## The quasi–interpolant as a tool in elementary polynomial spline theory

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This talk is intended to demonstrate with the help of some examples that the quasi–interpolant of [2] is very convenient when it comes to proving even very elementary old and new facts about polynomial splines. The key is a formula which gives each B–spline expansion coefficient for a given spline in terms of the value of its derivatives at a point.

### 1. Definitions

Let $k \in \mathbb{N}$, let $\mathbf{t} := (t_i)_{-\infty}^{\infty}$ be real, nondecreasing $t_i < t_{i+k}$, all $i$, and set

$$a := \inf_i t_i,$$

and

$$b := \sup_i t_i.$$

For $i \in \mathbb{Z}$, **the** $i^{th}$ **B–spline of order** $k$ **with** (or, for the) **knot sequence** $\mathbf{t}$ is given by the rule

$$\begin{aligned} N_{ik}(t) &:= g_k(t_i, \ldots, t_{i+k};\, t)\ (t_{i+k} - t_i) \\ g_k(s;t) &:= (s\ -\ t)_+^{k-1} \end{aligned}$$

taking, for each fixed $t$, the $k^{th}$ divided difference of $g(s) := g_k(s;t)$ at $t_i, \ldots t_{i+k}$ in the usual manner even when some or all of the $t_j$'s coincide. I leave unresolved any possible ambiguity when $t = t_j$ for some $j$, and concern myself only with left and right limits at such a point; i.e., I replace each $t = t_j$ by the "two points" $t_j^-$ and $t_j^+$.

As is well known,

$$N_{ik} > 0 \text{ on } (t_i, t_{i+k}), \text{ and } N_{ik} = 0 \text{ off } [t_i^+, t_{i+k}^-]$$

so that (since $t_i < t_{i+k}$, by assumption) $N_{ik}$ is not identically zero, while on the other hand, no more than $k$ of the $N_{jk}$'s are nonzero at any particular point. Consequently, for an arbitrary $\mathbf{a} \in \mathbb{R}^{\mathbb{Z}}$, the rule

$$f(t) := \sum_i a_i N_{ik}(t)$$

defines a function on $(a, b)$ if we take the sum to be *pointwise*. I call every such function a **polynomial spline of order** $k$ **with knot sequence** $\mathbf{t}$, and denote their collection by

$$\mathcal{S}_{k,\mathbf{t}}.$$

The "quasi–interpolator" $Q$ of interest here is given by the rule

$$Qf := \sum_i (\lambda_i f) N_{ik}$$

where

$$\lambda_i f := \lambda_{\tau_i, \psi_{ik}} f := \sum_{j<k} (-)^{k-1-j} \psi_{ik}^{(k-1-j)}(\tau_i) f^{(j)}(\tau_i)$$

$$\psi_{ik}(t) := (t_{i+1} - t)\ \ldots\ (t_{i+k-1} - t)/(k-1)!$$

and $\tau_i$ is an *arbitrary* point in $(t_i, t_{i+k})$. One verifies directly that [2]

$$\lambda_i N_{jk} = \delta_{ij}, \quad \text{all } i, j.$$

Consequently,

(i) $Q$ is a linear projector with range $\mathcal{S}_{k,\mathbf{t}}$;
(ii) every $f \in \mathcal{S}_{k,\mathbf{t}}$ has a unique representation as a B–spline series;
(iii) if $f = \sum_i a_i N_{ik}$, then
$$a_i = \lambda_{\tau_i, \psi_{ik}} f \text{ for arbitrary } \tau_i \in (t_i, t_{i+k}).$$

## 2. Existence and uniqueness of the B–spline expansion

The rather curious freedom in the choice of $\tau_i$ above leads to the following short proof of

**Theorem (Curry et Schoenberg [3]).** *$\mathcal{S}_{k,\mathbf{t}}$ consists of exactly those $f$ on (a,b) for which*

(i) *for all $i$, $f|_i \in \mathcal{P}_k (:=$ polynomials of degree $< k)$; and*

(ii) *if $t_s < t_{s+1} = \cdots = t_{s+r} < t_{s+r+1}$, then* $\mathrm{jump}_{t_{s+1}} f^{(k-j)} = 0$ *for all $j > r$.*

In particular, any such $f$ has exactly one B–spline expansion (in terms of the B–splines of order $k$ with knots $\mathbf{t}$).

Here and below, we denote by $f|_i$ the restriction of $f$ to $(t_i, t_{i+1})$. For the proof, I show that $Qf = f$ for all such $f$ :

(a) For all such $f$, and all $i$,

$$g(\tau) := \lambda_{\tau,\psi_{ik}} f = \sum_{j<k} (-)^{k-1-j} \psi_{ik}^{(k-1-j)}(\tau) f^{(j)}(\tau)$$

is constant on $\tau \in (t_i, t_{i+k}) =$ support $N_{ik}$, since

($\alpha$) for $\psi \in \mathcal{P}_k$ and smooth $f$,

$$(\lambda_{\tau,\psi} - \lambda_{\sigma,\psi}) f = \int_\sigma^\tau \psi \, df^{(k-1)} \quad (= 0 \text{ if } f|_{[\sigma,\tau]} \in \mathcal{P}_k)$$

hence, as $f|_{(t_j,t_{j+1})} \in \mathcal{P}_k$, *$g$ is constant on each* $(t_j, t_{j+1})$; and

($\beta$) if $t_i \le t_s < t_{s+1} = \cdots = t_{s+r} < t_{s+r+1} \le t_{i+k}$, then $t_{s+1}$ is an $r$-fold zero of $\psi_{ik}$, hence

$$\psi_{ik}^{(k-1-j)}(t_{s+1}) = 0, \quad \text{for } j = k-1,\ k-2, \ldots, k-r,$$

while, by assumption on $f$,

$$\mathrm{jump}_{t_{s+1}} f^{(j)} = 0, \quad \text{for } j = k-r-1,\ \ldots\ ,\ 0;$$

hence *$g$ is continuous across each $t_{s+1}$ with $t_i < t_{s+1} < t_{i+k}$.*

(b) For all such $f$, and all $j$ with $t_j < t_{j+1}$,

$$(Qf)|_j = f|_j.$$

For, $(Qf)\,|_j = \sum_{i=j+1-k}^{j} (\lambda_{\tau_i,\psi_{ik}} f)\,(N_{ik})|_j$. But I can assume by (a) without loss that $\tau_i \in (t_j, t_{j+1})$, $i = j+1-k, \ldots, j$; hence

$$(Qf)|_j = \sum_{i=j+1-k}^{j} \lambda_{\tau_i,\psi_{ik}} (f|_j)\,(N_{ik})|_j,$$

while

$$\delta_{ir} = \lambda_{\tau_i,\psi_{ik}} N_{rk} = \lambda_{\tau_i,\psi_{ik}} (N_{rk}|_j), \quad r = j+1-k, \ldots, j$$

shows the $k$–sequence $N_{ik}|_j$, $i = j+1-k, \ldots, j$, in $\mathcal{P}_k$ to be independent, hence a basis for $P_k$. Consequently,

$$\sum_{i=j+1-k}^{j} (\lambda_{\tau_i,\psi_{ik}} h)\,(N_{ik})|_j = h, \quad \text{for all } h \in \mathcal{P}_k.$$

## 3. Uniqueness of odd–degree spline interpolation

In discussing the smooth extension of a real valued function defined on some closed subset of $\mathbb{R}$ to all of $\mathbb{R}$, Golomb et Schoenberg [4] prove that, for $\mathbf{t}$ strictly increasing, every $f \in \mathcal{S}_{2k,\mathbf{t}}$ which vanishes at the points of $\mathbf{t}$ and has square–integrable $k^{th}$ derivative must vanish identically. Their proof is not simple. In particular, the straightforward argument

> $\forall_i f(t_i) = 0$, hence, $\forall_i\, 0 = f(t_i, \ldots, t_{i+k}) = \int N_{ik}(t) f^{(k)}(t)\, dt / c_{ik}$ with $c_{ik} := (k-1)!(t_{i+k} - t_i)$; i.e., $f^{(k)}$ is *orthogonal* to every $N_{ik}$, while at the same time being in $S_{k,\mathbf{t}}$ which is spanned by the $N_{ik}$'s; hence $f^{(k)} = 0$, and so $f = 0$.

was not open to them since it requires $(N_{ik})$ to be a Schauder basis for $\mathcal{S}_{k,\mathbf{t}} \cap L_2$, a fact they did not know.

**Theorem.** *Let $1 \le p \le \infty$, and $N_{ikp} := (k/(t_{i+k} - t_i))^{1/p} N_{ik}$. Then*

$$\sum_i b_i N_{ikp} \in L_p(a,b) \text{ iff } \|\mathbf{b}\|_p < \infty.$$

*Precisely, there exists $D_{kp} > 0$ (independent of $\mathbf{t}$) so that*

$$D_{kp}^{-1} \|\mathbf{b}\|_p \le \|\sum_i b_i N_{ikp}\|_p \le \|\mathbf{b}\|_p, \quad \text{for all } \mathbf{b} \in \mathbb{R}^{\mathbb{Z}}.$$

The second inequality is straightforward. As to the first, let $f := \sum_i a_i N_{ik} = \sum_i b_i N_{ikp}$, so that $a_i((t_{i+k} - t_i)/k)^{1/p} = b_i$, all $i$. Then, from Sec. 1, $|a_i| \le \sum_{j<k} |\psi_{ik}^{(k-1-j)}(\tau_i)|\, |f^{(j)}(\tau_i)|$.

Take $I$ to be a largest interval among $(t_i, t_{i+1}), \ldots, (t_{i+k-1}, t_{i+k})$, and choose $\tau_i \in I$. Then $|\psi_{ik}^{(k-1-j)}(\tau_i)| < A_{jk}|I|^j$ for some constants $A_{jk}$, while $|f^{(j)}(\tau_i)| \le B_{jkp}|I|^{-j-1/p} \cdot \left(\int_I |f(t)|^p dt\right)^{1/p}$ since $f|_I \in \mathcal{P}_k$. Hence

$$|b_i|^p = |a_i|^p (t_{i+k} - t_i)/k \le |a_i|^p |I| \le \left(\sum_j A_{jk} B_{jkp}\right)^p \int_I |f|^p$$
$$\le C_{kp} \int_{t_i}^{t_{i+k}} |f|^p$$

which, after summing over $i$, gives the required inequality with $D_{kp} = (kC_{kp})^{1/p}$.

For a *uniform* knot sequence $\mathbf{t}$, this theorem has already been proved by Schoenberg in [5] using a special case of the above formula for the B–spline coefficients.

**Corollary.** *For $1 \le p < \infty$, $(N_{ikp})_{-\infty}^{\infty}$ is a Schauder basis for $\mathcal{S}_{k,\mathbf{t}} \cap L_p(a,b)$.*

Bolstered by this Corollary, the earlier argument establishes uniqueness of odd–degree spline interpolation even in the limiting case of repeated or osculatory interpolation at multiple knots.

## 4. Bounds for least–squares approximation by splines

An attempt to bound the error in odd–degree spline interpolation to a smooth function in the uniform norm leads to the problem of bounding least–squares approximation by splines, considered as a map on $L_\infty$, independently of the knot sequence (cf. [1]), a question of interest in itself.

Let $n \in \mathbb{N}$, $\mathcal{S} = \operatorname{span}\{N_{1k}, \ldots, N_{nk}\}$, and denote by $Lf$ the least–squares approximation to an $f \in L_\infty[t_1, t_{n+k}]$ by elements of $\mathcal{S}$. Then, $L$ is a linear projector, characterized by the fact that

$$(*) \qquad Lf \in \mathcal{S}, \text{ and, for all } \lambda \in \Lambda,\ \lambda Lf = \lambda f$$

with the "interpolation conditions"

$$\Lambda := \{\lambda \in L_\infty^* |\text{ for some } \varphi \in \mathcal{S} \text{ and all } f,\ \lambda f = \int \varphi f\}.$$

one verifies that (∗) implies

$$\|L\| = \sup_{x\in\mathcal{S}} \inf_{\lambda\in\Lambda} \|\lambda\| \, \|x\|/|\lambda x|.$$

But, in order to compute, one needs to coordinatize. Letting $(\lambda_i)$ and $(\varphi_i)$ be bases for $\Lambda$ and $\mathcal{S}$, respectively, we get that

$$\|L\| = \sup_{\mathbf{a}} \inf_{\mathbf{b}} \|\sum_i b_i\lambda_i\| \|\sum_j a_j\varphi_j\| / \big|\sum_{ij} b_i\lambda_i\varphi_j a_j\big|.$$

Take $\varphi_i := N_{ik}, \lambda_i := k\int \cdot N_{ik}/(t_{i+k}-t_i), i = 1,\dots n$. From the earlier theorem,

$$D_{k1}^{-1} D_{k\infty}^{-1} \|\mathbf{b}\|_1 \|\mathbf{a}\|_\infty \le \|\sum_i b_i\lambda_i\| \, \|\sum_j a_j\varphi_j\| \le \|\mathbf{b}\|_1 \|\mathbf{a}\|_\infty$$

while

$$\sup_{\mathbf{a}} \inf_{\mathbf{b}} \|\mathbf{b}\|_1 \|\mathbf{a}\|_\infty / \big|\sum_{ij} b_i\lambda_i\varphi_j a_j\big| = \|(\lambda_i\varphi_j)^{-1}\|_\infty$$

with $\|A\|_p$ denoting the norm for the matrix $A$ induced by the $p$–norm on vectors. This proves

**Proposition.** *For some positive $C_k$ (independent of $\mathbf{t}$ and $n$),*

$$C_k \|(\lambda_i\varphi_j)^{-1}\|_\infty \le \|L\| \le \|(\lambda_i\varphi_j)^{-1}\|_\infty$$

*(considering $L$ as a map on $L_\infty[t_1, t_{n+k}]$), with*

$$(**) \qquad \lambda_i\varphi_j = k\int N_{ik}N_{jk}/(t_{i+k}-t_i), \quad i,j = 1,\dots,n.$$

It has been known for some time that $L$ could be bounded if only the Gramian $(\lambda_i\varphi_j)$ could be bounded below (in the max–norm). This proposition adds that such bounding below of the Gramian is also necessary for bounding $L$. For this reason, I offer the modest sum of $m$–1972 ten dollar bills to the first person who communicates to me a proof or a counterexample (but not both) of his or her own making for the following conjecture (known to be true when $k = 2$ or $k = 3$):

**Conjecture.** *For given $n$ and $\mathbf{t}$, let $(\lambda_i\varphi_j)$ be the $n \times n$ matrix whose entries are given by (∗∗). Then*

$$\sup_{n,\mathbf{t}} \|(\lambda_i\varphi_j)^{-1}\|_\infty < \infty.$$

Here, $m$ is the year A.D. of such communication.

## 5. Estimates for dist $(f, \mathcal{S}_{k,\mathbf{t}})$

Let $Qf$ be the quasi–interpolant to $f$ as defined in Section 1. For a sufficiently smooth $f$,

$$f(t) - (Qf)(t) = \int E(t,s)\, df^{(k-1)}(s)$$

with $E(t,\cdot)$ a *nonnegative* function of *small* support. This makes $Qf$ a convenient approximation when it comes to estimating the distance of such $f$ from splines with fixed and with variable knots. Lack of space precludes, unfortunately, any discussion of this important aspect of the quasi–interpolant here.

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### References

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[4] M. Golomb and I. J. Schoenberg (1968), "On $\mathcal{H}^m$–extension of functions and spline interpolation", MRC, U.Wisconsin-Madison.
This paper exists only as a reference in other papers, e.g., in [I. J. Schoenberg, On spline interpolation at all integer points of the real axis, Mathematica **10**(33), (1968), 151–170] where its proposed content is outlined, and in [M. Golomb and J. Jerome, Linear ordinary differential equations with boundary conditions on arbitrary point sets, Trans. AMS **153**, (1971), 235–264] in which it is incorrectly specified as a MRC TSR and where its proposed content is generalized.
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