On the (bi)infinite case of Shadrin's theorem concerning the L_∞-boundedness of the L_2-spline projector

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Dedicated to Professor Yurii Nikolaevich Subbotin on the occasion of his 75th birthday

Abstract. Some loose ends in Shadrin's remarkable paper are tied up.

Shadrin's theorem [S] settles a problem posed first in [B73] in the following setting.
For $k \in \mathbb{N}$, let $t := (t_i : i \in \mathbb{Z})$ be nondecreasing with $t_i < t_{i+k}$, all i, and set

$$a := \inf_i t_i, \quad \sup_i t_i =: b.$$

For each i, let

$$N_{ik}(x) := (t_{i+k} - t_i)\boldsymbol{\Delta}(t_i, \ldots, t_{i+k})(x - \cdot)_+^{k-1} = (\boldsymbol{\Delta}(t_{i+1}, \ldots, t_{i+k}) - \boldsymbol{\Delta}(t_i, \ldots, t_{i+k-1}))(x - \cdot)_+^{k-1}$$

be the **ith L_∞-normalized B-spline of order k for the knot sequence** t. For an arbitrary coefficient sequence $c = (c_i)$, the biinfinite sum $\sum_i N_{ik} c_i$ makes sense pointwise, i.e.,

$$\sum_i N_{ik}\, c_i : \mathbb{R} \to \mathbb{R} : x \mapsto \sum_{N_{ik}(x) \neq 0} N_{ik}(x) c_i$$

is well-defined since the last sum is finite due to the fact that $\operatorname{supp} N_{ik}$ lies in the interval $[t_i \,..\, t_{i+k}]$. Any such function $\sum_i N_{ik} c_i$ is called a **polynomial spline of order k with knot sequence** t. I'll denote the collection of all such functions by

$$\$_{kt}.$$

In particular, with

$$\ell_\infty := \ell_\infty(\mathbb{Z}), \quad L_\infty := L_\infty(a \,..\, b),$$

the map

$$\mathbf{N}_{kt} : \ell_\infty \to L_\infty : c \mapsto \sum_i N_{ik}\, c_i$$

is wellknown to be well-defined, of norm 1, and bounded below, hence boundedly invertible on its range

$$\$_{kt\infty} := \$_{kt} \cap L_\infty.$$

Now consider the linear map

$$\mathbf{M}_{kt}^{\mathrm{t}} : L_\infty \to \ell_\infty : f \mapsto (\int M_{ik} f : i \in \mathbb{Z})$$

with

$$M_{ik} := \frac{k}{t_{i+k} - t_i} N_{ik}$$

the **ith L_1-normalized B-spline of order k for the knot sequence** t, so called because

$$\int M_{ik} = \|M_{ik}\|_1 = 1,$$

hence $\|\mathbf{M}_{kt}^{\mathrm{t}}\| = 1$.

Now consider least-squares approximation to $g \in L_\infty$ from $\$_{kt\infty}$. To be sure, $g \in L_\infty$ need not have finite L_2-norm, hence it makes, offhand, no sense to talk about least-squares approximation to such g. But we can look for $f \in \$_{kt\infty}$ for which

$$\mathbf{M}_{kt}^{\mathrm{t}}(g - f) \ = \ 0,$$

a condition that characterizes f as the unique least-squares approximation to g from $\$_{kt\infty}$ for $g \in L_2(a\,..\,b)$. This raises the question whether the data map $\mathbf{M}_{kt}^{\mathrm{t}}$ is 1-1 on $\$_{kt\infty}$ which, in turn, raises the question whether the biinfinite matrix

$$A_{kt} := \mathbf{M}_{kt}^{\mathrm{t}}\mathbf{N}_{kt} = [M_{ik} : i \in \mathbb{Z}]^{\mathrm{t}}[N_{jk} : j \in \mathbb{Z}] = (\int M_{ik}N_{jk} : i, j \in \mathbb{Z}) \tag{1}$$

is invertible on ℓ_∞. If it is, then the linear map

$$P_{kt} := \mathbf{N}_{kt}(A_{kt})^{-1}\mathbf{M}_{kt}^{\mathrm{t}}$$

is a well-defined linear projector on L_∞ with range $\$_{kt\infty}$ and, for any $g \in L_\infty$, $P_{kt}g$ is the unique element f of $\$_{kt\infty}$ for which $g - f$ is **orthogonal to** $\$_{kt\infty}$ in the sense that $\mathbf{M}_{kt}^{\mathrm{t}}(g - f) = 0$. For this reason, I will call P_{kt} an **L_2-spline projector** and note that

$$\|P_{kt}\| \le \|\mathbf{N}_{kt}\|\,\|(A_{kt})^{-1}\|_\infty\,\|\mathbf{M}_{kt}^{\mathrm{t}}\| = \|(A_{kt})^{-1}\|_\infty,$$

hence the L_∞-boundedness of P_{kt} is assured once we know that A_{kt} is boundedly invertible as a map on ℓ_∞.

To be sure, while [B73] starts in this general setting, it considers L_2-spline projectors only in the *finite-dimensional* setting in which t is a *finite* knot sequence and, correspondingly, the invertibility of the Gramian is a standard result, and conjectures that

$$\forall\{k \in \mathbb{N}\} \quad \sup_t \|(A_{kt})^{-1}\|_\infty < \infty, \tag{2}$$

with the supremum taken over all finite knot sequences t.

It is this conjecture that Shadrin settles in [S] in the sense that he proves the following

Shadrin's Theorem ([S]). *For all $k \in \mathbb{N}$,*

$$s_k := \sup_t \|(A_{kt})^{-1}\|_\infty < \infty,$$

*with the supremum taken over all finite knot sequences t that are k-**complete**, meaning that the first and the last knot appear with maximal multiplicity k.*

Shadrin [S] also considers, in his Corollary II, the biinfinite case described above and deduces (2) for that case from the finite-dimensional case, using the observation that, therefore, all principal submatrices of A_{kt} are uniformly boundedly invertible, hence so must A_{kt} be, and with the same bound. It is the purpose of the present note to clarify this argument.

There are two points of concern.

(i) The bounded invertibility of A_{kt} is deduced from the uniformly bounded invertibility of its finite principal submatrices, but no reference is given for this (nontrivial) result.

(ii) Shadrin's Theorem is proved only for k-complete finite knot sequences hence says, offhand, nothing about finite principal submatrices of A_{kt} since there is no reason for any finite section $(t_i, \ldots, t_{n+k})$ of t to be k-complete. This was completely overlooked by me twelve years ago and only recently realized by Shadrin while trying to make the point that his paper [S] covers the periodic case.

Point (i) is taken care of by the following proposition which, while not explicitly stated, is established in [B82] during the proof of Theorem 4.1 there.

Proposition. *Let A be an ℓ_∞-bounded bi-infinite matrix that maps the closed subspace*

$$c_0 := \{a \in \mathbb{R}^{\mathbb{Z}} : \lim_{|i|\to\infty} a_i = 0\}$$

of the $\|\cdot\|_\infty$-normed space $\ell_\infty = \ell_\infty(\mathbb{Z})$ of bounded bi-infinite sequences into itself.

If for some r and for all sufficiently large integer intervals J, the submatrix

$$A_J := (A_{i,j} : i \in J, j \in r + J) \in \mathbb{R}^{J\times(r+J)}$$

is invertible and

$$s := \limsup_{J\to\mathbb{Z}} \|(A_J)^{-1}\|_\infty < \infty,$$

then also A is boundedly invertible as a map on ℓ_∞; in fact, $\|A^{-1}\|_\infty = s$.

Settling Point (ii) was the start of the present note. The argument given here for it also provides a proof of the Proposition for a totally positive matrix A.

It is well-known that the Gramian A_{kt} is **totally positive** (or **tp**, for short), meaning that all its minors are nonnegative. A good up-to-date reference regarding total positivity is [P]. We need only one of the many properties of invertible totally positive matrices A of order n, namely that *their inverse is a* **checkerboard matrix**, meaning that,

$$(-1)^{i-j} A^{-1}(i,j) \ge 0, \quad i,j = 1,\ldots,n. \tag{3}$$

This follows at once from Cramer's rule which gives

$$A^{-1}(i,j) = (-1)^{i-j} \det A(\backslash j, \backslash i)/\det A, \quad i,j = 1,\ldots,n.$$

Here, and in the following, it is convenient to denote the (i,j)-entry $A_{i,j}$ of the matrix A `MATLAB`-fashion by $A(i,j)$, and use the notation $A(\backslash j, \backslash i)$ for the matrix obtained from A by omitting the jth row and ith column.

The checkerboard nature of the inverse of a tp matrix implies the following remarkable property of tp matrices which was stated in [BJP] as a known fact, but without a reference and only a hint for how to prove it. Because of the importance of this property in the present context, I give a detailed and elementary proof.

Lemma ([BJP]). *If $B \in \mathbb{R}^{n\times n}$ is invertible and tp, then, for any integer interval $\mathbf{m} \subseteq \{1,2,\ldots,n\}$, so is the principal submatrix $C := B(\mathbf{m},\mathbf{m}) \in \mathbb{R}^{\mathbf{m}\times\mathbf{m}}$ of B involving only the rows and columns of B with index $i \in \mathbf{m}$, and*

$$0 \le (-1)^{i+j} C^{-1}(i,j) \le (-1)^{i+j} B^{-1}(i,j), \quad i,j \in \mathbf{m}. \tag{4}$$

Proof: By the generalized Hadamard inequality for tp matrices (see, e.g., Theorem 1.21 of [P]),

$$\det B \le \det C \det B(\backslash\mathbf{m}, \backslash\mathbf{m}),$$

with $B(\backslash\mathbf{m}, \backslash\mathbf{m})$ the principal submatrix of B complementary to C; hence the invertibility of B implies the invertibility of its principal submatrix C. The inequalities (4) follow by repeated application of the special case $\mathbf{m} = \{1,2,\ldots,n-1\}$ which in turn, by the checkerboard nature of the inverse of a tp matrix, follows from the formula

$$C^{-1} = B^{-1}(\mathbf{m},\mathbf{m}) - \frac{B^{-1}(\mathbf{m},n)B^{-1}(n,\mathbf{m})}{B^{-1}(n,n)}, \tag{5}$$

valid for that choice of $\mathbf{m}$. Finally, (5) for such $\mathbf{m}$ can be proved as follows. Since

$$\begin{bmatrix} \mathrm{id}_{\mathbf{m}} & 0 \\ 0 & 1 \end{bmatrix} = BB^{-1} = \begin{bmatrix} C & B(\mathbf{m},n) \\ B(n,\mathbf{m}) & B(n,n) \end{bmatrix} \begin{bmatrix} B^{-1}(\mathbf{m},\mathbf{m}) & B^{-1}(\mathbf{m},n) \\ B^{-1}(n,\mathbf{m}) & B^{-1}(n,n) \end{bmatrix},$$

therefore,

$$CB^{-1}(\mathbf{m},\mathbf{m}) = \mathrm{id}_{\mathbf{m}} - B(\mathbf{m},n)B^{-1}(n,\mathbf{m})$$

and

$$CB^{-1}(\mathbf{m},n) = -B(\mathbf{m},n)B^{-1}(n,n),$$

hence, with D the right side of (5),

$$\begin{array}{rcccc} CD = & C\Big(B^{-1}(\mathbf{m},\mathbf{m}) & - & B^{-1}(\mathbf{m},n)B^{-1}(n,\mathbf{m})/B^{-1}(n,n)\Big) \\ = & CB^{-1}(\mathbf{m},\mathbf{m}) & - & CB^{-1}(\mathbf{m},n)B^{-1}(n,\mathbf{m})/B^{-1}(n,n) \\ = & \mathrm{id}_{\mathbf{m}} - B(\mathbf{m},n)B^{-1}(n,\mathbf{m}) & + & B(\mathbf{m},n)B^{-1}(n,n)B^{-1}(n,\mathbf{m})/B^{-1}(n,n) \\ = & \mathrm{id}_{\mathbf{m}} & & \end{array}$$

which verifies (5) since it shows the right side of (5) to be a right inverse of C, hence necessarily its inverse since C is square. □

Corollary. *For all $k \in \mathbb{N}$, and all finite knot sequences t,*

(6) $$\|(A_{kt})^{-1}\|_\infty \le s_k < \infty.$$

Proof: Any finite knot sequence t can be embedded in a k-complete knot sequence r (in many ways), and, for any such choice, $A_{kt} = A_{kr}(\mathbf{m},\mathbf{m})$ for some integer interval $\mathbf{m}$, hence

$$\|(A_{kt})^{-1}\|_\infty \le \|(A_{kr})^{-1}\|_\infty \le s_k,$$

by (4) and Shadrin's Theorem. □

Such reasoning also obviates the discussion in [S: p. 70] of the case $N < 2k$ therein.

Note that Shadrin conjectures that

$$\forall\{k \in \mathbb{N}\} \quad \|P_{kt}\| \le 2k-1 \le \|(A_{kt})^{-1}\|_\infty\,.$$

We even have (6) for arbitrary infinite or biinfinite knot sequences t, as the following theorem, applied to A_{kt} in conjunction with the Corollary, implies.

Theorem. *Let $A \in \mathbb{R}^{I\times I}$ with I equal to $\mathbb{N}$ or $\mathbb{Z}$, and assume that A is tp, and* **banded** *in the sense that*

$$h := \max_{A(i,j)\ne 0} |i-j| < \infty.$$

If, for some finite s and all finite integer intervals $\mathbf{m} \subset I$, the corresponding principal submatrix $A_{\mathbf{m}} := A(\mathbf{m},\mathbf{m})$ is invertible and $\|(A_{\mathbf{m}})^{-1}\|_\infty \le s$, then also A is boundedly invertible as a linear map on $\ell_\infty(I)$, and $\|A^{-1}\|_\infty \le s$.

Proof: For any finite integer interval $\mathbf{m} \subset I$, let

$$A^{-1}_{\mathbf{m}}(i,j) := \begin{cases} A(\mathbf{m},\mathbf{m})^{-1}(i,j), & i,j \in \mathbf{m}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, by the Lemma, we know that, for any integer intervals $\mathbf{m}_1 \subset \mathbf{m}_2$,

$$0 \le (-1)^{i-j} A^{-1}_{\mathbf{m}_1}(i,j) \le (-1)^{i-j} A^{-1}_{\mathbf{m}_2}(i,j), \quad i,j \in \mathbf{m}_1,$$

which shows that, for any strictly increasing sequence $\mathbf{m}_1 \subset \mathbf{m}_2 \subset \cdots$ of integer intervals and any $i,j \in I$,

$$0 \le (-1)^{i-j} A^{-1}_{\mathbf{m}_r}(i,j), \quad r = r_0, r_0+1, r_0+2, \ldots$$

is a monotone increasing sequence bounded by s, hence converges monotonely to some limit value

$$(-1)^{i-j}D(i,j).$$

The resulting biinfinite matrix D has

$$\|D\|_\infty \le s$$

and is necessarily the inverse of A which can be worked out using the assumed bandedness of A, as follows. Since, for any $i \in I$, there is an $r_i \in \mathbb{N}$ so that $|\nu - i| \le h$ implies $\nu \in \mathbf{m}_{r_i}$, therefore

$$\begin{aligned}(AD)(i,j) &= \sum_{|\nu-i|\le h} A(i,\nu) \lim_{r_i\le r\to\infty} A_{\mathbf{m}_r}^{-1}(\nu,j)\\ &= \lim_{r_i\le r\to\infty} \sum_{|\nu-i|\le h} A(i,\nu)A_{\mathbf{m}_r}^{-1}(\nu,j)\\ &= \lim_{r_i\le r\to\infty} \left\{\begin{matrix}\delta_{i,j}, & j\in\mathbf{m}_r\\ 0, & \text{otherwise}\end{matrix}\right\} = \delta_{i,j}.\end{aligned}$$

Analogously, for any $j \in I$, there is an $r_j \in \mathbb{N}$ so that $|\nu - j| \le h$ implies $\nu \in \mathbf{m}_{r_j}$, therefore

$$\begin{aligned}(DA)(i,j) &= \sum_{|\nu-j|\le h} \lim_{r_j\le r\to\infty} A_{\mathbf{m}_r}^{-1}(i,\nu)A(\nu,j)\\ &= \lim_{r_j\le r\to\infty} \sum_{|\nu-j|\le h} A_{\mathbf{m}_r}^{-1}(i,\nu)A(\nu,j)\\ &= \lim_{r_j\le r\to\infty} \left\{\begin{matrix}\delta_{i,j}, & i\in\mathbf{m}_r\\ 0, & \text{otherwise}\end{matrix}\right\} = \delta_{i,j}.\end{aligned}$$

□

Note that, by the Lemma, it is sufficient to assume that $\|(A_\mathbf{m})^{-1}\|_\infty \le s$ for all sufficiently large $\mathbf{m}$. Note also that the conclusion is unchanged if $A_\mathbf{m} := A(\mathbf{m}, r + \mathbf{m})$ for some fixed r. Note finally that the argument would even work for a matrix A that is the norm-limit of banded matrices, something [L] calls **band-dominated**.

The Theorem is complementary to Theorem 1 of [BJP] which asserts that, for any ℓ_∞-invertible tp matrix $A \in \mathbb{R}^{I\times I}$, there exists r (necessarily unique) so that, for all finite intervals $\mathbf{m} \subset I$, $A_\mathbf{m} := A(\mathbf{m}, r+\mathbf{m})$ is invertible and $(A_\mathbf{m})^{-1}$ converges, monotonely in each entry, to A^{-1} as $\mathbf{m} \to I$.

Finally, the Lemma implies that, for any knot sequence t,

$$\|(A_{kt})^{-1}\|_\infty \ge 1/\inf_i \int M_{ik}N_{ik}. \tag{7}$$

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