On local linear functionals which vanish at all B-splines but one

Carl de Boor*)

1. Introduction

Let $k \in \mathbb{N}$, $\mathbf{t} := (t_i)$ nondecreasing (finite, infinite or biinfinite) with $t_i < t_{i+k}$, all i, and let (N_i) be the sequence of B-splines of order k for the knot sequence $\mathbf{t}$. This means that $N_i = N_{i,k,\mathbf{t}}$ is the B-spline of order k with knots $t_i, \ldots, t_{i+k}$, i.e., N_i is given by the rule

$$N_{i,k,\mathbf{t}}(t) := \Big([t_{i+1}, \ldots, t_{i+k}] - [t_i, \ldots, t_{i+k-1}]\Big)(\,\cdot - t)_+^{k-1} \tag{1.1}$$

with $[t_j, \ldots, t_{j+r}]f$ the r–th divided difference of f at the points $t_j, \ldots, t_{j+r}$. In particular,

$$N_i(t) > 0 \quad \text{on } [t_i, t_{i+k}] \quad \text{and } = 0 \quad \text{off } [t_i, t_{i+k}],$$

and, for $t \in [t_j, t_{j+1}]$,

$$\Big(\sum_i N_i\Big)(t) = \sum_{i=j-k+1}^{j} N_i(t) = 1,$$

i.e., such a B-spline sequence provides a partition of unity. For more information about B-splines, see Curry and Schoenberg's paper [9], and [5].

The present paper is concerned with linear functionals λ_i for which

$$\operatorname{supp} \lambda_i \subseteq [t_i, t_{i+k}], \ \lambda_i N_j = \delta_{ij}, \quad \text{all } j. \tag{1.2}$$

The first such linear functional seems to have been constructed in [1], for the purpose of demonstrating the linear independence over an interval of all B-splines which do not vanish identically on that interval. Since then, such linear functionals have been constructed in various ways and for a variety of jobs [2] – [7], [11], [13], [16], some of which are listed in Section 2.

In particular, it was shown in [6] that there exists a smallest number D_k so that, for all $\mathbf{t}$ and all i with $t_i < t_{i+k}$, an $h_i \in \mathbb{L}_\infty$ can be found with $\operatorname{supp} h_i \subseteq [t_i, t_{i+k}]$, $\|h_i\|_\infty \le D_k/(t_{i+k} - t_i)$, and $\int h_i N_j = \delta_{ij}$, all j. After a discussion in Section 3 as to how to construct linear functionals λ_i satisfying (1.2), it is shown in Section 4 that

$$(\pi/2)^k/2 \le D_k \le 2k\, 9^{k-1}.$$

Also, numerical evidence is presented to indicate that probably

$$D_k = O(2^k).$$

In Section 5, the related constant

$$D_{k,\infty} := \sup_{\mathbf{t}} \sup_i 1/\operatorname{dist}_{\infty,[t_{i+1}, t_{i+k-1}]}(N_i,\ \operatorname{span}(N_j)_{j \ne i})$$

is discussed. As pointed out in [2], this number is related to the condition number of the B-spline basis,

$$\operatorname{cond}_k := \sup_{\mathbf{t}} \operatorname{cond}_{k,\mathbf{t}}$$

since

*) Sponsored by the United States Army under Contract No. DAHC04–75–C–0024.

$$\mathrm{cond}_{k,\mathbf{t}} \;:=\; \frac{\sup \|\sum \alpha_j N_j\|_\infty / \|\underline{\underline{\alpha}}\|_\infty}{\inf \|\sum \alpha_j N_j\|_\infty / \|\underline{\underline{\alpha}}\|_\infty} \;=\; \frac{1}{\inf_i \mathrm{dist}_\infty(N_i, \mathrm{span}(N_j)_{j\neq i})}$$
$$\le\; D_{k,\infty}.$$

It is shown that

$$(\pi/2)^{k-1}/2 \le D_{k,\infty} \le D_k,$$

and numerical evidence is presented to suggest that

$$D_{k,\infty} \;\sim\; 2^{k-1}/\sqrt{2}.$$

2. Some results obtainable with the aid of such functionals.

In this section, we list some results obtainable through the explicit construction and analysis of specific local linear functionals which vanish at all B-splines but one.

(1) [1], [4]. For any open $I \subseteq \mathbb{R}$, $\{N_i \mid \mathrm{supp}\, N_i \bigcap I \neq \emptyset\}$ is linearly independent on I.

(2) [2]. Let $\$ = \$_{k,\mathbf{t}}$ denote the linear space of all splines of order k with knot sequence $\mathbf{t}$, i.e.,

$$\$:= \$_{k,\mathbf{t}} := \left\{ \sum_j \alpha_j N_{j,k,\mathbf{t}} \mid \alpha_j \in \mathbb{R}, \quad \text{all } j \right\}$$

with the sum taken pointwise in case $\mathbf{t}$ is not finite. There exists a constant $D_{k,\infty}$ depending only on k so that

$$\mathrm{dist}_\infty(f, \$) \le D_{k,\infty} \max_j \mathrm{dist}_{\infty,[t_{j+1-k}, t_{j+k}]}(f, \mathbb{P}_k).$$

Here, $\mathbb{P}_k$ denotes the collection of all polynomials of order k or degree $< k$, and the number $D_{k,\infty}$ is found as $\max_j \|\lambda_j\|$, with (λ_j) a sequence of local linear functionals dual to the B-spline sequence, i.e., $\lambda_i Nj = \delta_{ij}$, all i, j. This example raises the question of just how small one can make the norm of such linear functionals, a question taken up again in Section 5.

(3) [2], [4], [6]. There exists a smallest number D_k (depending only on k) so that, for all $\mathbf{t}$ and all i with $t_i < t_{i+k}$, an $h_i \in \mathbb{L}_\infty$ can be found satisfying

$$\begin{aligned}
&\mathrm{supp}\, h_i \subseteq [t_i, t_{i+k}],\\
&\|h_i\|_p \le D_k/(t_{i+k} - t_i)^{1/q}, \quad (1/p + 1/q = 1),\\
&\int h_i N_j = \delta_{ij}, \quad \text{all } j.
\end{aligned}$$

(Note that the constant D_k mentioned here is k times the number D_k mentioned in [6].)

This fact has many consequences, among them the following two.

(4) [4]. If $f = \sum \alpha_j N_j$, then $\alpha_i = \int h_i f \le \|h_i\|_q \|f\|_{p,[t_i,t_{i+k}]}$, hence

$$|\alpha_i|(t_{i+k} - t_i)^{1/p} \le D_k \|f\|_{p,[t_i,t_{i+k}]}, \tag{2.1}$$

therefore, with E the diagonal matrix given by

$$E := \lceil \ldots, (t_{i+k} - t_i)/k, \ldots \rfloor,$$

we have

$$\|E^{1/p} \underline{\underline{\alpha}}\|_p \le D_k \|\sum \alpha_j N_j\|_p.$$

(5) [8]. In particular, with

$$\overset{2}{N}_j := N_j \Big/ \left(\frac{t_{j+k} - t_j}{k} \right)^{1/2},$$

we get

$$\|\underline{\underline{\beta}}\|_2 \le D_k \| \sum \beta_j \overset{2}{N}_j \|_2.$$

Let L be $\mathbb{L}_2$–approximation by elements of \$, i.e.,

$$Lf \in \$, \quad \text{and} \quad f - Lf \perp \$.$$

Then $Lf = \sum \alpha_j \beta_j \overset{2}{N}_j$, with $G\underline{\underline{\beta}} = (\int \overset{2}{N}_i\, f)$ and $G := (\int \overset{2}{N}_i \overset{2}{N}_j)$. Let $G^{-1} =: (\alpha_{ij})$. Then G^{-1} decays exponentially away from the diagonal, i.e.,

$$|\alpha_{ij}| \le \text{const}\ \lambda^{|i-j|}$$

with $\lambda := (1 - D_k^{-2})^{1/(2k-2)} \in]0,1[$ and const $:= D_k^3/\lambda^{k-1}$ both independent of $\mathbf{t}$, as can be proved using a very nice idea of Douglas, Dupont and Wahlbin [10]. This implies that, as a map on $\mathbb{L}_\infty$,

$$\|L\| \le \text{const}_k\, (M_{\mathbf{t}}^{(k)})^{1/2},$$

a bound in terms of the global mesh ratio

$$M_{\mathbf{t}}^{(k)} := \max_{i,j} (t_{i+k} - t_i)/(t_{j+k} - t_j).$$

Finally, here are two applications which have, offhand, nothing to do with splines, but rather are concerned with the smooth interpolation of data.

(6) [6]. Suppose we are given $\mathbf{t} = (t_i)$ nondecreasing with $t_i < t_{i+k}$, all i. For given f, let $f\big|_{\mathbf{t}} := (f_i)$, with $f_i = f^{(j)}(t_i)$, where $j := \max\{r \mid t_{i-r} = t_i\}$. Then, given $\underline{\underline{\alpha}} = (\alpha_i)$, there is no difficulty in finding *some* smooth f so that $f\big|_{\mathbf{t}} = \underline{\underline{\alpha}}$, let $f_{\underline{\underline{\alpha}}}$ be one such, but it is not at all clear a priori under what circumstances such an f can be found in $\mathbb{L}_p^{(k)}(\mathbb{R})$. But, using (3) above, one can show that there exists $f \in \mathbb{L}_p^{(k)}(\mathbb{R})$ so that $f\big|_{\mathbf{t}} = \underline{\underline{\alpha}}$ if and only if $\big((t_{i+k} - t_i)^{1/p}[t_i, \ldots, t_{i+k}]f_{\underline{\underline{\alpha}}}\big) \in \ell_p$.

The argument is based on the observation that f, given by the conditions that it agree with $f_{\underline{\underline{\alpha}}}$ at k points and that

$$f^{(k)} := \sum_i c_i\big((t_{i+k} - t_i)/k\big)h_i$$

with

$$c_i := k![t_i, \ldots, t_{i+k}]f_{\underline{\underline{\alpha}}} = \frac{k}{t_{i+k} - t_i} \int N_i f_{\underline{\underline{\alpha}}}^{(k)},$$

necessarily agrees with $f_{\underline{\underline{\alpha}}}$ at $\mathbf{t}$ since $\int N_j h_i = \delta_{ij}$, i.e., since f and $f_{\underline{\underline{\alpha}}}$ have the same k–th divided differences.

(7) [6], [7]. In particular, with f the interpolant just constructed and $t_j < t_{j+1}$, at most k of the h_i are nonzero on $[t_j, t_{j+1}]$, therefore, from (3),

$$\begin{aligned} \|f^{(k)}\|_{\infty,[t_j,t_{j+1}]} &\le \| \sum_{h_i|_{[t_j,t_{j+1}]} \ne 0} |c_i| \frac{t_{i+k} - t_i}{k} |h_i| \,\|_\infty \\ &\le \max_{h_i|_{[t_j,t_{j+1}]} \ne 0} |c_i| D_k. \end{aligned}$$

This proves that, for given $\mathbf{t}$ and given $\underline{\underline{\alpha}}$, there exists $f \in \mathbb{L}_\infty^{(k)}$ so that $f\big|_{\mathbf{t}} = \underline{\underline{\alpha}}$ and, for all $t_j < t_{j+1}$,

$$\|f^{(k)}\|_{\infty,[t_j,t_{j+1}]} \le D_k \max_{[t_j,t_{j+1}] \subseteq [t_i,t_{i+k}]} k!|[t_i, \ldots, t_{i+k}]f_{\underline{\underline{\alpha}}}|,$$

a fact of interest when carrying out an a posteriori error analysis for a finite difference approximation to the solution of an ordinary differential equation.

3. Construction of λ_i.

The following observations seem to have been made first in [1]. They are also used implicitly by Jerome and Schumaker in [11].

Let

$$\psi_i(t) := (t - t_{i+1}) \cdots (t - t_{i+k-1})/(k-1)!,$$

$$\overset{+}{\psi}_i(t) := (t - t_{i+1})_+ \cdots (t - t_{i+k-1})_+/(k-1)!.$$

Then

$$[t_j, \ldots, t_{j+k}]\overset{+}{\psi}_i = \delta_{ij}/\big((t_{i+k} - t_i)\,(k-1)!\big)$$

since, for $j < i$, $\overset{+}{\psi}_i = 0$ on $t_j, \ldots, t_{j+k}$, while, for $j > i$, $\overset{+}{\psi}_i = \psi_i \in \mathbb{P}_k$ on $t_j, \ldots, t_{j+k}$, and, finally, for $j = i$, $\overset{+}{\psi}_i$ agrees on $t_j, \ldots, t_{j+k}$ with $\psi_i(t)\,(t - t_i)/(t_{i+k} - t_i)$, a polynomial of exact degree k with leading coefficient $1/\big((t_{i+k} - t_i)\,(k-1)!\big)$. Consequently,

$$(k-1)!\big([t_{j+1}, \ldots, t_{j+k}] - [t_j, \ldots, t_{j+k-1}]\big)\overset{+}{\psi}_i = \delta_{ij}.$$

On the other hand, from Taylor's expansion with integral remainder,

$$\big([t_{j+1}, \ldots, t_{j+k}] - [t_j, \ldots, t_{j+k-1}]\big)f = \int N_{j,k,\mathbf{t}}(t) f^{(k)}(t)\,dt/(k-1)!$$

for $f \in \mathbb{L}_1^{(k)}$. This proves the following lemma.

Lemma 3.1. *$\lambda \in \mathbb{L}_q \subseteq \mathbb{L}_p^*$ satisfies $\lambda N_j = \delta_{ij}$ iff $\lambda = f^{(k)}$ for some $f \in \mathbb{L}_q^{(k)}$ with $f = \overset{+}{\psi}_i$ on $\mathbf{t}$.*

If we require from λ, in addition, that $\operatorname{supp}\lambda \subseteq [t_i, t_{i+k}]$, then $f\big|_{t<t_i}$ and $f\big|_{t>t_{i+k}}$ are both polynomials of degree $< k$, hence then

$$f = \begin{matrix} 0, & t < t_i \\ & \\ \psi_i, & t > t_{i+k} \end{matrix},$$

at least for sufficiently long $\mathbf{t}$.

Corollary. *If $[a, b] \subseteq [t_i, t_{i+k}]$, and $f \in \mathbb{L}_q^{(k)}[a, b]$ with*

$$f = \begin{cases} 0\,, & k\text{-fold at } a, \\ 0 \;=\; \psi_i, & \text{at all } t_j \in]a, b[, \\ \psi_i, & k\text{-fold at } b, \end{cases}$$

then $\lambda_i \in \mathbb{L}_p^*$ given by

$$\lambda_i g := \int_{t_i}^{t_{i+k}} g\, f^{(k)}$$

has support in $[a, b]$ and satisfies

$$\lambda_i N_j = \delta_{ij}, \quad \text{all } i, j.$$

As a simple example, choose r so that $t_i \le t_r < t_{r+1} \le t_{i+k}$ and let $f \in \mathbb{L}_1^{(k)}[t_r, t_{r+1}]$ so that

$$f = \begin{cases} 0, & k\text{-fold at } t_r \\ \psi_i, & k\text{-fold at } t_{r+1} \end{cases}$$

Then $\lambda_i := f^{(k)}$ satisfies $\lambda_i N_j = \delta_{ij}$, all j, by the Corollary. Now note that, by assumption,

$$\psi_i^{(m)}(t_{r+1}) = f^{(m)}(t_{r+1}) = \int_{t_r}^{t_{r+1}} (t_{r+1} - s)^{k-m-1} f^{(k)}(s)\,ds/(k-m-1)!,$$

i.e.,

$$\lambda_i : (t_{r+1} - \cdot\,)^{k-m-1}/(k-m-1)! \;\mapsto\; \psi_i^{(m)}(t_{r+1}), \quad m = 0, \ldots, k-1.$$

Since $p(s) = \sum_{m=0}^{k-1} (-)^{k-m-1} p^{(k-m-1)}(t_{r+1})\,(t_{r+1}-s)^{k-m-1}/(k-m-1)!$, all $p \in \mathbb{P}_k$, this implies that

$$\lambda_i p = \sum_{m=0}^{k-1} (-)^{k-m-1} p^{(k-m-1)}(t_{r+1}) \psi_i^{(m)}(t_{r+1}), \quad \text{all } p \in \mathbb{P}_k.$$

But, for any p, $\psi \in \mathbb{P}$,

$$\begin{aligned}(d/d\tau) \sum_{m=0}^{k-1} (-)^{k-m-1} p^{(k-m-1)}(\tau)\psi^{(m)}(\tau) \\ &= (-)^{k-1} p^{(k)}(\tau)\psi(\tau) + p(\tau)\psi^{(k)}(\tau) \\ &= 0.\end{aligned}$$

Hence

$$\lambda_i p = \sum_{m=0}^{k-1} (-)^{k-m-1} p^{(k-m-1)}(\tau) \psi_i^{(m)}(\tau), \quad \text{all } \tau, \quad \text{all } p \in \mathbb{P}_k.$$

Further, for all j, $N_j|_{[t_r, t_{r+1}]} \in \mathbb{P}_k$. Therefore

$$\sum_{m=0}^{k-1} (-)^{k-m-1} N_j^{(k-m-1)}(\tau) \psi_i^{(m)}(\tau) = \delta_{ij}, \quad \text{all } \tau \in [t_i, t_{i+k}], \quad \text{all } j,$$

which is the identity on which the quasi-interpolant of [3] is based. The corresponding specific linear functional $\widehat{\lambda}_i$ given by the rule

$$\widehat{\lambda}_i g := \sum_{m=0}^{k-1} (-)^{k-m-1} g^{(k-m-1)}(\tau) \psi_i^{(m)}(\tau)$$

for some fixed $\tau \in [t_i, t_{i+k}]$ is the k–th derivative (in the weak sense) of the function

$$f := (\cdot - \tau)_+^0 \psi_i$$

which indeed agrees with $\overset{+}{\psi}_i$ at $\mathbf{t}$.

4. An estimate for D_k.

In this section, we get an estimate for the number D_k of (3) of Section 2 by constructing a specific linear functional λ_i with $\lambda_i N_j = \delta_{ij}$, all j, using Lemma 3.1 and its Corollary.

Let $a < b$ with

$$t_i \le a \le t_{i+1}, \quad t_{i+k-1} \le b \le t_{i+k}$$

and take $G \in \mathbb{L}_\infty^{(k)}$ to be such that

$$G = \begin{cases} 0, & k\text{-fold at } a \\ 1, & k\text{-fold at } b \end{cases} \tag{4.1}$$

Then, as was observed by D. J. Newman, the function $f := G\psi_i$ on $[a, b]$ satisfies the assumptions of the Corollary to Lemma 3.1. Therefore, the function $h_i := f^{(k)}$ on $[a, b]$ is in $\mathbb{L}_\infty$ and satisfies

$$\begin{aligned} \operatorname{supp} h_i \subseteq& [a, b] \subseteq [t_i, t_{i+k}] \\ \|h_i\|_p \le& \|h_i\|_\infty (b-a)^{1/p} \\ \int h_i N_j =& \delta_{ij}, \quad \text{all } j. \end{aligned} \tag{4.2}$$

Next, we estimate $\|h_i\|_\infty$. We have

$$\|h_i\|_\infty \le \sum_{m=0}^{k-1} \binom{k}{m} \|\psi_i^{(m)}\|_\infty \|G^{(k-m)}\|_\infty$$

and

$$\|\psi_i^{(m)}\|_\infty \le \frac{(k-1)\cdots(k-m)}{(k-1)!}\,(b-a)^{k-1-m}, \quad m = 0, \ldots, k. \tag{4.3}$$

Also,

$$G^{(k-m)}(t) = \int_a^b (t-s)_+^{m-1} G^{(k)}(s)\,ds/(m-1)!,$$

hence

$$\delta_{mk} = G^{(k-m)}(b) = \int_a^b (b-s)^{m-1} G^{(k)}(s)\,ds/(m-1)!, \quad m = 1, \ldots, k, \tag{4.4}$$

i.e., $G^{(k)}$ is orthogonal to $\mathbb{P}_{k-1}$ on $[a,b]$. This implies that

$$G^{(k-m)}(t) = \int_a^b \big[(t-s)_+^{m-1} - p(t,s)\big] G^{(k)}(s)\,ds/(m-1)!,$$
$$\text{all } p(t,\,\cdot\,) \in \mathbb{P}_{k-1}$$

and, choosing $p(t,\,\cdot\,)$, e.g. by interpolation, so that

$$\int_a^b \big|(t-s)_+^{m-1} - p(t,s)\,\big| ds \le 4\left(\frac{b-a}{4}\right)^m,$$

we conclude that

$$\|G^{(k-m)}\|_\infty \le 4\left(\frac{b-a}{4}\right)^m/(m-1)!\ \ \|G^{(k)}\|_\infty. \tag{4.5}$$

Next, we choose $G \in \mathbb{L}_k^{(k)}[a,b]$ so as to minimize $\|G^{(k)}\|_\infty$ subject to the conditions (4.1), i.e., subject to the conditions (4.4). This problem has been solved by Louboutin ten years ago and a solution is described by Schoenberg in [15]. Here is a simple argument:

Conditions (4.4) describe $G^{(k)} \in \mathbb{L}_\infty[a,b]$ as an extension to all of $\mathbb{L}_1[a,b]$ of the linear functional μ on $\mathbb{P}_k$ given by the rule

$$\mu(b - \,\cdot\,)^{m-1}/(m-1)! = \delta_{mk}, \quad m = 1, \ldots, k,$$

i.e.,

$$\mu p = (-)^{k-1} p^{(k-1)}, \quad \text{all } p \in \mathbb{P}_k.$$

Therefore, $\min \|G^{(k)}\|_\infty = \|\mu\|$, and $G^{(k)}$ is minimal iff $G^{(k)}$ takes its norm in $\mathbb{P}_k$. Let T_k be the Chebyshev polynomial of degree k. Then, sign $T_k^{(1)}$ is well known to be orthogonal to $\mathbb{P}_{k-1}$ on $[-1,1]$, while $T_k^{(k)} = k!\,2^{k-1}$. Hence, with

$$\widehat{T}_k(t) := (-)^{k-1} T_k\big(2\frac{t-a}{b-a} - 1\big),$$

sign $\widehat{T}_k^{(1)}$ is orthogonal to $\mathbb{P}_{k-1} = \ker \mu$ on $[a,b]$ while

$$\mu \widehat{T}_k = (-)^{k-1} \widehat{T}_k^{(k)} = \big(\frac{4}{b-a}\big)^k k!/2.$$

It follows that

$$\widehat{G}^{(k)} := \operatorname{sign} \widehat{T}_k^{(1)} \big(\frac{4}{b-a}\big)^k k!/\big(2\|\widehat{T}_k^{(1)}\|_\infty\big) \in \mathbb{L}_\infty = \mathbb{L}_1^*$$

extends μ to $\mathbb{L}_1$ and takes on its norm in $\mathbb{P}_k$ (at the point $\widehat{T}_k^{(1)} \in \mathbb{P}_k$), hence is minimal. Since

$$\|\widehat{T}_k^{(1)}\|_\infty = \operatorname{Var}_{[a,b]} \widehat{T}_k = 2k,$$

this shows the minimal $G^{(k)}$ to be

$$\widehat{G}^{(k)} := \operatorname{sign} \widehat{T}_k^{(1)} \big(\frac{4}{b-a}\big)^k \frac{(k-1)!}{4} \tag{4.6}$$

with

$$\min \|G^{(k)}\|_\infty = \|\widehat{G}^{(k)}\|_\infty = \big(\frac{4}{b-a}\big)^k \frac{(k-1)!}{4}. \tag{4.7}$$

Correspondingly, $\widehat{G}^{(1)}$ is the perfect B-spline of order k with simple knots at the $k+1$ extrema of $\widehat{T}_k$ in $[a,b]$ and normalized to have unit integral.

For this particular choice for G, (4.3), (4.5) and (4.7) give

$$\begin{aligned}
\|\psi_i^{(m)}\|_\infty \|G^{(k-m)}\|_\infty &\le \frac{(k-1)\cdots(k-m)}{(k-1)!}(b-a)^{k-1-m}\big(\frac{b-a}{4}\big)^{m-k}\frac{(k-1)!}{(m-1)!} \\
&= \frac{(k-1)\cdots(k-m)}{(m-1)!}4^{k-m}/(b-a), \quad m = 1,\ldots,k-1,
\end{aligned}$$

and

$$\|\psi_i\|_\infty \|G^{(k)}\|_\infty \le 4^{k-1}/(b-a),$$

hence

$$\begin{aligned}
(b-a)\,\|h_i\|_\infty &\le 4^{k-1} + \sum_{m=1}^{k-1}\binom{k}{m}\frac{(k-1)!}{(k-m-1)!(m-1)!}4^{k-m} \\
&< 4^{k-1} + 2(k-1)\left(\sum_{m=0}^{k}\binom{k}{m}2^{k-m} - 2^k - 1\right)\sum_{m=1}^{k-1}\binom{k-2}{m-1}2^{k-m-1} \\
&= 4^{k-1} + 2(k-1)\,(3^k - 2^k - 1)3^{k-2} \\
&< 2k\,9^{k-1}.
\end{aligned}$$

Theorem 4.1. *Let D_k be the smallest number with the property that, for every $\mathbf{t}$, every i and every $a < b$ with*

$$t_i \le a \le t_{i+1}, \quad t_{i+k-1} \le b \le t_{i+k},$$

there exists $h_i \in \mathbb{L}_\infty$ such that

$$\operatorname{supp}\, h_i \subseteq [a,b],\ \|h_i\|_\infty \le D_k/(b-a),\ \int h_i N_j = \delta_{ij}, \quad \text{all } j. \tag{4.8}$$

Then

$$(\pi/2)^k/2 \le D_k \le 2k\,9^{k-1}.$$

Proof: Only the first inequality still requires proof. For this, take Schoenberg's Euler spline [14], [16],

$$\mathcal{E}_k(t) := \gamma_k \sum_{j=-\infty}^{\infty}(-)^j N_{j,k+1,\mathbb{Z}}\Big(t - \frac{k+1}{2}\Big)$$

with

$$\gamma_k = 1/\varphi_{k+1}(\pi) = \left(\frac{\pi}{2}\right)^{k+1} / \sum_j \left(\frac{(-1)^j}{2j+1}\right)^{k+1} \ge \left(\frac{\pi}{2}\right)^k / 2 \tag{4.9}$$

so chosen that $\mathcal{E}_k(\nu) = (-)^\nu$, all $\nu \in \mathbb{Z}$. Then

$$\mathcal{E}_k^{(1)}(t) = 2\gamma_k \sum_j (-)^j N_{j,k,\mathbb{Z}}\left(t - \frac{k+1}{2}\right)$$

is a spline of order k, with knot sequence $\mathbb{Z} - s$ where $s := (k+1)/2$, hence, by (2.1), and since $\mathcal{E}_k$ is monotone between integers,

$$|2\gamma_k k| \le D_k \|\mathcal{E}_k^{(1)}\|_{1,[s,s+k]} = D_k \operatorname{Var}_{[s,s+k]} \mathcal{E}_k = 2k\, D_k$$

and so

$$\left(\frac{\pi}{2}\right)^k / 2 \le \gamma_k \le D_k; \qquad \textit{Q.E.D.}$$

It is possible to compute D_k for small k as follows. For $\underline{\underline{\sigma}} := (\sigma_i)_1^{3k-1}$ with

$$0 = \sigma_1 = \cdots = \sigma_k \le \sigma_{k+1} \le \cdots \le \sigma_{2k} = \cdots = \sigma_{3k-1} = 1,$$

compute the norm of the linear functional $\mu_{\underline{\underline{\sigma}}}$ given on $\$_{k,\underline{\underline{\sigma}}} \subseteq \mathbb{L}_1[0,1]$ by the rule

$$\mu_{\underline{\underline{\sigma}}} N_{j,k,\underline{\underline{\sigma}}} = \delta_{jk}, \quad j = 1, \ldots, 2k-1.$$

Much as in the computations reported in [7], this amounts to constructing (by Newton's method, say) an absolutely constant step function g on $[0,1]$ with $\dim \$_{k,\underline{\underline{\sigma}}}$ steps so that

$$\int_0^1 g\, N_j = \delta_{jk}, \quad \text{all } j.$$

Then $\|\mu_{\underline{\underline{\sigma}}}\| = \|g\|_\infty$, and

$$D_k = \sup_{\underline{\underline{\sigma}}} \|\mu_{\underline{\underline{\sigma}}}\|.$$

Somewhat more explicitly, the construction of such a g proceeds as follows. With

$$s := \dim \$_{k,\underline{\underline{\sigma}}},$$

and $0 = \rho_0 < \cdots < \rho_s = 1$, one computes $(\beta_j)_1^s$ such that

$$\sum_j \beta_j \int_{\rho_{j-1}}^{\rho_j} N_i = \delta_{ik}, \quad \text{all } i. \tag{4.10}$$

Now

$$\int^\rho N_{i,k} = \frac{\sigma_{i+k} - \sigma_i}{k} \int^\rho M_{i,k} = \frac{\sigma_{i+k} - \sigma_i}{k} \sum_{i \le n} N_{n,k+1}(p),$$

as one checks easily, therefore

$$\int_{\rho_{j-1}}^{\rho_j} N_i = \frac{\sigma_{i+k} - \sigma_i}{k} \sum_{i \le n} N_{n,k+1}(\rho_j) - N_{n,k+1}(\rho_{j-1}).$$

Since $\sigma_{2k} - \sigma_k = 1$, this shows that (4.10) is equivalent to

$$\sum_j \beta_j \sum_{i \le n} \big(N_{n,k+1}(\rho_j) - N_{n,k+1}(\rho_{j-1})\big) = k\delta_{ik}, \quad \text{all } i.$$

But, subtracting in order each equation in this system from all its predecessors, starting with the last, we obtain the equivalent system

$$\sum_j \beta_j \big(N_{i,k+1}(\rho_j) - N_{i,k+1}(\rho_{j-1})\big) = \begin{cases} -k, & i = k-1 \\ k, & i = k \\ 0, & \text{otherwise} \end{cases}, \tag{4.11}$$

which is very similar to the system dealt with in [7]. In particular, one proves that $\mu_{\underline{\underline{\sigma}}}$ *has exactly one extremal,* i.e., *there exists exactly one absolutely constant g with s steps on $[0,1]$ for which $\int_0^1 gN_i = \delta_{ik}$, all i.* This means that the nonlinear system for the β_j and ρ_j consisting of (4.11) and

$$\beta_{j-1} + \beta_j = 0, \quad j = 2, \ldots, s, \tag{4.12}$$

has exactly one solution.

For all k considered, such computations show $\sup_{\underline{\underline{\sigma}}} \|\mu_{\underline{\underline{\sigma}}}\|$ to be taken on at the middle vertex of the simplex over which $\underline{\underline{\sigma}}$ varies, i.e., at the point $\underline{\underline{\sigma}} = (\sigma_j)$ with

$$\sigma_j = \begin{matrix} 0, & j < k + k/2 \\ \\ 1, & j \ge k + k/2 \end{matrix}.$$

Computed values for D_k are

k	D_k	$\ln_2 D_k$
1	1	0
2	2.4142..	1.2715..
3	5.2044..	2.3797..
4	10.0290..	3.3261..
5	21.3201..	4.4141..
6	40.8972..	5.3539..
7	86.3688..	6.4324..
8	166.4052..	7.3785..
9	348.5582..	8.4452..
10	674.2949..	9.3972..
11	1402.9478..	10.4542..

These numbers strongly suggest that D_k grows like 2^k rather than like the upper bound 9^k established in Theorem 4.1.

5. An estimate for $D_{k,\infty}$.

If $a < b$ and $t_i \le a \le t_{i+1}$, $t_{i+k-1} \le b \le t_{i+k}$, then we can construct $h_i \in \mathbb{L}_\infty[a,b]$ so that $\int h_i Nj = \delta_{ij}$. In fact, such a function h_i with smallest possible ∞–norm can be constructed as a minimum norm extension to all of $\mathbb{L}_1[a,b]$ of the linear functional μ_i on $\$\big|_{[a,b]} \subseteq \mathbb{L}_1[a,b]$ given by the rule

$$\mu_i N_j = \delta_{ij}, \quad \text{all } j.$$

This fact was the basis for the computation of D_k reported in the preceding section.

In general, if we think of $\$|_{[a,b]}$ as a subspace of $\mathbb{L}_\infty[a,b]$, then a minimum norm extension of μ_i to all of $\mathbb{L}_\infty$ does not exist in the form $h_i \in \mathbb{L}_1$, i.e., in the form of a function on $[a,b]$. For this reason, it is more convenient to consider $\$|_{[a,b]}$ as a subspace of $C[a,b]$, – this requires the assumption

$$t_j < t_{j+k-1}, \quad \text{all } j, \quad - \tag{5.1}$$

and to consider a norm preserving extension of μ_i to all of $C[a,b]$ since the dual of $C[a,b]$, while still not representable by functions on $[a,b]$, is in some sense simpler than that of $\mathbb{L}_\infty$. In particular, it is always possible to find norm preserving extensions of μ_i of the form

$$\sum_{m=1}^{s} \alpha_m[\rho_m] \tag{5.2}$$

with

$$s := \dim \$_{k,\mathbf{t}}|_{[a,b]}$$

and

$$[p]\, f := f(p).$$

In this section, we estimate the number

$$\begin{aligned} D_{k,\infty} &:= \sup_{\mathbf{t}} \sup_i \; 1/\operatorname{dist}_{\infty,[t_{i+1},t_{i+k-1}]}(N_i, \operatorname{span}(N_j)_{j\neq i}) \\ &= \sup_{\mathbf{t}} \sup_i \|\lambda_i^*\|, \end{aligned} \tag{5.3}$$

with

$$\lambda_i^* := \text{minimizer of } \|\cdot\| \text{ over } \left\{\lambda_i \in C^*[t_{i+1}, t_{i+k-1}] \mid \lambda_i N_j = \delta_{ij}, \quad \text{all } j\right\}.$$

This number was shown to be finite in [2]. The argument relied on constructing explicitly a norm preserving extension of μ_i of the form $\sum \alpha_i[\rho_i]$ with $t_r \le \rho_1 < \cdots < \rho_k \le t_{r+1}$ and $[t_r, t_{r+1}]$ a largest interval of that form in $[t_{i+1}, t_{i+k-1}]$. But the resulting bound for $D_{k,\infty}$ seemed very pessimistic.

Theorem 5.1. *The constant $D_{k,\infty}$ defined by (5.3) satisfies*

$$(\pi/2)^{k-1}/2 \le D_{k,\infty} \le D_k. \tag{5.4}$$

Proof: By Theorem 4.1, the linear functional μ_i on $\$_{k,\mathbf{t}}$ given by $\mu_i N_j = \delta_{ij}$, all j, satisfies

$$|\mu_i f| \le D_k \|f\|_{\infty,[a,b]}$$

for any $a < b$ with $t_i \le a \le t_{i+1} \le t_{i+k-1} \le b \le t_{i+k}$. Hence, for $t_{i+1} < t_{i+k-1}$,

$$\operatorname{dist}_{\infty,[t_{i+1},t_{i+k-1}]}(N_i, \operatorname{span}(N_j)_{j\neq i}) = 1/\|\mu_i\| \ge D_k^{-1}$$

with $\|\mu_i\|$ the norm of μ_i with respect to $\|\cdot\|_{\infty,[t_{i+1},t_{i+k-1}]}$. For $t_{i+1} = t_{i+k-1}, N_j(t_{i+1}) = \delta_{ij}$, hence then $\operatorname{dist}_{\infty,[t_{i+1},t_{i+k-1}]}\bigl(N_i, \operatorname{span}(N_j)_{j\neq i}\bigr) = 1/\|\mu_i\| = 1$. This proves that $D_{k,\infty} \le D_k$. The inequality $\gamma_{k-1} \le D_{k,\infty}$ was already proved in [5], using Schoenberg's Euler spline. Q.E.D.

To be precise, it was shown in [5] that

$$\gamma_{k-1} = \operatorname{cond}_{k,\mathbb{Z}}$$

with

$$\operatorname{cond}_{k,\mathbf{t}} := \frac{\sup \|\sum \alpha_j N_j\|_\infty / \|\underline{\alpha}\|_\infty}{\inf \|\sum \alpha_j N_j\|_\infty / \|\underline{\alpha}\|_\infty} = \frac{1}{\inf_i \; \operatorname{dist}_\infty(N_i, \operatorname{span}(N_j)_{j\neq i})}$$
$$\le D_{k,\infty}$$

hence

$$\mathrm{cond}_k := \sup_{\mathbf{t}} \mathrm{cond}_{k,\mathbf{t}} \le D_{k,\infty}.$$

It is, of course, possible to prove that $D_{k,\infty} = O(9^k)$ directly without reference to Theorem 4.1: Let $[a,b] = [t_{i+1}, t_{i+k-1}]$ with $a < b$ and consider λ_i of the form $(G\psi_i)^{(k)}$ with

$$G(t) \;:=\; \begin{array}{ll} 0, & t < a \\ G^{(k)}\{(t - \cdot)_+^{k-1}/(k-1)!\}, & t \ge a, \end{array}$$

and $G^{(k)} \in C^*[a,b]$ so that

$$G^{(k)}\{(b - \cdot)^{k-j}/(k-j)!\} = \delta_{1j}, \quad j = 1, \ldots, k.$$

Then $G\psi_i$ agrees with $\overset{+}{\psi}_i$ at $\mathbf{t}$, hence $\lambda_i N_j = \delta_{ij}$, all j, i.e., $\lambda_i \in C^*[a,b]$ and λ_i extends μ_i. Next, choose $G(k)$ to have as small a norm as possible. This requires $G^{(k)}$ to be a norm preserving extension to all of $C[a,b]$ of the linear functional μ on $\mathbb{P}_k$ given by the rule

$$\mu(b - \cdot)^{k-j}/(k-j)! = \delta_{1j},\ j = 1, \ldots, k,$$

i.e.,

$$\mu p = (-)^{k-1} p^{(k-1)}, \quad \text{all } p \in \mathbb{P}_k.$$

Hence, with $a \le \rho_1 < \cdots < \rho_k \le b$,

$$(-)^{k-1}[\rho_1, \ldots, \rho_k] = \sum \alpha_j [\rho_j]$$

is an extension of μ. This extension is norm preserving provided it takes its norm in $\mathbb{P}_k$. Since the coefficients $\alpha_1, \ldots, \alpha_k$ strictly alternate in sign, this will happen iff $\rho_1, \ldots, \rho_k$ are chosen as the extrema of the Chebyshev polynomial of degree $k-1$ adjusted to the interval $[a,b]$. The resulting minimal G is an old acquaintance, viz. the integral of the perfect B-spline of order $k-1$ with support equal to $[a,b]$ and unit integral. We record this curious fact in the following

Proposition. *Let* $G_k(t) := \int_a^t B_k(s)\,ds$ *with* $B_k(s) := k[\rho_0, \ldots, \rho_k]\,(\cdot - s)_+^{k-1}$ *and*

$$\rho_j = \big(a + b + (a-b)\cos \pi j/k\big)/2, \quad j = 0, \ldots, k$$

the extrema of the k–th degree Chebyshev polynomial for $[a,b]$. Then, not only is $G_k^{(k)}$ the unique norm preserving extension to all of $\mathbb{L}_1[a,b]$ of the linear functional μ_k on $\mathbb{P}_k$ given by

$$\mu_k p = (-)^{k-1} p^{(k-1)}, \quad \textit{all } p \in \mathbb{P}_k,$$

and therefore $G_k^{(k)}$ is absolutely constant, hence B_k is perfect and

$$\|G_k^{(k)}\|_\infty = \|\mu_k\|_{\|\cdot\|_1} = \left(\frac{4}{b-a}\right)^k \frac{(k-1)!}{4}$$

– this much was shown already by Louboutin [15], – *but also $G_k^{(k+1)}$ is the unique norm preserving extension of the form $\sum \alpha_j[\rho_j]$ to all of $C[a,b]$ of μ_{k+1}, therefore*

$$\|G_k^{(k+1)}\|_{\text{“}1\text{''}} = \mathrm{Var}_{[a,b]} G_k^{(k)} = \|\mu_{k+1}\|_{\|\cdot\|_\infty} = \left(\frac{4}{b-a}\right)^k \frac{k!}{2}.$$

The rest of the argument for the estimate $D_{k,\infty} = O(9^k)$ now proceeds as in the proof of Theorem 4.1.

It is possible to compute $D_{k,\infty}$ for small k as

$$D_{k,\infty} = \sup_{\underline{\underline{\sigma}}} \|\mu_{\underline{\underline{\sigma}}}\|$$

with

$$0 = \sigma_1 = \cdots = \sigma_k < \sigma_{k+1} \le \cdots \le \sigma_n < \sigma_{n+1} = \cdots = \sigma_{n+k} = 1, \tag{5.5}$$

$$n := 2k - 3,$$

and $\mu_{\underline{\underline{\sigma}}}$ the linear functional on $S := \$_{k,\underline{\underline{\sigma}}}|_{[0,1]} \subseteq C[0,1]$ given by

$$\mu_{\underline{\underline{\sigma}}} N_{j,k,\underline{\underline{\sigma}}} = \delta_{j,k-1}.$$

In order to compute $\|\mu_{\underline{\underline{\sigma}}}\|$, one constructs $\varphi \in S\backslash\{0\}$ and $0 = \rho_1 < \cdots < \rho_n = 1$ so that

$$(-)^j \varphi(\rho_j) = \|\varphi\|_\infty, \quad \text{all } j.$$

This is possible since (N_j) is a weak Chebyshev system (see, e.g., [12]). Next, one constructs the extension of $\mu_{\underline{\underline{\sigma}}}$ of the form $\sum \alpha_j[\rho_j]$ to all of $C[0,1]$. Then $\sum N_r(\rho_j)\alpha_j = \delta_{r,k-1}$, hence $\alpha_{j-1}\alpha_j \le 0$, all j, since $(N_r(\rho_j))$ is totally positive (see, e.g., [12]). Therefore

$$|\mu_{\underline{\underline{\sigma}}}\varphi| = |\sum \alpha_j \varphi(\rho_j)| = \sum |\alpha_j| \, \|\varphi\|_\infty,$$

i.e.,

$$\|\mu_{\underline{\underline{\sigma}}}\| = \sum |\alpha_j|.$$

As with the earlier reported calculation of D_k, it appears from these computations that $\sup \|\mu_{\underline{\underline{\sigma}}}\|$ is taken on at the "middle" vertex of the simplex described by (5.5), i.e., at the point $\underline{\underline{\sigma}}$ with

$$\sigma_j = \begin{matrix} 0, & j \le k + k/2 - 1 \\ \\ 1, & j > k + k/2 - 1 \end{matrix} \ .$$

This would mean that

$$D_{k,\infty} = \|(N_{j,k,\underline{\underline{\tau}}}(\rho_i))^{-1}\|_\infty \tag{5.6}$$

with $\underline{\underline{\tau}} := (\tau_i)_1^{2k}$ given by

$$0 = \tau_1 = \cdots = \tau_k, \quad \tau_{k+1} = \cdots = \tau_{2k} = 1$$

and $0 = \rho_1 < \cdots < \rho_k = 1$ the extrema of the Chebyshev polynomial of degree $k-1$ for $[0,1]$. This gives the following values for $D_{k,\infty}$.

k	$D_{k,\infty}$	$\ln_2 D_{k,\infty}$
2	1	0
3	3	1.5849..
4	5	2.3219..
5	11 2/3	3.5443..
6	21	4.3923..
7	46 1/5	5.5298..
8	85 4/5	6.4229..
9	183 6/7	7.5224..
10	347 2/7	8.4399..
15	.1169E 5	13.5128..
20	.3635E 6	18.4715..
25	.1193E 8	23.5075..
30	.3747E 9	28.4813..
35	.1219E11	33.5053..
40	.3850E12	38.4861..

It is striking that the first few values of $D_{k,\infty}$ are such simple rational numbers and that these numbers conform so quickly to the pattern $D_{k,\infty} \sim 2^{k-1}/\sqrt{2}$, as can be seen by their logarithms to the base 2. This raises the hope that such a relation might be provable with a little effort.

References

[1] C. de Boor (1966), "The method of projections as applied to the numerical solution of two point boundary value problems using cubic splines", dissertation, Univ. Michigan.

[2] C. de Boor (1968), "On uniform approximation by splines", *J. Approx. Theory* **1**, 219–235.

[3] C. de Boor and G. J. Fix (1973), "Spline approximation by quasi-interpolants", *J. Approx. Theory* **8**, 19–45.

[4] C. de Boor (1973), "The quasi-interpolant as a tool in elementary polynomial spline theory", in *Approximation Theory* (G. G. Lorentz *et al.*, eds), Academic Press (New York), 269–276.

[5] C. de Boor (1972), "On calculating with B-splines", *J. Approx. Theory* **6**, 50–62.

[6] C. de Boor (1975), "How small can one make the derivatives of an interpolating function?", *J. Approx. Theory* **13**, 105–116.

[7] C. de Boor (1975), "A smooth and local interpolant with 'small' k-th derivative", in *Numerical Solutions of Boundary Value Problems for Ordinary Differential Equations* (A. Aziz, ed), Academic Press (New York), 177–197.

[8] C. de Boor (1976), "A bound on the L_∞-norm of L_2-approximation by splines in terms of a global mesh ratio", *Math. Comp.* **30(136)**, 765–771.

[9] H. B. Curry and I. J. Schoenberg (1966), "On Pólya frequency functions IV: the fundamental spline functions and their limits", *J. Analyse Math.* **17**, 71–107.

[10] Jim Douglas Jr., Todd Dupont, and Lars Wahlbin (1975), "Optimal L_∞ error estimates for Galerkin approximations to solutions of two-point boundary value problems", *Math. Comp.* **29(130)**, 475–483.

[11] J. Jerome and L. L. Schumaker (1969), "Characterizations of functions with higher order derivatives in L^p", *Trans. Amer. Math. Soc.* **143**, 363–371.

[12] S. Karlin (1968), *Total Positivity*, Stanford Univ. Press (Stanford).

[13] T. Lyche and L. L. Schumaker (1975), "Local spline approximation methods", *J. Approx. Theory* **15**, 294–325.

[14] I. J. Schoenberg (1969), "Cardinal interpolation and spline functions", *J. Approx. Theory* **2**, 167–206.

[15] I. J. Schoenberg (1971), "The perfect B-splines and a time-optimal control problem", *Israel J. Math.* **10**, 261–274.

[16] I. J. Schoenberg (1973), *Cardinal Spline Interpolation*, CBMS, SIAM (Philadelphia).