The limit at the origin of a smooth function space

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Abstract

The map $H \to H_\downarrow$ assigns to each finite-dimensional space of smooth functions a homogeneous polynomial space of the same dimension. We discuss applications of this map in the areas of multivariate polynomial interpolation, box spline theory and polynomial ideals.

§1. Introduction

Let A_0 be the space of all s-dimensional complex-valued functions analytic at the origin. For $f \in A_0$, we write its power series expansion at the origin as

$$f = f_0 + f_1 + f_2 + ..., \tag{1}$$

where, for each j, f_j is a homogeneous polynomial of degree j. We denote by $f_\downarrow$ the *least term of* f, i.e., the homogeneous polynomial f_k with $k := \max\{j : f_i = 0,\ \forall i < j\}$. For a finite-dimensional subspace H of A_0, we define

$$H_\downarrow := \operatorname{span}\{f_\downarrow :\ f \in H\}. \tag{2}$$

Note that $\deg f_\downarrow = k$ if and only if $T_k(f) = 0 \neq T_{k+1}(f)$, where $T_k : H \to \pi_{<k} : f \mapsto f_0 + f_1 + ... + f_{k-1}$. This means that

$$\dim\{f_\downarrow :\ f \in H,\ \deg f_\downarrow = k \text{ or } f = 0\} = \dim\ker T_k - \dim\ker T_{k+1}. \tag{3}$$

Summing this equation over all k, we obtain

Proposition 1. *The space $H_\downarrow$ is a homogeneous polynomial space of the same dimension as H.*

In [2], we provide the following simple algorithm for the computation of a basis for $H_\downarrow$ from a given basis for H. Its description uses the inner product

$$\langle p, q\rangle := p(D)q(0) = q(D)p(0) = \sum_{\alpha\in\mathbb{Z}_+^s} \frac{D^\alpha p(0) D^\alpha q(0)}{\alpha!}, \tag{4}$$

with $p(D) := \sum_\alpha \big(D^\alpha p\big)(0)/\alpha!\ D^\alpha$ the differential operator induced by the polynomial p.

Approximation Theory VI
C. K. Chui, L. L. Schumaker and J. D. Ward (eds.)

ISBN 0-12-17585-6:

Algorithm 2 [2]. *Given the basis (p_j) of the finite-dimensional subspace H of A_0.*

For $k = 1, 2, \ldots$, carry out the following three steps:

Step 1. $$q_k \leftarrow p_k - \sum_{j<k} q_j \frac{\langle r_j, p_k\rangle}{\langle r_j, q_j\rangle}$$

Step 2. $$r_k \leftarrow q_{k\downarrow}$$

Step 3. $$q_j \leftarrow q_j - q_k \frac{\langle r_k, q_j\rangle}{\langle r_k, q_k\rangle} \qquad \textit{if } \deg r_k > \deg r_j.$$

Then (r_j) is bi-orthogonal to (q_j) and provides a homogeneous orthogonal basis for $H_\downarrow$.

§2. $H_\downarrow$ and multivariate polynomial interpolation

Let $\mathbf{\Theta}$ be a finite subset of $\mathbb{R}^s$ ($\mathbb{C}^s$ will do as well). For each $\theta \in \mathbf{\Theta}$, let P_θ be a finite-dimensional polynomial space. In the interpolation problem $IP(\mathbf{\Theta}; P)$, we seek a polynomial space Q such that, for every smooth function f, there exists a unique $q_f \in Q$ satisfying

$$p(D)f(\theta) = p(D)q_f(\theta), \ \forall \theta \in \mathbf{\Theta}, \ p \in P_\theta. \tag{5}$$

We have

Theorem 3 [2]. *For given $IP(\mathbf{\Theta}; P)$, define $H := \operatorname{span}\{e_\theta p : \ \theta \in \mathbf{\Theta}, \ p \in P_\theta\}$. Then the space $H_\downarrow$ solves $IP(\mathbf{\Theta}; P)$, and is of least degree among all the solutions Q of that interpolation problem in the sense that*

$$\dim(\pi_j \cap H_\downarrow) \ge \dim(\pi_j \cap Q), \quad \forall j, Q.$$

§3. The polynomials in a box spline space

The polynomial space associated with a given (polynomial) box spline B_X is defined as follows: Let $X \subset \mathbb{R}^s \backslash 0$ be a spanning multiset for $\mathbb{R}^s$ and

$$\mathbb{K}(X) := \{K \subset X : \ \operatorname{span}(X \backslash K) \ne \mathbb{R}^s\}.$$

Also, for $Z \subset X$, define the homogeneous polynomial $p_Z := \prod_{x \in Z} \langle x, \cdot\rangle$, with $\langle x, y\rangle$ the scalar product of $x, y \in \mathbb{R}^s$. The polynomial space $\mathcal{H}(X)$, defined by

$$\mathcal{H}(X) = \{f \in \pi : \ p_K(D)f = 0, \ \forall K \in \mathbb{K}(X)\}, \tag{6}$$

is of importance in box spline theory since the box spline B_X is a piecewise-$\mathcal{H}(X)$ function.

In the context of *exponential* box splines, one deals with the following generalization of the above space: we associate with each direction $x \in X$

an (arbitrary) constant λ_x and, correspondingly, we define the possibly non-homogeneous polynomials $p_{Z,\lambda} := \prod_{x\in Z}(\langle x,\cdot\rangle - \lambda_x)$. The exponential space $\mathcal{H}(X,\lambda)$ is then defined analogously as

$$\mathcal{H}(X,\lambda) := \{f \text{ is entire}: \ p_{K,\lambda}f = 0,\ \forall K \in \mathbb{K}(X)\}. \tag{7}$$

With the aid of $\mathcal{H}(X,\lambda)$, we identify elements in $\mathcal{H}(X)$. For $f \in \mathcal{H}(X,\lambda)$ and $K \in K(X)$,

$$0 = p_{K,\lambda}(D)f = p_K(D)f_\downarrow + \text{ higher order terms}, \tag{8}$$

which implies that $p_K(D)f_\downarrow = 0$ and hence $f_\downarrow \in \mathcal{H}(X)$. Consequently,

$$\mathcal{H}(X,\lambda)_\downarrow \subset \mathcal{H}(X),$$

hence, by Proposition 1,

$$\dim \mathcal{H}(X,\lambda) \le \dim \mathcal{H}(X),$$

regardless of the choice of λ. Since, for a generic $\lambda \in \mathbb{C}^X$, $\mathcal{H}(X,\lambda)$ is spanned by $\#\mathbb{B}(X)$ different exponentials, [1], with $\mathbb{B}(X)$ the multiset of all bases for $\mathbb{R}^s$ from X, one concludes that

$$\#\mathbb{B}(X) \le \dim \mathcal{H}(X),$$

a result which is due to Dahmen and Micchelli, [4].

§4. A basis for $\mathcal{H}(X)$

We know from [3] that

$$\mathcal{H}(X,\lambda)_\downarrow = \mathcal{H}(X). \tag{9}$$

For a generic λ, the exponentials e_θ in $\mathcal{H}(X,\lambda)$ form a basis for it. The relevant set $\mathbf{\Theta}$ of frequencies θ is easily determined: Each $B \in \mathbb{B}(X)$ provides a $\theta = \theta_B$ as the unique solution of the linear system $\langle x,\theta\rangle = \lambda_x,\ \forall x \in B$. Thus, a basis for the polynomial space $\mathcal{H}(X)$ can be obtained as follows:

Step 1. *Compute the exponential basis for a suitable $\mathcal{H}(X,\lambda)$.*

Step 2. *Apply to this basis the Algorithm 2 for the construction of a basis for $H_\downarrow$ from a basis for H.*

Note that the algorithm requires the determination of the least term of functions. This presents no numerical problem in the present situation in case $X \subset \mathbb{Z}^s$. For, then λ can be chosen so that each θ is rational, and the algorithm's calculations can be carried out in exact (i.e., integer) arithmetic.

§5. Subspaces of $\mathcal{H}(X)$

The observations based on (8) made about the action of differential operators on $\mathcal{H}(X)$ and $\mathcal{H}(X,\lambda)$ can be formulated in terms of polynomial ideals and extended to more general settings, [3]. We omit here these details, yet describe the application of these extensions to subspaces of $\mathcal{H}(X)$.

Note that the elements of $\mathbb{K}(X)$ are exactly all subsets of X which intersect every element of $\mathbb{B}(X)$. Suppose now that $\mathbb{B}_1$ is a subset of $\mathbb{B}(X)$. Let $\mathbb{K}_1 := \{K \in X :\ K \cap B \neq \emptyset,\ \forall B \in \mathbb{B}_1\}$. Define

$$\mathcal{H}_1 := \{f :\ p_K(D)f = 0,\ \forall K \in K_1\}.$$

One checks that $\mathcal{H}_1 \subset \mathcal{H}(X)$.

Theorem 4 [3].

$$\#\mathbb{B}_1 \leq \dim \mathcal{H}_1. \tag{10}$$

The inequality in the theorem is sometimes strict. To guarantee equality, one may choose $\mathbb{B}_1$ to be **order-closed**: suppose that X is ordered, $X = \{x_1, ..., x_{\#X}\}$ say. This order induces a partial ordering on $\mathbb{B}(X)$:

$$B_1 = \{y_1, ..., y_s\} \leq B_2 = \{z_1, ..., z_s\} \iff y_j \leq z_j,\ \forall j.$$

We call $\mathbb{B}_1 \subset \mathbb{B}(X)$ *order-closed* if the condition

$$B_1 \leq B_2,\ B_2 \in \mathbb{B}_1 \Longrightarrow B_1 \in \mathbb{B}_1$$

holds for all $B_1, B_2 \in \mathbb{B}(X)$.

Theorem 5 [3]. *If* $\mathbb{B}_1$ *is an order-closed subset of* $\mathbb{B}(X)$, *then*

$$\#\mathbb{B}_1 = \dim \mathcal{H}_1.$$

Similar results hold for subspaces of the more general space $\mathcal{H}(X,\lambda)$. These results allow us to identify the local approximation order of subspaces of $\mathcal{H}(X,\lambda)$,[3].

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