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Smooth Refinable Functions Provide Good Approximation Orders

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ABSTRACT

We apply the general theory of approximation orders of shift-invariant spaces of [BDR1-3] to the special case when the finitely many generators $\Phi \subset L_2(\mathbb{R}^d)$ of the underlying space S satisfy an N-scale relation (i.e., they form a "father wavelet" set). We show that the approximation orders provided by such finitely generated shift-invariant spaces are bounded from below by the smoothness class of each $\psi \in S$ (in particular, each $\phi \in \Phi$), as well as by the decay rate of its Fourier transform. In fact, similar results are valid for refinable shift-invariant spaces that are *not* finitely generated.

Specifically, it is shown that, under some mild technical conditions on the scaling functions Φ, approximation order k is provided if either some $\psi \in S$ lies in the Sobolev space W_2^{k-1}, or its Fourier transform $\widehat{\psi}(w)$ decays near ∞ like $o(|w|^{1-k})$. No technical side-conditions are required if the spatial dimension is $d = 1$, and the functions in Φ are compactly supported.

For the special case of a singleton Φ, our first class of results (that are concerned with the condition $\phi \in W_2^{k-1}$) improve previously known results of Yves Meyer and of Cavaretta-Dahmen-Micchelli.

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1. Introduction and statement of the main results

We consider in this article the problem of determining *the approximation orders of refinable shift-invariant subspaces* of $L_2 := L_2(\mathbb{R}^d)$, $d \geq 1$. By definition, a subspace $S \subset L_2$ is **shift-invariant** (SI) if it is invariant under all **shifts** i.e., integer translations. We discuss only SI spaces that are *closed* (in L_2). The shift-invariant space S is usually defined with the aid of a *generating set* $\Phi \subset L_2$: we say that Φ **generates** S (and write $S = S(\Phi)$) if S is the smallest (closed) SI space that contains Φ. A **finitely generated SI** (FSI) space is a space $S(\Phi)$ generated by a *finite* Φ, and a **Principal SI** (PSI) space $S(\phi)$ is a space generated by a singleton ϕ. PSI and FSI spaces play a role in the theory and applications of multivariate splines, radial basis function approximation, sampling theory, wavelets and uniform subdivision schemes. The setup and problem addressed in the present paper is particularly relevant to the two latter areas.

In all the above mentioned applications, the SI space serves as a potential source of approximants. In this regard, then, it becomes important to analyse its "approximation power", preferably in terms of properties of its given generators. One convenient quantitative measurement of this "approximation power" (and the most standard one) is via the notion of *approximation orders*, defined with respect to a *ladder* of spaces. In the present context, the simplest ladder associated with S is the following *stationary* one.

Definition . The **stationary SI ladder generated by** Φ is the directed family

$$\mathcal{S} := \mathcal{S}(\Phi) := (S_h := \sigma_h S(\Phi))_{h>0},$$

with $S := S(\Phi)$ the SI space generated by Φ, and with σ_h the dilation operator

$$\sigma_h : f \mapsto f(\cdot/h).$$

Note that S_h is "spanned" by the $h\mathbb{Z}^d$-shifts of the dilated functions $\sigma_h\Phi$. (The adjective "stationary" refers here to the fact that finer spaces S_h are obtained from S_1 by dilation. Non-stationary ladders are obtained if one allows each S_h to be spanned by the $h\mathbb{Z}^d$-shifts of some functions $\Phi_h \neq \sigma_h\Phi$.)

Definition . Let $\mathcal{S} = (S_h)_h$ be an SI ladder generated by $\Phi \subset L_2$. We say that $\mathcal{S}$ **provides approximation order** k to the function space $F \subset W_2^k$, if, for every $f \in F$,

$$\mathrm{dist}(f, S_h) = O(h^k),$$

with "dist" being the usual L_2-distance between a function and a function set. If the stronger assumption

$$\mathrm{dist}(f, S_h) = o(h^k)$$

holds, we say that $\mathcal{S}$ *provides density order* k.

Here and hereafter, W_2^k, with k positive, is the usual Sobolev/potential space; i.e., if k is an integer, this is the space of all L_2-functions whose weak derivatives up to order k inclusive are in L_2,

The literature concerning L_2- (and, more generally, L_p-) approximation orders of PSI, FSI and general SI spaces is vast, and reviewing that literature to any extent in not within the scope of the present paper. We refer the reader to the introductions and bibliography of the papers [BR], [BDR1-3] and [Jo1,2].

The SI ladders that are employed in the context of (the multiresolution approximation approach to) wavelets satisfy an additional important property of refinability:

Definition . An SI ladder $\mathcal{S} = (S_h)_h$ is **refinable** (or, more explicitly N-refinable) if, for some $N > 1$, and for every integer j,

$$S_{N^{-j}} \subset S_{N^{-j-1}}.$$

Note that, if the ladder is *stationary*, refinability is implied by the single relation $S_1 \subset S_{1/N}$, and implies the relation $S_h \subset S_{h/N}$ for all h.

While, in general, the approximation orders of, say stationary, SI ladders $\mathcal{S}(\Phi)$ are unrelated to the smoothness of the generator(s) Φ, it is a known phenomenon that such a relation exists for certain stationary *refinable* PSI ladders. (It is worth mentioning that, in the case of a univariate refinable PSI ladder, approximation orders can sometimes be equivalently described in terms of the vanishing moment conditions of the corresponding wavelet). Results along these lines were proved by people interested in constructing wavelets via multiresolution, as well as by people studying uniform subdivision schemes.

As an example for the former, the following result can be obtained by combining Theorem 2.4 of [M] with the quasi-interpolation argument. (Warning: The more explicit result of [M], Theorem 2.6, cannot imply the full approximation order asserted below.)

Result 1.4 (Meyer, [M]). *Let k be a positive integer, and $\phi \in W_2^{k-1}$. Assume that all derivatives of ϕ of order $< k$ are bounded and rapidly decaying. If the shifts of ϕ are L_2-stable, and if the stationary PSI ladder $\mathcal{S}$ generated by ϕ is 2-refinable, then $\mathcal{S}$ provides approximation order k for W_2^k.*

In the context of subdivision, the following result was established by Cavaretta, Dahmen and Micchelli.

Result 1.5 ([CDM]). *Assume that ϕ is a compactly supported function in $C^{k-1}(\mathbb{R}^d)$, that $\mathcal{S}(\phi)$ is 2-refinable, that the refinement mask of ϕ is finite, and that the underlying subdivision scheme converges uniformly. Then, $\mathcal{S}(\phi)$ provides approximation order k for all sufficiently smooth functions.*

We note that the above C^{k-1}-assumption on ϕ appears to be too restrictive. For example, refinable polynomial B-splines as well as refinable polynomial box splines that are not in C^{k-1} (but are in C^{k-2}) provide approximation order k (cf. [BHR] for details). In any event, both results are restricted to PSI spaces, and impose fast decay rates on the generator ϕ, together with some kind of stability assumption on its shifts. (We have not defined the notions of L_2-stability or the uniform convergence of subdivision schemes. We do note that the former implies the latter and both imply that $\widehat{\phi}(0) \neq 0$.) We add here that Theorem 2.4 of [JM] can be combined with Theorem 1.15 of [BDR1], to show that, for the specific value $k = 1$, Result 1.4 is valid under very mild decay conditions on ϕ.

Note that all the above quoted results show that smoothness may imply approximation orders, and none addresses the converse statement. Indeed, there are various examples of high approximation orders of stationary refinable PSI ladders generated by functions of low smoothness (e.g., Daubechies' scaling functions, cf. [D]).

In the present paper we shall establish results concerning the connection between the existence of smooth functions in the refinable S, and the corresponding approximation order provided by the ladder $\mathcal{S}$. These results improve their literature counterparts in several ways: First, and foremost, they apply to FSI and even to arbitrary SI spaces, while the above-stated results are confined to PSI spaces only. Second, they hardly require any stability or related assumption on the shifts of a generating set Φ, and make no assumptions about a possible direct relation between Φ and their dilates (recall that the CDM-result, for example, assumes the relevant mask to be finite). Third, the results apply to functions Φ that decay only mildly, in fact, the most general results here do not even mention generating sets. Fourth, the results do not require the *generator(s)* to be smooth, but only that S contains one smooth function. That latter difference is critically important for SI spaces which are not principal. Further, none of our results restricts the integer value of N in the refinement condition; some of the results do not even require N to be an integer. Finally, the results of this paper remain valid if, instead of assuming that S is refinable and contains smooth functions, we drop the refinability, and assume only that $\cap_{i=1}^{\infty} S_{1/N^i}$ contains such smooth function.

The techniques employed apply also to L_p-approximation orders, $p \neq 2$, as well as to non-stationary ladders. These extensions, however, will be discussed elsewhere.

We have just mentioned that our results require S to contain smooth functions. We actually use three different conditions to describe "smoothness", and only one of which is a truly smoothness condition. We refer to the other two as *pseudo-smoothness conditions*. The first assumption is that for some k, $W_2^k \cap S \neq 0$. The other two criteria are in terms of the decay of the Fourier transform $\widehat{f}$ of f. Precisely, for some small neighborhood B of the origin, our pseudo-smoothness conditions will require the existence of $f \in S$ whose corresponding sequence

$$\lambda_f(m) := \|1 - \frac{|\widehat{f}|^2}{\sum_{j\in 2\pi m\mathbb{Z}^d} |\widehat{f}(\cdot+j)|^2}\|_{L_1(B)}^{1/2}, \tag{1.6}$$

or

$$\widetilde{\lambda}_f(m) := \|1 - \frac{|\widehat{f}|^2}{\sum_{j\in 2\pi m\mathbb{Z}^d} |\widehat{f}(\cdot+j)|^2}\|_{L_\infty(B)}^{1/2}, \tag{1.7}$$

decays at a certain rate. In the above expressions, $0/0 := 0$.

Discussion. Under mild conditions (e.g., $|\widehat{f}| \geq c >$ a.e. around the origin), there exists a function $g \in S(f)$ such that the decay rate of λ_f at ∞ is the same as the decay rate of the sequence

$$\mu_g : m \mapsto \| \sum_{j \in 2\pi m \mathbb{Z}^d \backslash 0} |\widehat{g}(\cdot + j)|^2 \|_{L_1(B)}^{1/2}.$$

In particular, if $g \in W_2^k$, then $\mu_g(m) = o(m^{-k})$, and hence also $\lambda_f(m) = o(m^{-k})$. Similarly, if $\widehat{g}(w) = O(|w|^{-k})$, $k > d/2$ (as $|w| \to \infty$), then both $\lambda_f(m)$ and $\widetilde{\lambda}_f(m)$ are $O(m^{-k})$. The discussion, thus, explains the point in choosing the terminology "pseudo-smoothness".

The most general result proved in this paper is as follows:

Theorem 1.8. *Let $\mathcal{S} = (S_h)_h$ be a stationary SI ladder, and let k' be a positive number. Assume that the following conditions hold:*
(a) "Pseudo-refinability": For some integer N, $S_0 := \cap_{i=1}^{\infty} S_{1/N^i} \neq 0$.
(b) "Pseudo-smoothness": there exists $\psi \in S_0$ such that, with $\widetilde{\lambda}_\psi$ defined as in (1.7), $\widetilde{\lambda}_\psi(N^j) = O(N^{-jk'})$.
Then $\mathcal{S}$ provides density order k for W_2^k, for every $k < k'$.

The highlight in this result lies in the fact that its requirements are almost the mere two very basic ones: *pseudo-refinability* and *pseudo-smoothness*. The space S can be PSI, FSI, or arbitrary SI, and the ability to find a "good" generating set for this space is not an issue. The pseudo-smoothness assumption on ψ in the theorem is satisfied if $\widehat{\psi}$ decays at ∞ like $O(|\cdot|^{-k'})$, provided that $k' > d/2$, and that $1/\widehat{\psi}$ is essentially bounded around the origin. This latter "side-condition" (i.e., the essential boundedness of $1/\widehat{\psi}$) cannot be dispensed with: approximation orders of the refinable $\mathcal{S}$ to W_2^k are *not* implied by the mere existence of smooth functions ψ in the refinable S. Here is a simple example.

Example: a refinable analytic PSI space that provides 0 approximation order. Let S be the space of all univariate band-limited functions with band in $[0..2\pi]$. S is a PSI space, and all functions in S are analytic, hence smooth. Moreover, S contains an abundance of rapidly decaying functions. However, the existence of smooth rapidly decaying functions in S cannot be converted to positive assertions concerning approximation orders provided by S: while $\mathcal{S}$ provides very good approximation order to *some* smooth functions, it provides 0 approximation order to many others. Consequently, already in the PSI context, refinability and smoothness alone cannot ensure good approximation properties. Note that, indeed, all decaying functions (say, L_1) in S must have a zero mean value.

It is quite safe to conjecture that the SI ladder $\mathcal{S}$ in the theorem provides approximation order k', and not only density orders $k < k'$.

The weak aspect of Theorem 1.8 is that the concluded approximation order in this result is bounded above by the "pseudo-smoothness parameter" k' of ψ. In comparison, in all other results of this paper, the asserted approximation order will be obtained by "rounding-up" this parameter to the next integer. It is impossible, however, to achieve such results without further assumptions, for the simple reason that there exist refinable spaces whose corresponding (maximal) approximation order is fractional.

We are now ready to present additional selected results from the paper. When reading these subsequent results, it will be convenient to classify the conditions assumed in them as follows: (a) (pseudo-)refinability, (b) (pseudo-)smoothness, and (c) extra "side-conditions".

The first result is a special case of Theorem 3.2.

Corollary 1.9. *Let $\mathcal{S}(\Phi)$ be a stationary FSI ladder, and let k be a positive integer. Assume that:*

(a) $S(\Phi)$ is refinable: $S_1 \subset S_{1/N}$.

(b) $S(\Phi) \cap W_2^{k-1} \neq 0$.

(c) The (finite) generating set Φ satisfies the following three "side-conditions":

(c1) $|\phi(x)| = O(|x|^{-\rho})$ (as $x \to \infty$), for some $\rho > k + d$, and for every $\phi \in \Phi$;

(c2) $\widehat{\phi}(0) \neq 0$, for some $\phi \in \Phi$;

(c3) The functions $(\sum_{\alpha \in \mathbb{Z}^d} \phi(\cdot - \alpha))_{\phi \in \Phi}$ are linearly independent.

Then, $\mathcal{S}(\Phi)$ provides approximation order k for W_2^k.

The complementary result, that invokes a pseudo-smoothness assumption, is the following corollary of Theorem 3.9.

Corollary 1.10. *Let $\mathcal{S}(\Phi)$ be a stationary FSI ladder, and let k be a positive integer. Assume that:*

(a,c) As in Corollary 1.9, but N is assumed here to be an integer.

(b) For some $\psi \in S(\Phi)$, $\lambda_\psi(N^j) = o(N^{-j(k-1)})$, where λ_ψ is defined by (1.6).

Then, $\mathcal{S}(\Phi)$ provides approximation order k for W_2^k.

We remark that for a PSI space $S(\phi)$ condition (c3) is redundant, since it is implied by (c2): for a PSI $S(\phi)$, (c3) is violated if and only if $\sum_{\alpha \in \mathbb{Z}^d} \phi(\cdot - \alpha) = 0$, and it is well-known that that can happen only if $\widehat{\phi}(0) = 0$. Thus, for the PSI case, Corollaries 1.9 and 1.10 lead to the following result:

Corollary 1.11. *Let $S(\phi)$ be a PSI stationary ladder, and let k be a positive integer. Assume that:*

(a) $S(\phi)$ is refinable: $S_1 \subset S_{1/N}$.

(b) Either of the two conditions holds:

(b1) $S(\phi) \cap W_2^{k-1} \neq 0$;

(b2) N is an integer, and $\lambda_\phi(N^j) = o(N^{-j(k-1)})$.

(c) The generating function ϕ satisfies the following two "side-conditions":

(c1) $|\phi(x)| = O(|x|^{-\rho})$ (as $x \to \infty$), for some $\rho > k + d$;

(c2) $\widehat{\phi}(0) \neq 0$.

Then, $\mathcal{S}(\phi)$ provides approximation order k for W_2^k.

Discussion. Corollary 1.11 implies that a refinable PSI $\mathcal{S}(\phi)$ provides approximation order k in case $\phi \in W_2^{k-1}$, and the side-condition (c1,2) are met. A comparison of these side-conditions with those assumed in Results 1.4 and 1.5 shows that the corollary requires less decay of ϕ, and frees the shifts of ϕ almost completely from any "stability" requirements (other than the basic assumption $\widehat{\phi}(0) \neq 0$). In terms of its smoothness requirement, it assumes about the same as Result 1.4, and less than Result 1.5.

Remark. At a late stage, when susbstantial modifications of this paper became prohibitive, R.Q. Jia brought to my attention the fact that, while I quote here Theorem 8.3 of [CDM], there is a significantly stronger result there, viz., Theorem 8.4. Indeed, that theorem seems to be on par with the (b1) variant of Corollary 1.11. One should note that our general PSI result (Theorem 3.9+ Proposition 4.1) applies to generating functions of slow decay (such as the sinc-function).

Example: B-splines and box splines. The simplest example of a refinable ϕ is the univariate B-spline of order k. It is compactly supported and non-negative hence trivially satisfies condition (c) of Corollary 1.11. It is well-known that $\mathcal{S}(\phi)$ provides approximation order k. That result is indeed reproduced (twice) by Corollary 1.11: First, $\widehat{\phi}(w) = w^{-k}\tau(w)$, for a certain trigonometric polynomial τ, hence $\widehat{\phi} = o(|\cdot|^{-k+\varepsilon})$, for any $\varepsilon > 0$, hence the sequence λ_ϕ decays at that rate, too. Second, ϕ can be shown to lie in W_2^s, for any $s < k - 1/2$. The situation for box splines (cf. [BHR] for definition and details) is similar.

The B-spline example shows also the sharpness of the pseudo-smoothness condition (b2) in Corollary 1.11: the B-spline ϕ of order $k-1$, whose ladder does *not* provide approximation order k, satisfies the condition $\lambda_\phi(m) = O(m^{-(k-1)})$. Corollary 1.11 thus fails to hold if we change the small o in (b2) to big O.

Example: band-limited functions. Let S be the PSI space of all univariate L_2-functions whose Fourier transform is supported on $[-\pi \mathinner{..} \pi]$. That space is well-known to provide all positive approximation orders. Result 1.4 cannot reproduce this fact since S can be easily shown to contain no L_1-function ϕ whose shifts are L_2-stable. In contrast, the space contains an abundance of functions that decay rapidly together with all their derivatives and have non-zero mean value. Hence, either one of (b1) and (b2) of Corollary 1.9 can be activated to yield these known spectral orders of approximation. Moreover, we remark that our more general result, Theorem 3.9, requires only the existence of $\psi \in S$ whose Fourier transform is C^∞ around the origin and does not vanish there.

Our next result deals with *univariate FSI ladders generated by compactly supported functions*. This case, though being very special, is of much practical interest. The point of the theorem is that, in the univariate case, "compact support" is already a "sufficient side-condition".

Theorem 1.12. *Let $\mathcal{S}(\Phi) = (S_h)_h$ be a stationary FSI ladder, and let k be a positive integer. Assume that*

(a) For some $N > 1$, $S_0 := \cap_{i=1}^{\infty} S_{1/N^i} \neq 0$.

(b) Either of the two conditions holds:

(b1) $S_0 \cap W_2^{k-1} \neq 0$;

(b2) N is an integer, and there exists $\psi \in S_0$ such that the sequence λ_ψ defined in (1.6) satisfies $|\lambda_\psi(N^j)| = o(N^{-j(k-1)})$.

(c) The spatial dimension d is 1, and Φ are compactly supported.

Then $\mathcal{S}(\Phi)$ provides approximation order k for W_2^k.

Example: C^1-cubics. This example is taken from [HSS]. Let S be the space of all univariate piecewise-cubic polynomials with breakpoints at the integers, and which are globally C^1. This space is obviously N-refinable (for all integers N). Less obviously, but quite well-known, it is a local FSI space of length 2, i.e., it is generated by two compactly supported functions. It is fairly obvious that the approximation order here is 4 (the subspace of C^2-cubics already does the job). That order is recovered from Theorem 1.12 as soon as one realizes that the (smooth) B-spline of order 4 is in our space. In contrast, standard generating sets for this space consist of functions each of which neither lies in W_2^3, nor satisfying the alternative requirement, (b2), that appears in Theorem 1.12. Thus, it is very important that our results are stated in terms of the smoothness of some function *in the space*, and *not* in terms of the smoothness of some function *in the generating set*.

Discussion. The example given after Theorem 1.8 shows that "compact support" in Theorem 1.12 cannot be replaced by "rapid algebraic decay". The theorem, though, *is* extendible to exponentially decaying generators.

All results as stated aim at providing *lower* bounds on the approximation order of the ladder $\mathcal{S}(\Phi)$ in terms of either the smoothness of the "smoothest" function in $S(\Phi)$, or the decay of its Fourier transform. Such presentation stems from the typical problem in *Spline Theory*, where smoothness is a more readily available property than the approximation orders (cf. [BHR]). However, for more general refinable functions, the readily available information (viz., the mask) may appear to be more adequate for computing approximation orders than estimating either the smoothness, say, of $\phi \in \Phi$ or the decay of its Fourier transform. From this point of view, the results of this paper can be regarded as providing *upper* bounds on the possible smoothness of functions in $S(\Phi)$ in terms of the known approximation order of the underlying ladder. As an illustration, we state the following immediate corollary:

Corollary 1.13. *Let $S(\Phi)$ be a univariate refinable FSI space generated by compactly supported functions. Then no compactly supported $\psi \in S(\Phi)$ is infinitely many times continuously differentiable.*

Proof: Let $\psi \in S(\Phi)$ be non-zero, compactly supported, and C^∞. Then, $\psi \in W_2^k$, for *every* k. By Theorem 1.12, $\mathcal{S}$ then provides all positive approximation orders. By Theorem 5 of [J], the shifts of Φ must then span all polynomials, an absurdity in view of the fact that these shifts have finite local dimension. □

The argument extends to more than one variable. One only needs to adopt further conditions: if Φ is a singleton (ϕ), the additional condition is that $\widehat{\phi}(0) \neq 0$ (so that we will be able to apply Corollary 1.11). For a finite Φ, the conditions should be (c2,3) of Corollary 1.9.

As alluded to before, there is a tight connection between the approximation orders of the refinable $\mathcal{S}$ and the vanishing moments of the wavelets. Many of our results, thus, can be stated in terms of such vanishing moments. Here is one illustration.

Corollary . *Let $\mathcal{S} = (S_h)_h$ be a univariate stationary FSI ladder generated by compactly supported functions, and let k be a positive integer. Assume further that for some $N > 1$,*

$\cap_{i=1}^{\infty} S_{1/N^i} \cap W_2^{k-1} \neq 0$. *Let f be compactly supported. If $f \perp S$, then $\widehat{f}$ has a zero of order k at the origin.*

Proof: From Theorem 1.12 we conclude that $\mathcal{S}$ provides approximation order k. Theorem 4.1 of [BDR2] then applies to yield that for some compactly supported supported $\phi \in S$, the ladder $\mathcal{S}(\phi)$ provides approximation order k, too. Theorem 3.7 of [R2], when combined with Theorem 1.1 of [R1], allows us to assume without loss that $\widehat{\phi}$ does not have any 2π-periodic zero. On the other hand, Theorem 1.14 of [BDR1] implies that $\widehat{\phi}$ must have a zero of order k at each $j \in 2\pi\mathbb{Z}\backslash 0$. This forces $\widehat{\phi}(0)$ to be non-zero, and, by a standard argument, we may assume $\widehat{\phi} - 1$ to have a k-fold zero at the origin.

The rest of the argument is routine. Let f be a compactly supported L_2-function, and set

$$H := \sum_{j \in 2\pi\mathbb{Z}} \widehat{f}(\cdot + j)\overline{\widehat{\phi}(\cdot + j)}.$$

In general, the above sum is L_1-convergent. However, since f and ϕ are compactly supported, one can show that the above sum can be differentiated term-by-term, and that each such differentiated sum converges uniformly on compactly sets. In view of the properties of $\widehat{\phi}$, this implies that $H - \widehat{f}$ has a zero of order k at the origin. Assuming, in addition, that $f \perp S(\phi)$ (certainly true if $f \perp S$), Poisson's summation formula yields that $H = 0$, and hence that $\widehat{f}$, indeed, has a zero of order k at the origin. □

Finally, a remark concerning the proofs of the main results. Theorem 3.2 and its corollary follow as a strikingly simple consequence of the general theory of [BDR1]. A totally different (and somewhat tricky) argument is employed in the proofs of Theorems 1.8 and 3.9; that latter argument leads to further consequences which will be discussed elsewhere. The two approaches differ also from a conceptual point of view: in the first approach, the function $f \in S \cap W_2^{k-1}$ is *approximated by* the ladder $\mathcal{S}$. In the second approach, the pseudo-smooth function ψ whose Fourier transform decays nicely is used *to provide approximants* to other smooth functions.

The paper is organized as follows: in §2 we collect general results concerning approximation orders of stationary SI ladders. None of the results of §2 assume refinability, and all results should be considered "auxiliary" from the standpoint of the present paper. The three basic results of this paper: Theorem 1.8, Theorem 3.2 and Theorem 3.9, comprise §3. The latter theorems of §3 assume a new, initially obscure, property of the space S, the $H(k)$ property. The study of this property is done in §4, where we show that PSI and FSI spaces with "reasonably good" generating sets satisfy it (the results there do not rely on refinability, and refinability buys no extra benefit, hence the decision to separate this discussion from that of §3).

2. Background on the approximation order of stationary SI ladders

We collect in this section all known and new results on approximation from stationary SI ladders that are used as *auxiliary* results in the present paper. We emphasize that none of these results use the *refinability* assumption, which is the pillar assumption in the main results of this article.

The following function plays the key role in the determination of the approximation order of the stationary PSI ladder $\mathcal{S}(\phi)$:

$$\Lambda_\phi := \Big(1 - \frac{|\widehat{\phi}|^2}{\sum_{j\in 2\pi\mathbb{Z}^d} |\widehat{\phi}(\cdot + j)|^2}\Big)^{1/2}. \tag{2.1}$$

In this definition, $0/0 := 0$.

The basic observation of [BDR1] is the following:

(2.2) Theorem 2.20 of [BDR1]. *Let f be a function whose Fourier transform is supported in the cube C/h, $C := [-\pi \,..\, \pi]^d$. Let $\phi \in L_2$. Then,*

$$\operatorname{dist}(f, \sigma_h S(\phi)) = (2\pi)^{-d/2} \|\Lambda_\phi(h\cdot)\widehat{f}\|_{L_2(C/h)}.$$

Whenever $\widehat{f}$ is not supported on C/h, [BDR1] splits f into $f_1 + f_2$, with $\widehat{f_1}$ coinciding with $\widehat{f}$ on B/h, $B \subset C$, and is zero elsewhere. This leads (almost immediately) to the following estimates:

Corollary 2.3. *Let $\mathcal{S}(\phi) = (S_h)_h$ be a stationary PSI ladder, and let B be a small neighborhood of the origin. Then, for every $f \in L_2$ and every $h > 0$,*

$$\operatorname{dist}(f, S_h) \le (2\pi)^{-d/2}(\|\Lambda_\phi(h\cdot)\widehat{f}\|_{L_2(B/h)} + \|\widehat{f}\|_{L_2(\mathbb{R}^d\setminus(B/h))}).$$

In particular, if $f \in W_2^k$, then

$$\operatorname{dist}(f, S_h) = (2\pi)^{-d/2}\|\Lambda_\phi(h\cdot)\widehat{f}\|_{L_2(B/h)} + o(1)h^k\|f\|_{W_2^k},$$

with the $o(1)$ expression bounded independently of f.

From that, the following characterization is provided in [BDR1]:

(2.4) Theorem 1.6 of [BDR1]. *The following conditions are equivalent for any $k > 0$:*

(a) The function $|\cdot|^{-k}\Lambda_\phi$ is in $L_\infty(B)$ for some 0-neighborhood B.

(b) The stationary PSI ladder $\mathcal{S}(\phi) = (S_h)_h$ provides approximation order k in the strong sense that

$$\operatorname{dist}(f, S_h) \le \text{const}\|f\|_{W_2^k} h^k, \tag{2.5}$$

for some const *independent of f and h.*

Under some favorable conditions on the generator ϕ, one is able to derive from the last characterization simpler ones. One such simplification in contained in the following statement.

(2.6) Corollary 5.15 of [BDR1]. *Assume that ϕ satisfies assumption (c) of Corollary 1.11. Then $\mathcal{S}(\phi)$ provides approximation order k if and only if ϕ satisfies the Strang-Fix conditions of order k, i.e., $\widehat{\phi}$ has a zero of order k at each $j \in 2\pi\mathbb{Z}^d\backslash 0$.*

The treatment of the approximation orders of FSI and SI spaces here is exclusively based on the powerful "superfunction" theory, that surrounds the existence of a "superfunction" $\psi \in S$, i.e., a function ψ whose corresponding $\mathcal{S}(\psi)$ provides the same approximation order as the larger ladder $\mathcal{S}$. A literature overview of the superfunction results prior to 1991 can be found in §1 of [BDR1]. §3 of [BDR1], §4 of [BDR2] and the entire [BDR3] constitute the most advanced progress on this problem.

(2.7) The stationary case of the "superfunction" results of [BDR1-3]. *Let $\mathcal{S} = (S_h)_h$ be a stationary SI ladder, and let $k > 0$. Then for any $\chi \in L_2$, there exists a stationary PSI ladder $\mathcal{T} = (T_h)_h$ such that $T_1 \subset S_1$ and:*

(a) For every $f \in L_2$,

$$\operatorname{dist}(f, T_h) \leq \operatorname{dist}(f, S_h) + 2\operatorname{dist}(f, \sigma_h S(\chi)).$$

(b) If S is finitely generated by compactly supported functions, and χ is compactly supported, T is generated by some compactly supported function.

(c) If $\mathcal{S}(\chi)$ provides approximation order $k+1$ to W_2^{k+1}, then, for every $f \in W_2^k$,

$$\operatorname{dist}(f, T_h) \leq \operatorname{dist}(f, S_h) + \varepsilon_f(h) h^k \|f\|_{W_2^k},$$

with ε_f bounded independently of f and h, and decaying to 0 with h.

(d) If $\mathcal{S}$ provides approximation order k in the sense that

$$\operatorname{dist}(f, S_h) \leq \operatorname{const}\|f\|_{W_2^k} h^k, \quad \forall f \in W_2^k,$$

then it provides density order k' for every $f \in W_2^{k'}$, and every $k' < k$.

Proof: (a) follows from of Theorem 3.3 of [BDR1]. (b) is the content of Theorem 4.1 of [BDR2]. We prove (c,d) simultaneously as follows. First, the PSI case of (d) follows from a simple comparison of the characterization of approximation orders (Theorem 1.6 of [BDR1], see above) and the characterization of density orders (Theorem 1.7 of [BDR1]). Claim (c) then follows by an application of (d) to the PSI ladder $\mathcal{S}(\chi)$, with $k+1$ replacing k.

It remains then to show that (d) is valid for a general SI space: assuming $\mathcal{S}$ to satisfy the assumption in (d), we choose χ such that $\mathcal{S}(\chi)$ provides approximation order $k+1$, and we then let $\mathcal{T}$ be the PSI ladder of (a) with respect to the present χ. By (c), the ladder $\mathcal{T}$ provides approximation order k (in the sense required in (d)). Since $\mathcal{T}$ is principal, and (d) is known to be valid with respect to principal ladders, we conclude that $\mathcal{T}$ provides density orders $k' < k$, a fortiori this is true for the original ladder $\mathcal{S}$. □

Finally, the following result, which is needed for the proof of Theorem 1.8, cannot be found in the literature.

Theorem 2.8. *Let $\mathcal{S}$ be a stationary SI ladder, let $m > k > 0$, and let $(h_i)_i$ be decreasing to zero and satisfying*

$$h_i/h_{i+1} \le A < \infty, \quad \forall i.$$

Assume that, and for every $f \in W_2^m$,

$$\text{dist}(f, S_{h_i}) \le c h_i^k \|f\|_{W_2^m}, \tag{2.9}$$

with c independent of f (and h). Then $\mathcal{S}$ provides approximation order k for W_2^k, in the sense of (2.5).

Proof: Invoking (2.9) together with (c) of (2.7) (with k there being our m here), we find $\phi \in S$ such that

$$\text{dist}(f, \sigma_{h_i} S(\phi)) \le c h_i^k \|f\|_{W_2^m}.$$

This reduces the problem to the PSI ladder $\mathcal{S}(\phi)$, since $\mathcal{S}$ certainly provides approximation order k the moment $\mathcal{S}(\phi)$ does so. Therefore, without loss, we may assume that $S = S(\phi)$, i.e., that our ladder is principal.

We now invoke Corollary 2.3 (with k there being our m here), to conclude that (2.9) implies that

$$\|\Lambda_\phi(h_i \cdot)\widehat{f}\|_{L_2(B/h_i)} \le c h_i^k \|f\|_{W_2^m}.$$

As f varies over W_2^m, $(1 + |\cdot|)^{2m} \widehat{f}^2$ varies over L_1, and we may then convert the last inequality to

$$\|(1 + |\cdot|)^{-2m} \Lambda_\phi(h_i \cdot)^2 f\|_{L_1(B/h_i)} \le c h_i^{2k} \|f\|_{L_1}, \quad \forall f \in L_1.$$

This implies the estimate

$$h_i^{2m} \|(h_i + |\cdot|)^{-2m} \Lambda_\phi^2\|_{L_\infty(B)} = \|(1 + |\cdot|)^{-2m} \Lambda_\phi(h_i \cdot)^2\|_{L_\infty(B/h_i)} \le c h_i^{2k}.$$

Assuming $h_{i+1} \le |x| \le h_i$, we obtain that

$$\Lambda_\phi(x)^2 \le c' h_i^{2k} \le c' A^{2k} |x|^{2k}.$$

Thus, the function $|\cdot|^{-k} \Lambda_\phi$ was proved to be bounded around the origin, which, in view of Theorem 1.6 of [BDR1] (cf. (2.4)) implies that $\mathcal{S}$ provides approximation order k to all the functions in W_2^k. $\square$

3. Core

Let S be an SI space. The main results of this paper are based on the presumption that the ladder $\mathcal{S}$ can either provide approximation order k to "almost no" smooth functions, or to "almost all" smooth functions. The most extreme statement of this nature is contained in the following definition.

Definition . Let $\mathcal{S} = (S_h)_h$ be a stationary (not necessarily refinable) SI ladder. Let k be a positive integer. We say that $\mathcal{S}$ has the **Property H(k)** if it provides approximation k for the entire W_2^k, whenever the following condition holds: "there exists a function $f \in W_2^{k-1} \backslash 0$ such that, for some sequence $(h_i)_i$ that decreases to zero,

$$\operatorname{dist}(f, S_h) = o(h^{k-1}), \ \ h = h_1, h_2, ...''$$

In the language of [BDR1], the satisfaction of Property H(k) implies that $\mathcal{S}$ provides *approximation* order k to all reasonably smooth functions as soon as it provides a *density* order $k-1$ to a single *smooth* non-zero function.

The main branch of our approach can be now clearly stated: we will show that "smoothness implies approximation orders" for SI ladders $\mathcal{S}$ that are *refinable* and *satisfy the* H(k) *property.* The two theorems that establish this fact are stated and proved in the present section, and apply to an arbitrary SI ladder (i.e., may not be an FSI one).

Of course, such results may be deemed useless unless we are able to find feasible side-conditions on S that guarantee the satisfaction of Property H(k). This complementary study is independent of the refinability or smoothness assumptions, and is even independent of the possible approximation order provided by $\mathcal{S}$. In fact, we show that some "mild technical conditions" (which are known to neither imply nor being implied by, nor related in any rigorous way to approximation orders) guarantee the satisfaction of the Property H(k). *However, this complementary analysis does require our space S to be a PSI or FSI space.*

Our first observation is, actually, trivial.

Theorem 3.2. *Let k be a positive integer, and let $\mathcal{S} = (S_h)_h$ be a stationary SI ladder. Assume further, that*
(a) For some $N > 1$, $\cap_{i=1}^{\infty} S_{1/N^i} \cap W_2^{k-1} \neq 0$.
(b) $\mathcal{S}$ has the Property $H(k)$.
Then the ladder $\mathcal{S}$ provides approximation order k for W_2^k.

Proof: Let f be a non-zero function in $\cap_{i=1}^{\infty} S_{1/N^i} \cap W_2^{k-1}$. Setting

$$V_i := S_{N^{-i}}, \quad \forall i \geq 0,$$

we know that $f \in V_i$, $\forall i \geq 0$. This means that $\operatorname{dist}(f, V_i) = 0$, for all $i \geq 0$. Thus, we may invoke Property H(k) with respect to this f, to conclude that $\mathcal{S}$ provides approximation order k to W_2^k, as asserted. □

The two other theorems of this section, Theorem 1.8 and Theorem 3.9, are very similar each to the other: not only in their pseudo-smoothness assumptions, but also in their proofs. However, they differ quite significantly in their conclusions, hence are entitled for the separate labeling.

Proof of Theorem 1.8. Let $k'' < k'$, and let $k > k''$ be any number of the form $k = \frac{r}{r+1}k'$, r integer. We will prove that there exists a sequence $(h_i = a^i)_i$ $(a < 1)$, and a constant c, such that, for every $f \in W_2^{(r+1)k}$,

$$\operatorname{dist}(f, S_h) \le c\|f\|_{W_2^{(r+1)k}} h^k, \quad h = h_1, h_2, \dots . \tag{3.3}$$

Theorem 2.8 would then yield that $\mathcal{S}$ provides approximation order k to W_2^k, and (d) of (2.7) would then complete the proof.

In what follows, we set $\widetilde{\lambda} := \widetilde{\lambda}_\psi$ (cf. (1.7)) and prove (3.3). However, since the method here will be needed, with a slight twist, in the proof of Theorem 3.9, we formalize it in a form of a separate lemma.

Lemma 3.4. *Let $\mathcal{S}$ be a stationary ladder, and let $\psi \in \cap_{i=1}^{\infty} S_{1/N^i}\backslash 0$, N a positive integer. Define the sequences $\lambda := \lambda_\psi$ and $\widetilde{\lambda} := \widetilde{\lambda}_\psi$ as in (1.6) and (1.7). Let n be a positive power of N, u a positive number, and $h := u/n$. Then, for every $f \in L_2$,*

$$\operatorname{dist}(f, S_h) \le \operatorname{const}(\|\widehat{f}\|_{L_2}\widetilde{\lambda}(n) + \|\widehat{f}\|_{L_2(\mathbb{R}^d\backslash(B/u))}),$$

and

$$\operatorname{dist}(f, S_h) \le \operatorname{const}(u^{-d/2}\|\widehat{f}\|_{L_\infty}\lambda(n) + \|\widehat{f}\|_{L_2(\mathbb{R}^d\backslash(B/u))}),$$

with const *depending on d only. Of course, the second estimate is meaningful only when $\widehat{f} \in L_\infty$.*

Proof: We split $f = f_1 + f_2$, with $\widehat{f_1}$ coinciding with $\widehat{f}$ on B/u, and $\widehat{f_2}$ coinciding with $\widehat{f}$ on $\mathbb{R}^d\backslash(B/u)$. Since, obviously,

$$\operatorname{dist}(f, S_h) \le \operatorname{dist}(f_1, S_h) + \|f_2\|,$$

and since the Fourier transform is an isometry on L_2, we see that the proof of the theorem is reduced to proving that

$$\operatorname{dist}(f_1, S_h) \le \operatorname{const}\|\widehat{f}\|_{L_2}\widetilde{\lambda}(n). \tag{4.5}$$

and

$$\operatorname{dist}(f_1, S_h) \le \operatorname{const} u^{-d/2}u^{-d/2}\|\widehat{f}\|_{L_\infty}\lambda(n).$$

We will prove (4.5). The proof will eventually establish the other bound, as well.

Let ϕ be any function in $S_{1/n}$. This latter space is invariant under $\mathbb{Z}^d/n$-shifts, and since n is an integer, it is also invariant under integer shifts. Thus, not only $\phi \in S_{1/n}$, but also $S(\phi) \subset S_{1/n}$. Applying dilation, we obtain that

$$\sigma_u S(\phi) \subset \sigma_u S_{1/n} = S_h.$$

Thus, instead of proving (4.5), we are entitled to prove that, for some $\phi \in S_{1/n}$,

$$\text{dist}(f_1, \sigma_u S(\phi)) \le \text{const}\|f\|_{L_2}\widetilde{\lambda}(n). \tag{3.6}$$

However, $\widehat{f_1}$ is supported on B/u, and therefore Theorem 2.20 of [BDR1] (cf. (2.2)) provides us with an explicit formula

$$\text{dist}(f_1, \sigma_u S(\phi)) = \text{const}\|\Lambda_\phi(u\cdot)\widehat{f_1}\|_{L_2(B/u)},$$

with Λ_ϕ defined as in (2.1). Comparing this with the desired (3.6), and taking into account that $\|f_1\|_{L_2} \le \|f\|_{L_2}$, we realize that our claim is established as soon as we can find $\phi \in S_{1/n}$ such that

$$\|\Lambda_\phi\|_{L_\infty(B)} \le \widetilde{\lambda}(n). \tag{}$$

(For the complementary case, we need an estimate $\|\Lambda_\phi\|_{L_2(B)} \le \lambda(n)$.) Since $\psi \in S_{N^{-i}}$, $i \ge 1$, and n is a power of N, $\psi \in S_{1/n}$. Theorem 2.14 of [BDR1] then entails that any L_2-function ϕ whose Fourier transform is of the form $\widehat{\phi} = \tau\widehat{\psi}$ is in $\sigma_{1/n}S(\psi) \subset S_{1/n}$, provided that τ is $2\pi n$-periodic. We take τ to be the $2\pi n$-periodization of the support function of B, and define ϕ accordingly. Then, $\widehat{\phi}$ vanishes on each domain of the form $j + B$, $j \in 2\pi(\mathbb{Z}^d \backslash n\mathbb{Z}^d)$, and $\widehat{\phi} = \widehat{\psi}$ on domains of the form $j + B$, $j \in 2\pi n\mathbb{Z}^d$. Therefore, on B,

$$\sum_{j\in 2\pi\mathbb{Z}^d} |\widehat{\phi}(\cdot + j)|^2 = \sum_{j \in 2\pi n\mathbb{Z}^d} |\widehat{\psi}(\cdot + j)|^2.$$

We then conclude that, on B,

$$\Lambda_\phi^2 = 1 - \frac{|\widehat{\psi}|^2}{\sum_{j\in 2\pi n\mathbb{Z}^d} |\widehat{\psi}(\cdot + j)|^2}.$$

This shows that $\widetilde{\lambda}(n) = \|\Lambda_\phi\|_{L_\infty(B)}$, thereby implying the desired result. □

We return now to the proof of the theorem. Remember that we ought to show that (3.3) holds. We apply the lemma with respect to $n := N^{ir}$ and $u := N^{-i}$, i integer. Since $\widetilde{\lambda}_\psi(n) \le cn^{-k'}$, then, with $h_i := N^{-i(r+1)} = u/n$,

$$\text{dist}(f, S_{h_i}) \le \text{const}(n^{-k'}\|f\|_{L_2} + \|\widehat{f}\|_{L_2(\mathbb{R}^d\backslash(N^iB))}). \tag{3.8}$$

We compute that

$$n^{-k'} = N^{-irk'} = N^{-i(r+1)k} = h_i^k,$$

which takes care of the first term in the right-hand-side of (3.8). As for the second term, since $f \in W_2^{(r+1)k}$, it is easy to see that

$$\|\widehat{f}\|_{L_2(\mathbb{R}^d\backslash(N^iB))} = o(1)N^{-ik(r+1)}\|f\|_{W_2^{(r+1)k}} = o(1)h_i^k\|f\|_{W_2^{(r+1)k}},$$

with $o(1)$ bounded independently of f. Thus, (3.3) follows from (3.8), and the proof is complete. □

Theorem 3.9. *Let k, N be positive integers, and let $\mathcal{S}$ be a stationary SI ladder. Assume that:*
(a) For some $\psi \in \cap_{i=1}^{\infty} S_{1/N^i}$, $\lambda_\psi(N^j) = o(N^{-(k-1)j})$.
(b) $\mathcal{S}$ has the Property H(k).
Then $\mathcal{S}$ provides approximation order k for W_2^k.

Proof: To invoke the assumed Property H(k), we take f to be any band-limited Schwartz function. We will show that, for a some sequence $(h_i)_i$,

$$\mathrm{dist}(f, S_{h_i}) = o(h_i^{k-1}).$$

Property H(k) would then yield the approximation order assertion.

We choose $(h_i)_i$ as follows: since $\lambda_\psi(N^j) = o(N^{(1-k)j})$, we have, for each integer i and for all sufficiently large j,

$$\lambda_\psi(N^j) \le \frac{1}{i} N^{i(1-k-d/2)} N^{(1-k)j}. \tag{3.10}$$

For each i, we choose j that satisfies (3.10), define

$$h_i := N^{-(i+j)},$$

and invoke Lemma 3.4, with u there being N^{-i} and n there being N^j. Since f is band-limited, we may take i sufficiently large to ensure that $\widehat{f}$ is supported on $N^i B = B/u$. Thus, the lemma together with (3.10) provides the estimate

$$\mathrm{dist}(f, S_{h_i}) \le \mathrm{const}|u|^{-d/2}\|\widehat{f}\|_{L_\infty} \lambda_\psi(N^j) \le \mathrm{const} N^{id/2} \frac{1}{i} N^{i(1-k-d/2)} N^{j(1-k)} = \mathrm{const}\frac{1}{i} h_i^{k-1}.$$

The last estimate implies the desired estimate

$$\mathrm{dist}(f, S_{h_i}) = o(h_i^{k-1}).$$

□

4. The H(k) property

While two of the three theorems of the previous section require $\mathcal{S}$ to satisfy Property H(k), some readers may suspect this property to be as complicated and demanding as the notion of approximation orders (after all, approximation orders appear in its statement). This, however, is not true: basic decay + regularity conditions on the generating set of S, that are totally unrelated to the approximation orders that space may provide, suffice for guaranteeing the satisfaction of H(k). The section contains three results along these lines: in the first we treat the PSI case, in the second we treat the FSI case, and in the last we treat univariate FSI spaces generated by compactly supported functions. The proofs of these results are postponed until we show how the three results, when combined with Theorem 3.2 and Theorem 3.9, yield the various corollaries stated in the introduction.

Proposition 4.1. *Let k be a positive integer and $\phi \in L_2$. Let $\rho > k + d$, and consider the following conditions:*
(a) $\phi = O(|\cdot|^{-\rho})$ near ∞, and $\widehat{\phi}(0) \neq 0$.
(b) Λ^2_ϕ (defined in (2.1)) is k-times continuously differentiable around the origin.
(c) The PSI ladder $\mathcal{S}(\phi)$ has the Property H(k).
Then (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c).

Proof of Corollary 1.11. We first assume (a),(b1),(c), and apply Theorem 3.2. Comparing the assumptions of the theorem to these of the corollary (i.e., (a,b1,c)), we immediately observe that we only need to show that assumption (c) of the corollary implies the Property H(k). That latter implication follows directly from Proposition 4.1.

The proof of the other case is identical, only that we invoke now Theorem 3.9. □

The only difference between the FSI case and its special PSI case is that it is much harder to verify Condition H(k) in the FSI context. We forgo generalizing completely Proposition 4.1, and prefer instead to focus on the main implication ($(a) \Longrightarrow (c)$) of that proposition.

Proposition 4.2. *Let $\Phi \subset L_2$ be a finite set that satisfies conditions (c) of Corollary 1.9, with respect to some positive integer k. Then $S(\Phi)$ satisfies the Property H(k).*

Corollary 1.9 now follows from Theorem 3.2 and Proposition 4.2, while Corollary 1.10 follows from Theorem 3.9 and Proposition 4.2.

Finally, Theorem 1.12 follows from Theorems 3.2 and 3.9 when combined with the following general observation:

Proposition 4.3. *A univariate FSI ladder which is generated by compactly supported functions satisfies the Property H(k) for all integer values k.*

We now turn to the proofs of Propositions 4.1, 4.2, and 4.3.

Proof of Proposition 4.1.
$(a) \Longrightarrow (b)$: Let ρ' be any number between $k + d/2$ and $\rho - d/2$. Since $\rho' < \rho - d/2$, it easily follows from Plancheral Theorem that $\widehat{\phi} \in W_2^{\rho'}$. Since $\rho' > k + d/2$, the Sobolev embedding theorem then implies that

$$\||\widehat{\phi}|^2\|_{C^k(j+B)} \leq \text{const}\|\widehat{\phi}\|^2_{W_2^{\rho'}(j+B)}.$$

Summing over all $j \in 2\pi\mathbb{Z}^d$, and using the subadditivity of the $W_2^{\rho'}$-norm (which is valid with respect to a set of disjoint cubes, as here, cf. [A; p. 225]), we obtain that

$$\sum_{j \in 2\pi\mathbb{Z}^d} \||\widehat{\phi}|^2\|_{C^k(j+B)} \leq \text{const}'\|\widehat{\phi}\|^2_{W_2^{\rho'}(B+2\pi\mathbb{Z}^d)} < \infty.$$

This readily implies that $\sum_{j\in 2\pi\mathbb{Z}^d} |\widehat{\phi}(\cdot + j)|^2$ is k-times continuously differentiable on B. Since that function does not vanish at 0 (since $\widehat{\phi}(0) \neq 0$), we finally conclude that $\Lambda^2_\phi \in C^k(B)$.
$(b) \Longrightarrow (c)$: The proof requires the following elementary lemma.

Lemma 4.4. *Let $g \in L_2(\mathbb{R}^d)$, and let B be open and bounded. Then, as $h \to 0$,*

$$\| |\cdot| g \|_{L_2(B/h)} = o(h^{-1}).$$

Proof: Without essential loss, we may assume that B is the Euclidean unit ball. We fix h, and abbreviate $B_1 := B/\sqrt{h}$, $B_2 := (B/h)\backslash B_1$. Then, obviously, $B/h = B_1 \cup B_2$, hence

$$\begin{aligned} \| |\cdot| g \|^2_{L_2(B/h)} &= \| |\cdot| g \|^2_{L_2(B_1)} + \| |\cdot| g \|^2_{L_2(B_2)} \\ &\le h^{-1} \|g\|^2_{L_2(\mathbb{R}^d)} + h^{-2} \|g\|^2_{L_2(\mathbb{R}^d \backslash B_1)} = O(h^{-1}) + o(h^{-2}) = o(h^{-2}). \end{aligned}$$

□

We now show how to derive (c) from (b): in order to prove that the H(k) Property holds, we assume that

$$\text{dist}(f, S_h) = o(h^{k-1}), \tag{4.5}$$

with $f \in W_2^{k-1} \backslash 0$, with $h = h_1, h_2, \ldots$, and with $(h_i)_i$ decreasing to 0. We will show that this assumption, together with assumption (b), implies that $\mathcal{S}(\phi)$ provides approximation order k for all W_2^k. In what follows, h is always selected from the sequence $(h_i)_i$.

First, substituting $k-1$ for k in the second display of Corollary 2.3, and using (4.5), we obtain that

$$\| \Lambda_\phi(h\cdot) \widehat{f} \|_{L_2(B/h)} = o(h^{k-1}). \tag{4.6}$$

We contend that (4.6) forces all derivatives of Λ_ϕ up to order $k-1$ to vanish at the origin. Since a similar argument is needed in the proof of Proposition 4.3, we prove that fact in a separate lemma.

Lemma 4.7. *Let M be any function that is defined on a neighborhood B of the origin, and is k-times continuously differentiable there. Let $f \in W_2^{k-1} \backslash 0$. If*

$$\| \mathrm{M}(h\cdot) \widehat{f} \|_{L_2(B/h)} = o(h^{(k-1)}), \tag{4.8}$$

then all derivatives of M up to order $k-1$ must vanish at the origin.

Proof of Lemma 4.7. Assume, to the contrary, that some derivatives of order $l < k$ of M do not vanish at the origin, and let l be the minimal integer with that property. Using the Taylor series expansion of M around 0, we find that, on B/h, we have an estimate of the form

$$|\mathrm{M}(h\cdot)| \ge h^l |p| - c h^{l+1} |\cdot|^{l+1}.$$

Here, c depends on ϕ, B, and l, but not on h, and p is a homogeneous polynomial of degree l. Thus, invoking that estimate, we obtain from (4.8) that

$$\begin{aligned} o(h^{(k-1)}) &= \| \mathrm{M}(h\cdot) \widehat{f} \|_{L_2(B/h)} \\ &\ge h^l \| p \widehat{f} \|_{L_2(B/h)} - c h^{(l+1)} \| |\cdot|^{l+1} \widehat{f} \|_{L_2(B/h)}. \end{aligned} \tag{4.9}$$

Since $f \in W_2^{k-1}$, and $l \le k-1$, we conclude that $g := |\cdot|^l \widehat{f} \in L_2$. Lemma 4.4 then applies to yield that

$$\||\cdot|^{l+1}\widehat{f}\|_{L_2(B/h)} = \||\cdot|g\|_{L_2(B/h)} = o(h^{-1}).$$

Substituting this estimate into (4.9), we finally obtain that

$$h_i^l \|p\widehat{f}\|_{L_2(B/h_i)} = o(h_i^l),$$

which can happen only if $p\widehat{f} = 0$, a contradiction (being in $L_2\backslash 0$, $\widehat{f}$ cannot be supported on the zero set of p since the latter is a null-set). □

We proceed now with the proof of the theorem. Since, as contended, all derivatives of Λ_ϕ of orders $< k$ vanish at the origin, we see that

$$\Lambda_\phi = O(|\cdot|^k),$$

around the origin. Hence Theorem 1.6 of [BDR1] (cf. (2.4)) implies that $\mathcal{S}(\phi)$ provides approximation order k to W_2^k. □

Proof of Proposition 4.2. Since we need to prove that Property H(k) holds, we assume that for some $f \in W_2^{k-1}\backslash 0$, and a subsequence $(h_i)_i$

$$\operatorname{dist}(f, \sigma_h S(\Phi)) = o(h^{k-1}),\ h = h_1, h_2, \ldots$$

We need then to show that $\mathcal{S}(\Phi)$ provides approximation order k.

For that, we will find in $S(\Phi)$ a function ψ that satisfies the following three conditions:

(i) The function

$$\Lambda_\psi^2 := 1 - \frac{|\widehat{\psi}|^2}{\sum_{\alpha\in 2\pi\mathbb{Z}^d} |\widehat{\psi}(\cdot+\alpha)|^2}$$

is k-times continuously differentiable around the origin;

(ii) $\operatorname{dist}(f, T_h) = o(h^{k-1})$, $h \in (h_i)_i$, with T_h the h-dilate of $S(\psi)$.

In view of Proposition 4.1, condition (i) above implies that $\mathcal{S}(\psi)$ satisfies the Property H(k), hence, in view of (ii) above, it must provide approximation order k (for all functions in W_2^k). Since $S(\psi) \subset S(\Phi)$, $\mathcal{S}(\Phi)$ must then provide this approximation order as well.

We need now to prove that ψ as above exists, indeed, in $S(\Phi)$. First, by (c) of (2.7), there exists $\psi \in S$ whose stationary PSI ladder $\mathcal{S}(\psi) := (T_h)_h$ satisfies

$$\operatorname{dist}(f, T_h) \le \operatorname{dist}(f, S_h) + o(h^{k-1}).$$

Therefore,

$$\operatorname{dist}(f, T_h) = o(h^{k-1}), \quad h = h_1, h_2, \ldots, \tag{4.10}$$

and hence ψ satisfies the required (ii).

The proof that ψ satisfies (i) is as follows. First, we define the bracket product $[f,g]$ of the L_2-functions f,g as follows:

$$[f,g] := \sum_{j\in 2\pi\mathbb{Z}^d} \widehat{f}(\cdot + j)\overline{\widehat{g}(\cdot + j)}.$$

The *Gramian* of Φ is then defined to be the $\Phi\times\Phi$ matrix whose (ϕ,ϕ')-entry is $[\widehat{\phi},\widehat{\phi'}]$. It is proved in [BDR3] that, under assumptions (a-c) here, G is k-times continuously differentiable, and $G(0)$ is non-singular. Thus, G^{-1} exists on a neighborhood B of the origin and is smooth there, too.

Now, with τ the 2π-periodization of the restriction of $\widehat{\Phi}$ to B, [BDR3] identifies (around the origin) $[\widehat{\psi},\widehat{\psi}]$ as $\tau^* G^{-1}\tau$. Since $\widehat{\Phi}\subset C^k$ (by virtue of the decay assumptions on Φ), it follows that $[\widehat{\psi},\widehat{\psi}]\in C^k(B)$. In addition, [BDR3] computes $\widehat{\psi}$ as $G^{-1}\widehat{\Phi}$ which, again, shows that $\widehat{\psi}\in C^k(B)$. Combining all these observations, one readily concludes that $\Lambda^2_\psi\in C^k(B)$, as well. □

Proof of Proposition 4.3. We follow the proof of Proposition 4.2, to obtain the "super-function" ψ as detailed in that proof. As in that proof, we need only show that $\Lambda^2_\psi\in C^k(B)$.

We argue the smoothness of Λ^2_ψ as follows. First, knowing that $S(\Phi)$ is generated by compactly supported functions (viz., Φ), (b) of (2.7) allows us to assume that $S(\psi)$ is generated by a compactly supported function, which we may assume without loss to be ψ itself. Further, since we are in a univariate situation, we may invoke Theorems 1.1 of [R1] and 3.7 of [R2]: combined, they say that every univariate PSI space which is generated by a compactly supported L_2-function ψ, is also generated by a compactly supported L_2-function whose Fourier transform does not have a 2π-periodic zero. Again, we may assume without loss that our generator ψ is already the "favorable" generator of [R1,2]. This implies that the denominator in the definition of Λ vanishes nowhere, and a simple application of Poisson's summation formula then yields that Λ^2_ψ is real analytic, and in particular analytic around the origin. □

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