A smooth and local interpolant with "small" k–th derivative

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1. Introduction. For nondecreasing $\mathbf{t} := (t_i)_1^{n+k}$ and sufficiently smooth f, denote by

$$f|_{\mathbf{t}} := (f_i)$$

the corresponding sequence given by the rule

$$f_i := f^{(j)}(t_i) \quad \text{with} \quad j := j(i) := \max\{m \mid t_{i-m} = t_i\}.$$

We will write "$f = g$ on $\mathbf{t}$", or, "f and g agree on $\mathbf{t}$" in case $f|_{\mathbf{t}} = g|_{\mathbf{t}}$. Assuming that $\operatorname{ran}\mathbf{t} \subseteq [a,b]$ and that $t_i < t_{i+k}$, all i, $f|_{\mathbf{t}}$ is defined for every f in the Sobolev space

$$L_p^{(k)}[a,b] := \{f \in C^{(k-1)}[a,b] \mid f^{(k-1)} \quad \text{abs.cont.}; \quad f^{(k)} \in L_p[a,b]\}.$$

In order to demonstrate that the number

$$K(k) := \sup_{f_0,\mathbf{t}} \frac{\inf\{\|f^{(k)}\|_\infty \mid f \in L_\infty^{(k)}, \quad f|_{\mathbf{t}} = f_0|_{\mathbf{t}}\}}{\max_i k!|[t_i,\ldots,t_{i+k}]f_0|} \tag{1}$$

is finite (with $[t_i,\ldots,t_{i+k}]g$ the k–th divided difference of g at the points $t_i,\ldots,t_{i+k}$), Favard [5] constructs, for each $\mathbf{t}$, a map $P_{\mathbf{t}}$ with the property that $P_{\mathbf{t}}f$ agrees with f on $\mathbf{t}$ while

$$\|(P_{\mathbf{t}}f)^{(k)}\|_\infty \le \text{const}_k \max_i |[t_i,\ldots,t_{i+k}]f|, \quad \text{all } f \in L_\infty^{(k)}$$

for some const_k depending only on k. But, Farvard's $P_{\mathbf{t}}$ can actually be shown to satisfy the following:

(i) $P_{\mathbf{t}} : L_\infty^{(k)} \to L_\infty^{(k)}$ is a linear projector of rank $n+k$ with $P_{\mathbf{t}}f = f$ on $\mathbf{t}$, all f.

(ii) For some constant C_k depending on k but not on $\mathbf{t}$ or n, and for all j,

$$\|(P_{\mathbf{t}}f)^{(k)}\|_{\infty,(t_j,t_{j+1})} \le C_k \max_{i\le j<i+k} k!|[t_i,\ldots,t_{i+k}]f|.$$

Hence, Farvard's construction can be used to demonstrate the finiteness of

$$K_0(k) := \inf\{C_k \mid C_k \text{ satisfies (i) and (ii)}\}. \tag{2}$$

Farvard shows that $K(2) = K_0(2) = 2$, but gives no quantitative information about $K_0(k)$ or $K(k)$ for $k > 2$.

A different construction, in [3], provides the explicit upper bound

$$K_0(k) \le k^2(2k+1)(2k-1)^{k-1} \tag{3}$$

which, already for $k = 5$, gives a uselessly large bound, i.e., $K_0(5) \le 1,804,275$. This is to be compared with the lower bound

$$K_0(k) \ge K(k) \ge \gamma_k := (\pi/2)^{k+1} / \sum_{j=-\infty}^{\infty} (-1/(2j+1))^{k+1}$$

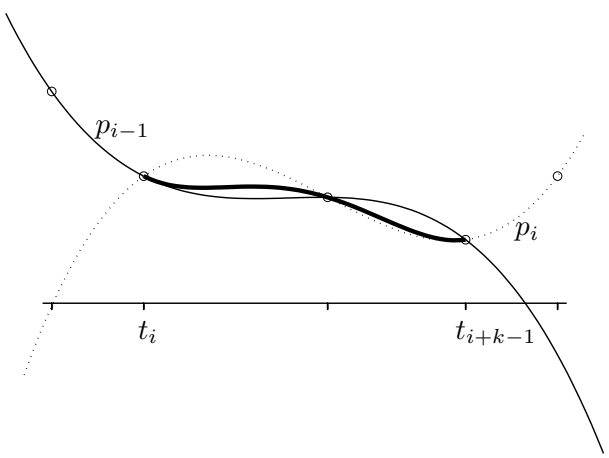
also proved in [3], giving, e.g., the lower bound $K_0(5) \geq 7.5$.

It is relatively easy to estimate Favard's C_k numerically, but the resulting bounds for $K_0(k)$ are not much better than those obtained from (3). A simple modification does improve the estimate somewhat, giving, e.g., $K_0(5) \leq 1,730$. In terms of Farvard's construction as described in [3], the modification consists in choosing, in Step 4, the break points for the piecewise constant function g_i *not* equally spaced but as the zeroes of the appropriate Chebyshev polynomial.

It is the purpose of this note to describe a more effective modification of Farvard's construction, resulting, e.g., in the computed bound $K_0(5) \leq 21.04$. In addition, the construction is described in a simpler way which makes its localness obvious. Finally, following up an idea of D.J.Newman [7], it is then possible to prove that

$$K_0(k) \leq (k-1)9^k. \tag{4}$$

The author's interest in these questions was sparked by work of H.-O. Kreiss reported in these Proceedings [6].



The construction of q from p_{i-1} and p_i for $k = 3$.

2. A modification of Farvard's construction. To recall, with p_i the polynomial of degree $\leq k$ which agrees with f_0 at $t_i, \ldots, t_{i+k}$, Farvard's construction consists in blending the n polynomials $p_1, \ldots, p_n$ together smoothly and without increasing the k–th derivative very much. Farvard carries out the transition from p_{i-1} to p_i over a largest subinterval (t_j, t_{j+1}) in (t_i, t_{i+k-1}). Our modification consists in carrying out this transition from p_{i-1} to p_i over the entire interval (t_i, t_{i+k-1}).

For this, consider the problem of constructing a function $q \in L_\infty^{(k)}$ for which

$$q = \begin{cases} p_{i-1} & \text{on} \quad t < t_i, \\ f_0(= p_{i-1} = p_i) & \text{on} \quad t_i, \ldots, t_{i+k-1}, \\ p_i & \text{on} \quad t > t_{i+k-1}. \end{cases}$$

Since

$$p_i - p_{i-1} = \alpha_i \psi_i$$

with

$$\psi_i(t) := (t - t_i) \cdots (t - t_{i+k-1}),$$
$$\alpha_i := ([t_i, \ldots, t_{i+k}] - [t_{i-1}, \ldots, t_{i+k-1}])f_0,$$

we can describe q equivalently as being of the form

$$q = p_{i-1} + \alpha_i h_i,$$

where h_i is any particular element of the class H_i consisting of those $h \in L_\infty^{(k)}$ for which

$$h = \begin{cases} 0 & \text{on} \quad t < t_i, \\ 0 = \psi_i & \text{on} \quad t_i, \ldots, t_{i+k-1}, \\ \psi_i & \text{on} \quad t > t_{i+k-1}. \end{cases}$$

For any $h_i \in H_i$, we have

$$([t_j, \ldots, t_{j+k}] - [t_{j-1}, \ldots, t_{j+k-1}])h_i = \delta_{ij}$$

since each such h_i agrees with 0 on $(t_r)_{r<i+k}$ and agrees with the monic k–th degree polynomial ψ_i on $(t_r)_{r\geq i}$. Since h_i agrees with 0 on $t_1, \ldots, t_{k+1}$ (for $i > 1$), the function

$$f := p_1 + \sum_{i=2}^{n} \alpha_i h_i \tag{5}$$

therefore agrees with p_1 on $t_1, \ldots, t_{k+1}$ and has the same k–th divided differences on points of $\mathbf{t}$ as does f_0, hence f and f_0 agree on $\mathbf{t}$. In fact, on (t_j, t_{j+1}),

$$\begin{aligned} f &= p_1 + \sum_{i \leq j+1-k} \alpha_i h_i + \sum_{i=j+2-k}^{j} \alpha_i h_i \\ &= p_{\max\{1, j+1-k\}} + \sum_{i \leq j < i+k} \alpha_i h_i \end{aligned}$$

since, on (t_j, t_{j+1}), $\alpha_i h_i = \alpha_i \psi_i = p_i - p_{i-1}$ for $i \leq j+1-k$ while $\alpha_i h_i = 0$ therefore $i > j$. Hence, f is a *local* interpolant to f_0, with f on (t_j, t_{j+1}) depending only on $p_{j+1-k}, \ldots, p_j$. In particular,

$$\begin{aligned} \|f^{(k)}\|_{\infty,(t_j,t_{j+1})} &\leq |p_{j+1-k}^{(k)}| + \sum_{i \leq j < i+k} |(p_i^{(k)} - p_{i-1}^{(k)})/k!| \|h_i^{(k)}\|_{\infty,(t_i,t_{i+k-1})} \\ &\leq (1 + 2(k-1) \max_i \|h_i^{(k)}\|_{\infty,(t_i,t_{i+k-1})}/k!) \max_{i \leq j < i+k} k!|[t_i, \ldots, t_{i+k}]f_0| \end{aligned} \tag{6}$$

We conclude that *each choice of* $h_i \in H_i$, $i = 2, \ldots, n$, *gives rise via* (5) *to a map* $P : f_0 \mapsto f$ *which is a linear projector on* $L_\infty^{(k)}$, *produces* Pf_0 *which agrees with* f_0 *on* $\mathbf{t}$, *and satisfies*

$$\|(Pf_0)^{(k)}\|_{\infty,(t_j,t_{j+1})} \leq C_{k,\mathbf{t},(h_i)} \max_{i \leq j < i+k} k!|[t_i, \ldots, t_{i+k}]f_0| \tag{7}$$

all j, *with*

$$C_{k,\mathbf{t},(h_i)} := 1 + 2(k-1) \max_i \|h_i^{(k)}\|_{\infty,(t_i,t_{i+k-1})}/k!. \tag{8}$$

3. The minimization of $C_{k,\mathbf{t},(h_i)}$ with respect to (h_i) is a *local* matter entirely as it involves the minimization of $\|h^{(k)}\|_{\infty,(t_i,t_{i+k-1})}$ over all $h \in H_i$ for each i separately. After a linear change of variables which takes (t_i, t_{i+k-1}) into $(0,1)$, the problem is one of minimizing $\|h^{(k)}\|_\infty/k!$ over all $h \in L_\infty^{(k)}[0,1]$ which satisfy

$$\begin{aligned} & h^{(j)}(0^+) = 0, \quad j = 0, \ldots, k-1 \\ & h \quad \text{agrees with} \quad \psi \quad \text{on} \quad \tau_0, \ldots, \tau_{k-1} \\ & h^{(j)}(1^-) = \psi^{(j)}(1^-), \quad j = 0, \ldots, k-1 \end{aligned} \tag{9}$$

for a certain $0 = \tau_0 \le \cdots \le \tau_{k-1} = 1$ and with

$$\psi(t) := (t - \tau_0) \cdots (t - \tau_{k-1}).$$

Denote the collection of all such h by $H_{\boldsymbol{\tau}}$ and set

$$\text{const}_{\boldsymbol{\tau}} := \inf_{h \in H_{\boldsymbol{\tau}}} \|h^{(k)}\|_\infty/k!.$$

Then, from the previous section,

$$K_0(k) \le 1 + 2(k-1) \sup_{0<\tau_1<\cdots<\tau_{k-2}<1} \text{const}_{\boldsymbol{\tau}}. \tag{10}$$

Let $\boldsymbol{\sigma} := (\sigma_i)_1^{r+k}$ be the smallest extension of $\boldsymbol{\tau}$ to a nondecreasing sequence containing both 0 and 1 exactly k times. Then, since ψ vanishes at $\tau_0, \ldots, \tau_{k-1}$, we can describe $H_{\boldsymbol{\tau}}$ more simply as the collection of all $h \in L_\infty^{(k)}[0,1]$ which agree with ψ_+ at $\boldsymbol{\sigma}$, where

$$\psi_+(t) := \psi(t)(t - \widehat{t})_+^0$$

for some (entirely arbitrary) $\widehat{t} \in (0,1)$. Our task then becomes to construct a "best" interpolant h to ψ_+, i.e., to find among the functions agreeing with ψ_+ one which has smallest k–th derivative as measured in the max–norm. As elaborated upon in [4], the normalized k–th derivative $\widehat{g} := \widehat{h}^{(k)}/k!$ of such an interpolant provides (and is provided by) a norm–preserving extension to all of $L_1[0,1]$ for the linear functional λ given on

$$\$_{k,\boldsymbol{\sigma}} := \text{span}(M_{1,k}, \ldots, M_{r,k}) \subseteq L_1[0,1]$$

by the rule

$$\lambda : \$_{k,\boldsymbol{\sigma}} \to \mathbb{R} : \varphi \mapsto \int_0^1 \varphi(t) h^{(k)}(t) dt/k! \qquad (\text{any } h \in H_{\boldsymbol{\tau}}). \tag{11}$$

Here, $M_{i,k}$ is the B–spline of order k with knots $\sigma_i, \ldots, \sigma_{i+k}$, normalized to have unit integral. Equivalently, $M_{i,k}$ represents the k–th divided difference at the points $\sigma_i, \ldots, \sigma_{i+k}$ in the same sense that

$$k!\ [\sigma_i, \ldots, \sigma_{i+k}]f = \int_0^1 M_{i,k}(t) f^{(k)}(t) dt, \quad \text{all } f \in L_1^{(k)}[0,1].$$

It follows that

$$\text{const}_{\boldsymbol{\tau}} = \|\lambda\| = \sup_{\varphi \in \$_{k,\boldsymbol{\sigma}}} \lambda\varphi/\|\varphi\|_1 \tag{12}$$

while

$$M_{i,k} = [\sigma_i, \dots, \sigma_{i+k}]\psi_+, \quad \text{all } i.$$

Now let σ_m be the entry of $\boldsymbol{\sigma}$ corresponding to τ_0 when $\boldsymbol{\tau}$ was extended to $\boldsymbol{\sigma}$. If $\tau_0 < \tau_1$, then $m = k$. More generally, m is such that $0 = \tau_0 = \cdots = \tau_{k-m} < \tau_{k-m+1}$. In any event, m is such that ψ_+ agrees with the monic polynomial ψ at $\sigma_i, \dots, \sigma_{i+k}$ for $i \geq m$ while ψ_+ agrees with 0 at $\sigma_i, \dots, \sigma_{i+k}$ for $i < m$. Hence

$$\lambda M_{i,k} = \begin{cases} 0, & i < m \\ 1, & i \geq m \end{cases} \tag{13}$$

and therefore

$$|\lambda \sum_i \alpha_i M_{i,k}| = |\sum_{i \geq m} \alpha_i| \leq \sum_i |\alpha_i| \leq D_k \| \sum_i \alpha_i M_{i,k} \|_1,$$

the last inequality valid, by the theorem in [1:Sec. 3], for some constant D_k depending only on k. Consequently, $\|\lambda\| \leq D_k$ and, combining this with (10) and (12), we get

$$K_0(k) \leq 1 + 2(k-1)D_k.$$

Unfortunately, the argument for the theorem in [1:Sec. 3] produces rather pessimistic bounds for D_k, as reflected in (3) above.

By contrast, D.J.Newman [7] gave the following very effective and simple argument for a bound on $K_0(k)$: Let

$$G(t) := \text{const} \int_0^t s^{k-1}(1-s)^{k-1} ds$$

with const $:= \frac{k}{2}\binom{2k}{k}$ so that $G(1) = 1$. Then

$$h(t) := G(t)\psi(t)$$

agrees with ψ_+ at $\boldsymbol{\sigma}$, hence

$$K_0(k) \leq 1 + 2(k-1)\|h^{(k)}\|_\infty / k!.$$

On the other hand, h is a polynomial of degree $3k-1$, hence

$$\|h^{(k)}\|_\infty / k! \leq T_{3k-1}^{(k)}(1) \|h\|_\infty 2^k / k!$$

by Markov's inequality, with T_{3k-1} the Chebyshev polynomial of degree $3k-1$. But

$$\|h\|_\infty \leq 1$$

since $G(t)$ increases monotonely from 0 to 1 as t goes from 0 to 1 while $\psi(t)$ on $[0,1]$ is a product of k factors all ≤ 1 in absolute value. Further,

$$\begin{aligned} T_{3k-1}^{(k)}(1) 4^{-k} / k! &\leq \sum_{j=0}^{3k-1} T_{3k-1}^{(j)}(1) 4^{-j} / j! \\ &= T_{3k-1}(5/4) = (2^{3k-1} + 2^{-(3k-1)})/2 \end{aligned}$$

therefore

$$\|h^{(k)}\|_\infty / k! < 8^k 2^{3k-1} < 64^k$$

or

$$K_0(k) = 0(64^k),$$

showing $K_0(k)$ to grow only exponentially with k.

Newman's argument can be refined as follows: Choose G, more generally, of the form

$$G(t) := \int_0^t g(s)ds$$

with g any function in $L_\infty^{(k-1)}[0,1]$ having a $(k-1)$ fold zero both at 0 and at 1 and such that $G(1) = 1$. By Leibniz' formula,

$$h^{(k)} = \sum_{i=0}^{k} \binom{k}{i} \psi^{(i)} G^{(k-i)}$$

while

$$\|\psi^{(i)}\|_\infty \leq k(k-1)\cdots(k-i+1)$$

and

$$G^{(k-i)}(t) = \int_0^t (t-s)^{i-1} G^{(k)}(s)ds/(i-1)!.$$

But $G^{(k)} = g^{(k-1)}$ is orthogonal to $\mathcal{P}_{k-1}$ on $[0,1]$ since

$$g^{(j)}(0) = g^{(j)}(1) = 0, \quad j = 0, \ldots, k-2,$$

by choice of g, therefore

$$G^{(k-i)}(t) = \int_0^1 [(t-s)_+^{i-1} - p(t,s)] G^{(k)}(s)ds/(i-1)!$$

with $p(t,.)$ an arbitrary element of $\mathcal{P}_{k-1}$. Choose, in particular, $p(t,.)$ to be the polynomial of degree $< i-1$ which agrees with $(t-.)_+^{i-1}$ at certain points $s_1, \ldots, s_{i-1}$. Then

$$|(t-s)_+^{i-1} - p(t,s)| \leq \prod_{j=1}^{i-1} |s - s_j|$$

while, by [9;2.9.31],

$$\min_{s_1,\ldots,s_{i-1}\in[0,1]} \int_0^1 \prod_{j=1}^{i-1} |s - s_j| ds = 4^{-i+1}.$$

Therefore

$$\|G^{(k-i)}\|_\infty \leq \|G^{(k)}\|_\infty 4^{-(i-1)}/(i-1)!.$$

Finally, by a theorem due to R.Louboutin (see [9; p.8]), among the functions $G \in L^{(k)}[0,1]$ having a k–fold zero at 0 and a k–fold one at 1, $\|G^{(k)}\|_\infty$ is uniquely minimized by the function

$$\widehat{G}(t) := \int_0^t M(s)ds$$

with M the B–spline of order k, normalized to have unit 1–norm and with the $k+1$ knots $(1-\cos(\pi j/k))/2$, $j=0,\ldots,k$. The minimum value is therefore $\|\widehat{G}^{(k)}\|_\infty = 2^{2k-2}(k-1)!$. With this choice $G=\widehat{G}$, we then get

$$\begin{aligned}
\|h^{(k)}\|_\infty/k! &\le \frac{(k-1)!}{k!}\Big(2^{2k-2}+\sum_{i=1}^{k}\binom{k}{i}\frac{k\cdots(k-i+1)}{(i-1)!}2^{2(k-i)}\Big)\\
&= 2^{2k-2}/k+\sum_{i=1}^{k}\binom{k}{i}\binom{k-1}{i-1}2^{2(k-i)}\\
&< 2^{2k-2}/k+\Big[\sum_{i=0}^{k}\binom{k}{i}2^i-2^k\Big]\sum_{i=1}^{k}\binom{k-1}{i-1}2^{i-1}\\
&= 2^{2k-2}/k+(3^k-2^k)3^{k-1}\\
&< 9^k/3-1/(2k-2).
\end{aligned}$$

Hence, finally we get

$$K_0(k) < (k-1)9^k \tag{4}$$

as mentioned in the introduction.

4. The explicit calculation of $\|\lambda\|$ seems to be the key to more realistic bounds for $K_0(k)$, at least for small k.

To begin with, one might try to compute $\|\lambda\|$ simply by maximizing $\lambda\varphi$ over the unit sphere $\{\varphi\in\$_{k,\boldsymbol{\sigma}} \mid \|\varphi\|_1=1\}$ in $\$_{k,\boldsymbol{\sigma}}$. This means, of course, finding an extremal for λ, i.e., a $\chi\in\$_{k,\boldsymbol{\sigma}}$ such that $\|\chi\|_1=1$ and $\lambda\chi=\|\lambda\|$. Unfortunately, the equivalent constrained maximization problem in $\mathbb{R}^r$ "Maximize $\sum_{i\ge m}\alpha_i$ over $S:=\{\boldsymbol{\alpha}\in\mathbb{R}^r \mid \|\sum_i \alpha_i M_{i,k}\|_1=1\}$" is not easily solved by standard techniques since S is only piecewise smooth. In any event, such computations result, strictly speaking, only in *lower* bounds for $\|\lambda\|$.

It seems more appropriate to compute *upper* bounds, by going back to the original problem of finding g with smallest possible sup–form for which $\int g\varphi=\lambda\varphi$, all $\varphi\in\$_{k,\boldsymbol{\sigma}}$, i.e., to the problem of finding norm preserving extensions for λ.

Lemma 1. *There exists exactly one norm preserving extension of λ to a linear functional $\widehat{\lambda}$ on all of $L_1[0,1]$. This extension is given by the rule*

$$\widehat{\lambda}\varphi=\int\widehat{g}\varphi,\quad \text{all } \varphi\in L_1,$$

with

$$\widehat{g}=\|\lambda\|\,\mathrm{signum}\,\chi$$

and χ any extremal for λ. In particular, $\widehat{g}$ is absolutely constant and has fewer than $r=\dim\$_{k,\boldsymbol{\sigma}}$ jumps.

Proof: We claim that

$$\|\lambda\|>1. \tag{14}$$

For, if not, then with $\widehat{g} \in L_\infty[0,1]$ a norm preserving extension of λ to all of $L_1[0,1]$, we would have

$$1 = \lambda M_{m,k} = \int_0^1 \widehat{g} M_{m,k} \le \|\widehat{g}\|_\infty \|M_{m,k}\|_1 = \|\lambda\| \cdot 1 \le 1,$$

therefore equality would hold in Hölder's inequality, hence, as $M_{m,k} > 0$ a.e. on $[0,1]$, $\widehat{g} = 1$ would follow, and so, with (13),

$$0 = \lambda M_{m-1,k} = \int \widehat{g} M_{m-1,k} = \int M_{m-1,k} = 1,$$

a contradiction.

Let $\chi = \sum_i \alpha_i M_{i,k}$ be an extremal for λ, i.e.,

$$\chi \in \$_{k,\boldsymbol{\sigma}}, \quad \|\chi\|_1 = 1, \quad \lambda\chi = \|\lambda\|.$$

Then, from (14),

$$\begin{aligned} \sum_{i<m} \alpha_i &= \sum_i \alpha_i - \lambda\chi \\ &= \int \chi - \|\lambda\| \\ &\le 1 - \|\lambda\| < 0 \end{aligned}$$

therefore $\alpha_i \ne 0$ for some $i < m$. But this implies that

$$\operatorname{supp} \chi = [0,1].$$

For, otherwise χ would vanish on (σ_{i-1}, σ_i) for some $i > k$ with $\sigma_{i-1} < \sigma_i$. Then $\alpha_{i-k} = \cdots = \alpha_{i-1} = 0$ and

$$\|\chi\|_1 = \int_0^{\sigma_{i-1}} |\sum_{j<i-k} \alpha_j M_{j,k}| + \int_{\sigma_i}^1 |\sum_{j\ge i} \alpha_j M_{j,k}|$$

while $\sum_{j<i-k} \alpha_j M_{j,k} \in \ker \lambda$, hence $\sum_{j<i-k} \alpha_j M_{j,k} = 0$ (since otherwise $\|\sum_{j\ge i} \alpha_j M_{j,k}\|_1 < \|\chi\|_1$ while $\lambda \sum_{j\ge i} \alpha_j M_{j,k} = \lambda\chi$, contradicting the fact that χ is an extremal for λ), hence then $\alpha_1 = \cdots = \alpha_{i-1} = 0$ for some $i > k$, contradicting the fact that $\alpha_j \ne 0$ for some $j < m$.

If now $\widehat{g}$ is any norm preserving extension of λ to all of $L_1[0,1]$, – (there exists at least one by the Hahn–Banach theorem), – i.e., if $\widehat{g} \in L_\infty[0,1]$ with $\|\widehat{g}\|_\infty = \|\lambda\|$ and $\lambda\varphi = \int \widehat{g}\varphi$, all $\varphi \in \$_{k,\boldsymbol{\sigma}}$, then, in particular,

$$\|\lambda\| = \lambda\chi = \int \widehat{g}\chi \le \|\widehat{g}\|_\infty \|\chi\|_1 = \|\widehat{g}\|_\infty = \|\lambda\|,$$

hence equality must hold in Hölder's inequality, therefore, as $\operatorname{supp} \chi = [0,1]$,

$$\widehat{g} = \|\widehat{g}\|_\infty \operatorname{signum} \chi$$

follows. This shows that $\widehat{g}$ is uniquely determined by χ. In particular, $\widehat{g}$ is absolutely constant. Further, $\widehat{g}$ changes sign only when χ does, while χ, as a linear combination of r B–splines, can change sign at most $r-1$ times. Q.E.D.

Lemma 1 suggests that we represent λ by a piecewise constant function g in such a way that $|g|$ is constant. If we succeed in constructing such a g, we may have found $\widehat{g}$, and therefore know $\|\lambda\|$. Such a g can only be found as the limit of some iterative process. The next lemma asserts that every iterate in such a process is apt to carry useful information about $\|\lambda\|$.

Lemma 2. *Let g be a piecewise constant function,*

$$g(t) = \beta_j \quad \text{on} \quad (\rho_{j-1}, \rho_j), \quad j = 1, \ldots, u,$$

for some sequence $(\beta_j)_1^u$ and some sequence $(\rho_j)_0^u$ with $0 = \rho_0 < \cdots < \rho_u = 1$. If g represents λ, i.e., if $\int g\varphi = \lambda\varphi$, all $\varphi \in \$_{k,\boldsymbol{\sigma}}$, and g has fewer than r sign changes, then

$$\min_j |\beta_j| \le \|\lambda\| \le \max_j |\beta_j|. \tag{15}$$

Proof: Only the first inequality requires proof, and this only in the case when $\min_j |\beta_j| > 1$, since $\|\lambda\| > 1$ by (14). Hence, assume that $\min_j |\beta_j| > 1$ and let $(v_j)_1^{s-1}$ be the points at which g changes sign. Then $s \le r$, by assumption. Further,

$$M_{i,k}(v_i) \ne 0, \quad i = 1, \ldots, m-1. \tag{16}$$

For, if (by way of contradiction) $M_{i,k}(v_i) \ne 0$ for $i = 1, \ldots, j-1$, but $M_{j,k}(v_j) = 0$ for some $j < m$, then one could find a nonzero $\varphi \in \operatorname{span}(M_{1,k}, \ldots, M_{j,k}) \subseteq \ker \lambda$ which changes sign only at $v_1, \ldots, v_{j-1}$, has $\operatorname{signum} \varphi = \operatorname{signum} g$ on $(0, v_1)$ and vanishes for $t \ge v_j$. But then,

$$0 = \lambda\varphi = \int_0^1 g(t)\varphi(t)dt = \int_0^{v_j} g\varphi \ge \min_{i \le j} |\beta_i| \int_0^1 |\varphi| > 0,$$

a contradiction. Further, since

$$\int_0^1 (1-g) \sum_j \alpha_j M_{j,k} = \sum_{j<m} \alpha_j$$

while $1-g$, like g, changes sign only at $(v_j)_1^{s-1}$, – (a consequence of our assumption that $\min_j |\beta_j| > 1$), – it follows similarly that

$$M_{r-i,k}(v_{s-1-i}) \ne 0, \quad i = 0, \ldots, r-m,$$

hence that

$$M_{i,k}(v_i) \ne 0, \quad i = m, \ldots, s-1, \tag{17}$$

since $\operatorname{supp} M_{i,k} \supseteq \operatorname{supp} M_{j,k}$ for $m \le i \le j$. Because of (16) and (17), we can therefore find $\varphi \in \operatorname{span}(M_{1,k}, \ldots, M_{s,k}) \subseteq \$_{k,\boldsymbol{\sigma}}$ which changes sign only at $v_1, \ldots, v_{s-1}$ and has the same sign as g in $(0, v_1)$. But then

$$\lambda\varphi = \int g\varphi \ge \min_j |\beta_j| \int |\varphi|$$

which proves that $\|\lambda\| \ge \min_j |\beta_j|$ since $\|\varphi\|_1 \ne 0$, by construction. Q.E.D.

Corollary. *If g is absolutely constant with fewer than r jumps and represents λ, then $g = \widehat{g}$ and $\|g\|_\infty = \|\lambda\|$.*

Consider now the problem of computing a piecewise constant representer g with s steps (i.e., $s-1$ breakpoints) for λ. For this g to be useful in bracketing $\|\lambda\|$, it should have $< r$ sign changes. This can be insured by choosing $s \le r$. On the other hand, once the $s-1$ breakpoints are picked, we have only s linear parameters available for matching λ on the r–dimensional space $\$_{k,\boldsymbol{\sigma}}$, hence s must be at least as big as r. For these reasons, we choose $s = r$, i.e.,

$$g(t) = \beta_j \quad \text{on} \quad (\rho_{j-1}, \rho_j), \quad j = 1, \ldots, r$$

with $0 = \rho_0 < \cdots < \rho_r = 1$, and determine $\boldsymbol{\beta}$ from the linear system

$$\sum_{j=1}^{r} \beta_j \int_{\rho_{j-1}}^{\rho_j} M_{i,k} = \begin{cases} 0, & i < m \\ 1, & i \geq m \end{cases}, \quad i = 1, \ldots, r, \tag{18}$$

(see (13)).

It turns out to be more convenient to solve a slightly different, equivalent system. Let $N_{i,k+1}$ be a B–spline of order $k+1$, with knots at $\sigma_1, \ldots, \sigma_{i+k+1}$, normalized in a certain way. Explicitly,

$$\begin{aligned} N_{i,k+1}(t) &:= ((\sigma_{i+k+1} - \sigma_i)/(k+1)) M_{i,k+1}(t) \\ &= ([\sigma_{i+1}, \ldots, \sigma_{i+k+1}] - [\sigma_i, \ldots, \sigma_{i+k}])(\cdot - t)_+^k. \end{aligned}$$

Then $N_{i,k+1}^{(1)} = -(M_{i+1,k} - M_{i,k})$, hence

$$\int_{\rho_{j-1}}^{\rho_j} (M_{i,k} - M_{i+1,k}) = N_{i,k+1}(\rho_j) - N_{i,k+1}(\rho_{j-1}).$$

Since

$$\int_{\rho_{j-1}}^{\rho_j} M_{r+1,k} = 0, \quad j = 1, \ldots, r,$$

– here we have added an arbitrary $\sigma_{r+k+1} > 1$ to $\boldsymbol{\sigma}$, – it follows that (18) is equivalent to

$$A\boldsymbol{\beta} = \mathbf{b} \tag{19a}$$

with

$$A := (N_{i,k+1}(\rho_j) - N_{i,k+1}(\rho_{j-1}))_{i,j=1}^r \tag{19b}$$

$$b_i := \begin{cases} -1, & i = m-1 \\ 1, & i = r \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, \ldots, r. \tag{19c}$$

Note that $N_{i,k+1}(\rho_0) = 0$, all i, hence A is column–equivalent to $(N_{i,k+1}(\rho_j))_{i,j=1}^r$, therefore invertible iff $N_{i,k+1}(\rho_i) \neq 0$, all i, i.e., iff $\sigma_i < \rho_i < \sigma_{i+k+1}$, all i, a condition on $\boldsymbol{\rho}$ easily enforced.

This settles the determination of $\boldsymbol{\beta}$. Consider next the question of how to choose $\boldsymbol{\rho}$ so as to make the resulting g absolutely constant.

Lemma 3. *Let $0 = \rho_0 < \cdots < \rho_r = 1$ be such that*

$$\begin{aligned} &N_{i-1,k+1}(\rho_i) \neq 0, \quad \text{i.e.,} \quad \rho_i < \sigma_{i+k}, \quad i = 2, \ldots, m-1, \\ &N_{i,k+1}(\rho_{i-1}) \neq 0, \quad \text{i.e.,} \quad \sigma_i < \rho_{i-1}, \quad i = m, \ldots, r. \end{aligned} \tag{20}$$

Then also $N_{i,k}(\rho_i) \neq 0$, $i = 1, \ldots, r$, hence (19) has a unique solution $\boldsymbol{\beta}$. This solution satisfies

$$(-)^{m+i}(\beta_i - \beta_{i-1}) > 0, \quad i = 2, \ldots, r.$$

Proof: By (19), the r–vector

$$\boldsymbol{\beta}' := (\beta_1 - \beta_2, \beta_2 - \beta_3, \ldots, \beta_{r-1} - \beta_r, \beta_r)$$

is the solution of

$$B\boldsymbol{\beta}' = \mathbf{b}$$

with

$$B := (N_{i,k+1}(\rho_j))_{i,j=1}^r$$

and $\mathbf{b} = (b_i)$ given by (19c). Therefore, $\boldsymbol{\beta}' = -\boldsymbol{\gamma}^{(m-1)} + \boldsymbol{\gamma}^{(r)}$, with $\boldsymbol{\gamma}^{(j)}$ the j–th column of B^{-1}. Further, since $N_{i,k+1}(\rho_r) = \delta_{ir}$, all i, the last column of B, and therefore also $\boldsymbol{\gamma}^{(r)}$, equals the unit vector with r–th entry equal to 1. Consequently,

$$\beta_i - \beta_{i-1} = \gamma_{i-1}^{(m-1)}, \quad i = 2, \ldots, r.$$

But $\gamma_{i-1}^{(m-1)}$, as the $(i-1, m-1)$–entry of B^{-1}, is given by

$$\gamma_{i-1}^{(m-1)} = (-)^{i+m} \det B_{(m-1,i-1)} / \det B,$$

with $B_{(r,s)}$ the matrix obtained from B by deleting row r and column s. Conditions (20) insure that $B_{(m-1,i-1)}$ has all diagonal entries nonzero which, by a slight extension [2;Theorem 2] of the wellknown fact that B is totally positive, implies that $\det B_{(m-1,i-1)} > 0$, $i = 2, \ldots, r$. Q.E.D.

Since $\det B_{(m-1,i-1)} = 0$ iff one of its diagonal entries is zero, it is now possible to describe the exact circumstances under which $\beta_i = \beta_{i-1}$, i.e., under which g has no jump at ρ_i. More importantly, we have the

Corollary 1. *The unique norm preserving extension $\widehat{g}$ for λ has exactly $r-1$ sign changes.*

Proof: Let $(v_j)_1^{s-1}$ be the increasing sequence of points at which $\widehat{g}$ changes sign. Then $s \le r$, by Lemma 1, and, by the proof for Lemma 2, (16) and (17) must hold. We can therefore extend $(v_j)_1^{s-1}$ to an increasing sequence $(\rho_j)_0^r$ with $\rho_0 = 0$ and $\rho_r = 1$ so that (20) holds, while $g = \beta_j$ on (ρ_{j-1}, ρ_j), $j = 1, \ldots, r$, for some absolutely constant $\boldsymbol{\beta}$. But then $\boldsymbol{\beta}$ satisfies (19), hence $\beta_i \ne \beta_{i-1}$, by Lemma 3, showing that $\widehat{g}$ must change sign at ρ_i, $i = 1, \ldots, r-1$. Q.E.D.

It follows that λ has exactly one extremal. Also, for the record,

Corollary 2. *The function $F(h) := \|h^{(k)}\|_\infty / k!$ discussed in Section 3 has exactly one minimum in $H_{\boldsymbol{\tau}}$ (see (9)). The minimum is a perfect spline of order $k+1$ with $r-1$ interior knots.*

Proof: The minimum is the unique $h \in H_{\boldsymbol{\tau}}$ with $h^{(k)} = k!\widehat{g}$. Q.E.D.

It follows that $\widehat{g}$, i.e., $\boldsymbol{\rho}$ and $\boldsymbol{\beta}$ for $\widehat{g}$, is the unique solution of the system (19a–c) together with the equations

$$\beta_i + \beta_{i-1} = 0, \quad i = 2, \ldots, r. \tag{19d}$$

For, $\widehat{g}$ certainly solves this system, while any solution to this system must give $\widehat{g}$, by the Corollary to Lemma 2.

We attempt to solve (19a–d) for the unknowns $\boldsymbol{\rho}$ and $\boldsymbol{\beta}$ by Newton's method. With $\boldsymbol{\beta}$ determined from (19a–c) for given $\boldsymbol{\rho}$, we compute the desired changes $\delta\rho_i$, $i = 1, \ldots, r-1$, from the condition that

$$\sum_{j=1}^{r-1} \left(\frac{\partial A}{\partial \rho_j} \delta\rho_j\right) \boldsymbol{\beta} = -A(c\boldsymbol{\varepsilon} - \boldsymbol{\beta})$$

where $\boldsymbol{\varepsilon} := (-1, +1, -1, \ldots)$. This gives

$$\begin{aligned} \delta\rho_i &= y_i/(\beta_i - \beta_{i+1}), \quad i = 1, \ldots, r-1, \\ c &= y_r \end{aligned} \tag{21a}$$

with $\mathbf{y}$ the solution of the linear system

$$C\mathbf{y} = \mathbf{b} \tag{21b}$$

where

$$C := \left(N^{(1)}_{i,k+1}(\rho_1) \,\vdots\, \cdots \,\vdots\, N^{(1)}_{i,k+1}(\rho_{r-1}) \,\vdots\, (A\boldsymbol{\varepsilon})_i \right)_{i=1}^{r}. \tag{21c}$$

5. The maximization of $\|\lambda\| = \mathrm{const}_{\boldsymbol{\tau}}$ over $\boldsymbol{\tau}$ is our final goal since, by (10) and (12),

$$K_0(k) \le 1 + 2(k-1) \sup_{0<\tau_1<\cdots<\tau_{k-2}<1} \mathrm{const}_{\boldsymbol{\tau}}.$$

For this, we calculated $\mathrm{const}_{\boldsymbol{\tau}}$, – a number between 1 and 37 for $k \le 10$, – to within an absolute error of .005 at a large number of points $(\tau_1, \ldots, \tau_{k-2})$ on

$$T_k := \{(\tau_1, \ldots, \tau_{k-2}) \mid 0 \le \tau_1 \le \cdots \le \tau_{k-2} \le 1\}$$

and for $k = 3, 4, 5$, using Newton's method as described in the previous section.

$\mathrm{const}_{\boldsymbol{\tau}}$ can be shown to be continuous on T_k and $k-1$ times differentiable in the interior of T_k, but does not appear to be convex. In view of the fact that Newton's method is only as good as the initial guess, it seemed most efficient to evaluate $\mathrm{const}_{\boldsymbol{\tau}}$ along rays, starting at the point $\tau_1 = \cdots = \tau_{k-2} = 1/2$ and using the $r = 2k-2$ Chebyshev points as the initial guess for $\rho_1, \ldots, \rho_r$, and then proceeding along the ray towards the boundary, using the previously computed $\boldsymbol{\rho}$ as the initial guess in the next step.

Details of these computations together with the `Fortran` program used can be found in the Mathematics Research Center TSR #1466.

For $k = 3, 4, 5$, we found the maximum of $\mathrm{const}_{\boldsymbol{\tau}}$ to occur at one of the vertices of T_k. Assuming this to be true for all k, we merely maximized $\mathrm{const}_{\boldsymbol{\tau}}$ for $k = 6, \ldots, 10$ over the vertices of T_k (and the rays leading from the midpoint to these vertices). The resulting upper bounds for $K_0(k)$ are listed in the table below together with the lower bounds for $K(k)$ obtained in [3]. The upper bounds seem to behave like c^k for some c slightly larger than 2, while the lower bounds are known to behave like $(\pi/2)^k$.

k	$\le K(k) \le$	$K_0(k) \le$
2	2	3.414
3	3	6.854
4	4.8	11.665
5	7.5	21.036
6	11.8	42.330
7	18.5	79.276
8	29.1	163.344
9	45.7	316.792
10	71.8	664.020

References

1. C. de Boor, The quasi-interpolant as a tool in elementary polynomial spline theory, *in* "Approximation Theory" (G. G. Lorentz *et al.* Eds.), pp. 269–276, Academic Press, New York, 1973.
2. C. de Boor, Total positivity of the spline collocation matrix, *Indiana Univ. Math. J.* **25** (1976), 541–551.
3. C. de Boor, How small can one make the derivatives of an interpolating function?, *J. Approx. Theory* **13** (1975), 105–116.
4. C. de Boor, On 'best' interpolation, *J. Approx. Theory* **16** (1976), 28–42.
5. J. Favard, Sur l'interpolation, *J. Math. Pures Appl.* **19** (1940), 281–306.
6. H.-O. Kreiss, Difference approximations for singular perturbation problems, *in* "Numerical Solutions of Boundary Value Problems for Ordinary Differential Equations" (A. Aziz Ed.), pp. xxx–xxx, Academic Press, New York, 1975.
7. Donald Newman, Private Communication, MRC, 1975.
8. I. J. Schoenberg, "Cardinal Spline Interpolation", CBMS, SIAM, Philadelphia, 1973.
9. A. F. Timan, "Theory of Approximation of Functions of a Real Variable", Pergamon Press, MacMillan, New York, 1963.