A bound on the L_∞-Norm of L_2-Approximation by Splines in Terms of a Global Mesh Ratio

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Abstract. Let $L_k f$ denote the least squares approximation to $f \in \mathbf{L}_1$ by splines of order k with knot sequence $\mathbf{t} = (t_i)_1^{n+k}$. In connection with their work on Galerkin's method for solving differential equations, Douglas, Dupont and Wahlbin have shown that the norm $\|L_k\|_\infty$ of L_k as a map on $\mathbf{L}_\infty$ can be bounded as follows,

$$\|L_k\|_\infty \le \mathrm{const}_k M_{\mathbf{t}},$$

with $M_{\mathbf{t}}$ a global mesh ratio, given by

$$M_{\mathbf{t}} := \max_i \Delta t_i / \min\{\Delta t_i : \Delta t_i > 0\}.$$

Using their very nice idea together with some facts about B-splines, it is shown here that even

$$\|L_k\|_\infty \le \mathrm{const}_k (M_{\mathbf{t}}^{(k)})^{1/2}$$

with the smaller global mesh ratio $M_{\mathbf{t}}^{(k)}$ given by

$$M_{\mathbf{t}}^{(k)} := \max_{i,j} (t_{i+k} - t_i)/(t_{j+k} - t_j).$$

A mesh independent bound for $\mathbf{L}_2$-approximation by continuous piecewise polynomials is also given.

1. Introduction. This note is an addendum to the clever paper by Douglas, Dupont and Wahlbin [2] in which these authors bound the linear map of least–squares approximation by splines of order k with knot sequence $\mathbf{t} := (t_i)$, as a map on $\mathbf{L}_\infty$, in terms of the particular global mesh ratio

$$M_{\mathbf{t}} := \max_i \Delta t_i / \min\{\Delta t_i : \Delta t_i > 0\}.$$

Their argument is very elegant. But their result is puzzling in one aspect: The ratio $M_{\mathbf{t}}$ is not a continuous function of $\mathbf{t}$. If, e.g., $\mathbf{t}$ is uniform, hence $M_{\mathbf{t}} = 1$, and we now let $\mathbf{t} \to \mathbf{t}^*$ by letting just one knot approach its neighbor, leaving all other knots fixed, then

$$\lim_{\mathbf{t} \to \mathbf{t}^*} M_{\mathbf{t}} = \infty, \quad \text{while } M_{\mathbf{t}^*} = 2.$$

Correspondingly, their bound goes to infinity as $\mathbf{t} \to \mathbf{t}^*$, yet is again finite for the particular knot sequence $\mathbf{t}^*$.

This puzzling aspect is removed below. It is shown that (as asserted in a footnote to [1]) their very nice argument can be used to give a bound in terms of the smaller global mesh ratio

$$M_{\mathbf{t}}^{(k)} := \max_i (t_{i+k} - t_i) / \min_i (t_{i+k} - t_i) \tag{1}$$

which does depend continuously on $\mathbf{t}$ in $\{\mathbf{t} \in \mathbb{R}^{n+k} : t_i \le t_{i+1}, t_i < t_{i+k}, \text{ all } i\}$.

2. Least-squares approximation by splines of order k. Let $\mathbf{t} := (t_i)_1^{n+k}$ be a nondecreasing sequence, with $t_i < t_{i+k}$, all i. A spline of order k with knot sequence $\mathbf{t}$ is, by definition, any function of the form

$$\sum_{i=1}^n \alpha_i N_i$$

with $\boldsymbol{\alpha} \in \mathbb{R}^n$ and N_i the normalized B-spline of order k with knots $t_i, \ldots, t_{i+k}$, i.e.,

$$N_i(t) := N_{i,k,\mathbf{t}}(t) := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}.$$

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In words, for each t, $N_i(t)$ is $(t_{i+k} - t_i)$ times the kth divided difference at $t_i, \ldots, t_{i+k}$ of $(s-t)_+^{k-1}$ as a function of s.

We denote the totality of all splines of order k with knot sequence $\mathbf{t}$ by $\mathbf{S}_{k,\mathbf{t}}$. More detail about $\mathbf{S}_{k,\mathbf{t}}$ is provided in [1] and its references.

Next, let L_k denote the linear projector on $\mathbf{L}_1$ defined by the condition that $L_k f \in \mathbf{S}_{k,\mathbf{t}}$, and, for all $g \in \mathbf{S}_{k,\mathbf{t}}$, $\int (f - L_k f) g = 0$, i.e., $L_k f$ is the $\mathbf{L}_2$-approximation to f in $\mathbf{S}_{k,\mathbf{t}}$. We are interested in estimating the norm $\|L_k\|_p$ of L_k as a map on $\mathbf{L}_p$. Since

$$\|L_k\|_p = \|L_k\|_q \quad \text{for } 1/p + 1/q = 1,$$

and $\|L_k\|_2 = 1$, interpolation will give a bound on $\|L_k\|_p$ in terms of $\|L_k\|_\infty = \|L_k\|_1$, as is pointed out in [2]. It therefore suffices to consider $\|L_k\|_\infty$.

Let $L_k f = \sum \alpha_j N_j$. Then $\|L_k f\|_\infty \le \|\boldsymbol{\alpha}\|_\infty$ since $N_i \ge 0$, all i, and $\sum_j N_j \le 1$, while

$$\sum_j \int N_i N_j \alpha_j = \int N_i f \le [(t_{i+k} - t_i)/k] \|f\|_\infty, \quad \text{all } i,$$

since $N_i \ge 0$ and $\int N_i = (t_{i+k} - t_i)/k$. Therefore,

$$\|L_k\|_\infty \le \|G^{-1}\|_\infty \tag{2}$$

with

$$G := G_\infty = E^{1/2} G_2 E^{-1/2}, \tag{3}$$

where E is a diagonal matrix,

$$E := \lceil k/(t_{k+1} - t_1), \ldots, k/(t_{k+n} - t_n) \rfloor, \tag{4}$$

and G_2 is the Gramian matrix for the basis $(\overset{2}{N}_i)$ of $\mathbf{S}_{k,\mathbf{t}}$, i.e.,

$$G_2 := \Big(\int \overset{2}{N}_i \overset{2}{N}_j \Big)_{i,j=1}^n \tag{5}$$

and

$$\overset{p}{N}_i := [k/(t_{i+k} - t_i)]^{1/p} N_i. \tag{6}$$

With this normalization, we are assured of the existence of a positive constant D_k depending only on k and not at all on $\mathbf{t}$ or n so that

$$D_k^{-1} \|\boldsymbol{\alpha}\|_p \le \Big\| \sum_j \alpha_j \overset{p}{N}_j \Big\|_p \le \|\boldsymbol{\alpha}\|_p, \quad \text{all } \boldsymbol{\alpha} \in \mathbb{R}^n \tag{7}$$

(see the theorem on p.539 of [1]). This inequality implies that

$$\|G_2^{-1}\|_\infty \le \text{const}_k \tag{8}$$

for some const_k depending only on k as we will show below; and, on combining this with (2)-(4), we obtain the desired conclusion

$$\|L_k\|_\infty \le \text{const}_k (M_{\mathbf{t}}^{(k)})^{1/2}. \tag{9}$$

3. A bound for $\|G_2^{-1}\|_\infty$. With $(\alpha_{ij})_{i,j=1}^n := G_2^{-1}$, let $f_i := \sum_j \alpha_{ij} \overset{2}{N}_j$. Then

$$\int f_i \overset{2}{N}_j = \delta_{ij}, \quad \text{all } j;$$

hence

$$\int \alpha_{ii} \overset{2}{N}_i f_i + \sum_{j \neq i} \alpha_{ij} \overset{2}{N}_j f_i = \alpha_{ii},$$

i.e.,

$$\|f_i\|_2^2 = \alpha_{ii}. \tag{10}$$

Therefore, by (7),

$$D_k^{-2} \alpha_{ii}^2 \le D_k^{-2} \sum_j |\alpha_{ij}|^2 \le \|f_i\|^2 = \alpha_{ii},$$

hence, as $\alpha_{ii} = \|f_i\|_2^2 \neq 0$ (G_2^{-1} is invertible!), we have $\alpha_{ii} \le D_k^2$; and so, $\|f_i\|_2 \le D_k$ and

$$\big(\sum_j |\alpha_{ij}|^2\big)^{1/2} \le D_k \|f_i\|_2 = D_k (\alpha_{ii})^{1/2} \le D_k^2. \tag{11}$$

This shows that

$$\|G_2^{-1}\|_\infty = \max_i \sum_j |\alpha_{ij}| \le n^{1/2} \max_i \big(\sum_j |\alpha_{ij}|^2\big)^{1/2} \le n^{1/2} D_k^2$$

and so bounds $\|G_2^{-1}\|_\infty$ in terms of only k and n. From this, one obtains

$$\|G^{-1}\|_\infty \le (n M_{\mathbf{t}}^{(k)})^{1/2} D_k^2,$$

a bound in terms of the desired global mesh ratio, except that the bound goes to infinity with the number of mesh points. Note that we can express $M_{\mathbf{t}}^{(k)}$ in terms of n and the local mesh ratio

$$m_{\mathbf{t}}^{(k)} := \max_{|i-j|=1} (t_{i+k} - t_i)/(t_{j+k} - t_j);$$

hence, we even have a bound on $\|G^{-1}\|_\infty$ in terms of that *local* mesh ratio but, alas, involving also n.

In order to remove this dependence on n, we use the ideas of Douglas, Dupont and Wahlbin [2] to prove the following lemma.

Lemma 1. *There exist* const_k *and* $\lambda_k \in (0,1)$ *independent of* n *or* $\mathbf{t}$ *so that, for all* i *and* j,

$$|\alpha_{ij}| \le \text{const}_k (\lambda_k)^{|i-j|}.$$

Proof: We observed earlier that the function $f_i = \sum_j \alpha_{ij} \overset{2}{N}_j$ is orthogonal to $\text{span}(N_j)_{j \neq i}$. Hence, for any $m > i$,

$$f_{i,m} := \sum_{m \le j} \alpha_{ij} \overset{2}{N}_j$$

is orthogonal to f_i and, therefore, also orthogonal to $f_{i,m-k+1}$ since the latter function agrees with f_i on the support of $f_{i,m}$. This proves that

$$\|f_{i,m-k+1}\|_2^2 + \|-f_{i,m}\|_2^2 \;=\; \|f_{i,m-k+1} - f_{i,m}\|_2^2 \tag{12}$$

from which we conclude that

$$\|\sum_{m-k<j} \alpha_{ij} \overset{2}{N}_j \|_2^2 \;\le\; \|\sum_{m-k<j<m} \alpha_{ij} \overset{2}{N}_j \|_2^2$$

or, with the inequality (7),

$$\sum_{m-k<j<m} |\alpha_{ij}|^2 \ge D_k^{-2} \sum_{m-k<j} |\alpha_{ij}|^2, \quad m = i+1, i+2, \ldots \tag{13}$$

Faced with a similar inequality, Douglas, Dupont and Wahlbin [2] make use of what amounts to the following discrete Gronwall inequality:

Lemma 2. *If the sequence a_0, a_1, … satisfies*

$$(14)\qquad |a_m| \ge c \sum_{m\le j} |a_j|, \quad m = 0,1,2\ldots,$$

for some $c \in (0,1)$, then $\lambda := 1-c \in (0,1)$ and

$$(15)\qquad |a_m| \le |a_0|\lambda^m/c, \quad m = 0,1,2,\ldots.$$

Proof: Let $A_m := \sum_{m\le j} |a_j|$. Then (14) reads

$$A_m - A_{m+1} \ge cA_m, \quad \text{all } m,$$

or, $A_{m+1} \le (1-c)A_m$, all m, therefore, with $\lambda := 1-c$,

$$A_{m+j} \le \lambda^j A_m, \quad \text{all } m, j,$$

and so,

$$|a_m| = A_m - A_{m+1} \le A_m \le \lambda^m A_0 \le |a_0|\lambda^m/c. \qquad \text{Q.E.D.}$$

In order to apply this lemma to (12), we pick $m_0 > i$ and let

$$J_m := \{j \in \mathbb{Z} : m_0 + (k-1)(m-1) \le j < m_0 + (k-1)m\}, \quad m = 0,1,\ldots.$$

Then, with

$$a_m := \sum_{j\in J_m} |\alpha_{ij}|^2, \quad \text{all } m,$$

we obtain from (12) that

$$a_m \ge D_k^{-2} \sum_{m\le j} a_j, \quad m = 0,1,2,\ldots;$$

hence, from the lemma,

$$\max_{j\in J_m} |\alpha_{ij}| \le a_m^{1/2} \le D_k(1-D_k^{-2})^{m/2} a_0^{1/2}$$

while, by (11),

$$a_0^{1/2} \le \big(\sum_j |\alpha_{ij}|^2\big)^{1/2} \le D_k^2.$$

This proves the asserted exponential decay of $|\alpha_{ij}|$ for $j > i$; but G_2 is symmetric. Q.E.D.

It follows at once that

$$(16)\qquad \|G_2^{-1}\|_\infty \le \text{const}_k 2/(1-\lambda_k).$$

In view of the discussion at the end of Section 2, we have therefore proved the following theorem.

Theorem 1. *There exists a constant c depending only on k so that the norm $\|L_k\|_\infty$ of $\mathbf{L}_2$-approximation by splines of order k with knot sequence $\mathbf{t}$, as a map on $\mathbf{L}_\infty$, satisfies*

$$\|L_k\|_\infty \le c(M_{\mathbf{t}}^{(k)})^{1/2}$$

with the global mesh ratio $M_{\mathbf{t}}^{(k)}$ given by

$$M_{\mathbf{t}}^{(k)} := \max_{i,j}(t_{i+k} - t_i)/(t_{j+k} - t_j).$$

There seems to be little hope that this argument would even support a bound in terms of $m_{\mathbf{t}}^{(k)}$, let alone a bound independent of the mesh $\mathbf{t}$.

4. A mesh independent bound for $\mathbf{L}_2$-approximation by C^0-piecewise polynomials. Pick $k>1$. Let $\boldsymbol{\xi}=(\xi_i)_1^r$ in (a,b) with $a=:\xi_0<\cdots<\xi_{r+1}:=b$, and let Pf be the $\mathbf{L}_2$-approximation to f by elements of $\mathbf{P}_{k,\boldsymbol{\xi}}\cap C^0 := \{f\in C[a,b] : f|_{(\xi_i,\xi_{i+1})}\in \mathbf{P}_k\}$. Todd Dupont [3] has shown some time ago that P can be bounded as a map on $\mathbf{L}_\infty$ independently of $\boldsymbol{\xi}$ by constructing a basis for $\operatorname{ran} P$ for which a certain matrix related to the Gramian is strictly diagonally dominant. We take the occasion to give a proof in terms of B-splines.

If $\mathbf{t}=(t_i)_1^{n+k}$ is the nondecreasing sequence which contains a and b exactly k times and each of $\xi_1,\ldots,\xi_r$, exactly $k-1$ times (and nothing else), then

$$\mathbf{P}_{k,\boldsymbol{\xi}}\cap C^0=\mathbf{S}_{k,\mathbf{t}},$$

hence then $P=L_k$ introduced in Section 2, therefore, $\|P\|\le\|G^{-1}\|$ with G given by (3)-(6) in terms of $\mathbf{t}$ as determined from $\boldsymbol{\xi}$.

Theorem 2. *Let $\widehat{G}:=(k\int_0^1\widehat{N}_i\widehat{N}_j)_{i,j=1}^k$ be the matrix G in the special case $r=0$, $[a,b]=[0,1]$. Then, for all $\boldsymbol{\xi}$, $\|G^{-1}\|_\infty=\|\widehat{G}^{-1}\|_\infty$. In particular, $\|P\|\le\|\widehat{G}^{-1}\|_\infty$ for all $\boldsymbol{\xi}$. Hence (T.Dupont) $\sup_{\boldsymbol{\xi}}\|P\|<\infty$.*

Proof: Let $\xi_{-1}=a$, $\xi_{r+2}=b$. Then, for $m=0,\ldots,r+1$, $N_{m(k-1)+1}$ has its support on the two intervals (ξ_{m-1},ξ_{m+1}) of $\boldsymbol{\xi}$. All other N_i have their support in just one interval. Correspondingly, the matrix G is almost block diagonal, with $r+1$ $k\times k$ blocks overlapping in just one row and column. For $k=4$ (the cubic case) and $r=2$ this looks like

$$\begin{matrix}
x & x & x & x & & & & & & \\
x & x & x & x & & & & & & \\
x & x & x & x & & & & & & \\
x & x & x & x & x & x & x & & & \\
 & & & x & x & x & x & & & \\
 & & & x & x & x & x & & & \\
 & & & x & x & x & x & x & x & x \\
 & & & & & & x & x & x & x \\
 & & & & & & x & x & x & x \\
 & & & & & & x & x & x & x
\end{matrix}$$

Since the linear change of the independent variable taking $[\xi_m,\xi_{m+1}]$ to $[0,1]$ carries

$$N_{m(k-1)+i}\ \text{ on } [\xi_m,\xi_{m+1}] \text{ to } \widehat{N}_i \text{ on } [0,1],\quad i=1,\ldots,k,$$

we have

$$(17)\qquad G_{m(k-1)+i,m(k-1)+j}=\left.\begin{cases}(\Delta\xi_m/(\xi_{m+1}-\xi_{m-1}))\widehat{G}_{1,j}, & i=1\\ \widehat{G}_{ij}, & i=2,\ldots,k-1\\ (\Delta\xi_m/(\xi_{m+2}-\xi_m))\widehat{G}_{kj}, & i=k\end{cases}\right\},\quad j=1,\ldots,k,$$

for $m=0,\ldots,r$. This says that each of the $r+1$ blocks of G is essentially equal to $\widehat{G}$.

G is totally positive by [1]. Its inverse is therefore a checkerboard matrix, hence (see [1, p. 541])

$$(18)\qquad \text{if } \mathbf{y} \text{ is such that } \sum_j G_{ij}(-)^{i+j}y_j=1,\quad \text{all } i,\quad \text{then } \|G^{-1}\|_\infty=\|\mathbf{y}\|_\infty.$$

But such a $\mathbf{y}$ is easily constructed. Take $\mathbf{x}=(x_1,\ldots,x_k)$ so that

$$(19)\qquad \sum_j\widehat{G}_{ij}(-)^{i+j}x_j=1,\quad \text{all } i,$$

and extend $\mathbf{x}$ to a $(k-1)$-periodic function $\mathbf{y} = (y_i)_1^n$ on all of $(1, \ldots, n)$. This is possible since $x_k = x_1$ by symmetry. Then, for $i = m(k-1) + I$, we have from (17) and (19) that

$$\sum_j G_{ij}(-1)^{i+j} y_j = \sum_{j=1}^{k} \widehat{G}_{Ij}(-)^{I+j} x_j = 1, \quad I = 2, \ldots, k-1; \quad m = 0, \ldots, r,$$

and also

$$\begin{aligned}\sum_j G_{ij}(-)^{i+j} y_j =& (\Delta\xi_{m-1}/(\xi_{m+1} - \xi_{m-1})) \sum_j \widehat{G}_{kj}(-)^{k+j} x_j \\ &+ (\Delta\xi_m/(\xi_{m+1} - \xi_{m-1})) \sum_j \widehat{G}_{1j}(-)^{1+j} x_j = 1 \\ &\qquad \text{for } I = 1; \quad m = 0, \ldots, r+1.\end{aligned}$$

This proves with (18) that

$$\|G^{-1}\|_\infty = \|\mathbf{y}\|_\infty = \|\mathbf{x}\|_\infty = \|\widehat{G}^{-1}\|_\infty. \qquad \text{Q.E.D.}$$

References

[1] C. de Boor, "Bounding the error in spline interpolation", *SIAM Rev.*, v.16, 1974, pp. 531–544. MR **50** # 13976.

[2] Jim Douglas Jr., Todd Dupont, and Lars Wahlbin, "Optimal L_∞ error estimates for Galerkin approximations to solutions of two-point boundary value problems", *Math. Comp.*, v.29, 1975, pp. 475–483. MR **51** # 7298.

[3] T. Dupont, Private Communication.