On Local Spline Approximation by Moments

CARL DE BOOR*

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1. This note is intended to generalize the statements of [1]. Incidentally it should justify some of the steps taken in [1].

2. Let m be a positive integer, $\pi : 0 = x_0 < x_1 < \cdots < x_n = 1$ a partition of the unit interval, and denote by $S = S_\pi$ the set of spline functions on $[0, 1]$ of degree $2m - 1$ with (interior) joints $x_1, \cdots, x_{n-1}$. We wish to investigate the behavior of

$$\text{dist}(f, S) = \min_{s \in S} \|f - s\|_\infty , \tag{1}$$

for $f \in C[0, 1]$, as the mesh of π, $|\pi| = \max_i |x_{i+1} - x_i|$, goes to zero. As is pointed out in [1],

$$\text{dist}(f, S_\pi) = O(|\pi|^k) \tag{2}$$

will not hold for $k > 2m$, except for the trivial case that f is a polynomial of degree $\leqq 2m - 1$. It is further stated there that if $f \in C^{2m}[0, 1]$ and if the numbers

$$M_\pi = \max_{|i-j|=1} (x_{i+1} - x_i)/(x_{j+1} - x_j) \tag{3}$$

stay bounded, then there exists K independent of f or π and $s_\pi \in S_\pi$ s.t.

$$|f(x) - s_\pi(x)| \leqq K |\pi|^{2m} \|f^{(2m)}\|_\infty , \quad \text{all} \quad x \in [x_m , x_{n-m}].$$

It is one result of this note that in fact

$$\text{dist}(f, S_\pi) = O(|\pi|^{2m}), \tag{4}$$

for $f \in C^{2m}[0, 1]$, and that (4) holds even without the assumption of bounded mesh ratios M_π.

The argument in [1] relies on a linear approximation scheme, called local spline approximation by moments, which realizes the convergence rate $O(|\pi|^{2m})$. Briefly, the approximation $P_\pi f$ to f is defined by

$$(P_\pi f)(x) = p(x) + \sum_i G(x, x_i) \int_0^1 W_i(t) f^{(2m)}(t)\, dt. \tag{5}$$

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Here, $p(x)$ is the polynomial of degree $2m - 1$ satisfying

$$p^{(j)}(0) = f^{(j)}(0),\ p^{(j)}(1) = f^{(j)}(1),\ \ j = 0, \cdots, m-1,$$

and $G(x, t)$ is Green's function for the boundary value problem $y^{(2m)}(t) = g(t)$, $y^{(j)}(0) = y^{(j)}(1) = 0$, $j = 0, \cdots, m - 1$, so that

$$f(x) \equiv p(x) + \int_0^1 G(x, t) f^{(2m)}(t)\, dt. \tag{6}$$

"The weight functions $W_i(t)$ are distributed over the 2m mesh points x_i nearest t in such a way as to have sum one and k-th moment $\Sigma(x_i - t)^k W_i = 0$ for $k = 1, \cdots, 2m - 1$." [1].

3. A little thought shows that, at least for "truly interior points" t, the $W_i(t)$ are the cardinal functions of an interpolation scheme which we will call, with [2], $(2m)$-point central (polynomial) interpolation. In this scheme, a function $g(x)$ is approximated on [0, 1] by

$$(Q_\pi g)(x) = p_{h-m}(x),\ x \in [x_{h-1}, x_h],\ h = 1, \cdots, n, \tag{7}$$

where $p_k(x)$ is the polynomial of degree $\leqq 2m - 1$ which interpolates $g(x)$ at the points $x_k, x_{k+1}, \cdots, x_{k+2m-1}$. This definition breaks down "near" $x = 0$ and $x = 1$. Since [1] gives no guidance in this matter, we pick one of the many supplemental definitions possible: Assume that $g(x)$ is defined in a neighborhood of [0, 1] and that π is supplemented by additional points satisfying

$$x_{-m+1} < x_{-m+2} < \cdots < x_{-1} < 0, \qquad 1 < x_{n+1} < \cdots < x_{n+m-1}.$$

Define

$$W_i(t) = Q_\pi \prod_{j \neq i} \frac{(t - x_j)}{(x_i - x_j)}, \qquad i = -m+1, \cdots, n+m-1. \tag{8}$$

Then

$$(Q_\pi g)(x) = \sum_{i=-m+1}^{n+m-1} g(x_i) W_i(x). \tag{9}$$

With this, we can use, more straightforwardly, Taylor's series with integral remainder,

$$f(x) = \sum_{i=0}^{2m-1} f^{(i)}(0) x^i / i! + \frac{1}{(2m-1)!} \int_0^1 (x - t)_+^{2m-1} f^{(2m)}(t)\, dt,$$

and slightly redefine P_π by

$$(P_\pi f)(x) = \sum_{i=0}^{2m-1} f^{(i)}(0) x^i / i! + \frac{1}{(2m-1)!} \sum_{i=-m+1}^{n+m-1} (x - x_i)_+^{2m-1} \int_0^1 W_i(t) f^{(2m)}(t)\, dt. \tag{10}$$

It is clear that this definition differs from (5) only by a polynomial of degree $\leqq 2m - 1$.

Set

$$e(x) = f(x) - (P_\pi f)(x). \tag{11}$$

Then, for $k = 0, 1, \cdots, 2m - 1$,

$$e^{(k)}(x) = \int_0^1 E_k(x, t) f^{(2m)}(t)\, dt, \tag{12}$$

with

$$E_k(x, t) = \left(\frac{\partial}{\partial x}\right)^k \frac{1}{(2m-1)!} [(x - t)_+^{2m-1} - \sum_i (x - x_i)_+^{2m-1} W_i(t)] \tag{13}$$
$$= (1 - Q_\pi)_{(t)} \frac{1}{(2m-1-k)!} (x - t)_+^{2m-1-k}.$$

4. The investigation of the behavior of $e^{(k)}(x)$ reduces, therefore, to the study of the error in applying central interpolation, Q_π, to

$$g(t) = (x - t)_+^{2m-1-k} / (2m - 1 - k)!. \tag{14}$$

Write $s = 2m - 1 - k$, for short. Let $\hat{x}, \hat{t} \in [0, 1]$, $x_{j-1} \leqq \hat{x} \leqq x_j$, $x_{h-1} \leqq \hat{t} \leqq x_h$, say, with $1 \leqq j, h \leqq n$.

It follows from the definition of Q_π that if $g(t)$ is a polynomial of degree $\leqq 2m - 1$ on $[x_{h-m}, x_{h+m-1}]$, then

$$(Q_\pi g)(t) = g(t), \quad \text{all} \quad t \in [x_{h-1}, x_h].$$

Hence

$$E_k(\hat{x}, \hat{t}) = 0 \quad \text{for} \quad |j - h| \geqq m. \tag{15}$$

Further, since also

$$g(t) = (-(x - t)_-^s + (x - t)^s)/s!, \tag{16}$$

we can assume without loss that $\hat{x} \leqq \hat{t}$ and $0 \leqq h - j < m$. Let $p_{ir}(t)$ denote the polynomial of degree $\leqq r$ which interpolates $g(t)$ at $x_i, x_{i+1}, \cdots, x_{i+r}$. Then

$$-(1 - Q_\pi) g(\hat{t}) = p_{h-m,2m-1}(\hat{t}). \tag{17}$$

To estimate $p_{h-m,2m-1}(t)$, we express it in terms of certain of the $p_{is}(t)$.

Lemma 1. *Let $t_0 < t_1 < \cdots < t_r$, and, for given $g(t)$, let p_{is} denote the s-th degree polynomial which interpolates $g(t)$ at the points $t_i, t_{i+1}, \cdots, t_{i+s}$, $i, s \geqq 0$, $i + s \leqq r$. Then*

$$p_{0,r}(t) = \sum_{i=0}^{r-s} p_{i,s}(t) L_{i,s}(t), \tag{18}$$

where the $L_{is}(t)$ depend on the points t_i but not on $g(t)$. Specifically,

(i) $L_{is}(t) = \alpha_{is}(t - t_0) \cdots (t - t_{i-1})(t - t_{i+s+1}) \cdots (t - t_r)$;

(ii) $1 \geqq L_{is}(t) \geqq 0$ *for all* $t \in [t_{i-1}, t_{i+s+1}]$.

Hence

(iii) $\sum_i |L_{is}(t)| = 1$, *all* $t \in [t_{r-s-1}, t_{s+1}]$;

(iv) $\sum_i |L_{is}(t)| \leqq C_{rs}(M)$, *all* $t \in [t_0, t_r]$,

with $C_{rs}(M)$ an increasing function of M and $M = \max_{|i-j|=1} (t_{i+1} - t_i)/(t_{j+1} - t_j)$.

Proof. Obviously, the $L_{is}(t)$ are generalizations of the Lagrange polynomials to which they reduce for $s = 0$. The $L_{is}(t)$ may be found recursively. Using Neville's formula,

$$p_{i,s+1}(t) = \frac{t - t_{i+s+1}}{t_i - t_{i+s+1}} p_{i,s}(t) + \frac{t - t_i}{t_{i+s+1} - t_i} p_{i+1,s}(t), \tag{19}$$

one gets

$$L_{is}(t) = \frac{t - t_{i-1}}{t_{i+s} - t_{i-1}} L_{i-1,s+1}(t) + \frac{t - t_{i+s+1}}{t_i - t_{i+s+1}} L_{i,s+1}(t), \tag{20}$$

with the convention that $L_{is}(t) \equiv 0$ for $i < 0$ or $i + s > r$. With $L_{0,r}(t) \equiv 1$, all statements of the lemma are clearly true for $s = r$. Using induction and the identity (20), (18) and (i), (ii), (iv) follow for all $s < r$. In particular, (ii) follows from the observation that the two "weights" in (20) are non negative for $t_{i-1} \leqq t \leqq t_{i+s+1}$, and that, by (18),

$$\sum_i L_{is}(t) \equiv 1. \tag{21}$$

This, together with (ii), also establishes (iii). Q.E.D.

Applying this lemma to (17), we get

$$|(1 - Q_\pi)g(\hat{t})| \leqq \max_i |p_{is}(\hat{t})| \sum_i |L_{i,s}(\hat{t})|, \tag{22}$$

where i runs from $h - m$ to $h + m - 1 - s$. The term

$$\max_i |p_{i,s}(\hat{t})|$$

is easily estimated. With $g(x_i, \cdots, x_{i+q})$ the q-th divided difference of $g(t)$ at the points $x_i, \cdots, x_{i+q}$, and $g^{(q)}(t) = (\hat{x} - t)_+^{s-q}/(s - q)!$, Newton's interpolation formula gives

$$|p_{is}(\hat{t})| \leqq \sum_{q=0}^{s} |g(x_i, \cdots, x_{i+q})| \prod_{r=0}^{q-1} |\hat{t} - x_{i+r}|$$

$$\leqq \sum_{q=0}^{s} \max_{t \in [x_i, x_{i+q}]} |g^{(q)}(t)| \frac{1}{q!} \alpha |\pi|^q \leqq C_s |\pi|^s,$$

where C_s is independent of π or g. Hence

$$|(1 - Q_\pi)g(\hat{t})| \leqq C_s |\pi|^s \sum_i |L_{is}(\hat{t})|. \tag{23}$$

Hence, with (iii) of the lemma,

$$|(1 - Q_\pi)g(\hat{t})| \leqq C_s |\pi|^s, \qquad s \geqq m - 1, \tag{24}$$

C_s independent of π.

If $s < m - 1$, some of the $L_{is}(\hat{t})$ will be negative, so that only (iv) of the lemma is at our disposal. With

$$M_\pi = \max_{|r-q|=1} (t_r - t_{r-1})/(t_q - t_{q-1}),$$

this gives

$$|(1 - Q_\pi)g(\hat{t})| \leqq C_s(M_\pi) |\pi|^s, \qquad s < m - 1, \tag{25}$$

with $C_s(\alpha)$ some increasing function of α.

5. It is now straightforward to prove the following

Theorem 1. *For $f \in C^{(2m)}[0, 1]$, and $\pi : x_{-m+1} < x_{-m+2} < \cdots < x_{n+m-1}$, with $x_0 = 0$, $x_n = 1$, and $P_\pi f$ as given by* (10), *we have*

$$\|f^{(k)} - (P_\pi f)^{(k)}\|_\infty \leqq N_k |\pi|^{2m-k} \|f^{(2m)}\|_\infty, \qquad k = 0, \cdots, 2m - 1, \tag{26}$$

with N_k independent of f. If $k \leqq m$, N_k is also independent of π, while for $k > m$, N_k can be bounded in terms of M_π.

Proof. With (12), (13) and (15),

$$|f^{(k)}(\hat{x}) - (P_\pi f)^{(k)}(\hat{x})| = \left| \int_{|t-\hat{x}| \leqq m|\pi|} E_k(\hat{x}, \hat{t}) f^{(2m)}(\hat{t})\, d\hat{t} \right|$$

$$\leqq 2m |\pi| \, \|E_k(\hat{x}, \cdot)\|_\infty \|f^{(2m)}\|_\infty .$$

For $s = 2m - 1 - k \geqq m - 1$, *i.e.*, for $k \leqq m$, (24) gives

$$\|E_k(\hat{x}, \cdot)\|_\infty \leqq C_s |\pi|^{2m-1-k} \tag{27}$$

with C_s independent of π; hence (26) follows with $N_k = 2mC_s$. If, else, $k > m$, (26) follows similarly from (25). Q.E.D.

Certain generalizations are possible. For one, it is sufficient in the above to assume merely that $f \in C^{(2m-1)}[0, 1]$ with $f^{(2m-1)}$ of bounded variation. Also, it is possible to let some of the joints coalesce. Precisely, we have the

Corollary. *Let $\pi : 0 = x_0 < x_1 < \cdots < x_n = 1$, and let S_π denote the set of piecewise polynomial functions of degree $2m - 1$ which have continuous derivatives up to and including the $(2m - d_i)$-th at x_i, d_i a positive integer not exceeding m, $i = 1, \cdots, n - 1$. If $f \in C^{(2m)}[0, 1]$, then there exists $s \in S_\pi$ s.t.*

$$(28) \qquad \|f^{(k)} - s^{(k)}\|_\infty \leqq N_k\, |\pi|^{2m-k}\, \|f^{(2m)}\|_\infty, \qquad k = 0, \cdots, m,$$

with N_k independent of f or π.

6. But, more important, the arguments leading up to Theorem 1 can be used to establish the analogous results for *even-degree* splines. Specifically, let $\pi : x_{-m} < x_{-m+1} < \cdots < x_{n+m}$ with $0 = x_0$, $1 = x_n$, and let S_π denote the set of spline functions of degree $2m$ on $[0, 1]$ with (interior) joints at $x_1, \cdots, x_{n-1}$. For $f \in C^{(2m+1)}[0, 1]$, define $P_\pi f \in S_\pi$ by

$$(29) \qquad (P_\pi f)(x) = \sum_{i=0}^{2m} f^{(i)}(0)x^i/i! + \frac{1}{(2m)!}\int_0^1 Q_{\pi(t)}(x - t)_+^{2m} f^{(2m+1)}(t)\, dt,$$

with Q_π denoting $(2m + 1)$-point central (polynomial) interpolation. Specifically, for $x \in [0, 1]$,

$$(Q_\pi g)(x) = p_{j-m}(x), \qquad x \in (x_{j-1/2}, x_{j+1/2}), \qquad j = 0, \cdots, n,$$

where $p_{j-m}(x)$ is the polynomial of degree $\leqq 2m$ interpolating $g(x)$ at $x_{j-m}, \cdots, x_{j+m}$, and $x_{r+1/2} = \frac{1}{2}(x_r + x_{r+1})$. To follow [2], the definition is completed by

$$(Q_\pi g)(x) = \tfrac{1}{2}[(Q_\pi g)(x+) + (Q_\pi g)(x-)], \quad \text{all} \quad x \in [0, 1],$$

although it would do just as well to define $Q_\pi g$ to be left-continuous or right-continuous everywhere.

With $e(x) = f(x) - (P_\pi f)(x)$, one gets, for $k = 0, \cdots, 2m$,

$$(30) \qquad e^{(k)}(x) = \int_0^1 E_k(x, t) f^{(2m+1)}(t)\, dt,$$

with

$$(31) \qquad E_k(x, t) = (1 - Q_\pi)_{(t)}(x - t)_+^{2m-k}/(2m - k)!.$$

Proceeding now just as in §4, set $s = 2m - k$ and consider

$$(32) \qquad g(t) = (x - t)_+^{2m-k}/(2m - k)!.$$

Let $\hat{x}, \hat{t} \in [0, 1]$, $x_{j-1} \leqq \hat{x} \leqq x_j$, $x_{h-1/2} \leqq \hat{t} \leqq x_{h+1/2}$, say. Then

$$(33) \qquad E_k(\hat{x}, \hat{t}) = 0 \quad \text{for} \quad |j - h| \geqq m.$$

Assume, without loss, that $\hat{x} \leqq \hat{t}$ and $0 \leqq h - j < m$. As before,

$$(34) \qquad |E_k(\hat{x}, \hat{t})| \leqq C_s\, |\pi|^s \sum_i |L_{is}(\hat{t})|,$$

with C_s independent of π. Here i runs from $h - m$ to $h + m - s$.

Using once again Lemma 1, this gives

Theorem 2. *For $f \in C^{(2m+1)}[0, 1]$, and $\pi : x_{-m} < \cdots < {}_{n+m}$, with $x_0 = 0$, $x_n = 1$, and $P_\pi f$ given by (29), we have*

$$(35) \qquad \|f^{(k)} - (P_\pi f)^{(k)}\|_\infty \leqq N_k\, |\pi|^{2m+1-k}\, \|f^{(2m+1)}\|_\infty, \qquad k = 0, \cdots, 2m,$$

with N_k independent of f. If $k \leqq m$, N_k is also independent of π, while for $k > m$, N_k can be bounded in terms of M_π.

Corollary. Let $\pi : 0 = x_0 < x_1 < \cdots < x_n = 1$, *and let S_π denote the set of piecewise polynomial functions of degree $2m$ which have continuous derivatives up to and including the $(2m + 1 - d_i)$-th at x_i, d_i a positive integer not exceeding m, $i = 1, \cdots, n - 1$. If $f \in C^{(2m+1)}[0, 1]$, then there exists $s \in S_\pi$ s.t.*

$$(36) \qquad \|f^{(k)} - s^{(k)}\|_\infty \leqq N_k\, |\pi|^{2m+1-k}\, \|f^{(2m+1)}\|_\infty, \qquad k = 0, \cdots, m,$$

with N_k independent of f or π.

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Purdue University

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