_{N\pi}^{t} \sin x \frac{x}{\sqrt{t^2 - x^2}}\, dx.$$

Since $x \mapsto x/\sqrt{t^2 - x^2}$ is increasing on $[0\,..\,t)$, it follows that $|A_n| < |A_{n+1}|$ for $n = 0, 1, \ldots, N-2$, and since the $A_n$'s are alternating in sign, it follows that $\left|\displaystyle\sum_{n=0}^{N-1} A_n\right| < |A_{N-1}|$. Therefore,

$$\begin{aligned}
|\widehat{\nu}(w)| &\le 4\,|A_{N-1}| + 4\int_{N\pi}^{t} |\sin x|\, \frac{x}{\sqrt{t^2 - x^2}}\, dx \\
&\le 4\int_{t-2\pi}^{t} \frac{x}{\sqrt{t^2 - x^2}}\, dx = 4\sqrt{4t\pi - 4\pi^2} \le 8\sqrt{\pi t}.
\end{aligned}$$

□

**Lemma 3.3.** *Let $g_1 \in L_1(B)$, let $g_2 \in L_1([-\pi \,..\, \pi])$, and let $g_3 \in C^\infty(\mathbb{R})$ be $2\pi$-periodic. If $\mu$ is the distribution given by*

$$\langle g, \mu \rangle := \int_B g(\xi) g_1(\xi)\, d\xi + \int_{-\pi}^{\pi} g(y(\theta)) g_2(\theta) + D_{y(\theta)} g(y(\theta)) g_3(\theta)\, d\theta,$$

*then $|\widehat{\mu}(w)| \le \mathrm{const}\,(1 + \sqrt{|w|})$, $w \in \mathbb{R}^2$.*

*Proof.* Let $\nu$ be as defined in Lemma 3.2, and let $q \in C_c^\infty(\mathbb{R}^2)$ be such that $q = 0$ on $\frac{1}{4}B$ and $q = 1$ on $\frac{5}{4}B \backslash \frac{3}{4}B$. Define $q_3 \in C_c^\infty(\mathbb{R}^2)$ by $q_3(x) := q(x) g_3(\arg(x_1 + i x_2))$. Note that

$$\begin{aligned}
\langle g, q_3 \nu \rangle = \langle g q_3, \nu \rangle &= \int_{-\pi}^{\pi} D_{y(\theta)}(g q_3)(y(\theta))\, d\theta \\
&= \int_{-\pi}^{\pi} D_{y(\theta)} g(y(\theta)) q_3(y(\theta))\, d\theta, \qquad \text{since } D_{y(\theta)} q_3(y(\theta)) = 0, \\
&= \int_{-\pi}^{\pi} D_{y(\theta)} g(y(\theta)) g_3(\theta)\, d\theta.
\end{aligned}$$

Hence, $\mu$ can be written as $\mu = \mu_1 + q_3 \nu$, where $\mu_1$ is given by

$$\langle g, \mu_1 \rangle := \int_B g(\xi) g_1(\xi)\, d\xi + \int_{-\pi}^{\pi} g(y(\theta)) g_2(\theta)\, d\theta.$$

Of course, $|\widehat{\mu}_1(w)| \le \|g_1\|_{L_1(B)} + \|g_2\|_{L_1([-\pi..\pi])}$. On the other hand, since $|\widehat{\nu}(w)| \le \mathrm{const}\,(1 + \sqrt{|w|})$ (by Lemma 3.2) and since $q_3 \in C_c^\infty(\mathbb{R}^2)$, it follows that

$$|\widehat{q_3 \nu}(w)| = (2\pi)^{-2}\, |\widehat{q}_3 * \widehat{\nu}(w)| \le \mathrm{const}\,(1 + \sqrt{|w|})$$

which completes the proof. □

The following statement of Green's second identity may be found in [5; page 17].

**Lemma 3.4.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a $C^1$ boundary, and let $u, v \in C^2(\overline{\Omega})$. Then*

$$\int_\Omega \Delta u\, v\, dm = \int_\Omega u \Delta v\, dm + \int_{\partial\Omega} [v D_{\overrightarrow{n}} u - u D_{\overrightarrow{n}} v]\, ds,$$

*where $\overrightarrow{n}$ denotes the outward unit normal to $\partial\Omega$.*

Our goal at present is to identify a distribution $\mu$ of the form described in Lemma 3.3 such that

$$f(x) = \phi * \mu(x), \quad \forall x \in B. \tag{3.5}$$

The following proposition displays (implicitly) a distribution whose convolution with $\phi$ agrees with $f$ on $B$. This distribution, however, is not of the desired form because its third term involves the Laplacian of the test function.

**Proposition 3.6.** *If $f \in C^\infty(\mathbb{R}^2)$, then for all $x \in B$,*

$$8\pi f(x) = \int_B \phi(\xi - x)\,\Delta^2 f(\xi)\,d\xi + \int_{\partial B} D_{\overrightarrow{n}}\phi(s-x)\,\Delta f(s) - \phi(s-x) D_{\overrightarrow{n}}\Delta f(s)\,ds$$
$$+ \int_{\partial B} D_{\overrightarrow{n}}\Delta\phi(s-x)\,f(s) - \Delta\phi(s-x)\,D_{\overrightarrow{n}} f(s)\,ds,$$

*where $\overrightarrow{n}$ is the outward unit normal to $\partial B$ at $\xi$.*

*Proof.* WLOG we may assume that $f$ is compactly supported since otherwise we could replace $f$ with a compactly supported $C^\infty$ function which agrees with $f$ on $B$. Let $a > 1$ be so large that $\operatorname{supp} f \subset aB$.

**Claim.** $\Delta^2 f * \phi = 8\pi f$.

*proof.* Since $\Delta^2 f \in C_c^\infty(\mathbb{R}^2)$ it follows that $\left(\Delta^2 f * \phi\right)\widehat{\ } = |\cdot|^4\, \widehat{f}\widehat{\phi}$. Hence $\left(\Delta^2 f * \phi\right)\widehat{\ } = 8\pi\widehat{f}$ on $\mathbb{R}^2\backslash 0$. Therefore $\Delta^2 f * \phi = 8\pi f + p$ for some polynomial $p$. In order to show that $p = 0$, it suffices to show that $\Delta^2 f * \phi(x) = 0$ for sufficiently large $|x|$. For that let $|x| > a$. Then

$$\begin{aligned}\Delta^2 f * \phi(x) &= \int_{aB} \Delta^2 f(\xi)\phi(x-\xi)\,d\xi \\ &= \int_{aB} f(\xi)\Delta^2\phi(x-\xi)\,d\xi, \qquad \text{by Lemma 3.4,} \\ &= 0, \qquad \text{as } \Delta^2\phi(x-\xi) = 0 \text{ for } \xi \in aB.\end{aligned}$$

Hence the claim.

Let $x \in B$. By the claim, we have

$$8\pi f(x) = \int_{\mathbb{R}^2} \Delta^2 f(\xi)\phi(x-\xi)\,d\xi = \int_{aB} \Delta^2 f(\xi)\phi(\xi - x)\,d\xi, \qquad \text{since } \phi(-\cdot) = \phi,$$

(3.7)

$$= \int_B \Delta^2 f(\xi)\phi(\xi-x)\,d\xi + \int_{aB\backslash B} \Delta^2 f(\xi)\phi(\xi-x)\,d\xi.$$

Since $\phi(\cdot - x) \in C^\infty(\mathbb{R}^2\backslash B)$, we may apply Lemma 3.4 twice to the latter integral above to obtain

$$\begin{aligned}&\int_{aB\backslash B} \Delta^2 f(\xi)\phi(\xi-x)\,d\xi \\ &= \int_{aB\backslash B} \Delta f(\xi)\Delta\phi(\xi-x)\,d\xi - \int_{\partial B} [D_{\overrightarrow{n}}\Delta f(s)\phi(s-x) - \Delta f(s) D_{\overrightarrow{n}}\phi(s-x)]\,ds \\ &= -\int_{\partial B} [D_{\overrightarrow{n}}\Delta f(s)\phi(s-x) - \Delta f(s) D_{\overrightarrow{n}}\phi(s-x)]\,ds \\ &\quad - \int_{\partial B} [D_{\overrightarrow{n}} f(s)\Delta\phi(s-x) - f(s) D_{\overrightarrow{n}}\Delta\phi(s-x)]\,ds,\end{aligned}$$

as $\int_{aB\backslash B} f(\xi)\Delta^2\phi(\xi - x)\,d\xi = 0$ since $\Delta^2\phi(\xi - x) = 0$ for $\xi \in aB\backslash B$. With (3.7) in view, this completes the proof. $\square$

In order to obtain a distribution of the desired form and satisfying (3.5), it suffices to find $C^\infty$ functions $g_1$ and $g_2$ such that

$$\begin{aligned}&\int_{\partial B} D_{\overrightarrow{n}}\Delta\phi(s-x)\,f(s) - \Delta\phi(s-x)\,D_{\overrightarrow{n}}f(s)\,ds\\ &= \int_{\partial B} D_{\overrightarrow{n}}\phi(s-x)\,g_1(s) + \phi(s-x)\,g_2(s)\,ds, \quad \forall x \in B.\end{aligned} \tag{3.8}$$

To do this we employ the Fourier series representations of $f_{|\partial B}$ and $D_{\overrightarrow{n}}f_{|\partial B}$, say $f(y(\theta)) = \sum_n a_n e^{in\theta}$ and $D_{y(\theta)}f(y(\theta)) = \sum_n b_n e^{in\theta}$. The purpose of the following proposition and two corollaries is to identify sequences $\{c_n\}$ and $\{d_n\}$ such that

$$\begin{aligned}&\int_{-\pi}^{\pi} D_{y(\theta)}\Delta\phi(y(\theta)-x)\,a_n e^{in\theta} - \Delta\phi(y(\theta)-x)\,b_n e^{in\theta}\,d\theta\\ &= \int_{-\pi}^{\pi} D_{y(\theta)}\phi(y(\theta)-x)\,c_n e^{in\theta} + \phi(y(\theta)-x)\,d_n e^{in\theta}\,d\theta, \quad \forall x \in B.\end{aligned}$$

That (3.8) holds will then follow with $g_1(y(\theta)) = \sum_n c_n e^{in\theta}$ and $g_2(y(\theta)) = \sum_n d_n e^{in\theta}$.

**Definition.** For $u \in C(\mathbb{R}^2)$ and $n \in \mathbb{Z}$, we define

$$R_n[u](t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}u(ty(\theta))\,d\theta,$$

where $y(\theta) := (\cos\theta, \sin\theta)$.

**Proposition 3.9.** *For $t > 0$, $x \in tB$, and $n \in \mathbb{Z}$, $|n| \geq 2$, the following hold:*

$$(i) \qquad R_0[\phi(\cdot - x)](t) = t^2\log t + |x|^2(1+\log t),$$

$$(ii) \qquad R_{\pm 1}[\phi(\cdot - x)](t) = -(x_1 \pm ix_2)(t\log t + t/2 + t^{-1}|x|^2/4),$$

$$(iii) \qquad R_n[\phi(\cdot - x)](t) = \frac{1}{2|n|}(x_1 + \operatorname{sign}(n)ix_2)^{|n|}\left(\frac{t^{2-|n|}}{|n|-1} - \frac{t^{-|n|}|x|^2}{|n|+1}\right).$$

*Proof.* (i) was proved in [6]. Since $R_{-n}[\phi(\cdot - x)](t)$ is simply the complex conjugate of $R_n[\phi(\cdot - x)](t)$, it suffices to prove (ii) and (iii) only for $n$ positive. Since, for $t > 0$ and $x \in \mathbb{R}^2$, $\phi(tx) = t^2|x|^2\log t + t^2\phi(x)$, it follows that

$$\begin{aligned}R_n[\phi(\cdot - x)](t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\phi(ty(\theta)-x)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\phi(t(y(\theta)-x/t))\,d\theta\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}[t^2|y(\theta)-x/t|^2\log t + t^2\phi(y(\theta)-x/t)]\,d\theta\\ &= t^2(\log t)R_n[|\cdot - x/t|^2](1) + t^2R_n[\phi(\cdot - x/t)](1).\end{aligned} \tag{3.10}$$

It is a simple matter to show that

$$R_n[|\cdot - \xi|^2](1) = \begin{cases} -(\xi_1 + i\xi_2) & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{3.11}$$

Define $G \in C(\mathbb{R}^2)$ by

$$G(\xi) := R_n[\phi(\cdot - \xi)](1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} \phi(y(\theta) - \xi)\, d\theta.$$

It is a straightforward matter to verify that $G_{|B} \in C^\infty(B)$, $\Delta^2 G = 0$ on $B$, and $G(\xi) = e^{in \arg(\xi_1 + i\xi_2)} G(|\xi|\, y(0))$ for $0 < |\xi| < 1$. In polar coordinates, if $\widetilde{G}(r, \theta) := G(ry(\theta))$, then this last condition can be expressed as $\widetilde{G}(r, \theta) = e^{in\theta} g(r)$, where $g(r) := \widetilde{G}(r, 0)$. The equation $\Delta^2 G = 0$, written in polar coordinates, reduces to the homogeneous differential equation

$$L^2 g = 0, \text{ where}$$
$$Lg := g'' + \frac{1}{r} g' - \frac{n^2}{r^2} g.$$

It is easy to verify that on the interval $(0 \,.\,.\, 1)$, this equation has the four linearly independent solutions $\{r, r^3, r^{-1}, r \log r\}$ (if $n = 1$) and $\{r^n, r^{2+n}, r^{-n}, r^{2-n}\}$ (if $n > 1$). It then follows from the classical theory of differential equations that there exist $a_n$, $b_n$, $c_n$, and $d_n$ such that

$$g(r) = \begin{cases} a_n r^n + b_n r^{2+n} + c_n r^{-n} + d_n r \log r & \text{if } n = 1, \\ a_n r^n + b_n r^{2+n} + c_n r^{-n} + d_n r^{2-n} & \text{if } n > 1. \end{cases}$$

It must be the case that $c_n = d_n = 0$ since otherwise $G$ would not be $C^\infty$ near the origin. Therefore, $g(r) = a_n r^n + b_n r^{2+n}$ and hence,

$$\begin{aligned} G(\xi) &= e^{in \arg(\xi_1 + i\xi_2)} (a_n |\xi|^n + b_n |\xi|^{n+2}) \\ &= (\xi_1 + i\xi_2)^n (a_n + b_n |\xi|^2), \quad \xi \in B. \end{aligned} \tag{3.12}$$

In order to find $a_n$ note that if $\xi = \tau y(0)$, then $G(\tau y(0)) = a_n \tau^n + b_n \tau^{n+2}$ and hence

$$a_n = \frac{1}{n!} \frac{d^n}{d\tau^n} G(\tau y(0))_{|\tau = 0}.$$

Put $k(\tau, \theta) := |y(\theta) - \tau y(0)|^2 = 1 - 2\tau \cos\theta + \tau^2$, $K(\tau, \theta) := \log(k(\tau, \theta))$, $F(\tau, \theta) := k(\tau, \theta) K(\tau, \theta)$, and let $'$ denote differentiation with respect to $\tau$ so that $k'(\tau, \theta) = -2\cos\theta + 2\tau$ and $k''(\tau, \theta) = 2$. Since $\phi(y(\theta) - \tau y(0)) = (1/2) F(\tau, \theta)$, it follows that

$$G(\tau y(0)) = \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{in\theta} F(\tau, \theta)\, d\theta, \quad \text{and}$$
$$a_n = \frac{1}{4\pi n!} \int_{-\pi}^{\pi} e^{in\theta} F^{(n)}(0, \theta)\, d\theta.$$

Let $\mathbb{T}_l$ denote the space of univariate trigonometric polynomials of degree $\le l$; that is, the space of functions $f$ which can be written as $f(\theta) = \sum_{j=-l}^{l} a_j e^{ij\theta}$, $\theta \in \mathbb{R}$, and let us agree that $\mathbb{T}_{-1} = \{0\}$.

**Claim.** There exists $q_n \in \mathbb{T}_{n-2}$ such that

$$K^{(n)}(0,\theta) = -(n-1)!2^n \cos^n \theta + q_n(\theta).$$

*proof.* Note that $K(0,\theta)=0$. Since $K'(\tau,\theta) = \dfrac{k'(\tau,\theta)}{k(\tau,\theta)}$, we have $K'(0,\theta) = -2\cos\theta$. A second differentiation shows that $K''(0,\theta) = -4\cos^2\theta + 2$. Thus the claim is true for $n=1,2$. Proceeding by induction, assume the claim for $n' \le n$ and consider $n+1 \ge 3$. Using Leibniz formula to differentiate the equality $K'(\tau,\theta)k(\tau,\theta) = k'(\tau,\theta)$ $n$ times yields $\sum_{l=0}^{n} \binom{n}{l} K^{(n+1-l)} k^{(l)} = k^{(1+n)} = 0$; hence

$$\begin{aligned} K^{(n+1)}(0,\theta) &= -\sum_{l=1}^{n} \binom{n}{l} K^{(n+1-l)}(0,\theta) k^{(l)}(0,\theta) \\ &= -nK^{(n)}(0,\theta)k'(0,\theta) - 2\binom{n}{2} K^{(n-2)}(0,\theta) \\ &= -n!2^{n+1}\cos^{n+1}\theta + 2nq_n(\theta)\cos\theta - 2\binom{n}{2} K^{(n-2)}(0,\theta), \qquad \text{by induction hyp.} \end{aligned}$$

Thus the claim is true for $n+1$ with $q_{n+1}(\theta) := 2nq_n(\theta)\cos\theta - 2\binom{n}{2} K^{(n-2)}(0,\theta)$.

Since $F'(0,\theta) = k'(0,\theta)K(0,\theta) + k(0,\theta)K'(0,\theta) = -2\cos\theta$ we have

$$a_1 = \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{i\theta}(-2\cos\theta)\,d\theta = -\frac{1}{2}.$$

If $n > 1$, then

$$\begin{aligned} F^{(n)}(0,\theta) &= \sum_{l=0}^{n} \binom{n}{l} k^{(l)}(0,\theta) K^{(n-l)}(0,\theta) \\ &= K^{(n)}(0,\theta) + nk'(0,\theta)K^{(n-1)}(0,\theta) + 2\binom{n}{2} K^{(n-2)}(0,\theta) \\ &= -(n-1)!2^n\cos^n\theta + q_n(\theta) + n(n-2)!2^n\cos^n\theta - 2n\cos\theta q_{n-1}(\theta) + 2\binom{n}{2} K^{(n-2)}(0,\theta) \\ &= (n-2)!2^n\cos^n\theta + \widetilde{q}_n(\theta), \qquad \text{where} \\ \widetilde{q}_n(\theta) &:= q_n(\theta) - 2n\cos\theta q_{n-1}(\theta) + 2\binom{n}{2} K^{(n-2)}(0,\theta). \end{aligned}$$

Note that $\widetilde{q}_n \in \mathbb{T}_{n-2}$. Since $\int_{-\pi}^{\pi} e^{in\theta} q(\theta)\,d\theta = 0$ for all $q \in \mathbb{T}_{n-2}$, it follows that

$$\begin{aligned} a_n &= \frac{1}{4\pi n!}\int_{-\pi}^{\pi} e^{in\theta}(n-2)!2^n\cos^n\theta\,d\theta \\ &= \frac{2^n}{2n(n-1)}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\cos^n\theta\,d\theta = \frac{1}{2n(n-1)}, \qquad n > 1. \end{aligned}$$

In order to determine the $b_n$'s, it is simpler if we first apply the Laplacian to (3.12) to obtain

$$\Delta G(\xi) = b_n 4(n+1)(\xi_1 + i\xi_2)^n, \quad \xi \in B.$$

If $\xi = \tau y(0)$, then $G(\tau y(0)) = b_n 4(n+1)\tau^n$ and hence

$$\begin{aligned} b_n &= \frac{1}{4(n+1)!}\frac{d^n}{d\tau^n}\Delta G(\tau y(0))_{|\tau=0} \\ &= \frac{1}{4(n+1)!}\frac{d^n}{d\tau^n} R_n[\Delta\phi(\cdot - \tau y(0))](1)_{|\tau=0} \end{aligned}$$

Since $\Delta\phi = 4 + 4\log|\cdot|$, it follows that $\Delta\phi(y(\theta) - \tau y(0)) = 4 + 2K(\tau,\theta)$ and hence

$$\begin{aligned} b_n &= \frac{1}{4(n+1)!}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}(4 + 2K^{(n)}(0,\theta))\,d\theta \\ &= \frac{1}{4(n+1)!}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}(4 - (n-1)!2^{n+1}\cos^n\theta + 2q_n(\theta))\,d\theta \\ &= -\frac{2^n}{2n(n+1)}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\cos^n\theta\,d\theta, \qquad \text{since } 4 + 2q_n \in \mathbb{T}_{n-2}, \\ &= -\frac{1}{2n(n+1)}. \end{aligned}$$

Considering first the case $n = 1$, we have by (3.12) that $G(\xi) = (\xi_1 + i\xi_2)(-\frac{1}{2} - \frac{1}{4}|\xi|^2)$, and hence by (3.10) and (3.11) it follows that

$$\begin{aligned} R_1[\phi(\cdot - x)](t) &= -t^2(\log t)(x_1/t + ix_2/t) + t^2 G(x/t) \\ &= -(x_1 + ix_2)t\log t + t^2(x_1/t + ix_2/t)(-1/2 - |x/t|^2/4) \\ &= -(x_1 + ix_2)(t\log t + t/2 + t^{-1}|x|^2/4). \end{aligned}$$

which proves (ii). Assume now that $n > 1$. Then by (3.12), $G(\xi) = (\xi_1 + i\xi_2)^n(\frac{1}{2n(n-1)} - \frac{1}{2n(n+1)}|\xi|^2)$, and by (3.10) and (3.11) we obtain

$$\begin{aligned} R_n[\phi(\cdot - x)](t) &= t^2 G(x/t) = t^2(x_1/t + ix_2/t)^n\left(\frac{1}{2n(n-1)} - \frac{1}{2n(n+1)}|x/t|^2\right) \\ &= \frac{1}{2n}(x_1 + ix_2)^n\left(\frac{t^{2-n}}{n-1} - \frac{t^{-n}|x|^2}{n+1}\right) \end{aligned}$$

which proves (iii). □

**Corollary 3.13.** *For all $x \in B$ and $n \in \mathbb{Z}$, $|n| \geq 2$, the following hold:*

(1) $$\begin{aligned} &R_0[D_{\vec{n}}\Delta\phi(\cdot - x)](1) = R_0[\Delta\phi(\cdot - x)](1) = 4 \\ &= 4(R_0[D_{\vec{n}}\phi(\cdot - x)](1) - R_0[\phi(\cdot - x)](1)), \end{aligned}$$

(2) $$\begin{aligned} &R_{\pm1}[D_{\vec{n}}\Delta\phi(\cdot - x)](1) = -R_{\pm1}[\Delta\phi(\cdot - x)](1) = 2(x_1 \pm ix_2) \\ &= -(R_{\pm1}[D_{\vec{n}}\phi(\cdot - x)](1) + R_{\pm1}[\phi(\cdot - x)](1)), \end{aligned}$$

(3) $$\begin{aligned} &R_n[D_{\vec{n}}\Delta\phi(\cdot - x)](1) = -|n|\,R_n[\Delta\phi(\cdot - x)](1) = 2(x_1 + \operatorname{sign}(n)ix_2)^{|n|} \\ &= 2|n|(|n| - 1)(R_n[D_{\vec{n}}\phi(\cdot - x)](1) + |n|\,R_n[\phi(\cdot - x)](1)). \end{aligned}$$

*where for brevity we have written* $R_n[D_{\overrightarrow{n}}u](1)$ *in place of* $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta} D_{\overrightarrow{n}}u(y(\theta))\,d\theta$, $\overrightarrow{n}$ *being the outward unit normal to* $\partial B$ *at* $y(\theta)$.

*Proof.* Let $\Delta_x$ denote the Laplacian with respect to $x$, ie. $\Delta_x := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. It is a straightforward matter to verify that

$$R_n[\Delta u(\cdot - x)](t) = \Delta_x R_n[u(\cdot - x)](t), \quad \text{and} \tag{3.14}$$

$$R_n[D_{\overrightarrow{n}}u(\cdot - x)](1) = \frac{\partial}{\partial t} R_n[u(\cdot - x)](t)_{|t=1}. \tag{3.15}$$

Evaluating (i) at $t = 1$ yields $R_0[\phi(\cdot - x)](1) = |x|^2$; while it follows from (i) and (3.15) that $R_0[D_{\overrightarrow{n}}\phi(\cdot - x)](1) = 1 + |x|^2$. Hence, $4R_0[D_{\overrightarrow{n}}\phi(\cdot - x)](1) - 4R_0[\phi(\cdot - x)](1) = 4$. It follows from (i) and (3.14) that

$$R_0[\Delta\phi(\cdot - x)](t) = 4(1 + \log t). \tag{3.16}$$

Evaluating (3.16) at $t = 1$ yields $R_0[\Delta\phi(\cdot - x)](1) = 4$; while it follows from (3.16) and (3.15) that $R_0[D_{\overrightarrow{n}}\Delta\phi(\cdot - x)](1) = 4$. Hence (1). In order to prove (2) and (3), it suffices to consider only $n \geq 1$ as the remaining cases follow simply by complex conjugation. In a similar manner to the above, it can be deduced from (ii), (3.14), and (3.15) that $R_1[\phi(\cdot - x)](1) = -(x_1 + ix_2)(\frac{1}{2} + \frac{1}{4}|x|^2)$, $R_1[D_{\overrightarrow{n}}\phi(\cdot - x)](1) = -(x_1 + ix_2)(\frac{3}{2} - \frac{1}{4}|x|^2)$, $R_1[\Delta\phi(\cdot - x)](1) = -2(x_1 + ix_2)$, and $R_1[D_{\overrightarrow{n}}\Delta\phi(\cdot - x)](1) = 2(x_1 + ix_2)$ from which (2) readily follows. Similarly, it follows from (iii), (3.14), and (3.15) that $R_n[\phi(\cdot - x)](1) = \frac{1}{2n}(x_1 + ix_2)^n(\frac{1}{n-1} - \frac{|x|^2}{n+1})$, $R_n[D_{\overrightarrow{n}}\phi(\cdot - x)](1) = \frac{1}{2n}(x_1 + ix_2)^n(\frac{2-n}{n-1} + \frac{n|x|^2}{n+1})$, $R_n[\Delta\phi(\cdot - x)](1) = -\frac{2}{n}(x_1 + ix_2)^n$, and $R_n[D_{\overrightarrow{n}}\Delta\phi(\cdot - x)](1) = 2(x_1 + ix_2)^n$ from which (3) readily follows. $\square$

**Corollary 3.17.** *Let* $f \in C^\infty(\mathbb{R}^2)$, *and define sequences* $\{c_n\}_{n\in\mathbb{Z}}$, $\{d_n\}_{n\in\mathbb{Z}}$ *by*

$$c_n := \begin{cases} 4(a_0 - b_0) & \text{if } n = 0, \\ -(a_n + b_n) & \text{if } |n| = 1, \\ 2(|n| - 1)(|n|\,a_n + b_n) & \text{if } |n| > 1; \end{cases} \qquad d_n := \begin{cases} -c_0 & \text{if } n = 0, \\ |n|\,c_n & \text{if } n \neq 0, \end{cases}$$

*where*

$$a_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(y(\theta))\,d\theta \quad \text{and } b_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} D_{y(\theta)} f(y(\theta))\,d\theta.$$

*Then for all* $x \in B$,

$$8\pi f(x) = \int_B \phi(\xi - x)\Delta^2 f(\xi)\,d\xi + \int_{-\pi}^{\pi} D_{y(\theta)}\phi(y(\theta) - x)[\Delta f(y(\theta)) + \sum_{n\in\mathbb{Z}} c_n e^{in\theta}]\,d\theta$$

$$+ \int_{-\pi}^{\pi} \phi(y(\theta) - x)[-D_{y(\theta)}\Delta f(y(\theta)) + \sum_{n\in\mathbb{Z}} d_n e^{in\theta}]\,d\theta.$$

*Proof.* In light of Proposition 3.6, it suffices to show that for all $x \in B$,

$$\int_{-\pi}^{\pi} D_{y(\theta)}\Delta\phi(y(\theta)-x)f(y(\theta)) - \Delta\phi(y(\theta)-x)D_{y(\theta)}f(y(\theta))\,d\theta$$
$$= \int_{-\pi}^{\pi} D_{y(\theta)}\phi(y(\theta)-x)\sum_{n\in\mathbb{Z}} c_n e^{in\theta} + \phi(y(\theta)-x)\sum_{n\in\mathbb{Z}} d_n e^{in\theta}\,d\theta.$$

Since $f \in C^\infty(\mathbb{R}^2)$, the following hold:

$$\lim_{|n|\to\infty} (|a_n| + |b_n|)\,|n|^m = 0, \quad \forall m > 0,$$
$$\lim_{|n|\to\infty} (|c_n| + |d_n|)\,|n|^m = 0 \quad \forall m > 0,$$
$$f(y(\theta)) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}, \quad \forall \theta \in [-\pi \,..\, \pi],$$
$$D_{y(\theta)}f(y(\theta)) = \sum_{n\in\mathbb{Z}} b_n e^{in\theta}, \quad \forall \theta \in [-\pi \,..\, \pi].$$

It follows from Corollary 3.13 that

$$a_n R_n[D_{\vec{n}}\Delta\phi(\cdot - x)](1) - b_n R_n[\Delta\phi(\cdot - x)](1)$$
$$= c_n R_n[D_{\vec{n}}\phi(\cdot - x)](1) + d_n R_n[\phi(\cdot - x)](1) \quad \forall n \in \mathbb{Z}, x \in B.$$

Hence, for $x \in B$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_{y(\theta)}\Delta\phi(y(\theta)-x)f(y(\theta)) - \Delta\phi(y(\theta)-x)D_{y(\theta)}f(y(\theta))\,d\theta$$
$$= \sum_{n\in\mathbb{Z}} (a_n R_n[D_{\vec{n}}\Delta\phi(\cdot - x)](1) - b_n R_n[\Delta\phi(\cdot - x)](1))$$
$$= \sum_{n\in\mathbb{Z}} (c_n R_n[D_{\vec{n}}\phi(\cdot - x)](1) + d_n R_n[\phi(\cdot - x)](1))$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_{y(\theta)}\phi(y(\theta)-x)\sum_{n\in\mathbb{Z}} c_n e^{in\theta} + \phi(y(\theta)-x)\sum_{n\in\mathbb{Z}} d_n e^{in\theta}\,d\theta.$$

□

*Proof of Theorem 3.1.* Let $f \in C^\infty(\mathbb{R}^2)$, and let $\{c_n\}_{n\in\mathbb{Z}}$ and $\{d_n\}_{n\in\mathbb{Z}}$ be as defined in Corollary 3.17. Define the distribution $\mu$ by

$$\langle g, \mu\rangle := \frac{1}{8\pi}\int_B g(\xi)\Delta^2 f(\xi)\,d\xi + \frac{1}{8\pi}\int_{-\pi}^{\pi} D_{y(\theta)}g(y(\theta))(\Delta f(y(\theta)) + \sum_{n\in\mathbb{Z}} c_n e^{in\theta})\,d\theta$$
$$+ \frac{1}{8\pi}\int_{-\pi}^{\pi} g(y(\theta))(-D_{y(\theta)}\Delta f(y(\theta)) + \sum_{n\in\mathbb{Z}} d_n e^{in\theta})\,d\theta.$$

It follows from Corollary 3.17 (and from the fact that $\phi = \phi(-\cdot)$) that $\phi * \mu(x) = f(x)$ for all $x \in B$. Define distributions $\mu_{(0,0)}$, $\mu_{(1,0)}$, and $\mu_{(0,1)}$ by

$$\langle g, \mu_{(0,0)} \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} D_{y(\theta)} g(y(\theta)) - g(y(\theta))\, d\theta,$$
$$\langle g, \mu_{(1,0)} \rangle := -\frac{1}{4\pi} \int_{-\pi}^{\pi} \big(D_{y(\theta)} g(y(\theta)) + g(y(\theta))\big) \cos\theta\, d\theta,$$
$$\langle g, \mu_{(0,1)} \rangle := -\frac{1}{4\pi} \int_{-\pi}^{\pi} \big(D_{y(\theta)} g(y(\theta)) + g(y(\theta))\big) \sin\theta\, d\theta.$$

It follows from Corollary 3.13 that for all $x \in B$, $\phi * \mu_{(0,0)}(x) = 1$, $\phi * \mu_{(1,0)}(x) = x_1$, and $\phi * \mu_{(0,1)}(x) = x_2$. And it can be shown with a simple integration that $\langle 1, \mu_{(0,0)} \rangle = -1$, $\langle ()^{(1,0)}, \mu_{(1,0)} \rangle = \langle ()^{(0,1)}, \mu_{(0,1)} \rangle = -1/2$, and $\langle ()^{\alpha}, \mu_{\beta} \rangle = 0$ whenever $\alpha, \beta \in \{(0,0), (1,0), (0,1)\}$ and $\alpha \neq \beta$. Define the distribution $\widetilde{\mu}$ and the polynomial $\widetilde{p} \in \Pi_1$ by

$$\widetilde{\mu} := \mu + \langle 1, \mu \rangle \mu_{(0,0)} + 2\langle ()^{(1,0)}, \mu \rangle \mu_{(1,0)} + 2\langle ()^{(0,1)}, \mu \rangle \mu_{(0,1)},$$
$$\widetilde{p} := -\langle 1, \mu \rangle - 2\langle ()^{(1,0)}, \mu \rangle ()^{(1,0)} - 2\langle ()^{(0,1)}, \mu \rangle ()^{(0,1)},$$

and note that $\operatorname{supp} \widetilde{\mu} \subset \overline{B}$. Put $\widetilde{f} := \phi * \widetilde{\mu} + \widetilde{p}$. Then, for $x \in B$,

$$\begin{aligned}\widetilde{f}(x) =& \phi * \mu(x) + \langle 1, \mu \rangle (\phi * \mu_{(0,0)}(x) - 1) + 2\langle ()^{(1,0)}, \mu \rangle (\phi * \mu_{(1,0)}(x) - x_1) \\ &+ 2\langle ()^{(0,1)}, \mu \rangle (\phi * \mu_{(0,1)} - x_2) = f(x).\end{aligned}$$

Since $f \in C^{\infty}(\mathbb{R}^2)$, it follows by Lemma 3.3 that $\left|\widehat{\widetilde{\mu}}(w)\right| \leq \mathrm{const}(1 + \sqrt{|w|})$, $w \in \mathbb{R}^2$. In the definition of $\widetilde{\mu}$, the coefficients of $\mu_{(0,0)}$, $\mu_{(1,0)}$, and $\mu_{(0,1)}$ were chosen to ensure that $\langle 1, \widetilde{\mu} \rangle = \langle ()^{(1,0)}, \widetilde{\mu} \rangle = \langle ()^{(0,1)}, \widetilde{\mu} \rangle = 0$. It follows from this that $\widehat{\widetilde{\mu}}(0) = D^{(1,0)} \widehat{\widetilde{\mu}}(0) = D^{(0,1)} \widehat{\widetilde{\mu}}(0) = 0$ from which we conclude $\left|\widehat{\widetilde{\mu}}(w)\right| \leq \mathrm{const}\, |w|^2$. Hence, $\left|\widehat{\widetilde{\mu}}(w)\right| \leq \mathrm{const}\, \dfrac{|w|^2}{1 + |w|^{3/2}}$, $w \in \mathbb{R}^2$. Therefore $\widetilde{f} \in \mathcal{F}$. $\square$

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