**Radial basis function approximation:**
**from gridded centers to scattered centers**

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ABSTRACT

The paper studies $L_\infty(\mathbb{R}^d)$-norm approximations from a space spanned by a discrete set of translates of a basis function $\phi$. Attention here is restricted to functions $\phi$ whose Fourier transform is smooth on $\mathbb{R}^d\backslash 0$, and has a singularity at the origin. Examples of such basis functions are the thin-plate splines and the multiquadrics, as well as other types of radial basis functions that are employed in Approximation Theory. The above approximation problem is well-understood in case the set of points $\Xi$ used for translating $\phi$ forms a lattice in $\mathbb{R}^d$, and many optimal and quasi-optimal approximation schemes can already be found in the literature. In contrast, only few, mostly specific, results are known for a set $\Xi$ of scattered points.

The main objective of this paper is to provide a general tool for extending approximation schemes that use integer translates of a basis function to the non-uniform case. We introduce a single, relatively simple, conversion method that preserves the approximation orders provided by a large number of schemes presently in the literature (more precisely, to almost all "stationary schemes"). In anticipation of future introduction of new schemes for uniform grids, an effort is made to impose only a few mild conditions on the function $\phi$, which still allow for a unified error analysis to hold. In the course of the discussion here, the recent results of [BuDL] on scattered center approximation are reproduced and improved upon.

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## 1. Introduction

### 1.1. General

The terminology "Radial Basis Function Approximation" usually refers to approximation to data defined on a set $\Xi$ of scattered points in $\mathbb{R}^d$, $d > 1$ (referred to as "centers") by linear combinations of a discrete set of translates (whose standard choice is the original $\Xi$) of "a radial basis function" $\phi$. Typical examples of a basis function $\phi$ are the bivariate *thin-plate spline*

$$x \mapsto |x|^2 \log |x|,$$

and the *Hardy's multiquadric*

$$x \mapsto (|x|^2 + c)^{1/2},$$

(where here and hereafter $|x|$ stands for the Euclidean norm of the vector $x \in \mathbb{R}^d$). The above functions are, indeed, radially symmetric (i.e., obtained by composing a univariate function with the map $x \mapsto |x|$), a feature which greatly facilitates computations with such functions. However, the title "radial basis functions approximation" is somewhat misleading: most of the radially symmetric functions are not considered as candidates for the basis function $\phi$, while, on the other hand, certain classes of non-radially symmetric functions do fit in. We elaborate on the actual conditions required of $\phi$ later on in the present paper, and also refer the reader to [R] (particularly, page 256 there).

While, as alluded to before, the major force of radial function approximation is in the area of scattered data approximation (and particularly interpolation, cf. the surveys [D1,2], [P], [Bu4]), M.J.D. Powell and I.R.H. Jackson (of Cambridge, England) initiated several years ago the study of approximation from **a principal shift-invariant space** (PSI, for short) $S$ generated by a radial basis function $\phi$. Such a space $S$ is, by definition, the smallest closed (in some underlying topology) space that contains all **shifts** (i.e., integer translates) of the basis function $\phi$. Together with the deepening of the understanding of the more mathematically accessible uniform case, emerged the hope that the tools and observations that are found and made in that latter case will serve for a better understanding of the practically important scattered situation. Indeed, the present paper is one of the first applications of the now-established theory of the uniform grid case to the scattered case.

The approximation properties of PSI and other shift-invariant spaces generated by radial basis functions are analysed in detail in a sequence of papers written by Powell's students I.R.H Jackson and M.D. Buhmann, by the two of us and by several other authors (cf. the above mentioned surveys and the references within. Specific results, hence references, which are relevant to this paper are sketched in §3). Moreover, those results on approximation orders of PSI spaces generated by radial basis functions served as one of the major forces behind the general results on approximation orders of [BR] and [BDeR]. In fact, the totality of results, techniques and observations concerning the approximation properties of PSI (and related) spaces can be fairly characterized as a "solid theory", at least from a qualitative point-of-view.

While the present state-of-art in the area of approximation from PSI spaces is satisfactory, the same cannot be said about the scattered case, (unless the spatial dimension is 1). As a matter of fact, the recent paper [BuDL] contains the *first* successful application of tools and results from the shift-invariant case to the scattered situation. It concerns the approximation by scattered shifts of any one of the basis functions of [DJLR], and employs quasi-interpolation as the approximation scheme. Apparently, the theory of the shift-invariant case is invoked in that paper only for the error analysis part (more precisely, in the polynomial reproduction argument), and not for the (quite involved) construction of the approximation map there, but, our post-analysis of the map from [BuDL] reveals that that construction can be understood as a specific instance of the following general three-step method. We use here, for some discrete $\Xi \subset \mathbb{R}^d$, the notation

$$S_\Xi(\phi)$$

to denote the "span" of $\{\phi(\cdot - \xi)\}_{\xi\in\Xi}$, where the precise meaning of "span" is yet to be defined.
**Step 1:** For each $\alpha \in \mathbb{Z}^d$, the translate $\phi(\cdot - \alpha)$ is approximated by a combination $\phi_\alpha$ of $\{\phi(\cdot - \xi)\}_{\xi\in\Xi}$. This gives rise to a linear map $A$ from $\mathbb{C}^\Xi$ into $\mathbb{C}^{\mathbb{Z}^d}$ (with $\delta_\xi$ the sequence which is 1 at $\xi$ and 0 on $\Xi\backslash\xi$, $A(\delta_\xi)(\alpha)$ is the coefficient $A(\xi, \alpha)$ of $\phi(\cdot - \xi)$ in the definition of $\phi_\alpha$. $A$ then extends by linearity to the extent that this is possible).
**Step 2:** An approximation scheme $L$ that is used to approximate from $S_{\mathbb{Z}^d}(\phi)$ is modified to obtain an approximation map $L_A$ into $S_\Xi(\phi)$ by simply replacing each appearance of $\phi(\cdot - \alpha)$ in $L$ by $\phi_\alpha$.
**Step 3:** The $L$ that corresponds to the specific $L_A$ of [BuDL] (the localization-quasi-interpolation map of [DJLR]) employs, for computing the approximant $Lf$ of $f$, the values of $f$ on $\mathbb{Z}^d$. Since, presumably, the available information on $f$ is $f_{|\Xi}$, the map $A$ (from (a)) is applied to $f_{|\Xi}$, to obtain approximations $\{f_\alpha\}_{\alpha\in\mathbb{Z}^d}$ to $\{f(\alpha)\}_{\alpha\in\mathbb{Z}^d}$. The actual approximation map of [BuDL] could then be obtained from $L_A$ by replacing each appearance of $f(\alpha)$ by $f_\alpha$.

The main objective of this paper is to provide a general tool for extending approximation maps that use uniform translates of a basis function, to the non-uniform case. We do that by showing that Steps 1 and 2 above can be combined to provide a general method for converting a map $L$ that acts into the shift-invariant $S_{\mathbb{Z}^d}(\phi)$ to a map $L_A$ that approximates from (i.e., maps into) $S_\Xi(\phi)$ and has approximation power similar to that of $L$. This, quite certainly, cannot be achieved without imposing some restrictions on the underlying basis function $\phi$: *we always assume throughout the paper that the function $\phi$, considered as a distribution in $\mathcal{D}'(\mathbb{R}^d)$, is tempered, and that its Fourier transform $\widehat{\phi}$ coincides on $\mathbb{R}^d\backslash 0$ with some continuous function (still denoted by $\widehat{\phi}$) while having a certain type of singularity (necessarily of finite order) at the origin.* In turn, such an assumption allows us to construct the map $A$ of Step 1 independently of the specific $\phi$ involved. Indeed, instead of approximating $\phi(\cdot - \alpha)$ by combinations of $\{\phi(\cdot - \xi)\}_{\xi\in\Xi}$, we will actually approximate at the origin the exponentials

$$e_\alpha : x \mapsto e^{i\alpha\cdot x}, \quad \alpha \in \mathbb{Z}^d$$

by combinations of $\{e_\xi\}_{\xi\in\Xi}$, under the requirement that $A$ is bounded as a map from $\ell_\infty(\Xi)$ to $\ell_\infty(\mathbb{Z}^d)$. This imposes constraints on the geometry of the center set $\Xi$ to which our method applies. The problem of constructing such a map $A$ is already considered in [BuDL].

As one should realize from the above description, we do not aim at establishing approximation orders for spaces generated by scattered translates of a suitable basis function, nor do we intend to introduce approximation maps especially designed for dealing with scattered approximation. Instead, we present here a general method that allows us to convert *any* known approximation scheme on uniform grids to the non-uniform case, while preserving (to the extent that this is possible) the approximation orders known in the former case. In anticipation of future introduction of new schemes for uniform grids, we made an effort to impose only few (mild) assumptions on

$\phi$ (in addition to the basic one specified in the previous paragraph). Remarkably, our conversion method based on Steps 1,2 above applies to a large number of cases presently in the literature, and our error analysis is also quite unified. In one regard we sacrifice generality for the sake of brevity and readability: only uniform-norm approximations are considered in the present paper. We also mention that the tools developed here do not seem to apply to non-stationary approximations (such as the ones that employ the Gaussian kernel, etc., cf. [MN], [BuD], [BR] and [BeLi]) that lead to spectral approximation orders.

### 1.2. The setup and a typical result

All functions considered in this paper are either real- or complex-valued functions defined on $\mathbb{R}^d$, for some fixed $d$. Norms of such functions $f$ are denote by $\|f\|_{\ldots}$, with the default norm in this paper is that of $L_\infty(\mathbb{R}^d)$, i.e.,

$$\|f\| := \|f\|_\infty := \|f\|_{L_\infty(\mathbb{R}^d)}.$$

Norms on $\mathbb{R}^d$ itself are denoted by

$$|v|_{\ldots}, \quad v \in \mathbb{R}^d,$$

with the default norm is the Euclidean one:

$$|v|^2 := \sum_{j=1}^d v_j^2.$$

Given an index set $\Xi$, and a function set $\{\phi_\xi\}_{\xi\in\Xi}$, we set

$$\phi_\Xi := \{\phi_\xi\}_{\xi\in\Xi}.$$

In the special case when $\Xi \subset \mathbb{R}^d$, and the function set is the set of all translates of a fixed $\phi$ by $\xi \in \Xi$, we use the more specific notation ${}_\Xi(\phi)$:

$${}_\Xi(\phi) := \{\phi(\cdot - \xi) : \ \xi \in \Xi\}.$$

Given a function set $F$, we let

$$SF$$

be the closure in the topology of uniform convergence on compact sets of the algebraic span of $F$. For example, this, in particular, defines precisely the space $S_\Xi(\phi)$, which is previously introduced in §1.1.

We are interested in approximating smooth functions, particularly in the space

$$W_\infty^k(\mathbb{R}^d), \quad k \in \mathbb{Z}_+$$

of all functions whose derivatives of orders $\le k$ are bounded. Some of our results deal with approximation to functions in the larger homogeneous Sobolev space

$$w_\infty^k(\mathbb{R}^d)$$

of all functions for which the semi-norm

$$|f|_{k,\infty} := \sum_{|\alpha|_1=k} \|D^\alpha f\|$$

is finite.

As is almost always the case in approximation theory, polynomials and exponentials are heavily employed here. We have already mentioned the symbol

$$e_\xi, \quad \xi \in \mathbb{R}^d,$$

for the complex exponential with frequency $\xi$. We use the symbol

$$\Pi_k$$

to denote the space of all polynomials of degree $\le k$.

Finally, given $f$ continuous on $\mathbb{R}^d\backslash 0$, we say that $f$ **has a singularity of order** $k'$ at the origin if

$$|\cdot|^{k'}|f|$$

is bounded above and below by positive constants in some origin-neighborhood.

For a given $\Xi \subset \mathbb{R}^d$, our interest is in approximation schemes from $S_\Xi(\phi)$ that are obtained by modifying a known approximation scheme from $S_{\mathbb{Z}^d}(\phi)$, and that preserve to the best possible extent the approximation power of that latter scheme.

A typical approximation scheme considered here is of the form

$$L_A : f \mapsto \sum_{\alpha\in\mathbb{Z}^d} \psi_\alpha \, \Lambda(f)(\alpha),$$

where the functions $(\psi_\alpha)_{\alpha\in\mathbb{Z}^d}$ are certain functions in $S_\Xi(\phi)$, known to satisfy (at least) the following "boundedness condition"

$$\sum_{\alpha\in\mathbb{Z}^d} |\psi_\alpha| \in L_\infty(\mathbb{R}^d). \tag{1.2.1}$$

$\Lambda$ is always taken to be a bounded operator from $L_\infty(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ into itself, hence, in particular, the sequence $((\Lambda f)(\alpha))_{\alpha\in\mathbb{Z}^d}$ is bounded, and, hence (due to (1.2.1)) $L_A$ is well-defined and bounded from $L_\infty(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ into $S_\Xi(\phi)$.

The actual construction of $L_A$ starts with a given a approximation scheme $L$ from $S_{\mathbb{Z}^d}(\phi)$ of the form

$$L : f \mapsto \sum_{\alpha\in\mathbb{Z}^d} \psi(\cdot - \alpha)\, \Lambda(f)(\alpha),$$

with $\Lambda$ as above, and with the function $\psi$ being a linear combination of the shifts of the original basis function $\phi$:

$$\psi = \sum_{\alpha\in\mathbb{Z}^d} \phi(\cdot - \alpha)\, \mu(\alpha).$$

To simplify the discussion at this introductory stage, we assume that the sequence $\mu$ is of finite support (the actual conditions imposed on $\mu$ in the paper are much milder). At a minimum, the function $\psi$ is assumed to satisfy the condition

$$\sum_{\alpha\in\mathbb{Z}^d} |\psi(\cdot - \alpha)| \in L_\infty(\mathbb{R}^d), \tag{1.2.2}$$

which guarantees the map $L$ to be well-defined and bounded on $L_\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. The reader should bear in mind that at the core of this paper is the assumption that *the map $L$, hence the sequence $\mu$, the function $\psi$, and the operator $\Lambda$, are all given, and the approximation properties of this map are known, too.* Our goal is to find an appropriate selection method for the functions $(\psi_\alpha)_{\alpha\in\mathbb{Z}^d}$ so that $L_A$ has approximation properties similar to those of $L$. Since in many cases the operator $\Lambda$ is the identity, we simplify further our introductory discussion here, by adopting such an assumption.

Our construction of the functions $(\psi_\alpha)_\alpha$ is done as follows. We first approximate each $\phi(\cdot-\alpha)$, $\alpha \in \mathbb{Z}^d$, by a linear combination

$$\phi_\alpha := \sum_{\xi\in\Xi} A(\alpha,\xi)\phi(\cdot-\xi). \tag{1.2.3}$$

The reader who feels uneasy about the lack of discussion of the convergence of this last sum, may assume without great loss that, for each $\alpha \in \mathbb{Z}^d$, the sequence $(A(\alpha,\xi))_{\xi\in\Xi}$ is finitely supported. We then define the functions $(\psi_\alpha)_\alpha$ by

$$\psi_\alpha = \sum_{\beta\in\mathbb{Z}^d} \phi_\beta\, \mu(\beta-\alpha).$$

The basic criterion of the approximation properties of $L$ and $L_A$ that we employ in this paper is that of *approximation orders*. We say that a map $L$ **provides an approximation order** $k > 0$ for a smoothness space $W$, if, for every $f \in W$,

$$\|\sigma_h f - L(\sigma_h f)\| = O(h^k), \tag{1.2.4}$$

with $\sigma_h$ the scaling operator

$$\sigma_h : f \mapsto f(h\cdot).$$

Note that, since the uniform norm is invariant under dilations, we could write (1.2.4) equivalently as

$$\|f - \sigma_{1/h} L(\sigma_h f)\| = O(h^k),$$

which is more frequently used in the literature for defining the notion of approximation orders.

We present now a result which is a variant of Theorem 2.2.16, and which can be regarded as a prototype for the main results of this paper.

**Theorem 1.2.5.** *Assume that the basis function $\phi$ is tempered (as distribution) and that its (distributional) Fourier transform is continuous on $\mathbb{R}^d\backslash 0$, has a singularity at the origin, and does not vanish identically on $2\pi\mathbb{Z}^d\backslash 0$. Let $\psi$, $(\phi_\alpha)_\alpha$, $(\psi_\alpha)_{\alpha\in\mathbb{Z}^d}$, $L$, and $L_A$ be defined as above, with* $\Lambda = \text{identity}$. *Assume that:*

*(a) $\mu$ is finitely supported.*

*(b) $\psi$ satisfies (1.2.2), and $\widehat{\psi}(0) \neq 0$. ($\widehat{\psi}$ is continuous, hence well-defined pointwise, because of (1.2.2).)*

*(c) $\sum_{\alpha\in\mathbb{Z}^d} |\phi(\cdot-\alpha) - \phi_\alpha| \in L_\infty(\mathbb{R}^d)$.*

*Then (1.2.1) holds and the hence-well-defined $L_A$ provides (to sufficiently smooth functions) at least the same approximation order as provided by $L$.*

Here, "sufficiently smooth functions" are functions in $W_\infty^k(\mathbb{R}^d)$, with $k$ depending on the underlying approximation order.

The outline of the paper is as follows. §2 is devoted to the developments of the general theory. The core of its discussion are §2.1, §2.2, and §2.7. In §2.1 we impose conditions on $\mu$, $\psi$ and $\Lambda$ which are sufficient for the development of the method of this paper. The main issue in that subsection is *localization* which amounts to the proper construction of the function $\psi$ and the approximation map $L$. §2.2 discusses the extension of $L$ to the scattered case, i.e., the construction of the "pseudo-shifts" $\phi_\alpha$ and $\psi_\alpha$, and the introduction of our approximation map $L_A$. It also includes the basic error analysis for $L_A$ (cf. Theorems 2.2.9 and 2.2.16). In §2.3, we modify the previous error analysis, in a way that relaxes the requirements on $\mu$ imposed in §2.2, but applies to a smaller set of approximands $f$. Since the standard results in the literature show that $L$ is a good approximation scheme not only for smooth bounded functions, but also to functions which are not bounded, but some of their derivatives are bounded, we show in §2.4 that, under some additional assumptions, such approximation properties are inherited by $L_A$. The issue of polynomial reproduction is at the heart of error analyses which are based on the quasi-interpolation argument, but plays no real role in the present paper. However, we show in §2.5 that $L_A$ reproduces polynomials to the expected degree, whenever such property can make sense. §2.8 suggests some relaxations in the various conditions assumed in §2.2 under which the main theorems of §2.2 are still valid. Finally, in §2.7 the following very important question is considered: what properties of the matrix $A$ imply the satisfaction of the crucial condition (c) in the above stated theorem.

Section 3 is mainly devoted to the application of the general theory to specific instances. In §3.1, some of the approximation schemes and approximation order results of [BR] are extended to the scattered case. The quasi-interpolation schemes of [DJLR] are extended in §3.3, while the extension of the interpolation schemes of [Bu1,2] is the issue of §3.2. In addition, we provide in §3.4 a detailed discussion of the scattered-center approximation schemes of [BuDL], in which we approach those schemes via a course different from the original one suggested in that reference. In this way, we provide an alternative error analysis of the [BuDL]-like schemes that is based on the results of §2.2 here, thereby providing extensions for the [BuDL] results in several different directions.

## 2. The approximation maps and their error analysis

### 2.1. Localization and approximation maps on uniform grids

As mentioned before, we assume that the Fourier transform $\widehat{\phi}$ of the basis function $\phi$ is regular and even continuous on $\mathbb{R}^d\backslash 0$, but is singular at the origin, and such assumption implies that $\phi(x)$ cannot decay fast to zero as $|x| \to \infty$; for instance $\phi \notin L_1(\mathbb{R}^d)$. In fact, in most examples $|\phi(x)|$ grows polynomially as $|x|$ tends to $\infty$. This makes it hard (although not impossible) to write explicit approximation schemes which are based on shifts of $\phi$. The standard way to circumvent this difficulty is via a **localization** process, which uses the shifts of $\phi$ to construct a new function $\psi$ with favorable decay properties at $\infty$, and subsequently implements approximation schemes which are based on the shifts of $\psi$.

Formally, the localization $\psi$ can be written as a linear combination of shifts of $\phi$ with coefficients $\mu : \mathbb{Z}^d \to \mathbb{C}$:

$$\psi := \sum_{\alpha\in\mathbb{Z}^d} \phi(\cdot - \alpha)\,\mu(\alpha). \tag{2.1.1}$$

Usually, the **localization sequence** $\mu$ decays fast at $\infty$ and the above sum converges uniformly on compact sets (as a matter of fact, $\mu$ is frequently chosen to have finite support). Careful attention is given in the literature to the rates of decay of $\psi$; in particular, the standard polynomial-reproduction / quasi-interpolation argument requires $\psi$ to decay at $\infty$ at a faster rate as the attempted approximation order increases.

For the purposes of this section, we require the localization process to satisfy the following three conditions:

**Localization Conditions 2.1.2.** *Given $\phi$, $\mu$, and $\psi$ as above, we assume throughout this section that:*

*(a) For some $m_\mu > d$,*

$$|\mu(\alpha)| = O(|\alpha|^{-m_\mu}), \quad \text{as } |\alpha| \to \infty. \tag{2.1.3}$$

*(b) For some $m_\psi > d$,*

$$|\psi(x)| = O(|x|^{-m_\psi}), \quad \text{as } |x| \to \infty. \tag{2.1.4}$$

*(c) The sum in (2.1.1) converges uniformly on compact sets.*

Several comments concerning the above assumptions are in order. First, the above three conditions should be regarded as very mild, and are satisfied by all specific localization processes that we are aware of. Second, in many cases condition (c) imposes decay rates on $\mu$ higher than those required in (a), making the latter condition redundant. Finally, condition (c) is not essential, and other, sometimes weaker, notions of convergence can be assumed.

Thanks to the decay properties of $\psi$ at $\infty$, it is easier to write explicit approximation maps in terms of its shifts than in terms of the shifts of $\phi$. Further, almost all analyses of approximation orders require the basis function to decay at $\infty$. In general, a linear approximation scheme $L$ which explicitly employs the shifts of $\psi$ has the form

$$L : f \mapsto \sum_{\alpha \in \mathbb{Z}^d} \psi(\cdot - \alpha)\,(\Lambda f)(\alpha), \tag{2.1.5}$$

with $f \mapsto \{(\Lambda f)(\alpha)\}_{\alpha \in \mathbb{Z}^d}$ some linear assignment. At a minimum, one assumes that $\Lambda$ is a bounded map from $L_\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ (equipped with the uniform norm) to $\ell_\infty(\mathbb{Z}^d)$. In such a case, the decay assumption imposed on $\psi$ in Localization Conditions 2.1.2 guarantees the sum in (2.1.5) to converge uniformly on compact sets. It is very convenient to assume further that $\Lambda$ commutes with integer shifts, i.e., that $\Lambda f(\alpha) = \Lambda(f(\cdot + \alpha))(0)$. For our purposes, it is important to assume slightly more: we need $\Lambda$ to commute with differentiation or, equivalently, to be a convolution operator:

**Uniform Scheme Condition 2.1.6.** *We assume that the approximation map $L$ is of the form*

$$Lf := \sum_{\alpha \in \mathbb{Z}^d} \psi(\cdot - \alpha)\,(\Lambda f)(\alpha),$$

*with $\Lambda$ a convolution operator which is well-defined and continuous from the subspace $L_\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ of $L_\infty(\mathbb{R}^d)$ into itself.*

As a matter of fact (cf. §3), in the examples that we consider here, we have either $\Lambda =$ identity, or $\Lambda = \lambda *$ for some band-limited $L_1$-function $\lambda$.

## 2.2. The map $L_A$ and its corresponding approximation orders

With $L$ as above, we define our approximation operator $L_A$ that uses $\{\phi(\cdot - \xi)\}_{\xi\in\Xi}$ as follows. First, recall from the introduction that for each $\alpha \in \mathbb{Z}^d$ we have available a "pseudo-shift" $\phi_\alpha$ of the form

$$\phi_\alpha = \sum_{\xi\in\Xi} A(\alpha,\xi)\, \phi(\cdot - \xi). \tag{2.2.1}$$

At the present stage, we do not discuss the exact type of the convergence in the definition of $\phi_\alpha$. Moreover, our error analysis would not make any use of the fact that $\phi_\alpha$ is a combination of scattered shifts of $\phi$, but only of the fact that $\phi_\alpha$ is "close" in some sense to $\phi(\cdot - \alpha)$. Still, of course, expressing $\phi_\alpha$ as a combination of the appropriate scattered shifts of $\phi$ is pertinent to the basic idea of approximating by scattered shifts of $\phi$.

The new pseudo-shifts $\{\phi_\alpha\}_{\alpha\in\mathbb{Z}^d}$ are also localized, and by the same localization sequence $\mu$ that is used in the uniform case. Precisely, we define

$$\psi_\alpha := \sum_{\beta\in\mathbb{Z}^d} \phi_\beta\, \mu(\beta-\alpha), \quad \alpha \in \mathbb{Z}^d, \tag{2.2.2}$$

and substitute these "localized pseudo-shifts" for the shifts of $\psi$ in the definition of $L$:

**(2.2.3) Definition of $L_A$.** *With $\phi$, $\psi$, $L$ and $\{\psi_\alpha\}$ as above, we define the scattered-center variant $L_A$ of $L$ by*

$$L_A : f \mapsto \sum_{\alpha\in\mathbb{Z}^d} \psi_\alpha\, \Lambda f(\alpha), \tag{2.2.4}$$

*with $\Lambda$ the same convolution operator that is used to define $L$.*

Prior to any analysis of the scheme $L_A$, we need to show that $\{\psi_\alpha\}_\alpha$ and $L_A$ are well-defined, i.e., that the infinite series used in their definitions converges in some topology. This is done with the aid of the following additional assumption, which is at the heart of our approach:

**Central Condition 2.2.5.** *For some $m_A > d$,*

$$|(\phi_\beta - \phi(\cdot-\beta))(x)| \le c(1+|x-\beta|)^{-m_A},\ \ \beta\in\mathbb{Z}^d,$$

*with $c$ independent of $x$ and $\beta$.*

We stress that *none of the results of this section require the representation (2.2.1)*, but only the above Central Condition.

Invoking this new condition, we settle in the next lemma the question of the well-definedness of $L_A$.

**Lemma 2.2.6.** *Assume that the Localization Conditions 2.1.2 and the Central Condition 2.2.5 hold. Then, for $\alpha\in\mathbb{Z}^d$,*

$$\sum_{\beta\in\mathbb{Z}^d} |\mu(\beta-\alpha)(\phi_\beta - \phi(\cdot-\beta))(x)| \le c'(1+|x-\alpha|)^{-m'},$$

*with $m' := \min\{m_\mu, m_A\}$, and $c'$ some $\alpha$-independent constant. Also*

$$|\psi_\alpha(x)| \le c''(1+|x-\alpha|)^{-m''}, \tag{2.2.7}$$

*with $m'' := \min\{m_\psi, m_\mu, m_A\}$ and $c''$ some $\alpha$-independent constant.*

**Proof:** We consider the difference

$$|\psi_\alpha(x) - \psi(x-\alpha)| \le \sum_{\beta\in\mathbb{Z}^d} |\mu(\beta-\alpha)(\phi_\beta(x) - \phi(x-\beta))|.$$

Because of Condition 2.2.5, the above sum is majorized by

$$c \sum_{\beta\in\mathbb{Z}^d} |\mu(\beta-\alpha)|\,(1+|x-\beta|)^{-m_A} = c\sum_{\beta\in\mathbb{Z}^d} |\mu(\beta)|\,(1+|(y+\alpha')-\beta|)^{-m_A}, \tag{2.2.8}$$

with $|y|_\infty \le 1/2$, and $\alpha' \in \mathbb{Z}^d$. By (b) of Localization Conditions 2.1.2, and for any fixed $y$, this latter sum is the evaluation at $\alpha'$ of the discrete convolution of sequences which decay at $\infty$ like the $m_\mu$th and $m_A$th inverse power. Since $y$ is restricted to a compact set, there exist constants such that

$$c_1(1+|y-\gamma|)^{-m_A} \le (1+|\gamma|)^{-m_A} \le c_2(1+|y-\gamma|)^{-m_A},$$

which allows us to bound the sum in (2.2.8) by

$$\mathrm{const}(1+|\alpha'|)^{-m'} \le \mathrm{const}'(1+|y+\alpha'|)^{-m'} = \mathrm{const}'(1+|x-\alpha|)^{-m'},$$

for $m' := \min\{m_\mu, m_A\}$, and for some $\alpha$-independent constants (cf. e.g., the argument in the proof of Proposition 6 in [BuD]). Combining this with (b) of Conditions 2.1.2 we obtain the decay assertion on $\psi_\alpha$. □

We are now ready to state and prove the main theorem of this section:

**Theorem 2.2.9.** *Let $\phi$, $\mu$, $\psi$, and $L$ be as above, and assume that the Localization Conditions 2.1.2, and the Uniform Scheme Condition 2.1.6 are satisfied. Let $(\phi_\alpha)_{\alpha\in\mathbb{Z}^d}$ be a collection of functions that satisfy the Central Condition 2.2.5, and let $L_A$ be their associated map defined as in (2.2.4). Assume further that for some positive integer $k$ and $0<\varepsilon\le 1$ the following hold:*
*(a) $|\mu(\alpha)| = O(|\alpha|^{-d-k-\varepsilon})$, as $|\alpha|\to\infty$.*
*(b) The functional $\mu$ defined by*

$$\mu p := \sum_{\alpha\in\mathbb{Z}^d} \mu(-\alpha)p(\alpha)$$

*annihilates $\Pi_k$.*
*Then, for every $f \in W^{k+1}_\infty(\mathbb{R}^d)\cap C^{k+1}(\mathbb{R}^d)$,*

$$\|(L-L_A)(\sigma_h f)\|_\infty \le \mathrm{const}(|f|_{\infty,k} + |f|_{\infty,k+1}) \begin{cases} h^{k+\varepsilon}, & \varepsilon<1,\\ h^{k+1}|\log h|, & \varepsilon = 1.\end{cases}$$

*Furthermore, if $\varepsilon=1$, and (a) above is replaced by the stronger condition $\sum_{\alpha\in\mathbb{Z}^d}|\mu(\alpha)||\alpha|^{k+1}<\infty$, then, for $f$ as above,*

$$\|(L-L_A)(\sigma_h f)\|_\infty \le \mathrm{const}\, h^{k+1}|f|_{\infty,k+1}.$$

**Proof:** We first prove the last (and simplest) case stated in the theorem, i.e., when $\sum_{\alpha\in\mathbb{Z}^d}|\mu(\alpha)||\alpha|^{k+1}<\infty$. We will then show how to modify the argument of that case to obtain the results stated with respect to the other cases.

Given $f \in W_\infty^{k+1}(\mathbb{R}^d) \cap C^{k+1}(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$, we first estimate

$$\mu(f(x+\cdot)) = \sum_{\alpha \in \mathbb{Z}^d} \mu(-\alpha) f(x+\alpha).$$

With $T_x f$ the Taylor expansion of degree $k$ of $f$ about $x$, the fact that $\mu$ annihilates $\Pi_k$ implies that $\mu(T_x f) = 0$. Since $|f(x+\alpha) - (T_x f)(x+\alpha)| \le \text{const}|f|_{\infty,k+1}\, |\alpha|^{k+1}$, we obtain the estimate

$$(2.2.10) \qquad |\mu(f(x+\cdot))| \le \text{const}|f|_{\infty,k+1} \sum_{\alpha \in \mathbb{Z}^d} |\mu(-\alpha)||\alpha|^{k+1} = \text{const}'|f|_{\infty,k+1}.$$

Now, we can write $(L - L_A)(\sigma_h f)$ as follows:

$$\begin{aligned} |(L-L_A)(\sigma_h f)(x)| &= |\sum_{\alpha \in \mathbb{Z}^d} (\psi(\cdot - \alpha) - \psi_\alpha)(x)\, \Lambda(\sigma_h f)(\alpha)| \\ &= |\sum_{\alpha \in \mathbb{Z}^d} \sum_{\beta \in \mathbb{Z}^d} (\phi(x-\beta) - \phi_\beta(x))\mu(\beta - \alpha)\, \Lambda(\sigma_h f)(\alpha)|. \end{aligned}$$

By Lemma 2.2.6 and Condition 2.1.6, the above double sum converges absolutely, and summation by parts yields that

$$(2.2.11) \qquad |(L-L_A)(\sigma_h f)(x)| = |\sum_{\beta \in \mathbb{Z}^d} (\phi(x-\beta) - \phi_\beta(x)) \sum_{\alpha \in \mathbb{Z}^d} \mu(\beta - \alpha)\Lambda(\sigma_h f)(\alpha)|.$$

Next, we identify the sum $\sum_{\alpha \in \mathbb{Z}^d} \mu(\beta-\alpha)\Lambda(\sigma_h f)(\alpha)$ in (2.2.11) as $\mu(\Lambda(\sigma_h f)(\beta + \cdot))$, and invoke (2.2.10) to estimate the latter expression as follows:

$$(2.2.12) \qquad |\mu(\Lambda(\sigma_h f)(\beta+\cdot))| \le \text{const}|\lambda * (\sigma_h f)|_{\infty,k+1} \le \text{const}\, h^{k+1} \|\Lambda\| |f|_{\infty,k+1},$$

where, in the second inequality, the fact that convolution commutes with differentiation and that dilation is an isometry on $L_\infty(\mathbb{R}^d)$ are used. Substituting (2.2.12) into (2.2.11), we arrive at the bound

$$|(L-L_A)(\sigma_h f)(x)| \le \text{const}\, h^{k+1}|f|_{\infty,k+1} \sum_{\beta \in \mathbb{Z}^d} |\phi(x-\beta) - \phi_\beta(x)|,$$

which completes the proof of the present case, since Condition 2.2.5 implies that the last sum is majorized by a (periodic) bounded function.

In case (a) and (b) hold, but the stronger assumption $\sum_{\alpha \in \mathbb{Z}^d} |\mu(\alpha)||\alpha|^{k+1} < \infty$ is not valid, we modify the argument that leads to (2.2.12) as follows. We partition $\mathbb{R}^d$ to a ball $B_h$ of radius $1/h$ centered at the origin and to the complement $B_h^c$ of that ball. If $\alpha \in B_h$, we estimate the difference $f(x+\alpha) - (T_x f)(x+\alpha)$ as before, i.e., obtain that

$$|f(x+\alpha) - (T_x f)(x+\alpha)| \le \text{const}|f|_{\infty,k+1}|\alpha|^{k+1}.$$

However, for $\alpha \in B_h^c$, we treat the zero $f(x+\cdot) - (T_x f)(x+\cdot)$ has at the origin as of order $k$ (though, it is of order $k+1$). This leads to an estimate of the form

$$|f(x+\alpha) - (T_x f)(x+\alpha)| \le \text{const}|f|_{\infty,k}|\alpha|^k.$$

Summing the first estimate over $\alpha \in B_h$ and the second one over $\alpha \in B_h^c$, and invoking the decay assumption (a) on $\mu$, we get that

$$|\mu(f(x+\cdot))| \le \mathrm{const}(|f|_{\infty,k+1} \sum_{\alpha\in B_h} |\alpha|^{-d-\varepsilon+1} + |f|_{\infty,k} \sum_{\alpha\in B_h^c} |\alpha|^{-d-\varepsilon}).$$

According to Lemma 4.2 of [DJLR], the sum in the second summand above behaves like $O(h^\varepsilon)$, while the first one is $O(h^{\varepsilon-1})$ and $O(|\log h|)$ for $0<\varepsilon<1$ and $\varepsilon=1$, respectively. We thus obtain an estimate analogous to (2.2.10), which, say for $\varepsilon<1$ reads as

$$|\mu(f(x+\cdot))| \le \mathrm{const}(|f|_{\infty,k+1} h^{\varepsilon-1} + |f|_{\infty,k} h^\varepsilon).$$

Following the argument before (2.2.12), we obtain that

$$|\mu(\Lambda(\sigma_h f)(\beta+\cdot))| \le \mathrm{const}\, h^{k+\varepsilon} \|\Lambda\| (|f|_{\infty,k+1} + |f|_{\infty,k}).$$

The proof is now obtained as in the previous case, with (2.2.12) being replaced by the above bound. The case $\varepsilon=1$ is similar. $\square$

At a first look, it is not clear that the theorem allows us to draw conclusions on the approximation orders provided by the map $L_A$, and because of two reasons: first, the theorem assumes that $\mu$ annihilates $\Pi_k$, and one may suspect this further condition on $\mu$ to compete with other requirements, or, at least, to exclude various localization processes which do not prepare for such a condition. Second, the powers $h^{k+\varepsilon}$ and $h^{k+1}$ that appear in the conclusion of the theorem seem unrelated to the possible approximation order provided by the map $L$. The next two results are meant to clarify these two important points. The first shows that the annihilation assumption on $\mu$ is usually a by-product of the Localization Conditions 2.1.2. The second shows that in most cases of interest the parameter $k+\varepsilon$ that appears in Theorem 2.2.9 is no smaller than the approximation order provided by $L$.

**Lemma 2.2.13.** *Let $k$ be a positive integer, and assume that $\widehat{\phi}$ is continuous on $\mathbb{R}^d\backslash 0$ and has a singularity of order $>k$ at the origin. Assume further that the Localization Conditions 2.1.2 hold, and that the linear functional*

$$\mu : p \mapsto \sum_{\alpha\in\mathbb{Z}^d} \mu(-\alpha)p(\alpha)$$

*is well-defined on $\Pi_k$ (i.e., the above sum converges absolutely for every $p\in\Pi_k$). Then $\mu$ annihilates $\Pi_k$.*

**Proof:** Condition (c) of Localization Conditions 2.1.2 allows us to compute the Fourier transform of $\psi$ in (2.1.1) term by term. Defining

$$\widehat{\mu} := \sum_{\alpha\in\mathbb{Z}^d} \mu(\alpha) e_{-\alpha}, \tag{2.2.14}$$

we thus obtain that, pointwise on $\mathbb{R}^d\backslash 0$, $\widehat{\psi} = \widehat{\phi}\widehat{\mu}$. Due to (b) of Localization Conditions 2.1.2, $\widehat{\psi}$ is continuous everywhere, and in particular at the origin, and hence $\widehat{\mu}$ necessarily has a zero of order $>k$ at the origin. Since $\mu$ is well-defined on $\Pi_k$, $\widehat{\mu}$ is $k$-times differentiable at the origin (and, as a matter of fact, everywhere), and can be differentiated term by term. Consequently, given any $p\in\Pi_k$, we obtain

$$0 = p(-iD)\widehat{\mu}(0) = \sum_{\alpha\in\mathbb{Z}^d} \mu(\alpha)p(-\alpha) = \mu p,$$

as claimed. $\square$

**Lemma 2.2.15.** *Assume that*

(a) *The Localization Conditions 2.1.2 hold;*

(b) $\widehat{\phi}$ *is continuous on* $\mathbb{R}^d\backslash 0$*, has a singularity of order* $k'$ *at the origin, and satisfies, for some* $j \in 2\pi\mathbb{Z}^d\backslash 0$, $\widehat{\phi}(j) \neq 0$;

(c) $\widehat{\psi}(0) \neq 0$.

*Then the approximation order provided by* $L$ *to* $W_\infty^\infty(\mathbb{R}^d)$ *is* $\leq k'$.

**Proof:** The proof follows from the general analysis of approximation orders of [BR], and the claim is essentially proved in Theorem 4.2 there, hence is only sketched here. First, Theorem 2.17 of [BR] implies that if, for some $\theta \in \mathbb{R}^d$, $L$ provides approximation order $n$ to the exponential function $e_\theta$, then, for any $j \in 2\pi\mathbb{Z}^d\backslash 0$, and in particular for the one specified in (b) above,

$$|\widehat{\psi}(j + h\theta)| = O(h^n).$$

Second, since $1/\widehat{\psi}$ is bounded in a neighborhood of zero, the last condition implies that

$$\widehat{\psi}(j + h\theta)/\widehat{\psi}(h\theta) = O(h^n).$$

Since $\widehat{\psi} = \widehat{\mu}\widehat{\phi}$, and $j$ is a period for $\widehat{\mu}$, we obtain that

$$\widehat{\phi}(j + h\theta)/\widehat{\phi}(h\theta) = O(h^n).$$

Since $\widehat{\phi}(j) \neq 0$, we finally conclude that, in case $L$ provides approximation order $n$ for all exponentials $e_\theta$, $\theta \in \mathbb{R}^d$,

$$1/\widehat{\phi}(h\theta) = O(h^n), \quad \forall \theta \in \mathbb{R}^d,$$

and since, by assumption, $\widehat{\phi}$ has a singularity of order $k'$ at the origin, we conclude that $n \leq k'$. □

The following theorem, which is a direct consequence of Theorem 2.2.9 and the last two lemmas, summarizes the results obtained for functions $\phi$ whose Fourier transform has a singularity at the origin.

**Theorem 2.2.16.** *Assume that:*

(a) $\widehat{\phi}$ *is continuous on* $\mathbb{R}^d\backslash 0$*, and has a singularity at the origin of exact order* $k'$.

(b) *The Localization Conditions 2.1.2 and the Uniform Scheme Condition 2.1.6 hold.*

(c) *For some integer* $k < k'$*, and* $\varepsilon \leq 1$, $\mu$ *is either well-defined on* $\Pi_{k+\varepsilon}$, $(\varepsilon = 1)$ *or satisfies*

$$|\mu(\alpha)| = O(|\alpha|^{-(k+d+\varepsilon)}), \quad \text{as } |\alpha| \to \infty,$$

$(\varepsilon < 1)$.

(d) *The Central Condition 2.2.5 holds.*

*Then:*

(i) $L_A$ *provides the same approximation order as* $L$ *provides to functions in* $W_\infty^{k+1}(\mathbb{R}^d) \cap C^{k+1}(\mathbb{R}^d)$*, unless this latter order exceeds* $k + \varepsilon$.

(ii) *If* $k + \varepsilon = k'$, $L_A$ *necessarily provides the same approximation order as provided by* $L$ *(to the space in (i)) in case* $\widehat{\phi}$ *is non-zero at some* $j \in 2\pi\mathbb{Z}^d\backslash 0$*, and* $\widehat{\psi}(0) \neq 0$.

**Remark 2.2.17.** Condition (c) in the above theorem distinguishes between an integral approximation order and fractional approximation order, and is more restrictive in the former case. Had we merely assumed that, for $\varepsilon = 1$, $|\mu(\alpha)| = O(|\alpha|^{-(k+d+1)})$, Theorem 2.2.9 would have yielded an approximation rate of $O(h^{(k+1)}|\log h|)$, rather than the $O(h^{(k+1)})$ obtained above. Though there are situations (see §3) where only the slower decay rate is known about $\mu$, the approximation rate $L$ is known to provide (in all such cases that we are aware of) is $O(h^{(k+1)}|\log h|)$, so that $L_A$ still inherits the approximation power of $L$.

**Remark 2.2.18.** Part (ii) of the last theorem deals with "the worst case" showing that, for *some* functions $f$, the approximation order provided by $L_A$ to $f$ does not lag behind the one provided by $L$. It does not exclude the possibility that, for *other* functions $f \in W_\infty^{k+1}(\mathbb{R}^d) \cap C^{k+1}(\mathbb{R}^d)$, $L$ provides approximation order $n > k'$ to $f$, and for such $f$, $L_A$ may fail to provide approximation order $n$ for $f$, since it is guaranteed to provide only order $k'$. However, we expect such a situation to be truly exceptional, and more precisely we conjecture that, under the assumptions of Theorem 2.2.16, no function $f \in W_\infty^{k'}(\mathbb{R}^d) \cap C^{k+1}(\mathbb{R}^d)$ other than the constants can be approximated to a rate better than $k'$. In this regard, we mention that [R] proves that in the $L_2$-analog of the present situation no non-zero function in the potential space $W_2^{k+1}(\mathbb{R}^d)$ can be approximated to an order $> k'$.

### 2.3. A complementary error analysis of the approximation order of $L_A$

Theorem 2.2.16 asserts that, under various conditions, the approximation order provided by $L_A$ is no smaller than the one provided by $L$. Most of the assumed conditions are acceptable in the sense that they hold in all the examples that are analysed in the next section. However, one of the conditions assumed in Theorem 2.2.16 may appear to be too restrictive, and excludes several examples of interest: we assume that $\mu$ decays fast enough to be absolutely summable against polynomials in $\Pi_k$. This condition can be relaxed by using Fourier analysis alternatives, as we discuss in the present subsection.

In the next theorem we use the notation

$$\widetilde{W}_\infty^k(\mathbb{R}^d)$$

to denote the space of all functions $f$ that (i): their Fourier transform $\widehat{f}$ is a Radon measure, and (ii): the total mass $\|(1+|\cdot|)^k\widehat{f}\|_1$ of $(1+|\cdot|)^k\widehat{f}$ is finite. The norm $\|(1+|\cdot|)^k\widehat{f}\|_1$ coincides with the $L_1(\mathbb{R}^d)$-norm of $(1+|\cdot|)^k\widehat{f}$, whenever $\widehat{f}$ is a function. It induces a norm on $\widetilde{W}_\infty^k(\mathbb{R}^d)$ which we denote by $\|\cdot\|'_{\infty,k}$, i.e.,

$$\|f\|'_{\infty,k} := \|(1+|\cdot|)^k\widehat{f}\|_1.$$

Similarly, we use

$$|f|'_{\infty,k}$$

to denote the total mass $\||\cdot|^k\widehat{f}\|_1$ of $|\cdot|^k\widehat{f}$. Note that, for integer $k$, $\widetilde{W}_\infty^k(\mathbb{R}^d)$ is continuously embedded into $W_\infty^k(\mathbb{R}^d) \cap C^k(\mathbb{R}^d)$.

**Theorem 2.3.1.** *Assume that Central Condition 2.2.5, Localization Conditions 2.1.2, and Uniform Scheme Condition 2.1.6 hold and that the convolution function $\lambda$ of 2.1.6 satisfies $\widehat{\lambda} \in L_\infty(\mathbb{R}^d)$.*

*(a) If, for some positive $k$, $|\cdot|^{-k}\widehat{\mu}$ is bounded, then, for every $f \in \widetilde{W}_\infty^k(\mathbb{R}^d)$,*

$$\|(L - L_A)(\sigma_h f)\|_\infty \le \text{const}\, h^k |f|'_{\infty,k}.$$

*In particular,*

*(b) $L_A$ provides to functions in $\widetilde{W}_\infty^{k'}(\mathbb{R}^d)$ the same approximation order provided by $L$ in case $\widehat{\phi}$ is continuous on $\mathbb{R}^d\backslash 0$, has a singularity of exact order $k'$ at the origin, does not vanish at some $j \in 2\pi\mathbb{Z}^d$, and $\widehat{\psi}(0) \neq 0$.*

**Proof:** We first claim that:

**Lemma.** *Under the conditions of part (a) of Theorem 2.3.1,*

$$|\mu g| \le \mathrm{const}|g|'_{\infty,k}, \quad g \in \widetilde{W}^k_\infty(\mathbb{R}^d).$$

**Proof of the Lemma.** We first assume that $g \in \widetilde{W}^k_\infty(\mathbb{R}^d)$ is compactly supported, and denote $g_| := g_{|_{\mathbb{Z}^d}}$. Then, with $\widehat{g}_|$ the Fourier series of $g_|$,

$$\mu g = \sum_{\alpha \in \mathbb{Z}^d} \mu(-\alpha) g(\alpha) = (\mu * g_|)(0) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \widehat{\mu}(t) \widehat{g}_|(t)\, dt. \tag{2.3.2}$$

Since $g \in \widetilde{W}^k_\infty(\mathbb{R}^d)$ and is compactly supported, $\widehat{g} \in L_1(\mathbb{R}^d)$, and therefore we may invoke Poisson's summation formula to obtain that

$$\sum_{\alpha \in 2\pi\mathbb{Z}^d} \widehat{g}(\cdot + \alpha) = (2\pi)^{-d} \sum_{\beta \in \mathbb{Z}^d} g(\beta) e_{-\beta} = (2\pi)^{-d} \widehat{g}_|.$$

Combining the above with (2.3.2), we arrive at

$$\mu g = \int_{[-\pi,\pi]^d} \widehat{\mu}(t) \sum_{\alpha \in 2\pi\mathbb{Z}^d} \widehat{g}(t+\alpha)\, dt = \sum_{\alpha \in 2\pi\mathbb{Z}^d} \int_{[-\pi,\pi]^d} \widehat{\mu}(t) \widehat{g}(t+\alpha)\, dt = \int_{\mathbb{R}^d} \widehat{\mu}\widehat{g}. \tag{2.3.3}$$

(The summation by parts in the second equality can be justified by dominated convergence arguments, since $\widehat{\mu}$ is bounded and the sum there is $L_1$-convergent.)

Next, we extend (2.3.3) to a general $g \in \widetilde{W}^k_\infty(\mathbb{R}^d)$. For that, we take $(\eta_h)_h$ to be a band-limited rapidly decaying approximate identity, with $\widehat{\eta}_1$ equals 1 around the origin, and define $g_h := g\eta_h^\vee$. By (2.3.3),

$$\mu g_h = \int_{\mathbb{R}^d} \widehat{\mu}\widehat{g}_h = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{\mu}(\widehat{g} * \eta_h) = (2\pi)^{-d} \eta_h * (\widehat{g} * \widehat{\mu}(-\cdot))(0).$$

Clearly, $\mu g_h \to (2\pi)^{-d} \mu g$ as $h \to 0$. Also, since $\widehat{g} * \widehat{\mu}(-\cdot)$ is continuous and bounded, its convolution with the approximate identity converges pointwise to itself, and hence

$$\eta_h * (\widehat{g} * \widehat{\mu}(-\cdot))(0) \to (\widehat{g} * \widehat{\mu}(-\cdot))(0) = \int_{\mathbb{R}^d} \widehat{\mu}\,\widehat{g}, \quad \text{as } h \to 0,$$

and the desired extension of (2.3.3) easily follows.

Since, by assumption, $|\widehat{\mu}| \le \mathrm{const}|\cdot|^k$, we can deduce now that

$$|\mu g| = |\sum_{\alpha \in \mathbb{Z}^d} \mu(-\alpha) g(\alpha)| \le \mathrm{const} \int_{\mathbb{R}^d} |\cdot|^k\, |\widehat{g}| = \mathrm{const}|g|'_{\infty,k}.$$

$\square$

We now continue with the proof of Theorem 2.3.1. Similarly to the proof of Theorem 2.2.9, we estimate $\mu(\Lambda(\sigma_h f)(\beta+\cdot))$. Invoking the lemma with $g := \Lambda(\sigma_h f)(\beta+\cdot)$, we obtain from the translation invariance of the semi-norm $|\cdot|'_{\infty,k}$ that

$$|\mu(\Lambda(\sigma_h f)(\beta+\cdot))| \le \mathrm{const}|\lambda * (\sigma_h f)|'_{\infty,k}.$$

Since $\widehat{\lambda}$ is bounded, $\lambda*$ can be easily seen to be continuous on $\widetilde{W}^k_\infty(\mathrm{I\!R}^d)$, and hence, by elementary properties of Fourier transform

$$|\mu(\Lambda(\sigma_h f)(\beta+\cdot))| \le \mathrm{const}|\sigma_h f|'_{\infty,k} = \mathrm{const} h^k |f|'_{\infty,k}.$$

The rest of the proof of (a) is identical with its counterpart in Theorem 2.2.9.

In the proof of (b), we employ Lemma 2.2.15. Note that the lemma is stated with respect to $W^k_\infty(\mathrm{I\!R}^d) \cap C^k(\mathrm{I\!R}^d)$, but is valid also with respect to the smaller space $\widetilde{W}^k_\infty(\mathrm{I\!R}^d)$, since the exponential functions $e_\theta$, $\theta \in \mathrm{I\!R}^d$ that are used in the proof there are contained in $\widetilde{W}^k_\infty(\mathrm{I\!R}^d)$, too. Under the further assumption in (b), we can use an argument as in the first part of the proof of Lemma 2.2.13 to conclude that $\widehat{\mu}$ has a zero of order $k'$ at the origin, and therefore the conditions, hence the consequences of part (a) here are valid, for the choice $k := k'$. Combining part (a) with Lemma 2.2.15, we obtain (b). □

### 2.4. Approximation to unbounded functions

The kind of approximation which is considered in the previous section is **stationary**. That means, by definition, that the space from which our approximant to $\sigma_h f$ is selected does not change with $h$. On uniform grids, stationary approximation have the following important advantage: in many cases the approximation map $L$ can be extended in a natural way to a map $\overline{L}$ that acts from

$$N_k := \{f : \mathrm{I\!R}^d \to \mathbb{C} :\ \|f\|_{N_k} := \|(1+|\cdot|)^{-k} f\|_{L_\infty(\mathrm{I\!R}^d)} < \infty\} \tag{2.4.1}$$

into $S_{\mathbb{Z}^d}(\phi)$, and still provides to $k$-time continuously differentiable functions in

$$w^k_\infty(\mathrm{I\!R}^d) := \{f \in N_k :\ |f|_{k,\infty} < \infty\}$$

the same approximation order as provided by $L$ to $W^k_\infty(\mathrm{I\!R}^d) \cap C^k(\mathrm{I\!R}^d)$. If $L$ has the form assumed in Uniform Scheme Condition 2.1.6, i.e.,

$$L : f \mapsto \sum_{\alpha\in\mathbb{Z}^d} \psi(\cdot-\alpha)(\Lambda f)(\alpha), \tag{2.4.2}$$

the "natural extension" $\overline{L}$ still satisfies the same rule, and one then needs to ensure that $\lambda*$ is well-defined and continuous on $N_k$, and that the sum in (2.4.2) is meaningful (i.e., converges in some reasonable sense) even for $f \in N_k$.

In case $\overline{L}$ is well-defined and is known then to provide approximation order $k$ not only to $k$-time differentiable functions in $W^k_\infty(\mathrm{I\!R}^d)$, but also in the larger space $w^k_\infty(\mathrm{I\!R}^d)$, it is desirable to extend $L_A$ appropriately. However, for such an extension, we need to assume slightly more than the assumptions made in Theorem 2.2.9. The modifications are mainly concerned with the imposition of faster decay rates on $\mu$ and $\psi$ in Localization Conditions 2.1.2, and on $\phi_\alpha - \phi(\cdot-\alpha)$ in Central Condition 2.2.5. It is worth mentioning that, at least as far as the decay rates of $\psi$ are concerned, the modification seems to be right in place: for some $f \in w^k_\infty(\mathrm{I\!R}^d)$, we expect the coefficients $\{\Lambda(f)(\alpha)\}_{\alpha\in\mathbb{Z}^d}$ to grow like $|\alpha|^k$, hence need the faster decay rates for the convergence of the series

$$\overline{L} f := \sum_{\alpha\in\mathbb{Z}^d} \psi(\cdot-\alpha)(\Lambda f)(\alpha). \tag{2.4.3}$$

**Theorem 2.4.4.** *Assume that Localization Conditions 2.1.2, as well as Central Condition 2.2.5 hold for*

$$m_\mu, m_\psi, m_A > d + r, \tag{2.4.5}$$

*for some positive $r$. Assume further that $\Lambda$ of (2.1.5) extends continuously to a convolution map from $N_r$ into itself. Then, the series (2.2.4) that defines $L_A f$ converges absolutely and uniformly on compact sets, for every $f \in N_r$. Further, assume that conditions (a) and (b) in Theorem 2.2.9 hold, with $k$ there $\geq r$. Then, the assertions made in Theorem 2.2.9 about $(L - L_A)(\sigma_h f)$ hold for functions in $w_\infty^{k+1}(\mathbb{R}^d) \cap w_\infty^k(\mathbb{R}^d) \cap C^{k+1}(\mathbb{R}^d) \cap N_r$.*

**Remark.** The standard choice for $r$ above is $r = k$. The fact that we allow here $r < k$ as well, is aimed at deriving partial improvements of Theorem 2.2.9 for the case when the decay rates $m_\psi, m_A$ are only partially improved.

**Proof:** Since we are assuming that $m_\mu, m_\psi, m_A > d + r$, we conclude from Lemma 2.2.6 that

$$|\psi_\alpha(x)| \leq \mathrm{const}(1 + |x - \alpha|)^{-m} \tag{2.4.6}$$

with $m > d + r$, and since, further, $\Lambda f \in N_r$, the map $L_A$ is well-defined on $N_r$.

We may then follow the proof of Theorem 2.2.9 *verbatim*, with straightforward modifications of some of the justifications of the various steps. For example, the fact that dilation is an isometry on $L_\infty(\mathbb{R}^d)$ is still effective in (2.2.12), since the estimate of $|\lambda * \sigma_h f|_{\infty,k}$ involves the convolution of $\lambda$ with bounded functions only (despite of the fact that $f$ itself is not necessarily bounded). As another instance, the summation by parts, which is applied to the double sum in the (2.2.11), is also justified by the modified decay assumptions on $|\phi(\cdot - \beta) - \phi_\beta|$ combined with the fact that $\Lambda(\sigma_h f) \in N_r$. $\square$

Theorem 2.4.4 leads to an improvement of Theorem 2.2.16 by extending the range of approximants from $W_\infty^{k+1}(\mathbb{R}^d) \cap C^{k+1}(\mathbb{R}^d)$ to $w_\infty^{k+1}(\mathbb{R}^d) \cap w_\infty^k(\mathbb{R}^d) \cap C^{k+1}(\mathbb{R}^d) \cap N_r$. We omit these straightforward details.

## 2.5. Polynomial reproduction

With $\psi$ as in Localization Conditions 2.1.2, and $L$ as in (2.1.5), we say that $L$ **reproduces** $\Pi_r$ ($r \geq 0$), if $L_{|\Pi_r}$ is the identity. Embedded in this is the requirement that $m_\psi$ of Conditions 2.1.2 would be no smaller than $d + r$. The polynomial reproduction property is an essential ingredient in the error analysis of the so-called *quasi-interpolation* schemes, and in turn, makes the approximation orders that can be established by such schemes restricted by the decay rate of $\psi$.

In our context here, we do not need to know the argument used for the derivation of the approximation order of $L$, and more significantly, do not need a corresponding polynomial reproduction property from $L_A$. This is very important because of the following: if $L$ is proved to provide approximation order $k$ via a polynomial reproduction argument, then we know that the localization $\psi$ employed by $L$ satisfies $m_\psi \geq d + k - 1$. In order to show that $L_A$ also reproduces polynomials, we need to know that $L_A$ is well-defined on $\Pi_{k-1}$, which, if Lemma 2.2.6 is used for that, will require further assumptions, primarily that $m_A$ of Condition 2.2.5 would satisfy, at a minimum,

$$m_A \geq d + k - 1.$$

This, by all means, is not a minor requirement, since it makes the construction of the matrix $A$ more involved (see §2.7). Fortunately, even though the approximation orders of $L$ might have been derived via polynomial reproduction, our conversion results (Theorem 2.2.9 and Theorem 2.2.16) do not require $L_A$ to reproduce polynomials, and hence we are not forced to look for $A$ that improves upon the basic condition $m_A > d$.

However, it is worth mentioning that the polynomial reproduction property of $L$ is inherited by $L_A$ if the various decay rates $m_\psi, m_\mu, m_A$ are *known* to be large enough:

**Corollary 2.5.1.** *Under the notations and assumptions of Theorem 2.4.4, if $L$ reproduces $\Pi_r$ ($r \le k$) so does $L_A$.*

**Proof:** Let $p \in \Pi_r$. Had we known that $\deg p < k$, the conclusion would have been immediate, since, by Theorem 2.4.4, $\|(L - L_A)p\| \le \text{const}(|p|_{\infty,k} + |p|_{\infty,k+1}) = 0$. The argument, however, for the case $\deg p = k$, is not much more involved: since we assume that $\mu$ annihilates $\Pi_k$, hence $\Pi_r$ (that assumption is made in Theorem 2.2.9, and is adopted in Theorem 2.4.4, hence here), we may derive from the proof of Theorem 2.2.9 (cf. (2.2.11) and its following line) that

$$(L - L_A)p = \sum_{\beta \in \mathbb{Z}^d} (\phi(\cdot - \beta) - \phi_\beta)(\mu(\Lambda p)(\cdot + \beta)). \tag{2.5.2}$$

Since $\Lambda$ is convolution, $\Pi_r$ is an invariant subspace of it, and therefore $\Lambda p \in \Pi_r$. Since $\mu$ annihilates $\Pi_r$, (2.5.2) implies that $L_A p = Lp = p$, the second equality by the polynomial reproduction assumption made with respect to $L$. □

The crucial assumptions in the above corollary are (a) and (b) of Theorem 2.2.9, stating that $\mu$ is well-defined on $\Pi_r$ and annihilates it. It is possible to replace these two assumptions by the condition $m_\mu > d + r$, if we assume, as we do in the present paper, that $\widehat{\phi}$ is continuous on $\mathbb{R}^d \backslash 0$, and that $\widehat{\phi}$ does not vanish identically on $2\pi\mathbb{Z}^d \backslash 0$. Indeed, it is well-known then, that the $\Pi_r$-reproduction property of $L$ implies that $|\cdot|^r \widehat{\phi}$ is singular at the origin, which means that, if $\widehat{\phi}$ has a singularity of some order at the origin, that order is $> r$. Consequently, Lemma 2.2.13 implies that $\mu$ annihilates $\Pi_r$.

The argument used in the proof of Corollary 2.5.1 supports also the following stronger claim.

**Corollary 2.5.3.** *Let $P$ be a translation-invariant space of polynomials, which is reproduced by $L$ and is annihilated by $\mu$. If, further, the double sum*

$$\sum_{\alpha, \beta \in \mathbb{Z}^d} (\phi(\cdot - \beta) - \phi_\beta)\mu(\beta - \alpha)\Lambda p(\alpha)$$

*(cf. (2.2.11)) converges absolutely, for every $p \in P$, then $L_A$ reproduces $P$, as well.*

### 2.6. Remarks on the location of the centers

In actual approximation schemes, we do not dilate the approximand $f$. Instead, we dilate the space $S_\Xi(\phi)$ from which the approximants are selected. Thus, the "true" error to be considered should have been

$$f - \sigma_{1/h} L(\sigma_h f).$$

The change to the study of $\sigma_h f - L(\sigma_h f)$ is due to technical convenience and is made available by the invariance of the max-norm under dilation.

However, in order to understand the nature of our approximants as $h$ changes, it is instructive to look closer at the approximant $\sigma_{1/h}L(\sigma_h f)$ and its scattered center analogue. First, we see that, adopting the convolution assumption on $\Lambda$ (cf. Uniform Scheme Conditions 2.1.6),

$$\sigma_{1/h}L(\sigma_h f) = \sum_{\alpha\in h\mathbb{Z}^d} (\sigma_{1/h}\psi)\,(\cdot-\alpha)(\Lambda_h f)(\alpha),$$

where

$$\Lambda_h : f \mapsto h^{-d}(\sigma_{1/h}\lambda) * f.$$

The scattered center version $\sigma_{1/h}L_A(\sigma_h f)$ is obtained by replacing each $(\sigma_{1/h}\psi)(\cdot-\alpha)$ by $\sigma_{1/h}\psi_\alpha$. Finally, the function $\sigma_{1/h}\psi_\alpha$ is obtained by an application of the same matrix $A$ to the scattered shifts of $\sigma_{1/h}\phi$, only that the center set $\Xi$ is replaced by the scaled set $h\Xi$. Thus, in summary, the scattered center approximant $\sigma_{1/h}L_A(\sigma_h f)$ makes use of the $h\Xi$-translates of $\sigma_{1/h}\phi$.

The fact that at the $h$-level the dilated center set $h\Xi$ is employed should not be regarded as an essential ingredient of our approach and arguments. As a matter of fact, one might employ at the $h$-level a center set $\Xi_h$ which may resemble no relation to $h\Xi$. This, in turn, forces the search of different matrices $A_h$ for different values of $h$. Applying $A_h$ to ${}_{\Xi_h}(\phi)$, one obtains a set of functions $\{\phi_{\alpha,h}\}_{\alpha\in\mathbb{Z}^d}$, and the Central Condition 2.2.5 should then be modified to

$$|(\sigma_h\phi_{\alpha,h})(x) - \phi(x-\alpha)| \le \text{const}(1+|x-\alpha|)^{-m_A}, \quad \alpha\in\mathbb{Z}^d,$$

with const independent not only of $\alpha$ and $x$ but also of $h$.

Another remark is concerned with the *stationary* nature of the approximation schemes discussed in this section. That notion refers to the fact that the sequence of approximants $\{L(\sigma_h f)\}_h$ to $\{\sigma_h f\}_h$ employs translations of the *same* function $\phi$ at all $h$-levels. There are important situations in spline theory when such simplification in the construction is unacceptable (primarily exponential box splines, cf. [DR]). But, in the context of radial basis function approximation the only presently known case when non-stationary schemes are used is concerned with *spectral approximation* (cf. [BeLi], [BuD], [BR], [R]), a case that cannot be covered by the approach developed here.

### 2.7. On the Central Condition 2.2.5

Condition 2.2.5 is the major condition that is assumed in both Theorem 2.2.9 and Theorem 2.2.16. As a matter of fact, any set $\{\phi_\alpha\}_\alpha$ that satisfies Condition 2.2.5 can be used to define the approximation map $L_A$. However, since our interest is in approximating from the span of the scattered translates of $\phi$, we certainly restrict our attention to functions $\{\phi_\alpha\}$ which are expressed in the form (2.2.1).

In the present subsection, we briefly discuss the following important (and natural) question: what sufficient conditions on the coefficient matrix $A$ imply Condition 2.2.5? Here we show that, under some basic assumptions that we adopt here with respect to the structure of the Fourier transform of $\phi$, one should aim at approximating (around the origin) the constant function 1 by exponentials of the form

$$e^*_\alpha = \sum_{\xi\in\Xi} A(\alpha,\xi)e_{\alpha-\xi},$$

for each $\alpha\in\mathbb{Z}^d$. The (formal) derivatives of $e^*_\alpha$ are of the form

$$D^\beta e^*_\alpha = \sum_{\xi\in\Xi} i^{|\beta|_1}(\alpha-\xi)^\beta A(\alpha,\xi)e_{\alpha-\xi},$$

and therefore, given an integer $j$, if the sequence $\xi \mapsto (1+|\alpha-\xi|^j)A(\alpha,\xi)$ is in $\ell_1(\Xi)$, then $e^*_\alpha$ is $j$-times boundedly differentiable, can be differentiated term by term, and its $W^j_\infty(\mathbb{R}^d)$-norm satisfies

$$\|e^*_\alpha\|_{W^j_\infty(\mathbb{R}^d)} \le c_j \|((1+|\cdot-\alpha|^j)A(\alpha,\cdot))\|_{\ell_1(\Xi)}.$$

**Theorem 2.7.1.** *Let $A=(A(\alpha,\xi))_{\mathbb{Z}^d\times\Xi}$ be the coefficient matrix of (2.2.1), and assume that each of the rows of $A$ is in $\ell_1(\Xi)$. Let $e^*_\alpha$ be defined as above. Suppose the following conditions hold:*

*(a) The set*

$$\{\Xi \ni \xi \mapsto (1+|\xi-\alpha|^j)A(\alpha,\xi): \ \alpha\in\mathbb{Z}^d\}$$

*lies in $\ell_1(\Xi)$ and is bounded there, for all $j\le s$, for some non-negative integer $s$.*

*(b) For each $\alpha\in\mathbb{Z}^d$, all derivatives of $1-e^*_\alpha$ of orders $\le m$ vanish at the origin, for some non-negative integer $m<s$.*

*(c) For yet another positive integer $r$, $\widehat{\phi}$ is $r$-times differentiable on $\mathbb{R}^d\backslash 0$, and each $D^\alpha\widehat{\phi}$, $|\alpha|_1\le r$, is summable around $\infty$.*

*(d) Each $D^\alpha\widehat{\phi}$, $|\alpha|_1\le r$, (calculated on $\mathbb{R}^d\backslash 0$) has a singularity of order $k'+|\alpha|_1$ at the origin, for some positive $k':=k'(\phi)$, and for the same $r$ of (c).*

*(e) The distribution $\widehat{\phi}$ is of order $\le m$.*

*Then, Condition 2.2.5 is satisfied with $m_A=m-[k']+d$, with $[k']$ the greatest integer $\le k'$, and provided that this $m_A$ exceeds neither $r$ nor $s$.*

Roughly speaking, $r$ can be regarded as a smoothness parameter for $\widehat{\phi}$. Thus, in these terms, the threshold for the "order of matching" $m$ (in (b) above), in order to achieve $m_A>d$, is the order of the singularity of $\widehat{\phi}$ at the origin. Above that threshold, a better matching (i.e., higher $m$) would result in a larger $m_A$ in Condition 2.2.5 (which might be important, e.g., if we approximate unbounded functions, cf. §2.4), but overmatching, i.e., matching which exceeds $r+[k']-d$, may not yield any improvement in $m_A$.

**Proof:** Condition 2.2.5 is equivalent to the statement that $\{(1+|\cdot|)^{m_A}(\phi-\phi_\alpha(\cdot+\alpha)):\ \alpha\in\mathbb{Z}^d\}$ is bounded in $L_\infty(\mathbb{R}^d)$. Instead, we prove that the Fourier transforms of these functions are bounded in $L_1(\mathbb{R}^d)$. Since $m_A=m-[k']+d$, it suffices, for that latter purpose, to prove that, for each integer $j\le m-[k']+d$, the Fourier transforms of $|\cdot|^j(\phi-\phi_\alpha(\cdot+\alpha))$, $\alpha\in\mathbb{Z}^d$ are bounded in $L_1(\mathbb{R}^d)$. Since the Fourier transform of $\phi-\phi_\alpha(\cdot+\alpha)$ is $(1-e^*_\alpha)\widehat{\phi}$, we may prove that the functions

$$\{D^\beta((1-e^*_\alpha)\widehat{\phi}):\ \alpha\in\mathbb{Z}^d,\ |\beta|_1\le m-[k']+d\}$$

are bounded in $L_1(\mathbb{R}^d)$. By applying Leibnitz' rule, we, finally, may replace this last set of functions by

$$\{D^\gamma(1-e^*_\alpha)D^{\beta-\gamma}\widehat{\phi}:\ \alpha\in\mathbb{Z}^d,\ \gamma\le\beta,\ |\beta|_1\le m-[k']+d\}. \tag{2.7.2}$$

Since $m-[k']+d\le r$, by assumption, condition (c) implies that $D^{\beta-\gamma}\widehat{\phi}$ is summable around $\infty$. In addition, since $m-[k']+d\le s$, condition (a) implies that $\{D^\gamma(1-e^*_\alpha)\}_\alpha$ $(|\gamma|_1\le m-[k']+d)$ are bounded in $L_\infty(\mathbb{R}^d)$. Thus, we obtain that the functions in (2.7.2) are bounded in $L_1(\Omega)$, for some neighborhood $\Omega$ of $\infty$, uniformly in $\alpha\in\mathbb{Z}^d$.

Next, we show that the functions in (2.7.2) are bounded in $L_1(B)$, for some origin-neighborhood $B$. Here, recall first that, on $\mathbb{R}^d\backslash 0$, the Fourier transform of $\phi-\phi_\alpha(\cdot+\alpha)$ is

$$(1-e^*_\alpha)\widehat{\phi}. \tag{2.7.3}$$

Since, by (b), $1-e^*_\alpha$ has a zero of order $m+1$ at the origin, while, by (e), the order of the distribution $\widehat{\phi}$ at the origin is $\le m$, it follows that the representation (2.7.3) for the Fourier transform of $\phi - \phi_\alpha(\cdot + \alpha)$ extends to the entire $\mathbb{R}^d$.

Assumptions (d) and (b), when combined, prove that $D^\gamma(1-e^*_\alpha)D^{\beta-\gamma}\widehat{\phi}$ is bounded around the origin as long as $|\beta|_1 \le m - k' + 1$, and has a singularity of order $k' + |\beta|_1 - m - 1$ for larger $\beta$. This implies that, as long as $|\beta|_1 \le m - [k'] + d$, $D^\gamma(1-e^*_\alpha)D^{\beta-\gamma}\widehat{\phi}$ has a singularity at the origin of order $< d$, and hence is integrable there. The required uniformity in the $L_1(B)$-norm follows from the uniformity in assumption (a). □

Under a slightly stronger assumption on the behaviour of $\widehat{\phi}$ around the origin, the assertions of the last theorem can be improved as follows.

**Corollary 2.7.4.** *Let $A = (A(\alpha,\xi))_{\mathbb{Z}^d\times\Xi}$ be the coefficient matrix of (2.2.1), and assume that each of the rows of $A$ is in $\ell_1(\Xi)$. Let $e^*_\alpha$ be defined as before. Suppose the following conditions hold:*

(a,b,c,d) *Same as in Theorem 2.7.1, only that we require $m < s - 1$.*
*Around the origin, $\widehat{\phi}$ can be represented as a sum $\rho_0 + \rho_1$, such that*

($d_0$) *On $\mathbb{R}^d\backslash 0$, $\rho_0$ is continuous and is a homogeneous function of order $-k'$, i.e., for every $t > 0$, the support of the distribution $\sigma_t\rho_0 - t^{-k'}\rho_0$ is the origin.*

($d_1$) *Each $D^\alpha\rho_1$, $|\alpha|_1 \le r$, (calculated on $\mathbb{R}^d\backslash 0$) has a singularity of order $k'(\rho_1)+|\alpha|_1$ at the origin, for some positive $k'(\rho_1) < [k']$.*

(e) *Each $\rho_i$ is a tempered distribution of order $\le m$.*

*Then Condition 2.2.5 is satisfied with $m_A = m - k' + d + 1$, provided that $m_A \le r, s$.*

The improvement here can be observed as follows: assumptions ($d_0$), ($d_1$) imply that our function $\widehat{\phi}$ has a singularity of order $k'$ at the origin. Therefore, a direct application of Theorem 2.7.1 can establish Condition 2.2.5 only for $m_A := m - [k'] + d$. However, the present corollary asserts that the Central Condition is valid also for the larger $m_A := m - k' + d + 1$.

**Proof:** Let $\eta$ be a compactly supported $C^\infty(\mathbb{R}^d)$-function, which is 1 around the origin, and such that the decomposition $\widehat{\phi} = \rho_0 + \rho_1$ is valid on $\operatorname{supp}\eta$. We write

$$\widehat{\phi} = \eta\rho_0 + \eta\rho_1 + (1-\eta)\widehat{\phi}.$$

One easily checks that the inverse transform $\tau$ of $\eta\rho_1 + (1-\eta)\widehat{\phi}$ satisfies conditions (c-e) of Theorem 2.7.1, with $k'(\tau) = k'(\rho_1)$, and hence with $[k'(\tau)] < [k']$. Defining $\tau_\alpha$, $\alpha \in \mathbb{Z}^d$ by

$$\tau_\alpha := \sum_{\xi\in\Xi} A(\alpha,\xi)\tau(\cdot - \xi),$$

we may appeal to Theorem 2.7.1 to conclude that Condition 2.2.5 is satisfied by $\tau$, and with $m_A(\tau) = m - [k'(\rho_1)] + d \ge m - [k'] + d + 1 \ge m - k' + d + 1$.

Consequently, in order to prove the corollary, we may assume without loss that $\widehat{\phi} = \eta\rho_0$, as we do in the rest of the proof. This trivially implies that $\phi$ satisfies condition (c) of Theorem 2.7.1 for all $r$, (since $\widehat{\phi}$ is now compactly supported) and also satisfies (e) and (d) there, with $k'(\phi) = k'$ by the assumptions on $\rho_0$. This, however, would lead us to an unsatisfactory estimate for $m_A$, and thus, a finer analysis is required.

We let $T_\alpha$ be the homogeneous polynomial of degree $m+1$ in the Taylor expansion of $1 - e^*_\alpha$ around the origin, and write

$$(1-e^*_\alpha)\widehat{\phi} = (1-e^*_\alpha)\eta\rho_0 = T_\alpha\eta\rho_0 + (1 - e^*_\alpha - T_\alpha)\eta\rho_0.$$

Since $\rho_0$ is a distribution of order $\le m$, and is homogeneous of order $-k'$, and since $T_\alpha$ is homogeneous of order $m+1$, the product $T_\alpha\rho_0$ is a homogeneous *function* of order $m+1-k'$. Thus, its inverse transform $g$ decays at $\infty$ like $O(|\cdot|^{-(m+1-k'+d)})$. As $(T_\alpha\eta\rho_0)^\vee = \eta^\vee * g$, and $\eta^\vee$ decays rapidly, we conclude that $(T_\alpha\eta\rho_0)^\vee = O(|\cdot|^{-(m+1-k'+d)})$.

The other term in the representation of $(1-e_\alpha^*)\widehat{\phi}$ is $(1-e_\alpha^*-T_\alpha)\widehat{\phi}$. Here, all derivatives of $1-e_\alpha^*-T_\alpha$ of orders $\le m+1 < s$ vanish at the origin. Therefore, the argument used in the proof of Theorem 2.7.1 can be followed (with $(1-e_\alpha^*)\widehat{\phi}$ replaced by $(1-e_\alpha^*-T_\alpha)\widehat{\phi}$) to yield that $((1-e_\alpha^*-T_\alpha)\widehat{\phi})^\vee$ decays at $\infty$ like $O(|\cdot|^{-(m+1-[k']+d)})$, which is (at worst) the required rate.

The uniformity required in Condition 2.2.5 follows, once again, from the uniformity in condition (a) here. □

Theorem 2.7.1 as well as Corollary 2.7.4 require the matrix $A$ to satisfy conditions that may impose constraints on the distribution of $\Xi$, and may be hard to obtain. However, it is shown in [BuDL] (see also §3.4) that, for every fixed $m$, and for all sufficiently small $c \le c_0(m)$ if each ball of radius $c$ in $\mathbb{R}^d$ intersects with $\Xi$, there exists a matrix $A$ that satisfies conditions (a-b) of Theorem 2.7.1.

### 2.8. On the decay rates $m_A$, $m_\psi$, $m_\mu$

The derivation of the main results of this section required us to assume decay rates on the localization sequence $\mu$, the localization function $\psi$, and the difference $\phi(\cdot-\alpha)-\phi_\alpha$ (cf. Conditions 2.1.2 and 2.2.5). We preferred to write these assumptions in terms of the $O(|\cdot|^{-k})$ (for an appropriate $k$) since that allows us to derive quantitative results concerning the behaviour of $(\psi_\alpha)_{\alpha\in\mathbb{Z}^d}$ at $\infty$ (cf. Lemma 2.2.6), which is important in several places, especially in §2.4. However, as Theorem 1.2.5 actually asserts, Theorems 2.2.9, 2.2.16, and 2.3.1 remain valid under slower decay rates. These improvements are not very important with respect to the decay rates of $\mu$ (for example, the decay rates on $\mu$ assumed in Theorem 2.2.9 are more restrictive than those assumed in the Localization Conditions 2.1.2). Also, such improvements do not seem to be very significant with respect to the decay rates of $\psi$ (in 2.1.2), since, in the known examples, $\psi$ decays at $\infty$ at faster rates. However, as far as the behaviour of $\phi(\cdot-\alpha)-\phi_\alpha$ is concerned, every relaxation of the Central Condition 2.2.5 is important, since it might simplify the construction of the matrix $A$, hence eventually simplify the actual approximation scheme $L_A$. Therefore, we devote the present subsection to a brief discussion of these relaxations.

Our modified conditions are as follows:

(a) In lieu of (a) in Conditions 2.1.2, we assume only that $\mu \in \ell_1(\mathbb{Z}^d)$.
(b) In lieu of (b) in Conditions 2.1.2, we assume only that $\sum_{\alpha\in\mathbb{Z}^d} |\psi(\cdot-\alpha)| \in L_\infty(\mathbb{R}^d)$.
(c) In lieu of Condition 2.2.5, we assume that $\sum_{\alpha\in\mathbb{Z}^d} |\phi(\cdot-\alpha)-\phi_\alpha| \in L_\infty(\mathbb{R}^d)$.

The new variant of Lemma 2.2.6 then reads as follows:

**Lemma 2.2.6∗.** *Assume that conditions (a-c) above hold. Then*

$$\sum_{\alpha\in\mathbb{Z}^d} |\psi_\alpha| \in L_\infty(\mathbb{R}^d),$$

*and, hence, the operator*

$$c \mapsto \sum_{\alpha\in\mathbb{Z}^d} \psi_\alpha\, c(\alpha)$$

*is bounded from $\ell_\infty(\mathbb{Z}^d)$ to $L_\infty(\mathbb{R}^d)$.*

**Proof:** By the definition of $\psi$ and $\psi_\alpha$,

$$|\psi_\alpha(x) - \psi(x-\alpha)| \le \sum_{\beta\in\mathbb{Z}^d} |\mu(\beta-\alpha)||\phi_\beta(x) - \phi(x-\beta)|.$$

Therefore,

$$\sum_{\alpha\in\mathbb{Z}^d} |\psi_\alpha(x) - \psi(x-\alpha)| \le \sum_{\alpha,\beta\in\mathbb{Z}^d} |\mu(\beta-\alpha)||\phi_\beta(x) - \phi(x-\beta)| \le \|\mu\|_{\ell_1(\mathbb{Z}^d)} \sum_{\beta\in\mathbb{Z}^d} |\phi_\beta(x) - \phi(x-\beta)|.$$

Consequently, by (a) and (c) above, $\sum_{\alpha\in\mathbb{Z}^d} |\psi_\alpha - \psi(\cdot-\alpha)| \in L_\infty(\mathbb{R}^d)$, and combining that with (b) above we obtain the lemma's claim. □

The results of Theorems 2.2.9, 2.2.16, and 2.3.1 remain unchanged, and with the same proofs. In fact, one observes that Condition 2.2.5 is not invoked in the proof of Theorem 2.2.9 in its full power, and only the $L_\infty$-boundedness of $\sum_{\alpha\in\mathbb{Z}^d} |\phi_\alpha - \phi(\cdot-\alpha)|$ is used there.

## 3. Examples

### 3.1. The approximation schemes of [BR]

The reference [BR] contains a general study of $L_\infty(\mathbb{R}^d)$-approximation orders from spaces generated by the shifts of one basis functions $\psi$. It assumes $\psi$ to decay fast enough at $\infty$ to make the map

$$\psi *' : c \mapsto \psi *' c := \sum_{\alpha\in\mathbb{Z}^d} \psi(\cdot-\alpha)c(\alpha)$$

well-defined and continuous from $\ell_\infty(\mathbb{R}^d)$ to $L_\infty(\mathbb{R}^d)$ (see (1.2.2)). Note that such a condition implies that $\psi \in L_1(\mathbb{R}^d)$, and is implied by (b) of Localization Conditions 2.1.2. After establishing upper bounds on approximation orders from the "span" of the shifts of $\psi$ (cf. Theorem 2.8 there, and also Lemma 2.2.15 above), [BR] attempts to realize these bounds by employing the approximation map

$$L : f \mapsto \psi *' \Lambda(f)_{|_{\mathbb{Z}^d}}, \tag{3.1.1}$$

where $\Lambda$ is the convolution map $\lambda * f$, with

$$\widehat{\lambda} = \eta/\widehat{\psi}. \tag{3.1.2}$$

Here, $\eta$ is any smooth compactly supported function which is 1 around the origin. Results which are relevant to the basis functions of this paper are obtained in Theorems 3.6 and 4.2 of [BR]. In particular, the following result follows from Theorem 4.2 there:

**Result 3.1.3.** *Assume that $\psi *'$ is bounded, $\widehat{\psi}(0) \ne 0$, and that, on $\mathbb{R}^d\backslash 0$, $\widehat{\psi} = \widehat{\mu}\widehat{\phi}$, with $\widehat{\mu}$ continuous and $2\pi$-periodic and $\widehat{\phi}$ continuous on $\mathbb{R}^d\backslash 0$. Assume further that for some neighborhood $\Omega$ of the origin and some positive $k$,*

$$\sum_{\alpha\in 2\pi\mathbb{Z}^d\backslash 0} \|\widehat{\phi}(\cdot+\alpha)\|_{L_\infty(\Omega)} < \infty,$$

*and*

$$\| |\cdot|^{-k}/\widehat{\phi}\|_{L_\infty(\Omega)} < \infty.$$

*Then $L$ provides approximation order (at least) $k$ for every $f \in \widetilde{W}^k_\infty(\mathbb{R}^d)$.*

In order to extend Result 3.1.3 to non-uniform grids, we want to apply Theorem 2.3.1, hence need to verify its various conditions. First, $\Lambda$ of the present $L$ is indeed a convolution operator $\lambda *$. Since the result assumes $\widehat{\psi}$ to be non-zero at 0, and $\widehat{\psi}$ is continuous (since $\psi \in L_1(\mathbb{R}^d)$ by virtue of the boundedness of $\psi *'$), we conclude that $\widehat{\lambda}$ is bounded. Furthermore, around the origin

$$| \, |x|^{-k}\widehat{\mu}(x)| = |(|x|^{-k}/\widehat{\phi}(x))\widehat{\psi}(x)| \leq \text{const},$$

and hence $|\cdot|^{-k}\widehat{\mu}$ is bounded (on $\mathbb{R}^d$). Therefore, among the various conditions assumed for (a) of Theorem 2.3.1, we need only assume the Localization Conditions 2.1.2 and the Central Condition 2.2.5:

**Corollary 3.1.4.** *Let $L$ be defined as in (3.1.1) and (3.1.2), and assume that $\psi$ satisfies the assumptions of Result 3.1.3. If, in addition, the Conditions 2.1.2 and 2.2.5 are fulfilled, the corresponding $L_A$ provides approximation order at least $k$ to every $f \in \widetilde{W}^k_\infty(\mathbb{R}^d)$, with $k$ as in Result 3.1.3.*

Under the further assumption that $\widehat{\phi}_{|_{2\pi\mathbb{Z}^d\backslash 0}} \neq 0$, [BR] proves that (3.1.1) attains the highest possible approximation order. Thus, the above corollary shows that, under all the assumptions made there, *we are able to realize in the scattered case the best possible approximation orders available in the uniform case.*

### 3.2. Interpolation Schemes

Approximation via cardinal interpolation that employs radial basis functions is studied in Buhmann's thesis (cf. [Bu1,2]). The interpolation operator is of the form

$$Lf = \sum_{\alpha \in \mathbb{Z}^d} \psi(\cdot - \alpha) f(\alpha),$$

with $\psi$ the fundamental solution of the cardinal interpolation problem, i.e., a function that vanishes on $\mathbb{Z}^d\backslash 0$, and assumes the value 1 at the origin. The function $\psi$ is obtained as an infinite combination of shifts of the original basis function $\phi$:

$$\psi = \sum_{\alpha \in \mathbb{Z}^d} \phi(\cdot - \alpha)\mu(\alpha).$$

The sequence $\mu$, though not constructed for the mere sake of localization, is indeed a localization sequence since $\psi$ is shown to decay at $\infty$, in contrast with the original $\phi$.

The existence of the above fundamental solution is proved in [Bu1,2] under certain conditions on the Fourier transform $\widehat{\phi}$ of the basis function $\phi$. In particular, $\widehat{\phi}$ is assumed to be *radially symmetric*, to be smooth and positive on $\mathbb{R}^d\backslash 0$, to have derivatives that decay at $\infty$ at a $O(|\cdot|^{-(d+\varepsilon)})$-rate, and, most importantly, to have a singularity of a certain type and of some positive order $k'$ at the origin. The sequence $\mu$ is then proved there to decay at $\infty$ at a rate $O(|\cdot|^{-(d+k')})$, and the same decay rate is then established with respect to the fundamental interpolant $\psi$. The sum that defines $\psi$ is proved to converge uniformly on compact sets. Thus, we see that the Localization Conditions 2.1.2 as well as the Uniform Scheme Condition 2.1.6 which are required in our conversion theorems, are valid in Buhmann's interpolation schemes.

The approximation orders associated with cardinal interpolation that are derived in [Bu1,2] are of the following type. In case $k' = k + \varepsilon$, with $0 < \varepsilon < 1$, $k$ integer, $L$ is proved to provide approximation order $k'$ to functions in $C^{k+1}(\mathbb{R}^d)$ whose derivatives of order $k, k+1$ are bounded. For an integer $k'$, $L$ is proved to satisfy $\|\sigma_h f - L(\sigma_h f)\|_\infty = O(h^{k'}|\log h|)$, for functions in $C^{k'}(\mathbb{R}^d)$, with bounded derivatives of order $k'-1, k'$. Furthermore, for an *even* integer $k'$, and under a stronger assumption on the behaviour of $\widehat{\phi}$ at the origin, the $O(h^{k'}|\log h|)$ order is improved to approximation order $k'$. Though we do not provide here the details of those extra conditions (nor we provide the full details of Buhmann's original conditions), we mention that, with the aid of the extra conditions, $\mu$ is proved to decay like $O(|\cdot|^{-(d+k'+\nu)})$, for some positive integer $\nu$.

In the conversion of the above results to the scattered case, we need not appeal to the results of section 2.3, simply because $\mu$ decays here at a rate which allows the application of Theorem 2.2.16 in its full power. Indeed, we already realized that Conditions 2.1.2 and 2.1.6 are valid here, and the requirements on $\widehat{\phi}$ needed in [Bu1,2] are by far more restrictive than our mere assumption (a) of Theorem 2.2.16. The decay conditions on $\mu$ required in (c) of Theorem 2.2.16 are satisfied only in the case of a non-integer $k'$. Still, this is in agreement with Buhmann's results, since he obtains only the order $O(h^{k'+1}|\log h|)$ in the integer case, exactly as can be derived for $L_A$ (cf. the remark after the proof of Theorem 2.2.16). Finally, in the case when $k'$ is an integer and, nonetheless, a full approximation order is obtained in [Bu1,2], $\mu$ decays fast enough to be well-defined on $\Pi_{k'}$, and Theorem 2.2.16 applies again here. In summary, assuming that the Central Condition 2.2.5 is satisfied, all the approximation orders that are established in [Bu1,2] for cardinal interpolation $L$ are valid also for its "scattered variant" $L_A$ (though, $L_A$, of course, is not a cardinal interpolant, nor is guaranteed to be any kind of interpolant), when the approximand satisfies the conditions required in [Bu1,2], and, in addition, is bounded.

An extension of the $L_A$-approximation orders to unbounded functions, follows from the analysis of §2.4. For example, if $k' = k+\varepsilon$, $0 < \varepsilon < 1$, then, the decay rate of $\psi$ and of $\mu$ are $m_\psi = m_\mu = d + k'$, which allows us to invoke Theorem 2.4.4 for the value $r := k$ there, provided that $m_A$ of Condition 2.2.5 is known to be larger than $d+k$. Consequently, the approximation order $k'$ of $L_A$ holds, by Theorem 2.4.4, not only for bounded functions, but for functions in $C^{k+1}(\mathbb{R}^d)$ whose derivatives of orders $k$ and $k+1$ are bounded. As mentioned before, this is exactly the class of functions that are approximated in the uniform grid results of [Bu1,2]. Thus, the conversion here fully preserves both the approximation order and the space of approximands. Analogous results are valid in the case of an integer $k'$.

### 3.3. Quasi-interpolation schemes

Quasi-interpolation schemes which are based on uniform shifts of localized radial basis functions are discussed in [J], [DJLR], [P], [Ra], [Bu1,3]. In this section, we show that the uniform grid results of [DJLR] can be extended to non-uniform grids, with the aid of the machinery of §2 here.

In [DJLR], the Fourier transform of the basis function $\phi$ is assumed to be represented, on $\mathbb{R}^d\backslash 0$, as a quotient

$$\widehat{\phi} = F/G, \tag{3.3.1}$$

with $F, G$ satisfying several conditions, among which we mention here the following two:

(a) $G$ is a homogeneous polynomial of degree $k' \in 2\,\mathbb{N}$, with no zeros in $\mathbb{R}^d\backslash 0$.
(b) $F \in C^\infty(\mathbb{R}^d\backslash 0)$, is smooth (but not necessarily infinitely smooth) at the origin and does not vanish there.

We forgo mentioning the detailed conditions assumed in [DJLR] (and refer the reader to that reference), since most of those details will not be needed here. We do mention that the basic examples of $\phi$ that satisfy all the conditions assumed there are the fundamental solutions of the iterated Laplacian (in particular, the thin–plate splines in even dimensions) and their "shifted" version, where $|x|$ is replaced by $(|x|^2+c^2)^{1/2}$ (in particular, the multiquadrics in odd dimensions).

The localization process of [DJLR] is done with the aid of a *finitely supported* localization sequence $\mu$. This means that, for the sake of extending the [DJLR]-schemes to the scattered case, the required conditions (a) and (c) of 2.1.2 are automatically satisfied. Since the error analysis of [DJLR] is based on polynomial reproduction, a careful attention is given there to the decay rates $m_\psi$ of $\psi$ at $\infty$. In general, these decay rates are $m_\psi = d+\ell+1$ for some non-negative integer $\ell$, which means that condition (b) of 2.1.2 is also satisfied in the present case.

The approximation scheme used in [DJLR] is the simple one

$$Lf := \sum_{\alpha\in\mathbb{Z}^d} \psi(\cdot-\alpha)\, f(\alpha), \tag{3.3.2}$$

and the quasi-interpolation argument is employed there to yield the error estimate

$$\|\sigma_h f - L\sigma_h f\| = O(h^{\ell'+1}|\log h|), \quad \ell' := \min\{\ell, k'-1\}, \tag{3.3.3}$$

provided that $f$ is $\ell'+1$-times differentiable, and that $|f|_{\ell',\infty}$ and $|f|_{\ell'+1,\infty}$ are finite. For functions $f$ whose derivatives of orders $\ell'+2$ are also continuous and bounded, and under more subtle information on the decay of $\psi$, the above $O(h^{\ell'+1}|\log h|)$ is improved to $O(h^{\ell'+1})$. We refer to section 4 of [DJLR] for the full details.

It is now easy to verify that all the conditions on $\phi$, $\psi$, and $\mu$ imposed in Theorem 2.2.9 and Theorem 2.2.16 are satisfied by the schemes considered in [DJLR]. Therefore, the only condition that we really need to assume is the Central Condition 2.2.5, in order to obtain

**Theorem 3.3.4.** *Let $\phi$ be any of the basis functions considered in [DJLR] (whose Fourier transform has a singularity of even order $k'$ at the origin). Let $\psi$ be its localization (localized with the aid of a finite $\mu$), which decays at $\infty$ like $O(|\cdot|^{-(d+\ell+1)})$. Let $L$ be the approximation scheme of (3.3.2), and let $L_A$ be its scattered version. If Condition 2.2.5 is satisfied, then*

$$\|(L-L_A)\sigma_h f\|_{L_\infty(\mathbb{R}^d)} = O(h^k),$$

*for every $f \in W^k_\infty(\mathbb{R}^d)\cap C^k(\mathbb{R}^d)$, provided that $k \le k'$.*

Note that the result is valid for $k := k'$, even when $\ell < k'$.

Thus, while the approximation orders of (3.3.3) and their variants are restricted by the rate of decay of $\psi$, i.e., the parameter $\ell$, the estimate for $(L-L_A)(\sigma_h f)$ is optimal already for $\ell = 0$. The extension in the above theorem from uniform grids to scattered grids required us to pay some price: while the uniform results are obtained with respect to functions whose derivatives of order $\ell$ and $\ell+1$ are continuous and bounded, the non-uniform extensions require $f$ to be bounded, too. However, this further restriction can be relaxed with the aid of Theorem 2.4.4: choosing $k$ and $r$ in that theorem to be our $\ell$ here, we see that since in our present case $m_\mu = \infty$, and $m_\psi = d+\ell+1$, we need only know that in Central Condition 2.2.5 $m_A > d+\ell$, in order to obtain that

$$\|(L-L_A)\sigma_h f\|_{L_\infty(\mathbb{R}^d)} = O(h^{\ell+1}),$$

for functions $f$ whose derivatives of order $\ell$ and $\ell+1$ are continuous and bounded.

Our last remark here deals with the polynomials in $S_\Xi(\phi)$. It is shown in [DJLR] (cf. §3 there) that if $F$ of (3.3.1) is the constant function (as is the case for the thin-plate splines), then we can construct the function $\psi$ to have an arbitrary high decay rate $m_\phi$, and to reproduce $\Pi_k \cap \mathcal{P}$, with $\mathcal{P}$ the kernel of $G(D)$, and $k = m_\phi - d - 1$. It is further shown there that the corresponding sequence $\mu$ would then annihilate $\Pi_k \cap \mathcal{P}$. In such a case, Corollary 2.5.3 can be invoked to show that $L_A$ reproduces $\Pi_k \cap \mathcal{P}$, and, consequently, that $S_\Xi(\phi)$ contains $\Pi \cap \mathcal{P}$.

### 3.4. Review and extensions of the scheme used in [BuDL]

As we already mentioned in the introduction, the reference [BuDL] discusses approximation schemes and approximation orders that are based on the scattered shifts of one basic function $\phi$. In this subsection we review the results of [BuDL] with the aid of the techniques and observations of the present paper and, in this course, extend those results in several directions.

The aim of [BuDL] is to extend the approximation orders on uniform grids of [DJLR] to non-uniform grids. While we did the same in the previous subsection, the error analysis we have used here and the one used in [BuDL] are entirely different. Furthermore, apparently, the approximation schemes used here and there seem to be very different, too. We will show, however, that the [BuDL] scheme can be viewed as a variation of a special case of the method we suggest here. In particular, implicitly, the approximation scheme of [BuDL] is engaged, too, with the approximation of uniform shifts by scattered shifts.

The setup in [BuDL] starts with a basis function $\phi$ that obeys the conditions of [DJLR] (some of these conditions are quoted in the previous subsection). Then, for each center $\xi \in \Xi$, a function of the form

$$\psi_\xi := \sum_{\eta\in\Xi} \mu_{\xi,\eta}\, \phi(\cdot - \eta),$$

with $(\mu_{\xi,\eta})_{\eta\in\Xi}$ a sequence of finite support (supported on centers which are close to $\xi$), is constructed. The functions $(\psi_\xi)_{\xi\in\Xi}$ are constructed such that they decay (uniformly) at $\infty$ at rates similar to those of $\psi$ of [DJLR], i.e., $O(|\cdot|^{-d-\ell-1})$ for some non-negative integer $\ell$. The approximation map $\widetilde{L}_A$ is then defined as

$$\widetilde{L}_A : f \mapsto \sum_{\xi\in\Xi} f(\xi)\psi_\xi.$$

A very careful choice of the coefficients $(\mu_{\xi,\eta})$ enables [BuDL] to extend the results of [DJLR] to the scattered case.

We make two preliminary observations concerning the difference between the map $\widetilde{L}_A$ of [BuDL] and the map $L_A$ here. First, the map $\widetilde{L}_A$ uses the values of $f$ at $\Xi$ only, while $L_A$ involves evaluation at $\mathbb{Z}^d$ of $\lambda * f$. Second, the localization sequence $\mu$ which is used in [DJLR] in order to construct the localization $\psi$ from the original $\phi$ (and which is used in the approach of the present paper in an essential way) is not invoked directly in [BuDL]. Instead, it is imitated in the sequences $(\mu_{\xi,\eta})_{\eta\in\Xi}$, $\xi \in \Xi$.

The problem of selecting appropriate coefficients $(\mu_{\xi,\eta})$ is approached in [BuDL] as follows. First, it assumes the existence of a matrix

$$A = (A(\alpha,\xi))_{\alpha\in\mathbb{Z}^d,\,\xi\in\Xi}$$

that satisfies the following three conditions:

**Assumptions 3.4.1.**

(a) *There exists a positive constant $c_1$ such that $A(\alpha,\xi)=0$, whenever $|\alpha-\xi|>c_1$.*

(b) *The sequence set $\{A(\alpha,\xi)_{\xi\in\Xi}:\ \alpha\in\mathbb{Z}^d\}$, lies in $\ell_1(\Xi)$ and is bounded there.*

(c) *For some positive integer $m$, and for every $\alpha\in\mathbb{Z}^d$, $\sum_{\xi\in\Xi}A(\alpha,\xi)\,\delta_\xi=\delta_\alpha$ on $\Pi_m$, i.e., $\sum_{\xi\in\Xi}A(\alpha,\xi)\,p(\xi)=p(\alpha)$, for every $p\in\Pi_m$.*

Note that the above assumptions imply conditions (a-b) of Theorem 2.7.1. Indeed, most of the subsequent results here that assume Assumptions 3.4.1, are valid under weaker assumptions (such as (a-b) of Theorem 2.7.1).
The value of $m$ required in [BuDL] is $m=k'+\ell$, with $\ell$ as above, and $k'$ the degree of $G$ in (3.3.1).

The coefficients $\mathrm{M}:=(\mu_{\xi,\eta})$ are then defined by

$$\mu_{\xi,\eta}:=\sum_{\alpha,\beta\in\mathbb{Z}^d}A(\alpha,\xi)\,A(\beta,\eta)\,\langle N(\cdot-\alpha),N(\cdot-\beta)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is a certain bilinear-form that depends on $\phi$, and where $N$ is a sufficiently smooth compactly supported multivariate spline which reproduces $\Pi_{k'+\ell}$ (i.e., $\sum_{\alpha\in\mathbb{Z}^d}N(\cdot-\alpha)\,p(\alpha)=p$, $\forall p\in\Pi_{k'+\ell}$), (cf. [BuDL] for the full details). With $\mathcal{N}$ being the Gramian matrix

$$\mathcal{N}:=(\nu(\alpha-\beta))_{\alpha,\beta}:=(\langle N(\cdot-\alpha),N(\cdot-\beta)\rangle), \tag{3.4.2}$$

we see that

$$\mathrm{M}=A^T\mathcal{N}A.$$

Furthermore, it is proved in [BuDL] that the sequence $\nu$ is a suitable localization sequence for the [DJLR] construction.

The crucial properties of M required in [BuDL] are:

(i) $\sum_{\eta\in\Xi}\mu_{\xi,\eta}\,p(\eta)=0$, for every $p\in\Pi_{k'+\ell}\cap\mathcal{P}$, and every $\xi\in\Xi$, where $\mathcal{P}$ is the kernel of the differential operator $G(D)$, with $G$ the denominator of $\widehat{\phi}$ (see (3.3.1)).

(ii) $\sum_{\xi\in\Xi}\mu_{\xi,\eta}\,p(\xi)=0$, for every $p\in\Pi_{\min\{\ell,k'-1\}}$, and every $\eta\in\Xi$.

The [BuDL] analysis insists on a symmetric $\mu$, since, in that way, condition (ii) above is implied by (i), as $G$ is a homogeneous polynomial of degree $k'$.
From our factorization $A^T\mathcal{N}A$ of M, one can observe that (i) and (ii) hold with $\mathcal{N}=(\mu(\alpha-\beta))_{\alpha,\beta}$, for *any* localization sequence $\mu$ that satisfies the [DJLR] requirements, and not just for the specific $\nu$, (3.4.2), that is employed by [BuDL].

In order to see how the [BuDL]-scheme is related to the approach of the present paper, we denote

$$\Psi:=\psi_\Xi=\{\psi_\xi\}_\xi,\quad \Phi:={}_\Xi(\phi)=\{\phi(\cdot-\xi):\ \xi\in\Xi\}.$$

Then, we may write

$$\Psi=A^T\mathcal{N}A\,\Phi.$$

Also, since $\nu$ can serve as a localization sequence of [DJLR], it follows that

$$\psi:=\sum_{\alpha\in\mathbb{Z}^d}\nu(\alpha)\phi(\cdot-\alpha)$$

decays at $\infty$ like $O(|\cdot|^{-(d+\ell+1)})$. Therefore, we see that the Localization Conditions 2.1.2 are satisfied for the choice $\mu:=\nu$.

Given, now, an approximand $f$, we may write the approximant $\widetilde{L}_A f$ that is provided by [BuDL] in matrix form as

$$\widetilde{L}_A f = (f_{|\Xi})^T A^T \mathcal{N} A\,\Phi,$$

with $f_{|\Xi}$ the restriction of $f$ to $\Xi$, treated as a column vector. On the other hand, the approximation map $L_A$ of the present paper, when written in matrix form, too, reads, in case the convolution $\lambda *$ is the identity, as

$$L_A f = (f_{|_{\mathbb{Z}^d}})^T \mathcal{N} A\,\Phi,$$

since $A\,\Phi$ are exactly our pseudo-shifts $(\phi_\alpha)_\alpha$, and $\mathcal{N} A\,\Phi$ are our pseudo-shifts $(\psi_\alpha)_\alpha$.

In summary, while our map $L_A$ is of the form

$$L_A f = \sum_{\alpha\in\mathbb{Z}^d} f(\alpha)\psi_\alpha,$$

the [BuDL] map, written with the aid of our $(\psi_\alpha)$, is of the form

$$\widetilde{L}_A f = \sum_{\alpha\in\mathbb{Z}^d} (Af_{|\Xi})(\alpha)\psi_\alpha.$$

Consequently, as soon as we can show that $Af_{|\Xi}$ provides a "reasonable" approximation to $f_{|_{\mathbb{Z}^d}}$, we will be able to derive the [BuDL] approximation results directly from the results of the present paper. Since the results of the present paper are derived under assumptions which are substantially more general than the ones of [BuDL], we will obtain in such a way an extension of the ideas and the results of that paper beyond their stated limits.

The fact that $(Af_{|\Xi})(\alpha)$ approximates $f(\alpha)$ and in the "right" way, can be established in a far more general setup than the one discussed above, as we now describe. First, we adopt Assumptions 3.4.1, with a general integer $m$ whose desired values will be clarified in the sequel. Next, let $f\in C^{m+1}(\mathbb{R}^d)$, and $\alpha\in\mathbb{Z}^d$. Let $T_\alpha f$ be the Taylor polynomial of degree $m$ of $f$ about $\alpha$. By Assumption 3.4.1 (c), $A(p_{|\Xi}) = p_{|_{\mathbb{Z}^d}}$ for every $p\in\Pi_m$, and therefore

$$A((T_\alpha f)_{|\Xi})(\alpha) = (T_\alpha f)(\alpha) = f(\alpha).$$

This implies that, in view of Assumptions 3.4.1 (a,b) on $A$,

$$\begin{aligned}|(Af_{|\Xi})(\alpha) - f(\alpha)| =& |\sum_{|\alpha-\xi|\le c_1} A(\alpha,\xi)(f(\xi) - (T_\alpha f)(\xi))|\\ \le& \mathrm{const}_{m,c_1}\|A(\alpha,\cdot)\|_{\ell_1(\Xi)}|f|_{\infty,m+1} \le \mathrm{const}\,|f|_{\infty,m+1}\end{aligned}$$

Consequently, we obtain

**Lemma 3.4.3.** *For $f\in C^{m+1}(\mathbb{R}^d)$, under Assumptions 3.4.1, Localization Conditions 2.1.2, and Central Condition 2.2.5,*

$$|(\widetilde{L}_A - L_A)(\sigma_h f)| \le \mathrm{const}\,h^{m+1}|f|_{\infty,m+1}.$$

**Proof:** From the arguments that precede the lemma, we conclude that

$$|(\widetilde{L}_A - L_A)(\sigma_h f)| \le \mathrm{const}\,h^{m+1}|f|_{\infty,m+1}\sum_{\alpha\in\mathbb{Z}^d}|\psi_\alpha(x)| \le \mathrm{const}\,h^{m+1}|f|_{\infty,m+1},$$

where, in the second inequality, we used the fact that $\sum_{\alpha\in\mathbb{Z}^d}|\psi_\alpha|\in L_\infty(\mathbb{R}^d)$, which, thanks to our assumptions here, follows from Lemma 2.2.6. □

Therefore, in order to extend the approximation order results concerning the map $L_A$ to the map $\widetilde{L}_A$ we need the number $m+1$ of Lemma 3.4.3 to match or exceed the orders for $\|(L_A - L)(\sigma_h f)\|$ derived in the theorems of section 2. This can be guaranteed if we take $m$ as the largest integer smaller than the order of the singularity $\widehat{\phi}$ has at the origin. For example, the following is the extension of Theorem 2.2.16 to the map $\widetilde{L}_A$:

**Corollary 3.4.4.** *Assume that conditions (a-d) of Theorem 2.2.16 hold. Assume further that the matrix $A$ satisfies Assumptions 3.4.1, with $m$ taken to be $k$ of Theorem 2.2.16. Then, the assertions of that theorem remain valid with respect to the map $\widetilde{L}_A$ defined above.*

**Proof:** Since all the approximation orders asserted in Theorem 2.2.16 with respect to $L_A$ are $\le k+1$, the present assertion follows from Lemma 3.4.3. □

**Remark.** With $k, k'$ as in Theorem 2.2.16, if $k' \notin \mathbb{Z}_+$, then Condition 2.2.5 (that is, condition (d) in Theorem 2.2.16), which is the crucial condition exploited in §2, becomes redundant in the above corollary, under mild conditions on $\phi$. This follows by an application of Corollary 2.7.4: conditions (a) and (b) of that corollary are a consequence of Assumptions 3.4.1, and therefore, if $\phi$ satisfies assumptions (c-e) of that corollary, we obtain that $m_A$ of Condition 2.2.5 can be taken as $m - k' + d + 1 = k - k' + d + 1 > d$, since $k - k' > -1$, as $k' \notin \mathbb{Z}_+$. In the other case, when $k'$ *is* an integer, we obtain a similar result if $m$ of Assumption (c) of 3.4.1 is taken as $k'$.

We show now, that the approximation order results of [BuDL] follow from our results in §3.3 and §3.4. Indeed, for *bounded* approximands this is obtained directly from Theorem 3.3.4 and Corollary 3.4.4. As for *unbounded* approximands, the reasoning goes as follows. First, by the discussion in the last paragraph of §3.3, our scattered analog $L_A$ of the [DJLR] $L$ provides to functions with bounded $\ell$- and $\ell+1$-order derivatives the desired approximation orders, provided that $m_A$ of Condition 2.2.5 is $> d + \ell$. In particular, $L_A$ is well-defined on $N_\ell$, and, adopting (a) and (b) of Assumptions 3.4.1, we easily conclude that $\widetilde{L}_A$ is well-defined on $N_\ell$, as well. Thus, by Lemma 3.4.3, $\widetilde{L}_A$ provides the desired approximation orders as long as $m$ is at least $\ell$. In summary, the [BuDL]-scheme approximates unbounded functions to the desired rates, as soon as (i): $m_A > d + \ell$, (ii) $m$ of Assumptions 3.4.1 is at least $\ell$.

We can obtain (i) above from Assumptions 3.4.1 by invoking Corollary 2.7.4, which applies to all $\phi$ considered in [DJLR] for the choice $\rho_0 := F(0)/G$ (with $F, G$ as in (3.3.1)). Condition (i) follows from that corollary if $m - k' + d + 1 > d + \ell$, i.e., if $m \ge k' + \ell$. Such condition covers the required (ii), as well. Hence, the choice $m := k' + \ell$ in Assumptions 3.4.1 (which indeed is the value of $m$ used in [BuDL]) suffices to reproduce the [BuDL] results.

We have just shown that Corollary 3.4.4 reproduces the results of [BuDL]. In fact, it extends those results quite substantially. First, it applies to a much larger family of basis functions $\phi$, and in particular, the order of the singularity of $\widehat{\phi}$ need not be an even integer. Second, neither do we need the localization sequence $\mu$ to be the specific one employed in [BuDL], nor we even need $\mu$ to be finitely supported. Third, though we did not take advantage of that option, the matrix $A$ that is used to define $(\phi_\alpha)_\alpha$ need not be identical to the matrix $A$ used in the expression $(Af_{|\Xi})(\alpha)$ (they only need to satisfy, both, Assumptions 3.4.1 with $m = k' + \ell$, and $m = \ell$ respectively). Also, for bounded approximands, requirement (c) in Assumptions 3.4.1 is weaker in the last corollary, as compared to the analogous requirement in [BuDL] (we need $m = \ell$ instead of $m = k' + \ell$).

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