A NOTE ON MATRIX REFINEMENT EQUATIONS

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Abstract. Refinement equations involving matrix masks are receiving a lot of attention these days. They can play a central role in the study of refinable finitely generated shift-invariant spaces, multiresolutions generated by more than one function, multi-wavelets, splines with multiple knots, and matrix subdivision schemes — including Hermite-type subdivision schemes. Several recent papers on this subject begin with an assumption on the eigenstructure of the mask, pointing out that this assumption is heuristically "natural" or "preferred". In this note, we prove that stability of the shifts of the refinable function requires this assumption.

Key words. Matrix refinement equation, matrix subdivision scheme, refinable function vector, stability, Riesz basis, multi-wavelet, shift-invariant space, FSI space

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1. Introduction.

Several desirable properties are not available with compactly supported orthogonal wavelets, e.g., symmetry and piecewise polynomial structure. Presently, multi-wavelets seem to offer a satisfactory alternative (cf., e.g., [3], [5]). Multi-wavelets are wavelets constructed from a refinable function vector Φ which satisfies a matrix refinement equation of the form

$$\Phi = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)\Phi(M^T \cdot - \alpha).$$

Here, each coefficient $a(\alpha)$ is a $\Phi \times \Phi$ matrix, and $M \in \mathbb{Z}^{d \times d}$. Refinable function vectors have also appeared in the study of matrix subdivision schemes, which play an important role in the analysis of multivariate subdivision schemes (cf. [4]).

As in the case of a single refinable function, it is often impossible to study a refinable function vector directly. In such a case, its properties are analyzed indirectly via the coefficient sequence $(a(\alpha))_{\alpha \in \mathbb{Z}^d}$ (cf., e.g., [2], [6], [7], [8], [10], [11]). For example, the eigenstructure of the matrix

$$A(0) = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)$$

has played an important role in such analyses. In particular, it is assumed in [7] and [11], that 1 is a simple eigenvalue of $A(0)$ and that all other eigenvalues have modulus strictly less than 1. In this paper, we demonstrate that this is a very reasonable assumption by proving that without such an assumption, the refinable function vector Φ can not possibly have stable shifts.

A slightly weaker statement has already been proved by Cohen, Dyn, and Levin in [1] for ℓ^∞-stability. In that paper, they assumed that $a(\alpha) = 0$ for all but finitely many α and

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that the associated subdivision scheme is C^0, i.e., convergent. In this paper we strengthen and extend their results.

To present our results in a general setting, we recall the following definition from [9]:

$$\mathcal{L}^p := \mathcal{L}^p(\mathbb{R}^d) := \left\{ \phi : \mathbb{R}^d \to \mathbb{C} \;\middle|\; |\phi|_p := \|\widetilde{\phi}\|_{L^p([0..1)^d)} < \infty \right\}$$

for $1 \le p \le \infty$, where $\widetilde{\phi} := \sum_{\alpha \in \mathbb{Z}^d} |\phi(\cdot - \alpha)|$. As pointed out in [9], $\mathcal{L}^p$ is a Banach space with norm $|\cdot|_p$ and

$$\mathcal{L}^p \subset L^1 \cap L^p.$$

Now, let $M \in \mathbb{Z}^{d\times d}$ be an integer matrix satisfying $\lim_{k\to\infty} M^{-k} = 0$. And let $\phi_1, \ldots, \phi_m \in \mathcal{L}^p$. We say that $\Phi := (\phi_j)_{j=1}^m$ is M**-refinable** if there exist sequences $a_{j,k} \in \ell^1(\mathbb{Z}^d) \quad (1 \le j, k \le m)$ such that

$$\phi_j = \sum_{k=1}^m \sum_{\alpha \in \mathbb{Z}^d} a_{j,k}(\alpha)\phi_k(M^T \cdot - \alpha) \qquad (j = 1, \ldots, m).$$

Equivalently, Φ is refinable if

$$\widehat{\Phi}(M\xi) = A(\xi)\widehat{\Phi}(\xi) \text{ for all } \xi \in \mathbb{R}^d, \tag{1.1}$$

where the matrix $A := (A_{j,k})_{1\le j,k\le m}$ of (continuous 2π-periodic) functions is defined by

$$A_{j,k}(\xi) := \frac{1}{|\det M|} \sum_{\alpha \in \mathbb{Z}^d} a_{j,k}(\alpha) e^{-i\langle \alpha, \xi\rangle}.$$

The matrix A is referred to as the **(refinement) mask**.

It is already well-known that equation (1.1) has only the trivial solution $\Phi = 0$ if the spectral radius $\rho(A(0)) < 1$. It is also well-known that convergence of the infinite product

$$P := \prod_{j=1}^\infty A(M^{-j}\cdot) \tag{1.2}$$

requires that (i) $\rho(A(0)) \le 1$, (ii) 1 be the only eigenvalue of modulus 1, and (iii) the algebraic and geometric multiplicities of the eigenvalue 1 be the same. When this product does converge, the function Φ defined by $\widehat{\Phi}(\xi) = P(\xi)x$ is a solution to equation (1.1) for any $x \in \mathbb{C}^m$. Convergence of the matrix subdivision scheme associated with equation (1.1) requires similar assumptions on A (cf. [1]). Nonetheless, solutions to equation (1.1) may exist even without such assumptions (as pointed out in [6] and [2]). However, the existence of solutions with stable shifts will require these and more.

The shifts of $\phi_1, \ldots, \phi_m \in \mathcal{L}^p$ are said to be ℓ^p**-stable** if there exist constants $0 < c_1 \le c_2 < \infty$ such that

$$c_1 \sum_{j=1}^m \|a_j\|_{\ell^p} \le \|\sum_{j=1}^m \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\phi_j(\cdot - \alpha)\|_{L^p} \le c_2 \sum_{j=1}^m \|a_j\|_{\ell^p}$$

for any $a_1, \dots, a_m \in \ell^p(\mathbb{Z}^d)$. In [9], Jia and Micchelli proved that the shifts of any $\phi_1, \dots, \phi_m \in \mathcal{L}^p$ are ℓ^p-stable if and only if the sequences

$$\left(\widehat{\phi}_j(\xi + 2\alpha\pi)\right)_{\alpha \in \mathbb{Z}^d} \qquad (j = 1, \dots, m)$$

are linearly independent for every $\xi \in \mathbb{R}^d$. This will play the major role in our proofs.

In the statement of our theorems, we use the following terminology. An eigenvalue is **non-degenerate** if its algebraic and geometric multiplicities agree. A **simple** eigenvalue is a non-degenerate eigenvalue of multiplicity one.

To facilitate our proofs, we define

$$V := \{ v \in \mathbb{Z}^d \mid v = Mt \text{ for some } t \in [0 \,..\, 1)^d \}.$$

Then V is a complete set of representatives for the quotient group $\mathbb{Z}^d / M\mathbb{Z}^d$. In particular, $\mathbb{Z}^d$ is the disjoint union of the sets $v + M\mathbb{Z}^d$ $(v \in V)$. We will actually only ever make use of the set $V' := V \backslash 0$.

2. Stability imposes structure.

THEOREM 2.1. *Let $\phi_1, \dots, \phi_m \in \mathcal{L}^p$. Suppose $\Phi := (\phi_j)_{j=1}^m$ is M-refinable with mask A. If the shifts of $\phi_1, \dots, \phi_m$ are ℓ^p-stable, then 1 is a simple eigenvalue of $A(0)$ and all other eigenvalues have modulus strictly less than 1. Moreover, $\widehat{\Phi}(0)$ is a right 1-eigenvector.*

Proof. We assume stability to demonstrate the eigenvalue assertions.

By the refinement equation (1.1), we have for any $\xi \in \mathbb{R}^d$ and $n \in \mathbb{N}$,

$$\widehat{\Phi}(\xi) = \prod_{j=1}^{k} A(M^{-j}\xi)\widehat{\Phi}(M^{-k}\xi)$$

since $M^{-k} \to 0$ (and since A and $\widehat{\Phi}$ are both continuous), $\rho(A(0)) < 1$ would imply that Φ is identically zero — contradicting the assumption that the shifts of Φ are stable. So $\rho(A(0)) \geq 1$.

Now, suppose $y \in \mathbb{C}^m$ satisfies $y^T A(0) = \mu y^T \neq 0$ for some $\mu \in \mathbb{C}$ with $|\mu| \geq 1$. Then the refinement equation (1.1) implies that

$$\begin{aligned} y^T \widehat{\Phi}(2M^k(M\alpha + v)\pi) &= y^T A^k(0) A(2M^{-1}v\pi)\widehat{\Phi}(2M^{-1}v\pi + 2\alpha\pi) \\ &= \mu^k y^T A(2M^{-1}v\pi)\widehat{\Phi}(2M^{-1}v\pi + 2\alpha\pi) \end{aligned} \tag{2.1}$$

for any $k \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}^d$, $v \in V'$. Since $v \in V'$ (hence $M\alpha + v \neq 0$), our assumptions on M imply that $\lim_{k\to\infty} |M^k(M\alpha + v)| = \infty$. Since $y^T\Phi \in \mathcal{L}^p \subset L^1$, the left-hand side of equation (2.1) then tends to zero as k tends to infinity. And, since $|\mu| \geq 1$, this implies that

$$y^T A(2M^{-1}v\pi)\widehat{\Phi}(2M^{-1}v\pi + 2\alpha\pi) = 0$$

for every $\alpha \in \mathbb{Z}^d$ (which implies that $y^T A(2M^{-1}v\pi) = 0$ for every $v \in V'$, since the shifts of Φ are stable). Together with equation (2.1), this implies that $y^T\widehat{\Phi}(2\beta\pi) = 0$ for all

$\beta \in \mathbb{Z}^d \backslash 0$, since every such β has a (unique) representation of the form $\beta = M^k(M\alpha + v)$ for some $k \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}^d$, $v \in V'$. The stability of the shifts of Φ then implies that $y^T\widehat{\Phi}(0) \neq 0$ and, *a fortiori*, $\widehat{\Phi}(0) \neq 0$. The refinement equation (1.1) then implies that $\widehat{\Phi}(0)$ is a right 1-eigenvector of $A(0)$.

Now, suppose that $y_1^T A(0) = \mu_1 y_1^T \neq 0$ and $y_2^T A(0) = \mu_2 y_2^T \neq 0$ with $|\mu_i| \geq 1$ ($i = 1, 2$). The above arguments imply that $y_i^T\widehat{\Phi}(2\beta\pi) = 0$ $\forall \beta \in \mathbb{Z}^d \backslash 0$ and $y_i^T\widehat{\Phi}(0) \neq 0$ — without loss of generality, we may assume that $y_i^T\widehat{\Phi}(0) = 1$. Then $(y_2 - y_1)^T\widehat{\Phi}(2\beta\pi) = 0$ for every $\beta \in \mathbb{Z}^d$. The stability of the shifts of Φ now implies that $y_1 = y_2$.

We conclude that 1 is an eigenvalue of $A(0)$ of *geometric* multiplicity one. It is the only eigenvalue outside the open unit disc. Its (unique up to multiplicity) right eigenvector is $\widehat{\Phi}(0)$ and its (unique up to multiplicity) left eigenvector, y^T, satisfies $y^T\widehat{\Phi}(2\beta\pi) = 0$ for all $\beta \in \mathbb{Z}^d \backslash 0$. If the algebraic multiplicity of the eigenvalue 1 were greater than one, then the (one dimensional) left and right eigenspaces would be orthogonal one to the other (this follows easily by considering the Jordan canonical form of $A(0)$). That is, $y^T\widehat{\Phi}(0)$ would be zero — contradicting the assumption that the shifts of Φ are stable. ∎

Remark. The above proof in fact implies that the left-eigenvector y^T of $A(0)$ actually satisfies the "sum rules"

$$y^T \sum_{\alpha \in \mathbb{Z}^d} a(\beta + M^T\alpha) = y^T \quad \forall \beta \in \mathbb{Z}^d,$$

as well as the so-called Strang-Fix conditions of order 1

$$y^T\widehat{\Phi}(0) \neq 0, \qquad y^T\widehat{\Phi}(2\beta\pi) = 0 \quad \forall \beta \in \mathbb{Z}^d \backslash 0.$$

So stability implies accuracy of order 1 (or density) as expected.

3. Stability of matrix functions. A generalized stability notion for matrix functions has been recently considered in [1]. In the spirit of that paper, we will say that the shifts of any $m \times n$ matrix $\Phi = (\phi_{j,k})$ of $\mathcal{L}^p$-functions are **ℓ^p-stable** if there exist constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \sum_{j=1}^m \|a_j\|_{\ell^p} \leq \sum_{k=1}^n \| \sum_{j=1}^m \sum_{\alpha \in \mathbb{Z}^d} a_j(\alpha)\phi_{j,k}(\cdot - \alpha)\|_{L^p} \leq c_2 \sum_{j=1}^m \|a_j\|_{\ell^p}$$

for any $a_1, \ldots, a_m \in \ell^p(\mathbb{Z}^d)$.

Many of the results from [9] can be generalized to cover this notion. To state some pertinent ones, we first recall some of their notation. We denote the d-dimensional torus

$$\left\{ (z_1, \ldots, z_d) \in \mathbb{C}^d \;\middle|\; |z_1| = \cdots = |z_d| = 1 \right\}$$

by $\mathbb{T}^d$. Then, for any $f, g \in \mathcal{L}^2$, define

$$[f, g](z) := \sum_{\alpha \in \mathbb{Z}^d} \langle f, g(\cdot - \alpha)\rangle z^\alpha \qquad (z \in \mathbb{T}^d),$$

where $\langle f, g\rangle := \int_{\mathbb{R}^d} f\overline{g}$ for $f, g \in L^2(\mathbb{R}^d)$. And, lastly, for $\phi_1, \ldots, \phi_m \in \mathcal{L}^p$, define

$$\mathcal{S}^1(\phi_1, \ldots, \phi_m) := \Big\{ \sum_{j=1}^{m} \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\phi_j(\cdot - \alpha) \;\Big|\; a_j \in \ell^1(\mathbb{Z}^d) \text{ for } j = 1, \ldots, m \Big\}.$$

It is worth pointing out that $[f, g](z)$ is a continuous function of z on $\mathbb{T}^d$, and that $\mathcal{S}^1(\phi_1, \ldots, \phi_m)$ is a subspace of $\mathcal{L}^p(\mathbb{R}^d)$.

A generalized statement of [9, Theorem 4.1] is

THEOREM 3.1. *Let $\phi_{j,k} \in \mathcal{L}^2(\mathbb{R}^d)$, $(j = 1, \ldots, m; k = 1, \ldots, n)$. Then the shifts of $\Phi = (\phi_{j,k})$ are ℓ^2-stable if and only if one of the following conditions holds:*

(i) *For any $\xi \in \mathbb{R}^d$, the sequences $(\widehat{\phi}_{j,k}(\xi + 2\alpha\pi))_{k=1,\alpha\in\mathbb{Z}^d}^n$ $(j = 1, \ldots, m)$ are linearly independent.*

(ii) *The matrix $(\sum_{k=1}^n [\phi_{j,k}, \phi_{\ell,k}](z))_{1\le j,\ell\le m}$ is positive definite for every $z \in \mathbb{T}^d$.*

(iii) *There exist $g_{j,k} \in \mathcal{S}^1(\phi_{1,k}, \ldots, \phi_{m,k})$ $(j = 1, \ldots, m; k = 1, \ldots, n)$ such that*

$$\sum_{k=1}^{n} \langle g_{j,k}, \phi_{\ell,k}(\cdot - \alpha)\rangle = \delta_{j\ell}\delta_{0\alpha} \text{ for } 1 \le j, \ell \le m \text{ and } \alpha \in \mathbb{Z}^d.$$

And [9, Theorem 4.2] is generalized as follows.

THEOREM 3.2. *Let $\phi_{j,k} \in \mathcal{L}^p(\mathbb{R}^d)$, $(j = 1, \ldots, m; k = 1, \ldots, n)$. Then the shifts of $\Phi = (\phi_{j,k})$ are ℓ^p-stable if and only if condition* (3.1.i) *holds.*

The proofs of these theorems are clear from the proofs of [9, Theorem 3.3], [9, Theorem 3.5], and [9, Theorem 4.1]. We now state a generalization of Theorem 2.1. The proof is similar, so we provide only the major distinctions below.

THEOREM 3.3. *Let $\phi_{j,k} \in \mathcal{L}^p(\mathbb{R}^d)$, $(j = 1, \ldots, m; k = 1, \ldots, n)$. Suppose $\Phi = (\phi_{j,k})$ is M-refinable with mask A. If the shifts of Φ are ℓ^p-stable, then* 1 *is a non-degenerate eigenvalue of $A(0)$; its multiplicity is the rank of the matrix $\widehat{\Phi}(0) = \big(\widehat{\phi}_{i,j}(0)\big)$; and all other eigenvalues have modulus strictly less than* 1*. In particular, the columns of $\widehat{\Phi}(0)$ must span the right* 1*-eigenspace of $A(0)$.*

Proof. Define

$$W := \{\, y \in \mathbb{C}^m \mid y^T A(0) = \mu y^T \text{ for some } |\mu| \ge 1 \,\} \text{ and } X := \{\, x \in \mathbb{C}^m \mid A(0)x = x \,\}.$$

Then $\operatorname{rank}\widehat{\Phi}(0) \le \dim X \le \dim W$, since every non-zero column of $\widehat{\Phi}(0)$ is a right 1-eigenvector of $A(0)$.

As in the proof of Theorem 2.1, if $y \in W$ then $y^T\widehat{\Phi}(2\beta\pi) = 0$ for all $\beta \in \mathbb{Z}^d\backslash 0$. If the shifts of Φ are stable, then $y^T\widehat{\Phi}(0) \ne 0$ for every $y \in W$. This implies that $\dim W \le \operatorname{rank}\widehat{\Phi}(0)$, hence both must equal the geometric multiplicity of the eigenvalue 1. In particular, all other eigenvalues have modulus strictly less than one, and the columns of $\widehat{\Phi}(0)$ span the right 1-eigenspace.

If the algebraic multiplicity of the eigenvalue 1 is greater than its geometric multiplicity, then there exists a left 1-eigenvector y for which $y^T x = 0$ for all $x \in X$. Such y satisfies $y^T\widehat{\Phi}(2\alpha\pi) = 0$ for all $\alpha \in \mathbb{Z}^d$ and the shifts of Φ are not stable. $\square$

We can say even more under slightly more restrictive assumptions on the sequences $(a_{j,k}(\alpha))_{\alpha\in\mathbb{Z}}$. Suppose, for example, that each of these sequences decays exponentially fast, then the entries of the matrix A are analytic functions. Now, if Φ is a matrix solution to the refinement equation (1.1) and the shifts of Φ are stable, then the arguments of [6] (and the consequences of Theorem 3.3) imply that the infinite matrix product (1.2) is convergent, and that the map $v \mapsto Pv$ is an isomorphism from the right 1-eigenspace of $A(0)$ onto the (vector) solution space of the refinement equation (1.1). Hence this solution space is already spanned by some N of the columns of Φ, where N is the multiplicity of the eigenvalue 1 of $A(0)$. This leads to

THEOREM 3.4. *Suppose some (matrix) solution of the refinement equation* (1.1) *has* ℓ^p*-stable shifts. Then a given solution* Φ *has* ℓ^p*-stable shifts if and only if the columns of* $\widehat{\Phi}(0)$ *span the right* 1*-eigenspace of* $A(0)$.

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