THE L_2-APPROXIMATION ORDER
OF SURFACE SPLINE INTERPOLATION

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Abstract. We show that if the open, bounded domain $\Omega \subset \mathbb{R}^d$ has a sufficiently smooth boundary and if the data function f is sufficiently smooth, then the $L_p(\Omega)$-norm of the error between f and its surface spline interpolant is $O(\delta^{\gamma_p+1/2})$ $(1 \le p \le \infty)$, where $\gamma_p := \min\{m, m-d/2+d/p\}$ and m is an integer parameter specifying the surface spline. In case $p = 2$, this lower bound on the approximation order agrees with a previously obtained upper bound, and so we conclude that the L_2-approximation order of surface spline interpolation is $m + 1/2$.

1. Introduction

Let $d, m \in \mathbb{N} := \{1, 2, 3, \dots\}$ with $m > d/2$. Let H^m be the space of all $f \in C(\mathbb{R}^d)$ such that $D^\alpha f \in L_2(\mathbb{R}^d)$ (in the distrubutional sense) for all $|\alpha| = m$. We define the semi-norm $|||\cdot|||_{H^m}$ on H^m by

$$|||f|||_{H^m} := \left\| |\cdot|^m \widehat{f} \right\|_{L_2(\mathbb{R}^d\backslash 0)},$$

where $\widehat{f}$ denotes the Fourier transform of f. Let Π_k denote the space of all d-variate polynomials whose total degree is less or equal to k. It is known [Du1] that if $f \in H^m$ and $\Xi \subset \mathbb{R}^d$ satisfies

$$p(\Xi) \ne \{0\} \quad \forall p \in \Pi_{m-1}\backslash 0, \tag{1.1}$$

then there exists a unique $s \in H^m$ which minimizes $|||s|||_{H^m}$ subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. The function s is called the *surface spline interpolant to f at Ξ* and will be denoted by $T_\Xi f$. In case Ξ is a finite subset of $\mathbb{R}^d$ satisfying (1.1), $T_\Xi f$ has the

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concrete representation as the unique function in $S(\phi, \Xi)$ which satisfies $s_{|\Xi} = f_{|\Xi}$. Here $\phi : \mathbb{R}^d \to \mathbb{R}$ is the radially symmetric function given by

$$\phi := \begin{cases} |\cdot|^{2m-d} & \text{if } d \text{ is odd} \\ |\cdot|^{2m-d} \log |\cdot| & \text{if } d \text{ is even,} \end{cases}$$

and $S(\phi, \Xi)$ denotes the space of all functions of the form

$$q + \sum_{\xi \in \Xi} \lambda_\xi \phi(\cdot - \xi),$$

where $q \in \Pi_{m-1}$ and the λ_ξ's satisfy

$$\sum_{\xi \in \Xi} \lambda_\xi p(\xi) = 0, \quad \forall p \in \Pi_{m-1}. \tag{1.2}$$

Surface spline interpolation is a prominent member of a family of interpolants known as *radial basis function interpolants.* The approximation properties of these interpolants have received considerable attention in the literature (for a sampling see [Du2], [Bu1], [WS], [MN], [WS], [DR], [BDL], [P2], [J1], [S1], [J2], [S2], [Bej], and the surveys [P1], [Bu2]).

In order to discuss the approximation properties of surface spline interpolation, we assume that $\Omega \subset \mathbb{R}^d$ is bounded and open and that the interpolation points Ξ are contained within $\overline{\Omega} :=$ closure(Ω). The 'density' of Ξ in Ω is measured by

$$\delta(\Xi, \Omega) := \sup_{x \in \Omega} \inf_{\xi \in \Xi} |x - \xi| .$$

Roughly speaking, we say that surface spline interpolation provides *L_p-approximation of order γ* if for all bounded, open $\Omega \subset \mathbb{R}^d$ having a sufficiently smooth boundary and for all sufficiently smooth functions f,

$$\|f - T_\Xi f\|_{L_p(\Omega)} = O(\delta^\gamma) \quad \text{as} \quad \delta := \delta(\Xi, \Omega) \to 0.$$

The largest (or supremum of all) such γ is called *the L_p-approximation order of surface spline interpolation.* Duchon [Du2] has shown that the L_p-approximation order of surface spline interpolation is at least $\gamma_p := \min\{m, m - d/2 + d/p\}$ for all $1 \le p \le \infty$. The precise details are as follows:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^d$ be bounded, open and have the cone property. Then there exists $\delta_0 > 0$ (depending only on Ω, m) such that if $f \in H^m$ and $\delta := \delta(\Xi, \Omega) \le \delta_0$, then*

$$\|f - T_\Xi f\|_{L_p(\Omega)} \le \text{const}(\Omega, m)\, \delta^{\gamma_p} |||T_\Omega f - T_\Xi f|||_{H^m}, \quad \textit{and}$$
$$|||T_\Omega f - T_\Xi f|||_{H^m} \to 0 \textit{ as } \delta \to 0.$$

On the other hand, it is known [J1] that the L_p-approximation order of surface spline interpolation is at most $m + 1/p$ for all $1 \le p \le \infty$. Specifically, it is known that if Ω is the open unit ball $B := \{x \in \mathbb{R}^d : |x| < 1\}$, then there exists $f \in C^\infty(\mathbb{R}^d)$ such that

$$\|f - T_\Xi f\|_{L_p(\Omega)} \ne o(\delta^{m+1/p}) \quad \text{as} \quad \delta := \delta(\Xi, \Omega) \to 0.$$

For the sake of comparison, we mention that in the ideal case $\Omega = \mathbb{R}^d$, $\Xi = h\mathbb{Z}^d$, (which of course violates our present setup) it is known ([Bu1], [JL]) that the L_p-approximation order of surface spline interpolation is $2m$, a value at least twice γ_p.

The purpose of the present work is to show that the L_p-approximation order of surface spline interpolation is at least $\gamma_p + 1/2$ for all $1 \leq p \leq \infty$. In case $p = 2$, this new lower bound matches the upper bound of $m + 1/p$, and so we conclude that the L_2-approximation order of surface spline interpolation is $m + 1/2$. In order to state our main result, we need the following definition which is taken from [Ad, p.67]. Our statement of the definition has been specialized (simplified) to the case when A has a bounded boundary.

Definition 1.4. Let $k \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ and let $A \subset \mathbb{R}^d$ be an open set having a bounded boundary. A has the *uniform C^k-regularity property* if there exists a finite open cover $\{U_j\}$ of ∂A, and a corresponding collection of one-to-one transformations $\{\Phi_j\}$ with Φ_j taking U_j onto B, such that:

(i) For each j, the components of Φ_j belong to $C^k(\overline{U_j})$.

(ii) For each j, the components of Φ_j^{-1} belong to $C^k(\overline{B})$.

(iii) For some $h > 0$, $(\partial A + hB) \subset \bigcup_j \Phi_j^{-1}(B/2)$.

(iv) For each j, $\Phi_j(U_j \cap A) = \{y \in B : y_d > 0\}$.

Our main result is the following:

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^d$ be bounded, open and have the uniform C^{2m}-regularity property. Then there exists $\delta_0 > 0$ (depending only on Ω, m) such that if $f \in B_{2,1}^{m+1/2}$ and $\delta := \delta(\Xi, \Omega) \leq \delta_0$, then*

$$|||T_\Omega f - T_\Xi f|||_{H^m} \leq \text{const}(\Omega, m)\delta^{1/2} \|f\|_{B_{2,1}^{m+1/2}}$$

and hence by Theorem 1.3,

$$\|f - T_\Xi f\|_{L_p(\Omega)} \leq \text{const}(\Omega, m)\delta^{\gamma_p + 1/2} \|f\|_{B_{2,1}^{m+1/2}}.$$

Here, $B_{2,1}^{m+1/2}$ denotes a certain Besov space which we define in section 2.

An outline of the paper is as follows: In section 2, we recall previous work on this problem and state in Theorem 2.3 precisely what will be proven in the present paper. In section 3, we estimate the size of $\phi * \mu$ in various function spaces under various assumptions on the compactly supported distribution μ. A general representation of $T_A f$ is then obtained in section 4 assuming only that A is bounded and $f \in H^m$. The regularity of $T_\Omega f$ in the exterior domain $\Omega_{\text{ext}} := \mathbb{R}^d \backslash \overline{\Omega}$ is studied in section 5 and the global regularity of $T_\Omega f$ is then deduced in section 6. Finally, in section 7, the representation and global regularity of $T_\Omega f$ are employed to prove Theorem 2.3.

Throughout this paper we use standard multi-index notation: $D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$. The natural numbers are denoted $\mathbb{N} := \{1, 2, 3, \dots\}$, and the non-negative integers are denoted $\mathbb{N}_0$. For multi-indices $\alpha \in \mathbb{N}_0^d$, we define $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$, while for $x \in \mathbb{R}^d$, we define $|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$. For multi-indices α, we employ the notation $()^\alpha$ to represent the monomial $x \mapsto x^\alpha$, $x \in \mathbb{R}^d$. The space of polynomials of total degree $\leq k$ can then be expressed as $\Pi_k := \operatorname{span}\{()^\alpha : |\alpha| \leq k\}$. The Fourier transform of an integrable function f is defined by $\widehat{f}(w) := \int_{\mathbb{R}^d} e^{-iw\cdot x} f(x)\,dx$. The space of compactly supported C^∞ functions whose support is contained in $A \subset \mathbb{R}^d$ is denoted $C_c^\infty(A)$. If μ is a distribution and g is a test function, then the application of μ to g is denoted $\langle g, \mu\rangle$. We employ the notation const to denote a generic constant in the range $(0\,..\,\infty)$ whose value may change with each occurence. An important aspect of this notation is that const depends only on its arguments if any, and otherwise depends on nothing.

2. A Reduction of the Problem

The Besov spaces, which we now define, play an essential role in our theory.

Definition 2.1. Let $A_0 := \overline{B}$, and for $k \in \mathbb{N}$, let $A_k := 2^k\overline{B}\backslash 2^{k-1}B$. The Besov space $B_{2,q}^\gamma$, $\gamma \in \mathbb{R}$, $1 \leq q \leq \infty$, is defined to be the set of all tempered distributions f for which $\widehat{f}$ is a locally integrable function and

$$\|f\|_{B_{2,q}^\gamma} := \left\| k \mapsto 2^{k\gamma} \left\|\widehat{f}\right\|_{L_2(A_k)} \right\|_{\ell_q(\mathbb{N}_0)} < \infty.$$

We also employ the Sobolev spaces $W^{n,p}(A)$ defined for open $A \subset \mathbb{R}^d$ and $n \in \mathbb{N}_0$, $p \in [1\,..\,\infty]$ by

$$W^{n,p}(A) := \{f \in L_2(A) : \|f\|_{W^{n,p}(A)} < \infty\},$$

where $\|f\|_{W^{n,p}(A)} := (\sum_{|\alpha|\leq n} \|D^\alpha f\|_{L_p(A)}^p)^{1/p}$ if $1 \leq p < \infty$ and $\|f\|_{W^{n,p}(A)} := \max_{|\alpha|\leq n} \|D^\alpha f\|_{L_\infty(A)}$ if $p = \infty$. The closure of $C_c^\infty(A)$ in $W^{m,p}(A)$ is denoted

$$W_0^{m,p}(A) := \operatorname{closure}(C_c^\infty(A); W^{m,p}(A)).$$

For $s \geq 0$, the Sobolev space W^s is defined by

$$W^s := \{f \in L_2 : \|f\|_{W^s} := \left\|(1 + |\cdot|^2)^{s/2}\widehat{f}\right\|_{L_2} < \infty\}.$$

All of the above defined spaces are Banach spaces. The following continuous embeddings can be found in [Pe] (they are also easy to prove from the definitions):

$$\begin{aligned} &B_{2,q_1}^{s_1} \hookrightarrow B_{2,q_2}^{s_2} \quad \text{if } s_1 > s2, \\ &B_{2,q_1}^{s} \hookrightarrow W^s \hookrightarrow B_{2,q_2}^{s} \quad \text{if } q_1 \leq 2 \leq q_2,\ s \geq 0, \text{ and} \\ &W^{s_1} \hookrightarrow B_{2,q}^{s} \hookrightarrow W^{s_2} \quad \text{if } s_1 > s > s_2 \geq 0. \end{aligned}$$

Moreover, if $s \geq 0$, then $W^s = B_{2,2}^s$ (with equivalent norms), and if $n \in \mathbb{N}_0$, then $W^{n,2}(\mathbb{R}^d) = W^n$ (with equivalent norms).

A significant part of our task (proving Theorem 1.5) has already been established in [J3]. Before stating the relevant result, we must define the convolution between ϕ and a compactly supported distribution. The Fourier transform of ϕ can be identified on $\mathbb{R}^d\backslash 0$ with the locally integrable function $c_\phi\left|\cdot\right|^{-2m}$, where is c_ϕ is a nonzero real constant which depends only on d, m (see [GS]). If μ is any compactly supported distribution, then we define the convolution $\phi * \mu$ in the Fourier transform domain via

$$(\phi * \mu)\hat{}:= \widehat{\phi}\widehat{\mu}.$$

That this is well-defined stems from the fact that $\widehat{\phi}\widehat{\mu}$ is a tempered distribution (as can be seen from the fact that $\widehat{\mu} \in C^\infty(\mathbb{R}^d)$ and $|\widehat{\mu}(x)|$ has at most polynomial growth as $|x| \to \infty$). The following has been proven (in greater generality) in [J3]:

Theorem 2.2. *Let Ω be a bounded, open subset of $\mathbb{R}^d$ having the cone property. There exists $\delta_0 > 0$ (depending only on Ω, m) such that if $f \in C(\mathbb{R}^d)$ is such that there exists $q \in \Pi_{m-1}$, $\mu \in B_{2,\infty}^{-m+1/2}$ satisfying $\operatorname{supp}\mu \subset \overline{\Omega}$, $\langle \Pi_{m-1}, \mu\rangle = \{0\}$, and $q + \phi * \mu = f$ on Ω, then*

(i) $\quad T_\Omega f = q + \phi * \mu$, *and*

(ii) $\quad |||T_\Omega f - T_\Xi f|||_{H^m} \le \operatorname{const}(\Omega, m)\delta^{1/2}\left\|\mu\right\|_{B_{2,\infty}^{-m+1/2}}$ *whenever* $\delta := \delta(\Xi, \Omega) \le \delta_0$.

In view of Theorem 1.3 and Theorem 2.2, the task of proving Theorem 1.5 reduces to proving the following:

Theorem 2.3. *Let Ω be a bounded, open subset of $\mathbb{R}^d$ having the uniform C^{2m}-regularity property. If $f \in B_{2,1}^{m+1/2}$, then there exists $q \in \Pi_{m-1}$ and $\mu \in B_{2,\infty}^{-m+1/2}$ such that $\operatorname{supp}\mu \subset \overline{\Omega}$, $\langle \Pi_{m-1}, \mu\rangle = \{0\}$, $q + \phi * \mu = f$ on Ω, and*

$$\left\|\mu\right\|_{B_{2,\infty}^{-m+1/2}} \le \operatorname{const}(\Omega, m)\left\|f\right\|_{B_{2,1}^{m+1/2}}. \tag{2.4}$$

We mention that in the special case $d = m = 2$, $\Omega = B$, it has already been shown in [J2] that such a q and μ exist (without (2.4)) whenever $f \in C^\infty(\mathbb{R}^2)$. In this special case, it is possible to express μ explicitly in terms of the boundary data and normal derivatives of f on ∂B; however, such an approach would be hopeless for general Ω.

3. AN EXAMINATION OF $\phi * \mu$

The purpose of this section is to prove the following:

Proposition 3.1. *Let $r > 0$ and let $\mu \in B_{2,2}^{-m}$ be supported in $r\overline{B}$. The following hold:*
(i) If $\langle \Pi_{m-1}, \mu\rangle = \{0\}$, then $\phi * \mu \in H^m$ and

$$\operatorname{const}(d, m)\left\|\mu\right\|_{B_{2,2}^{-m}} \le |||\phi * \mu|||_{H^m} \le \operatorname{const}(d, m, r)\left\|\mu\right\|_{B_{2,2}^{-m}}.$$

(ii) If $\langle \Pi_{2m-1}, \mu\rangle = \{0\}$, then $\phi * \mu \in W^m$ and

$$\operatorname{const}(d, m)\left\|\mu\right\|_{B_{2,2}^{-m}} \le \left\|\phi * \mu\right\|_{W^m} \le \operatorname{const}(d, m, r)\left\|\mu\right\|_{B_{2,2}^{-m}}.$$

(iii) $\phi * \mu \in W^{m,2}(rB)$ and $\left\|\phi * \mu\right\|_{W^{m,2}(rB)} \le \operatorname{const}(d, m, r)\left\|\mu\right\|_{B_{2,2}^{-m}}$.
(iv) If $\mu \in L_2$, then $\phi * \mu \in W^{2m,2}(rB)$ and $\left\|\phi * \mu\right\|_{W^{2m,2}(rB)} \le \operatorname{const}(d, m, r)\left\|\mu\right\|_{L_2}$.

Our proof of Proposition 3.1 requires the following two lemmata.

Lemma 3.2. *If* $g \in C_c^\infty(\mathbb{R}^d)$ *satisfies* $|g(w)| = O(|w|^{2m-d+1})$ *as* $|w| \to 0$, *then*

$$\langle g, \widehat{\phi} \rangle = c_\phi \int_{\mathbb{R}^d} g(w)\, |w|^{-2m}\, dw.$$

Proof. The proof can be adapted from that of [J2, Lem. 2.3] in a straightforward fashion.

Lemma 3.3. *Let* $r > 0$, $\gamma \geq 0$, $n \in \mathbb{N}$, *and let* $\mu \in B_{2,2}^{-\gamma}$ *be supported in* $r\overline{B}$. *Then*

$$\|\widehat{\mu}\|_{W^{n,\infty}(B)} \leq \mathrm{const}(d, \gamma, n, r)\, \|\mu\|_{B_{2,2}^{-\gamma}},$$

and if $\langle \Pi_{n-1}, \mu \rangle = \{0\}$, *then*

$$\left\| |\cdot|^{-n}\, \widehat{\mu} \right\|_{L_\infty(B)} \leq \mathrm{const}(d, \gamma, n, r)\, \|\mu\|_{B_{2,2}^{-\gamma}}.$$

Proof. Since μ is compactly supported, $\widehat{\mu}$ is entire. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be such that $\eta = 1$ on rB and for $\alpha \in \mathbb{N}_0^d$, let $\eta_\alpha := ()^\alpha \eta \in C_c^\infty(\mathbb{R}^d)$. Note that

$$D^\alpha \widehat{\mu} = i^{-|\alpha|}\, (()^\alpha \mu)\widehat{\ } = i^{-|\alpha|}\, (\eta_\alpha \mu)\widehat{\ } = i^{-|\alpha|} (2\pi)^{-d} \widehat{\eta_\alpha} * \widehat{\mu}.$$

Hence, for $w \in B$,

$$\begin{aligned} |D^\alpha \widehat{\mu}(w)| &= (2\pi)^{-d} \left| \int_{\mathbb{R}^d} \widehat{\mu}(t) \widehat{\eta_\alpha}(w - t)\, dt \right| \\ &\leq (2\pi)^{-d} \left\| \frac{\widehat{\mu}}{1 + |\cdot|^\gamma} \right\|_{L_2} \left\| (1 + |\cdot|^\gamma)\, \widehat{\eta_\alpha}(w - \cdot) \right\|_{L_2} \leq \mathrm{const}(\eta, \gamma, \alpha)\, \|\mu\|_{B_{2,2}^{-\gamma}}. \end{aligned}$$

Therefore, after a suitable choice of η, $\|\widehat{\mu}\|_{W^{n,\infty}(B)} \leq \mathrm{const}(d, \gamma, n, r)\, \|\mu\|_{B_{2,2}^{-\gamma}}$. Now assume that $\langle \Pi_{n-1}, \mu \rangle = \{0\}$. It follows that $D^\alpha \widehat{\mu}(0) = 0\ \forall\, |\alpha| < n$. Hence, by Taylor's Theorem, $|\widehat{\mu}(w)| \leq \mathrm{const}(d, n)\, |w|^n\, \|\widehat{\mu}\|_{W^{n,\infty}(B)}\ \forall w \in B$. Therefore, $\left\| |\cdot|^{-n}\, \widehat{\mu} \right\|_{L_\infty(B)} \leq \mathrm{const}(d, \gamma, n, r)\, \|\mu\|_{B_{2,2}^{-\gamma}}$. □

Proof of Proposition 3.1. Assume $\langle \Pi_{m-1}, \mu \rangle = \{0\}$. Put $f := \phi * \mu$. Let $|\alpha| = m$. Then $(D^\alpha f)\widehat{\ } = i^m ()^\alpha \widehat{\phi} \widehat{\mu}$. If $g \in C_c^\infty(\mathbb{R}^d)$, then $g_1 := i^m ()^\alpha \widehat{\mu} g \in C_c^\infty(\mathbb{R}^d)$ satisfies $|g_1(w)| = O(|w|^{2m})$ as $|w| \to 0$ and hence by Lemma 3.2,

$$\langle g, (D^\alpha f)\widehat{\ } \rangle = \langle g_1, \widehat{\phi} \rangle = c_\phi \int_{\mathbb{R}^d} |w|^{-2m}\, g_1(w)\, dw = c_\phi i^m \int_{\mathbb{R}^d} |w|^{-2m}\, w^\alpha \widehat{\mu}(w) g(w)\, dw.$$

The assumptions on μ ensure that $|\cdot|^{-2m}\, ()^\alpha \widehat{\mu} \in L_2$; hence, $(D^\alpha f)\widehat{\ } \in L_2$ and by the Plancherel Theorem, $D^\alpha f \in L_2$. Therefore, $f \in H^m$. Now,

$$|||f|||_{H^m}^2 = \left\| |\cdot|^m\, \widehat{f} \right\|_{L_2(\mathbb{R}^d \setminus 0)}^2 = c_\phi^2 \sum_{k=0}^{\infty} \left\| |\cdot|^{-m}\, \widehat{\mu} \right\|_{L_2(A_k)}^2.$$

For $k > 0$ we have $2^{-mk}\left\|\widehat{\mu}\right\|_{L_2(A_k)} \le \left\||\cdot|^{-m}\,\widehat{\mu}\right\|_{L_2(A_k)} \le 2^m 2^{-mk}\left\|\widehat{\mu}\right\|_{L_2(A_k)}$ while for $k = 0$ we have $\left\|\widehat{\mu}\right\|_{L_2(B)} \le \left\||\cdot|^{-m}\,\widehat{\mu}\right\|_{L_2(B)} \le \mathrm{const}(d,m,r)\left\|\mu\right\|_{B^{-m}_{2,2}}$ by Lemma 3.3. It now follows that $\mathrm{const}(d,m)\left\|\mu\right\|_{B^{-m}_{2,2}} \le |||f|||_{H^m} \le \mathrm{const}(d,m,r)\left\|\mu\right\|_{B^{-m}_{2,2}}$ which proves (i). For (ii) assume $\langle \Pi_{2m-1}, \mu\rangle = \{0\}$. The argument used to prove (i) can be easily adapted to show that $D^\alpha f \in L_2$ for all $|\alpha| \le m$. Hence $f \in W^m$. Now

$$\left\|f\right\|^2_{W^m} = \left\|\left(1+|\cdot|^2\right)^{m/2}\widehat{f}\right\|^2_{L_2} = c_\phi^2\sum_{k=0}^{\infty}\left\|\left(1+|\cdot|^2\right)^{m/2}|\cdot|^{-2m}\,\widehat{\mu}\right\|^2_{L_2(A_k)}.$$

For $k > 0$ we have $2^{-m}2^{-mk}\left\|\widehat{\mu}\right\|_{L_2(A_k)} \le \left\|\left(1+|\cdot|^2\right)^{m/2}|\cdot|^{-2m}\,\widehat{\mu}\right\|_{L_2(A_k)} \le 2^{3m}2^{-mk}\left\|\widehat{\mu}\right\|_{L_2(A_k)}$, and for $k = 0$ we have $\left\|\widehat{\mu}\right\|_{L_2(B)} \le \left\|\left(1+|\cdot|^2\right)^{m/2}|\cdot|^{-2m}\,\widehat{\mu}\right\|_{L_2(B)} \le \mathrm{const}(d,m,r)\left\|\mu\right\|_{B^{-m}_{2,2}}$ by Lemma 3.3. It now follows that $\mathrm{const}(d,m)\left\|\mu\right\|_{B^{-m}_{2,2}} \le \left\|f\right\|_{W^m} \le \mathrm{const}(d,m,r)\left\|\mu\right\|_{B^{-m}_{2,2}}$ which proves (ii). Turning now to (iii)–(iv), we no longer assume $\langle \Pi_{m-1}, \mu\rangle = \{0\}$. There exist $\mu_\alpha \in C_c^\infty(rB)$ such that for all $|\alpha|, |\beta| < 2m$, $\langle ()^\beta, \mu_\alpha\rangle = \delta_{\alpha,\beta}$, $\left\|\mu_\alpha\right\|_{L_2} \le \mathrm{const}(d,m,r)$, and $\left\|\phi * \mu_\alpha\right\|_{W^{2m,2}(rB)} \le \mathrm{const}(d,m,r)$. For $|\alpha| < 2m$ we have

$$\left|\langle ()^\alpha, \mu\rangle\right| = \left|D^\alpha\widehat{\mu}(0)\right| \le \left\|\widehat{\mu}\right\|_{W^{2m,\infty}(B)} \le \mathrm{const}(d,m,r)\left\|\mu\right\|_{B^{-m}_{2,2}}$$

by Lemma 3.3. Put $\nu := \mu - \sum_{|\alpha|<2m}\langle ()^\alpha, \mu\rangle\mu_\alpha$. Then $\operatorname{supp}\nu \subset r\overline{B}$, $\langle \Pi_{2m-1}, \nu\rangle = \{0\}$, and
(3.4)

$$\left\|\nu\right\|_{B^{-m}_{2,2}} \le \left\|\mu\right\|_{B^{-m}_{2,2}}\left(1 + \mathrm{const}(d,m,r)\sum_{|\alpha|<2m}\left\|\mu_\alpha\right\|_{B^{-m}_{2,2}}\right) \le \mathrm{const}(d,m,r)\left\|\mu\right\|_{B^{-m}_{2,2}}.$$

Therefore,

$$\begin{aligned}\left\|\phi * \mu\right\|_{W^{m,2}(rB)} &\le \left\|\phi * \nu\right\|_{W^{m,2}(rB)} + \left\|\phi * \sum_{|\alpha|<2m}\langle ()^\alpha, \mu\rangle\mu_\alpha\right\|_{W^{m,2}(rB)}\\ &\le \mathrm{const}(d,m,r)\left(\left\|\phi * \nu\right\|_{W^m} + \left\|\mu\right\|_{B^{-m}_{2,2}}\right) \le \mathrm{const}(d,m,r)\left\|\mu\right\|_{B^{-m}_{2,2}}\end{aligned}$$

by (ii) and (3.4). Hence (iii). In order to prove (iv), we assume $\mu \in L_2$. It follows from Lemma 3.3 that $\left|\langle ()^\alpha, \mu\rangle\right| \le \mathrm{const}(d,m,r)\left\|\mu\right\|_{L_2}$ $\forall\, |\alpha| < 2m$ and consequently,

$$\left\|\nu\right\|_{L_2} \le \left\|\mu\right\|_{L_2}\left(1 + \mathrm{const}(d,m,r)\sum_{|\alpha|<2m}\left\|\mu_\alpha\right\|_{L_2}\right) \le \mathrm{const}(d,m,r)\left\|\mu\right\|_{L_2}. \tag{3.5}$$

Hence,

$$\begin{aligned}\|\phi*\mu\|_{W^{2m,2}(rB)} &\le \|\phi*\nu\|_{W^{2m,2}(rB)} + \left\|\phi*\sum_{|\alpha|<2m}\langle ()^{\alpha},\mu\rangle\mu_{\alpha}\right\|_{W^{2m,2}(rB)}\\ &\le \operatorname{const}(d,m,r)\left(\|\phi*\nu\|_{W^{2m}}+\|\mu\|_{L_2}\right).\end{aligned}\tag{3.6}$$

Now,

$$\begin{aligned}\|\phi*\nu\|_{W^{2m}} &= c_{\phi}^{2}\left(\left\|\left(1+|\cdot|^{2}\right)^{m}|\cdot|^{-2m}\,\widehat{\nu}\right\|_{L_2(B)}^{2}+\left\|\left(1+|\cdot|^{2}\right)^{m}|\cdot|^{-2m}\,\widehat{\nu}\right\|_{L_2(\mathbb{R}^d\setminus B)}^{2}\right)\\ &\le \operatorname{const}(d,m)\left(\left\||\cdot|^{-2m}\,\widehat{\nu}\right\|_{L_2(B)}^{2}+\|\widehat{\nu}\|_{L_2(\mathbb{R}^d\setminus B)}^{2}\right)\le \operatorname{const}(d,m,r)\,\|\nu\|_{L_2}^{2}\end{aligned}$$

by Lemma 3.3 and the Plancherel Theorem which, in view of (3.6) and (3.5), proves (iv). □

4. A REPRESENTATION OF $T_A f$

The following representation of $T_A f$ is probably known, particularly by Duchon, but to the best of my knowledge has yet to be clearly stated and proved. Since our subsequent development relies heavily on this representation, we give it a careful treatment.

Theorem 4.1. *Let $A \subset \mathbb{R}^d$ be bounded and satisfy (1.1). For all $f \in H^m$, there exists a unique polynomial q and compactly supported distribution μ such that*

$$T_A f = q + \phi*\mu.$$

Moreover, the following hold

$$\begin{aligned}&(i) && q\in\Pi_{m-1},\ \mu\in B_{2,2}^{-m},\ \textit{and}\ \operatorname{supp}\mu\subset\overline{A}.\\ &(ii) && \langle\Pi_{m-1},\mu\rangle=\{0\}.\\ &(iii) && \|\mu\|_{B_{2,2}^{-m}}\le\operatorname{const}(A,m)|||T_Af|||_{H^m}.\\ &(iv) && \mu\in\operatorname{closure}(\operatorname{span}\{\delta_\xi:\xi\in A\};B_{2,2}^{-m}),\end{aligned}$$

where δ_ξ denotes the Dirac delta distribution defined by $\langle f,\delta_\xi\rangle = f(\xi)$.

Proof. An important property of surface spline interpolation is that for all $\Xi\subset\mathbb{R}^d$ and $g\in H^m$,

$$|||g-T_{\Xi}g|||_{H^m}^{2} = |||g|||_{H^m}^{2} - |||T_{\Xi}g|||_{H^m}^{2}.\tag{4.2}$$

Let Ξ_n be an increasing sequence of finite subsets of A, each satisfying (1.1), such that $\delta(\Xi_n, A)\to 0$ as $n\to\infty$. Let $f\in H^m$. Duchon [Du1] has shown that there exists

$q_n \in \Pi_{m-1}$ and $\mu_n \in \operatorname{span}\{\delta_\xi : \xi \in \Xi_n\}$, satisfying $\langle \Pi_{m-1}, \mu_n \rangle = \{0\}$, such that $T_{\Xi_n} f = q_n + \phi * \mu_n$. Since $\Xi_n \subset \Xi_{n+1}$, it follows that $T_{\Xi_n} f = T_{\Xi_n}(T_{\Xi_{n+1}} f)$. Hence, by (4.2),

$$0 \le |||T_{\Xi_{n+1}} f - T_{\Xi_n} f|||_{H^m}^2 = |||T_{\Xi_{n+1}} f|||_{H^m}^2 - |||T_{\Xi_n} f|||_{H^m}^2.$$

The sequence $\{|||T_{\Xi_n} f|||_{H^m}\}_{n\in\mathbb{N}}$ is therefore monotonically increasing and bounded above by $|||f|||_{H^m}$ and hence convergent. By choosing a subsequence of $\{\Xi_n\}$, if necessary, we may assume without loss of generality that $|||T_{\Xi_{n+1}} f|||_{H^m} - |||T_{\Xi_n} f|||_{H^m} \le 2^{-n}$ $\forall n \in \mathbb{N}$. Let $r > 0$ be the smallest positive real number satisfying $\overline{A} \subset r\overline{B}$. By Proposition 3.1 (i),

$$\begin{aligned} \|\mu_{n+1} - \mu_n\|_{B_{2,2}^{-m}} &\le \operatorname{const}(d,m) |||\phi * (\mu_{n+1} - \mu_n)|||_{H^m} \\ &= \operatorname{const}(d,m) |||T_{\Xi_{n+1}} f - T_{\Xi_n} f|||_{H^m} \le \operatorname{const}(d,m) 2^{-n}. \end{aligned}$$

It follows that $\{\mu_n\}$ is a Cauchy sequence in the Banach space $B_{2,2}^{-m}$, and hence there exists $\mu \in B_{2,2}^{-m}$ such that $\mu_n \to \mu$ in $B_{2,2}^{-m}$. Since the space of distributions in $B_{2,2}^{-m}$ which are supported in $\overline{A}$ and annihilate Π_{m-1} is a closed subspace of $B_{2,2}^{-m}$, it follows that $\operatorname{supp} \mu \subset \overline{A}$ and $\langle \Pi_{m-1}, \mu \rangle = \{0\}$. It follows from Proposition 3.1 (iii) that $\phi * \mu_n \to \phi * \mu$ in $W^{m,2}(rB)$. Since $m > d/2$, the Sobolev Imbedding Theorem [Ad, p.97] asserts that $W^{m,2}(rB)$ is continuously imbedded in $C(rB)$ (taken with the $L_\infty(rB)$-norm). Consequently $f - \phi * \mu_n \to f - \phi * \mu$ in $C(rB)$. But $f - \phi * \mu_n = q_n$ on Ξ_n. Hence, there exists $q \in \Pi_{m-1}$ such that $q_n \to q$ in Π_{m-1}. It follows now that $f = q + \phi * \mu$ on A. By Proposition 3.1 (i), $q + \phi * \mu \in H^m$, and by (4.2), $|||q + \phi * \mu|||_{H^m} = \lim_{n\to\infty} |||\phi * \mu_n|||_{H^m} \le |||T_A f|||_{H^m}$. Therefore $T_A f = q + \phi * \mu$. Note that (i), (ii), and (iv) hold, and (iii) follows from Proposition 3.1 (i). It remains to show that q and μ are unique. Assume that the polynomial $\widetilde{q}$ and the compactly supported distribution $\widetilde{\mu}$ are such that $T_A f = \widetilde{q} + \phi * \widetilde{\mu}$. Then $q - \widetilde{q} + \phi * (\mu - \widetilde{\mu}) = 0$ and consequently, $(q - \widetilde{q})\widehat{\ } + \widehat{\phi}(\mu - \widetilde{\mu})\widehat{\ } = 0$. Since $(q - \widetilde{q})\widehat{\ }$ is supported on $\{0\}$ and $\widehat{\phi} = c_\phi |\cdot|^{-2m}$ on $\mathbb{R}^d \backslash 0$, it follows that $(\mu - \widetilde{\mu})\widehat{\ } = 0$ on $\mathbb{R}^d \backslash 0$ and hence $\mu = \widetilde{\mu}$. Thus $(q - \widetilde{q})\widehat{\ } = 0$ which implies $q = \widetilde{q}$. □

5. The Regularity of $T_\Omega f$ in Ω_{ext}

At this point we know that the μ in the representation $T_\Omega f = q + \phi * \mu$ belongs to $B_{2,2}^{-m}$ whenever $f \in H^m$. The main hurdle in proving Theorem 2.3 is to show that if Ω has a sufficiently smooth boundary and $f \in B_{2,1}^{m+1/2}$, then the regularity of μ increases to that of $B_{2,\infty}^{-m+1/2}$. As will become clear in section 7, there is an intimate relation between the regularity of μ and the regularity of $T_\Omega f$. We begin by studying the regularity of $T_\Omega f$ in the exterior domain

$$\Omega_{\text{ext}} := \mathbb{R}^d \backslash \overline{\Omega}.$$

We assume, throughout this section, that $\Omega \subset \mathbb{R}^d$ is open, bounded and has the uniform C^{2m}-regularity property. It follows from this that Ω_{ext} has a bounded boundary and the uniform C^{2m}-regularity property. Our purpose in this section is to prove the following:

Proposition 5.1. *If $f \in W^{2m}$, then for all $|\alpha| = m$, $D^\alpha T_\Omega f \in W^{m,2}(\Omega_{\text{ext}})$ and*

$$\|D^\alpha T_\Omega f\|_{W^{m,2}(\Omega_{\text{ext}})} \le \operatorname{const}(\Omega, m) \|f\|_{W^{2m}}.$$

We will employ a regularity result regarding a solution of a linear elliptic partial differential equation. Since we are concerned only with the differential operator Δ^m, we will state a simplified result which applies to constant coefficient differential operators. The following result appears as a remark generalizing [Ag, Th. 9.8].

Theorem 5.2. *Let $A \subset \mathbb{R}^d$ be an open set having a bounded boundary and having the uniform C^{2m}-regularity property. Let $\{a_{\alpha,\beta}\}_{|\alpha|,|\beta|\le m}$ be complex numbers satisfying*

$$Re \sum_{|\alpha|,|\beta|=m} a_{\alpha,\beta}\xi^{\alpha+\beta} \ge E_0 \, |\xi|^{2m} \quad \forall \xi \in \mathbb{R}^d \tag{5.3}$$

for some constant $E_0 > 0$. Let b be the Dirichlet bilinear form

$$b[u,v] := \sum_{|\alpha|,|\beta|\le m} a_{\alpha,\beta} \int_A D^\alpha u(x)\overline{D^\beta v(x)}\, dx. \tag{5.4}$$

If $u \in W_0^{m,2}(A)$ and $g \in L_2(A)$ are such that

$$b[u,v] = \int_A g(x)\overline{v(x)}\, dx \quad \forall v \in C_c^\infty(A), \tag{5.5}$$

then $u \in W^{2m,2}(A)$ and

$$\|u\|_{W^{2m,2}(A)} \le \operatorname{const}(A, m, \{a_{\alpha,\beta}\}) \left(\|g\|_{L_2(A)} + \|u\|_{L_2(A)} \right).$$

Proof. The case when A is bounded is covered by [Ag, Th. 9.8] so we assume A is unbounded. Let r_0 be the smallest positive real number such that $\mathbb{R}^d \backslash r_0\overline{B} \subset A$ and put $r := r_0 + 4\sqrt{d}$. By [Ag, Th. 9.8],

$$\|u\|_{W^{2m,2}(A\cap rB)} \le \operatorname{const}(A, m, \{a_{\alpha,\beta}\}) \left(\|g\|_{L_2(A)} + \|u\|_{L_2(A)} \right). \tag{5.6}$$

The proof of this is done in two steps. First, it is shown that (5.6) holds with $\|u\|_{L_2(A)}$ replaced by $\|u\|_{W^{m,2}(A)}$, and then Garding's inequality is employed to show that

$$\|u\|_{W^{m,2}(A)} \le \operatorname{const}(d, m, \{a_{\alpha,\beta}\}) \left(\|g\|_{L_2(A)} + \|u\|_{L_2(A)} \right). \tag{5.7}$$

We turn now to $A\backslash rB$. For $j \in \mathbb{Z}^d$, put $G_j := j + 2\sqrt{d}B$ and $\widetilde{G}_j := j + \sqrt{d}B$, and let $\mathcal{N} := \{j \in \mathbb{Z}^d : \widetilde{G}_j \cap (A\backslash rB) \ne \emptyset\}$. Since $A\backslash rB = \mathbb{R}^d\backslash rB$, the choice of r ensures that $G_j \subset A \; \forall j \in \mathcal{N}$. By [Ag, Th. 9.6], for each $j \in \mathcal{N}$, $\|u\|_{W^{2m,2}(\widetilde{G}_j)} \le \operatorname{const}(d, m, \{a_{\alpha,\beta}\}) \left(\|g\|_{L_2(G_j)} + \|u\|_{W^{m,2}(G_j)} \right)$. Hence,

$$\begin{aligned} \|u\|^2_{W^{2m,2}(A\backslash rB)} \le \sum_{j\in\mathcal{N}} \|u\|^2_{W^{2m,2}(\widetilde{G}_j)} &\le \operatorname{const}(d, m, \{a_{\alpha,\beta}\}) \sum_{j\in\mathcal{N}} \left(\|g\|^2_{L_2(G_j)} + \|u\|^2_{W^{m,2}(G_j)} \right) \\ &\le \operatorname{const}(d, m, \{a_{\alpha,\beta}\}) \left(\|g\|^2_{L_2(A)} + \|u\|^2_{W^{m,2}(A)} \right) \end{aligned}$$

which, in view of (5.7) and (5.6), completes the proof. ☐

In our proof of Proposition 5.1, we employ the following two lemmata in establishing the hypothesis of Theorem 5.2. The first lemma is an immediate consequence of [Ad, Th. 7.55].

Lemma 5.8. *Let A be as in Theorem 5.2. If $f \in W^m$ equals 0 on $\mathbb{R}^d \backslash \overline{A}$, then $f \in W_0^{m,2}(A)$.*

Lemma 5.9. *If μ is a compactly supported distribution, then*

$$\Delta^m(\phi * \mu) = (-1)^m c_\phi \mu,$$

where $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ denotes the Laplacian operator.

Proof. Let $g \in C_c^\infty(\mathbb{R}^d)$. Then

$$\begin{aligned}\langle g, (\Delta^m(\phi * \mu))\widehat{\ }\rangle &= (-1)^m \langle g, |\cdot|^{2m}\, \widehat{\mu}\widehat{\phi}\rangle = (-1)^m \langle g\, |\cdot|^{2m}\, \widehat{\mu}, \widehat{\phi}\rangle \\ &= (-1)^m c_\phi \int_{\mathbb{R}^d} g(w)\, |w|^{2m}\, \widehat{\mu}(w)\, |w|^{-2m}\, dw, \qquad \text{by Lemma 3.2,} \\ &= (-1)^m c_\phi \langle g, \widehat{\mu}\rangle.\end{aligned}$$

□

Before embarking on the proof below, an explanation is in order. Ideally, we would like to choose u, in Theorem 5.2, to be $f - T_\Omega f$. Unfortunately, we only know that $T_\Omega f \in H^m$ which means that $|T_\Omega f(x)|$ may grow as $|x| \to \infty$; hence we cannot assert that $f - T_\Omega f$ belongs to $W^{m,2}(\Omega_{\text{ext}})$. Fortunately, the offending part of $T_\Omega f$ ($q + \phi * \nu$ in the language of the proof below) can be subtracted off and treated seperately.

Proof of Proposition 5.1. For $|\alpha| < 2m$, let $\mu_\alpha \in C_c^\infty(\Omega)$ be such that $\langle ()^\beta, \mu_\alpha\rangle = \delta_{\alpha,\beta}$ $\forall\, |\alpha|, |\beta| < 2m$ and $\sum_{|\alpha|<2m} \|\mu_\alpha\|_{L_2} \le \text{const}(\Omega, m)$. Let $f \in W^{2m}$ and let $q \in \Pi_{m-1}$, $\mu \in B_{2,2}^{-m}$ be as in Theorem 4.1. Then $\operatorname{supp} \mu \subset \overline{\Omega}$, $\langle \Pi_{m-1}, \mu\rangle = \{0\}$, $T_\Omega f = q + \phi * \mu$, and

$$(5.10) \quad \|\mu\|_{B_{2,2}^{-m}} \le \text{const}(\Omega, m) |||T_\Omega f|||_{H^m} \le \text{const}(\Omega, m) |||f|||_{H^m} \le \text{const}(\Omega, m)\, \|f\|_{W^m}\,.$$

Let r be the smallest positive real number for which $\Omega \subset (r/2)B$. Note that since $q = f - \phi * \mu$ on Ω and Π_{m-1} is finite dimensional, it follows that

$$\begin{aligned}\|q\|_{W^{2m,2}(rB)} \le \text{const}(\Omega, m)\, \|q\|_{W^{m,2}(\Omega)} &\le \text{const}(\Omega, m) \left(\|f\|_{W^m} + \|\phi * \mu\|_{W^{m,2}(rB)}\right) \\ &\le \text{const}(\Omega, m) \left(\|f\|_{W^m} + \|\mu\|_{B_{2,2}^{-m}}\right) \le \text{const}(\Omega, m)\, \|f\|_{W^m}\end{aligned}$$

by Proposition 3.1 (iii) and (5.10). Put $\nu := \sum_{m \le \alpha < 2m} \langle ()^\alpha, \mu\rangle \mu_\alpha$ and note that $\langle \Pi_{2m-1}, \mu - \nu\rangle = \{0\}$. Hence we can write $T_\Omega f = q + \phi * \nu + \phi * (\mu - \nu)$ with $\phi * (\mu - \nu) \in W^m$ by Proposition 3.1 (ii). It follows from Lemma 3.3 that for all $|\alpha| < 2m$,

$$|\langle ()^\alpha, \mu\rangle| = |D^\alpha \widehat{\mu}(0)| \le \|\widehat{\mu}\|_{W^{2m-1,\infty}(B)} \le \text{const}(\Omega, m)\, \|\mu\|_{B_{2,2}^{-m}} \le \text{const}(\Omega, m)\, \|f\|_{W^m}$$

by (5.10). Hence,

$$(5.11) \qquad \|\nu\|_{L_2} \le \sum_{m \le |\alpha| < 2m} |\langle ()^\alpha, \mu\rangle|\, \|\mu_\alpha\|_{L_2} \le \text{const}(\Omega, m)\, \|f\|_{W^m}\,.$$

Consequently, we have by Proposition 3.1 (iv), that $\|\phi * \nu\|_{W^{2m}(rB)} \le \mathrm{const}(\Omega, m)\, \|\nu\|_{L_2} \le \mathrm{const}(\Omega, m)\, \|f\|_{W^m}$. Let $\sigma \in C_c^\infty(rB)$ be such that $\sigma = 1$ on Ω and $\|\sigma\|_{W^{2m,\infty}(rB)} \le \mathrm{const}(\Omega, m)$. We then obtain the estimate

$$\begin{aligned} \|f - \sigma(q + \phi * \nu)\|_{W^{2m}} &\le \|f\|_{W^{2m}} + \|\sigma(q + \phi * \nu)\|_{W^{2m}} \\ &\le \|f\|_{W^{2m}} + \mathrm{const}(\Omega, m)\, \|q + \phi * \nu\|_{W^{2m,2}(rB)} \le \mathrm{const}(\Omega, m)\, \|f\|_{W^{2m}} . \end{aligned} \tag{5.12}$$

Put $u := f - \sigma(q + \phi * \nu) - \phi * (\mu - \nu)$. Note that $u = 0$ on Ω and

$$\begin{aligned} \|u\|_{W^m} &\le \|f - \sigma(q + \phi * \nu)\|_{W^m} + \|\phi * (\mu - \nu)\|_{W^m} \\ &\le \mathrm{const}(\Omega, m) \left(\|f\|_{W^{2m}} + \|\mu - \nu\|_{B_{2,2}^{-m}} \right), \qquad \text{by (5.12) and Proposition 3.1,} \\ &\le \mathrm{const}(\Omega, m)\, \|f\|_{W^{2m}} \end{aligned}$$

by (5.10) and (5.11). By Lemma 5.9, $\Delta^m(\phi * (\mu - \nu)) = (-1)^m(\mu - \nu) = 0$ on Ω_{ext}. Hence, if $g := (-1)^m \Delta^m (f - \sigma(q + \phi * \nu))$, then $(-1)^m \Delta^m u = g$ on Ω_{ext}. Note that by (5.12),

$$\|g\|_{L_2} \le \mathrm{const}(\Omega, m)\, \|f\|_{W^{2m}} .$$

Let $\{c_\alpha\}_{|\alpha|=m}$ be the positive integers defined by $|\xi|^{2m} = \sum_{|\alpha|=m} c_\alpha \xi^{2\alpha}$, $\xi \in \mathbb{R}^d$. We wish now to employ Theorem 5.2 on the exterior domain Ω_{ext} with
$a_{\alpha,\beta} := \begin{cases} c_\alpha & \text{if } \alpha = \beta \text{ and } |\alpha| = m, \\ 0 & \text{otherwise.} \end{cases}$ Condition (5.3) is satisfied with $E_0 = 1$ since the quantity on the left side of (5.3) equals $|\xi|^{2m}$. Since $u \in W^m$ and $u = 0$ on Ω, it follows by Lemma 5.8 that $u \in W_0^{m,2}(\Omega_{\mathrm{ext}})$. Note that the Dirichlet form in (5.4) simplifies to $b[u, v] = \sum_{|\alpha|=m} c_\alpha \int_{\Omega_{\mathrm{ext}}} D^\alpha u \overline{D^\alpha v}$. To see that (5.5) holds, let $v \in C_c^\infty(\Omega_{\mathrm{ext}})$. Then

$$\begin{aligned} b[u, \overline{v}] &= \sum_{|\alpha|=m} c_\alpha \langle D^\alpha v, D^\alpha u \rangle = (-1)^m \sum_{|\alpha|=m} c_\alpha \langle D^{2\alpha} v, u \rangle \\ &= (-1)^m \langle \Delta^m v, u \rangle = (-1)^m \langle v, \Delta^m u \rangle = \int_{\Omega_{\mathrm{ext}}} v(x) g(x)\, dx \end{aligned}$$

where the first and last equality hold since $\mathrm{supp}\, v \subset \Omega_{\mathrm{ext}}$. Therefore, by Theorem 5.2, $u \in W^{2m,2}(\Omega_{\mathrm{ext}})$ and

$$\|u\|_{W^{2m,2}(\Omega_{\mathrm{ext}})} \le \mathrm{const}(\Omega, m) \left(\|g\|_{L_2(\Omega_{\mathrm{ext}})} + \|u\|_{L_2(\Omega_{\mathrm{ext}})} \right) \le \mathrm{const}(\Omega, m)\, \|f\|_{W^{2m}} .$$

Now $T_\Omega f$ can be written as $T_\Omega f = q + \phi * \nu + f - \sigma(q + \phi * \nu) - u$. Let $|\alpha| = m$ and note that $D^\alpha(\phi * \nu) = \phi * D^\alpha \nu$. Since $D^\alpha \nu \in B_{2,2}^{-m}$ and $\langle \Pi_{2m-1}, D^\alpha \nu \rangle = \{0\}$, we have by Proposition 3.1 (ii) that $\phi * D^\alpha \nu \in W^m$ and

$$\|\phi * D^\alpha \nu\|_{W^m} \le \mathrm{const}(\Omega, m)\, \|D^\alpha \nu\|_{B_{2,2}^{-m}} \le \mathrm{const}(\Omega, m)\, \|\nu\|_{L_2} \le \mathrm{const}(\Omega, m)\, \|f\|_{W^m}$$

by (5.11). Therefore,

$$\begin{aligned} \|D^\alpha T_\Omega f\|_{W^{m,2}(\Omega_{\mathrm{ext}})} \le \|D^\alpha(\phi * \nu)\|_{W^m} + \|f - \sigma(q + \phi * \nu)\|_{W^{2m}} + \|u\|_{W^{2m,2}(\Omega_{\mathrm{ext}})} \\ \le \mathrm{const}(\Omega, m)\, \|f\|_{W^{2m}} . \end{aligned}$$

□

6. THE GLOBAL REGULARITY OF $T_\Omega f$

As in the previous section, we assume throughout this section that $\Omega \subset \mathbb{R}^d$ is open, bounded and has the uniform C^{2m}-regularity property. Our purpose in this section is to prove the following:

Theorem 6.1. *If $f \in B^{m+1/2}_{2,1}$, then for all $|\alpha| = m$, $D^\alpha T_\Omega f \in B^{1/2}_{2,\infty}$ and*

$$\|D^\alpha T_\Omega f\|_{B^{1/2}_{2,\infty}} \le \mathrm{const}(\Omega, m)\, \|f\|_{B^{m+1/2}_{2,1}}.$$

The following definition and theorem are taken from [Ad, p.83–86].

Definition. Let $A \subset \mathbb{R}^d$ be open. For given k and p, a linear operator $E : W^{k,p}(A) \to W^{k,p}(\mathbb{R}^d)$ is called a *simple (m,p)-extension operator for A* if for all $u \in W^{k,p}(A)$,

$$\begin{array}{ll} (i) & Eu(x) = u(x) \quad \text{a.e. in } A \text{ and} \\ (ii) & \|Eu\|_{W^{k,p}(\mathbb{R}^d)} \le \mathrm{const}(A,k,p)\, \|u\|_{W^{k,p}(A)}. \end{array}$$

E is called a *strong n-extension operator for A* if E is a linear operator mapping functions defined a.e. in A into functions defined a.e. in $\mathbb{R}^d$ and if for every $k \in \{0, 1, \dots, n\}$ and for every $p \in [1 \,..\, \infty)$, the restriction of E to $W^{k,p}(A)$ is a simple (k,p)-extension operator for A.

The following theorem is proved in [Ad, p.84].

Theorem 6.2. *If $A \subset \mathbb{R}^d$ is open, has a bounded boundary, and has the uniform C^n-regularity property, then there exists a strong n-extension operator E for A.*

The assumptions on Ω ensure that $\Omega_{\mathrm{ext}} := \mathbb{R}^d \backslash \overline{\Omega}$ has a bounded boundary and the uniform C^{2m}-regularity property. Hence, by Theorem 6.2, there exists a strong m-extension operator E for Ω_{ext}.

Lemma 6.3. *If $|\alpha| = m$ and $f \in B^{m+1/2}_{2,1}$, then $ED^\alpha T_\Omega f \in B^{1/2}_{2,1}$ and*

$$\|ED^\alpha T_\Omega f\|_{B^{1/2}_{2,1}} \le \mathrm{const}(\Omega, m)\, \|f\|_{B^{m+1/2}_{2,1}}.$$

Proof. We employ a result regarding real interpolation of Banach spaces. If X_1, X_2 are two Sobolev spaces, then Peetre's K-functional is defined for $t > 0$, $f \in X_1 + X_2$ by

$$K(t,f) := \inf\{\|f_1\|_{X_1} + t\,\|f_2\|_{X_2} : f = f_1 + f_2, f_1 \in X_1, f_2 \in X_2\}.$$

For $0 < \theta < 1$ and $1 \le q < \infty$, let

$$(X_1, X_2)_{\theta,q} := \{f \in X_1 + X_2 : \|f\|_{(X_1,X_2)_{\theta,q}} := (\int_0^\infty t^{-\theta q - 1} K(t,f)^q \, dt)^{1/q} < \infty\}.$$

It is known [Tr1, p.39–40] that if $s_1, s_2 \in \mathbb{N}_0$, then $(W^{s_1}, W^{s_2})_{\theta,q} = B^s_{2,q}$ (with equivalent norms) where $s := s_1(1-\theta) + s_2\theta$. Taking $\theta = 1/(2m)$, $q = 1$ yields $(W^m, W^{2m})_{\theta,q} =$

$B^{m+1/2}_{2,1}$ and $(L_2, W^m)_{\theta,q} = B^{1/2}_{2,1}$. To see that the operator $ED^\alpha T_\Omega$ is a bounded linear operator from W^m into L_2, we observe that

$$\begin{aligned}\|ED^\alpha T_\Omega f\|_{L_2} \le \mathrm{const}(\Omega, m)\,\|D^\alpha T_\Omega f\|_{L_2(\Omega_{\mathrm{ext}})} &\le \mathrm{const}(\Omega, m)\,\|D^\alpha T_\Omega f\|_{L_2}\\ &\le \mathrm{const}(\Omega, m)|||T_\Omega f|||_{H^m} \le \mathrm{const}(\Omega, m)\,\|f\|_{W^m}.\end{aligned}$$

In addition, $ED^\alpha T_\Omega$ is a bounded linear operator from W^{2m} into W^m. Indeed,

$$\|ED^\alpha T_\Omega f\|_{W^m} \le \mathrm{const}(\Omega, m)\,\|D^\alpha T_\Omega f\|_{W^{m,2}(\Omega_{\mathrm{ext}})} \le \mathrm{const}(\Omega, m)\,\|f\|_{W^{2m}}$$

by Proposition 5.1. It follows by the *interpolation property* (see [Tr1, p.38]) that $ED^\alpha T_\Omega$ is a bounded linear operator from $B^{m+1/2}_{2,1}$ into $B^{1/2}_{2,1}$. □

Our point of view now is the following: Assuming $f \in B^{m+1/2}_{2,1}$ and $|\alpha| = m$, we have that both $D^\alpha f$ and $ED^\alpha T_\Omega f$ belong to $B^{1/2}_{2,1}$. The function $D^\alpha T_\Omega f$ equals $D^\alpha f$ on Ω and equals $ED^\alpha T_\Omega f$ on Ω_{ext}, and based on this we wish to show that $D^\alpha T_\Omega f \in B^{1/2}_{2,\infty}$. The purpose of the following three lemmata is to relate the $B^{1/2}_{2,\infty}$-norm of a function g with the rate at which an approximate identity convolved with g converges to g in the L_2-norm.

Lemma 6.4. *Let $r \ge 1$. There exists $\varepsilon > 0$ (depending only on Ω, r) such that if $n \in \mathbb{N}_0$ and $0 < h \le \varepsilon 2^{-n}$, then*

$$m_d((\partial\Omega + hB) \cap (x + 2^{-n}rB)) \le \mathrm{const}(\Omega, r)h2^{-n(d-1)} \quad \forall x \in \mathbb{R}^d,$$

where m_d denotes Lesbegue measure in $\mathbb{R}^d$.

Proof. The assumptions on Ω ensure that there exists $c_1 > 0$ (depending only on Ω) such that for all j

$$|\Phi_j(x) - \Phi_j(y)| \le c_1\,|x - y| \quad \forall x, y \in U_j \text{ and} \tag{6.5}$$

$$m_d(A) \le c_1 m_d(\Phi_j(A)) \quad \forall \text{ measurable } A \subset U_j. \tag{6.6}$$

Put $\widetilde{U}_j := \Phi_j^{-1}(B/2)$. By Definition 1.4 (iii), there exists $\delta > 0$ such that $\partial\Omega + \delta B \subset \cup_j \widetilde{U}_j$. Let ε be the largest positive real number satisfying $\varepsilon \le \delta$ and $\widetilde{U}_j + 6\varepsilon B \subset U_j$ $\forall j$. Let $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$ and assume $0 < h \le \varepsilon 2^{-n}$. Put $F := (\partial\Omega + hB) \cap (x + 2^{-n}rB)$. It is a straightforward matter to show that $m_d(\partial\Omega + hB) \le \mathrm{const}(\Omega)h$. Hence, if $2^{-n}r \ge \varepsilon$, then $m_d(F) \le m_d(\partial\Omega + hB) \le \mathrm{const}(\Omega)h \le \mathrm{const}(\Omega, r)h2^{-n(d-1)}$. So assume $2^{-n}r < \varepsilon$. Let $a \in F$. Then there exists $a' \in \partial\Omega$ such that $|a - a'| < h$. Put $F_1 := (\partial\Omega \cap [a' + 2(h + 2^{-n}r)B]) + hB$ and note that $F \subset F_1$. Indeed, if $y \in F$, then there exists $y' \in \partial\Omega$ such that $|y - y'| < h$. Since $|y - x| < 2^{-n}r$, we have $|x - y'| < h + 2^{-n}r$. Hence,

$$|y' - a'| \le |y' - x| + |x - a| + |a - a'| < (h + 2^{-n}r) + 2^{-n}r + h = 2(h + 2^{-n}r).$$

Thus $y' \in \partial\Omega \cap (a' + 2(h + 2^{-n}r)B)$ and consequently $y \in F_1$. Let $\mathcal{N} := \{j : F_1 \cap \widetilde{U}_j \ne \emptyset\}$. We note that if $j \in \mathcal{N}$, say $y \in F_1 \cap \widetilde{U}_j$, then $|a' - y| \le 2(h + 2^{-n}r) + h \le 3(h + 2^{-n}r)$ and hence $a' \in \widetilde{U}_j + 3(h + 2^{-n}r)B \subset \widetilde{U}_j + 6\varepsilon B \subset U_j$. Consequently, $\Phi_j(a')$ is defined whenever $j \in \mathcal{N}$.

Claim. If $j \in \mathcal{N}$, then

$$\Phi_j(F_1 \cap \widetilde{U}_j) \subset \{w \in \mathbb{R}^d : |w_d| \le c_1 h,\ |(w_1, \ldots, w_{d-1}, 0) - \Phi_j(a')| \le 3c_1(h + 2^{-n}r)\}.$$

proof. Let $z \in F_1 \cap \widetilde{U}_j$ and put $w = \Phi_j(z)$. Then there exists $z' \in \partial\Omega \cap (a' + 2(h + 2^{-n}r)B)$ such that $|z - z'| < h$. Note that $z' \in z + hB \subset \widetilde{U}_j + \varepsilon B \subset U_j$ and hence $w' := \Phi_j(z')$ is defined. Since $w'_d = 0$, we have $|w_d| \le |w - w'| \le c_1 |z - z'| \le c_1 h$ by (6.5). And

$$\begin{aligned} &|(w_1, \ldots, w_{d-1}, 0) - \Phi_j(a')| \le |w - \Phi_j(a')| \le c_1 |z - a'|, \quad \text{by (6.5)}, \\ &\le c_1(|z - z'| + |z' - a'|) \le c_1(h + 2(h + 2^{-n}r)) \le 3c_1(h + 2^{-n}r) \end{aligned}$$

which proves the claim.

Since $F \subset F_1 \subset \partial\Omega + hB \subset \partial\Omega + \delta B \subset \cup_j \widetilde{U}_j$, it follows that

$$\begin{aligned} m_d(F) &\le \sum_{j\in\mathcal{N}} m_d(F_1 \cap \widetilde{U}_j) \le c_1 \sum_{j\in\mathcal{N}} m_d(\Phi_j(F_1 \cap \widetilde{U}_j)), \quad \text{by (6.6)}, \\ &\le c_1 \sum_{j\in\mathcal{N}} m_d(\{w \in \mathbb{R}^d : |w_d| \le c_1 h,\ |(w_1, \ldots, w_{d-1}, 0) - \Phi_j(a')| \le 3c_1(h + 2^{-n}r)\}) \\ &= c_1 \sum_{j\in\mathcal{N}} \text{const}(d) c_1 h (3c_1(h + 2^{-n}r))^{d-1} \le \text{const}(\Omega, r) h 2^{-n(d-1)}. \end{aligned}$$

□

Lemma 6.7. *For all* $f \in B^{1/2}_{2,1}$ *and* $h > 0$,

$$\|f\|_{L_2(\partial\Omega + hB)} \le \text{const}(\Omega) h^{1/2} \|f\|_{B^{1/2}_{2,1}}.$$

Proof. We employ the atomic decomposition of $B^{1/2}_{2,1}$ (see [Tr2, p.70–81]). It is known that there exists $r \ge 1$ and functions $a_{n,j} \in C^1(\mathbb{R}^d)$, $n \in \mathbb{N}_0$, $j \in \mathbb{Z}^d$, (depending only on d) satisfying

$$\operatorname{supp} a_{n,j} \subset 2^{-n}(j + rB) \quad \text{and} \tag{6.8}$$

$$\|D^\alpha a_{n,j}\|_{L_\infty} \le 2^{n(|\alpha| + (d-1)/2)} \quad \forall\, |\alpha| \le 1 \tag{6.9}$$

such that for all $f \in B^{1/2}_{2,1}$, there exists $\{\lambda_{n,j}\}$ such that

$$\sum_{n=0}^{\infty} \left(\sum_{j\in\mathbb{Z}^d} |\lambda_{n,j}|^2 \right)^{1/2} \le \text{const}(d) \|f\|_{B^{1/2}_{2,1}}, \quad \text{and} \tag{6.10}$$

$$f = \sum_{n=0}^{\infty} \sum_{j\in\mathbb{Z}^d} \lambda_{n,j} a_{n,j} \quad (\text{convergence in } L_2). \tag{6.11}$$

It follows from (6.8) that for all $n \in \mathbb{N}_0$,

$$\left\| \sum_{j\in\mathbb{Z}^d} \lambda_{n,j} a_{n,j} \right\|^2_{L_2(\partial\Omega+hB)} \le \operatorname{const}(d) \sum_{j\in\mathbb{Z}^d} |\lambda_{n,j}|^2 \, \|a_{n,j}\|^2_{L_2(\partial\Omega+hB)} . \tag{6.12}$$

We estimate $\|a_{n,j}\|^2_{L_2(\partial\Omega+hB)}$ in two cases. Let $\varepsilon > 0$ be as in Lemma 6.4. If $h \le \varepsilon 2^{-n}$, then by Lemma 6.4, $m_d((\partial\Omega + hB) \cap 2^{-n}(j + rB)) \le \operatorname{const}(\Omega) h 2^{-n(d-1)}$ and hence,

$$\begin{aligned} \|a_{n,j}\|^2_{L_2(\partial\Omega+hB)} &\le \|a_{n,j}\|^2_{L_\infty(\partial\Omega+hB)}\, m_d((\partial\Omega + hB) \cap 2^{-n}(j + rB)) \\ &\le 2^{n(d-1)} \operatorname{const}(\Omega) h 2^{-n(d-1)} = \operatorname{const}(\Omega) h. \end{aligned}$$

On the other hand, if $h > \varepsilon 2^{-n}$, then $m_d(2^{-n}(j + rB)) \le \operatorname{const}(d) 2^{-nd}$ and hence

$$\begin{aligned} \|a_{n,j}\|^2_{L_2(\partial\Omega+hB)} &\le \|a_{n,j}\|^2_{L_2} \le \|a_{n,j}\|^2_{L_\infty}\, m_d(2^{-n}(j + rB)) \\ &\le 2^{n(d-1)} \operatorname{const}(d) 2^{-nd} \le \operatorname{const}(d) 2^{-n} \le \operatorname{const}(\Omega) h. \end{aligned}$$

It therefore follows by (6.12) that $\left\| \sum_{j\in\mathbb{Z}^d} \lambda_{n,j} a_{n,j} \right\|^2_{L_2(\partial\Omega+hB)} \le \operatorname{const}(\Omega) h \sum_{j\in\mathbb{Z}^d} |\lambda_{n,j}|^2$. Hence by (6.11),

$$\begin{aligned} \|f\|_{L_2(\partial\Omega+hB)} &\le \sum_{n=0}^{\infty} \left\| \sum_{j\in\mathbb{Z}^d} \lambda_{n,j} a_{n,j} \right\|_{L_2(\partial\Omega+hB)} \\ &\le \operatorname{const}(\Omega) h^{1/2} \sum_{n=0}^{\infty} \left(\sum_{j\in\mathbb{Z}^d} |\lambda_{n,j}|^2 \right)^{1/2} \le \operatorname{const}(\Omega) h^{1/2} \|f\|_{B^{1/2}_{2,1}} \end{aligned}$$

by (6.10). □

Lemma 6.13. *Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be such that $\widehat{\psi}(0) = 1$ and put $\psi_h := h^{-d}\psi(\cdot/h)$, $h > 0$. Then for all $g \in L_2$,*

(i) $\|g - \psi_h * g\|_{L_2} \le \operatorname{const}(\psi) h^{1/2} \|g\|_{B^{1/2}_{2,\infty}}$ $\forall h > 0$ *and*

(ii) $\|g\|_{B^{1/2}_{2,\infty}} \le \operatorname{const}(\psi,\varepsilon) \left(\|g\|_{L_2} + \sup_{0<h\le\varepsilon} h^{-1/2} \|g - \psi_h * g\|_{L_2} \right)$ $\forall \varepsilon > 0$.

Proof. Let $g \in L_2$ and $h > 0$. We first prove (i). If $h \ge 1$, then $\|g - \psi_h * g\|_{L_2} \le (1 + \|\psi\|_{L_1}) \|g\|_{L_2} \le \operatorname{const}(\psi) \|g\|_{B^{1/2}_{2,\infty}} \le \operatorname{const}(\psi) h^{1/2} \|g\|_{B^{1/2}_{2,\infty}}$. So assume $0 < h < 1$. Let k be the least integer such that $2^k \ge h^{-1}$. Then

$$\begin{aligned} (2\pi)^d \|g - \psi_h * g\|^2_{L_2} &= \left\| (1 - \widehat{\psi}(h\cdot))\widehat{g} \right\|^2_{L_2} = \sum_{n=0}^{\infty} \left\| (1 - \widehat{\psi}(h\cdot))\widehat{g} \right\|^2_{L_2(A_n)} \\ &\le \sum_{n=0}^{\infty} \left\| 1 - \widehat{\psi}(h\cdot) \right\|^2_{L_\infty(A_n)} \|\widehat{g}\|^2_{L_2(A_n)} \le \operatorname{const}(\psi) \|g\|^2_{B^{1/2}_{2,\infty}} \left(\sum_{n=0}^{k} |h2^n|^2 2^{-n} + \sum_{n=k+1}^{\infty} 2^{-n} \right) \\ &\le \operatorname{const}(\psi) \|g\|^2_{B^{1/2}_{2,\infty}} h \end{aligned}$$

which proves (i). Let $\varepsilon > 0$ and put $M := \sup_{0<h\le\varepsilon} h^{-1/2} \|g - \psi_h * g\|_{L_2}$. Let k be the least positive integer such that $2^{-k} < \varepsilon$ and $\left\|\widehat{\psi}\right\|_{L_\infty(\mathbb{R}^d\setminus 2^k B)} \le 1/2$. For $n \in \{0, 1, \dots, 2k\}$ we have $2^{n/2} \|\widehat{g}\|_{L_2(A_n)} \le 2^k \|\widehat{g}\|_{L_2} \le \operatorname{const}(\psi, \varepsilon) \|g\|_{L_2}$. For $n > 2k$, put $h := 2^{k-n+1} < \varepsilon$. Then

$$\begin{aligned}2^{n/2} \|\widehat{g}\|_{L_2(A_n)} \le 2^{1+n/2} \left\|(1 - \widehat{\psi}(h\cdot))\widehat{g}\right\|_{L_2} &= 2^{1+n/2}(2\pi)^{d/2} \|g - \psi_h * g\|_{L_2} \\ &\le 2^{1+n/2}(2\pi)^{d/2} M h^{1/2} \le \operatorname{const}(\psi, \varepsilon) M.\end{aligned}$$

Therefore, $\|g\|_{B^{1/2}_{2,\infty}} = \sup_{n\in\mathbb{N}_0} 2^{n/2} \|\widehat{g}\|_{L_2(A_n)} \le \operatorname{const}(\psi, \varepsilon)(\|g\|_{L_2} + M)$. $\square$

Proof of Theorem 6.1. Let $f \in B^{m+1/2}_{2,1}$ and $|\alpha| = m$. Put $g := D^\alpha T_\Omega f$ and note that $\|g\|_{L_2} \le (2\pi)^{-d/2} |||T_\Omega f|||_{H^m} \le \operatorname{const}(d, m) \|f\|_{B^{m+1/2}_{2,1}}$. Put $\Omega_h := \partial\Omega + hB$, $h > 0$. Let ε be the largest positive real for which $m_d(\Omega\setminus\Omega_{2\varepsilon}) \ge m_d(\Omega)/2$. Let $\psi \in C_c^\infty(B)$ be such that $\widehat{\psi}(0) = 1$. We intend to estimate $\|g\|_{B^{1/2}_{2,\infty}}$ using Lemma 6.13. Let $h \in (0\,..\,\varepsilon]$. Then

$$\begin{aligned}\|g - \psi_h * g\|_{L_2(\Omega\setminus\Omega_h)} = \|D^\alpha f - \psi_h * (D^\alpha f)\|_{L_2(\Omega\setminus\Omega_h)} &\le \|D^\alpha f - \psi_h * (D^\alpha f)\|_{L_2} \\ &\le \operatorname{const}(\psi) h^{1/2} \|D^\alpha f\|_{B^{1/2}_{2,\infty}} \le \operatorname{const}(m, \psi) h^{1/2} \|f\|_{B^{m+1/2}_{2,1}}\end{aligned}$$

by Lemma 6.13 (i). Similarly,

$$\begin{aligned}&\|g - \psi_h * g\|_{L_2(\Omega_{\mathrm{ext}}\setminus\Omega_h)} = \|E D^\alpha T_\Omega f - \psi_h * (E D^\alpha T_\Omega f)\|_{L_2(\Omega_{\mathrm{ext}}\setminus\Omega_h)} \\ &\le \|E D^\alpha T_\Omega f - \psi_h * (E D^\alpha T_\Omega f)\|_{L_2} \le \operatorname{const}(\psi) h^{1/2} \|E D^\alpha T_\Omega f\|_{B^{1/2}_{2,\infty}}\end{aligned}$$

$\le \operatorname{const}(m, \psi) h^{1/2} \|f\|_{B^{m+1/2}_{2,1}}$ by Lemma 6.3. Define $G := g\chi_{\Omega_{2h}}$. Then

$$\begin{aligned}&\|g - \psi_h * g\|_{L_2(\Omega_h)} \le \|G - \psi_h * G\|_{L_2} \le \operatorname{const}(\psi) \|G\|_{L_2} = \operatorname{const}(\psi) \|g\|_{L_2(\Omega_{2h})} \\ &\le \operatorname{const}(\psi) \left(\|D^\alpha f\|_{L_2(\Omega_{2h})} + \|E D^\alpha T_\Omega f\|_{L_2(\Omega_{2h})}\right) \\ &\le \operatorname{const}(\Omega, \psi) h^{1/2} \left(\|D^\alpha f\|_{B^{1/2}_{2,1}} + \|E D^\alpha T_\Omega f\|_{B^{1/2}_{2,1}}\right) \le \operatorname{const}(\Omega, m, \psi) h^{1/2} \|f\|_{B^{m+1/2}_{2,1}}\end{aligned}$$

by Lemma 6.7 and Lemma 6.3. Therefore, $\|g - \psi_h * g\|_{L_2} \le \operatorname{const}(\Omega, m, \psi) h^{1/2} \|f\|_{B^{m+1/2}_{2,1}}$ $\forall 0 < h \le \varepsilon$. Hence, by Lemma 6.13, $\|g\|_{B^{1/2}_{2,\infty}} \le \operatorname{const}(\Omega, m, \psi) \|f\|_{B^{m+1/2}_{2,1}}$ which, after a suitable choice of ψ, completes the proof. $\square$

7. The Proof of the Main Result

Proof of Theorem 2.3. Let $\{c_\alpha\}_{|\alpha|=m}$ be the positive integers defined by $|\xi|^{2m} = \sum_{|\alpha|=m} c_\alpha \xi^{2\alpha}$, $\xi \in \mathbb{R}^d$. Then $\left\||\cdot|^m \widehat{f}\right\|^2_{L_2(A_k)} = \sum_{|\alpha|=m} c_\alpha \|\widehat{(D^\alpha f)}\|^2_{L_2(A_k)}$ $\forall f \in L_2$, $k \in \mathbb{N}_0$. Let

$f \in B^{m+1/2}_{2,1}$. Then $\|D^\alpha T_\Omega f\|_{B^{1/2}_{2,\infty}} \le \mathrm{const}(\Omega, m)\,\|f\|_{B^{m+1/2}_{2,1}}$ $\forall\,|\alpha| = m$ by Theorem 6.1. Let μ, q be as in Theorem 4.1. Since $T_\Omega f = q + \phi * \mu$, it follows that $\widehat{\mu} = \dfrac{1}{c_\phi}\,|\cdot|^{2m}\,(T_\Omega f)^\wedge$ on $\mathbb{R}^d\backslash 0$. By Theorem 4.1 (iii),

$$\|\widehat{\mu}\|_{L_2(A_0)} \le \|\mu\|_{B^{-m}_{2,2}} \le \mathrm{const}(\Omega, m)|||T_\Omega f|||_{H^m} \le \mathrm{const}(\Omega, m)\,\|f\|_{B^{m+1/2}_{2,1}}.$$

For $k \ge 1$ we have

$$\begin{aligned}
\|\widehat{\mu}\|_{L_2(A_k)} &\le \frac{1}{|c_\phi|} 2^{mk}\, \||\cdot|^m\,(T_\Omega f)\rceil\|_{L_2(A_k)} = \frac{1}{|c_\phi|}2^{mk}\left(\sum_{|\alpha|=m} c_\alpha\,\|(D^\alpha T_\Omega f)\rceil\|^2_{L_2(A_k)}\right)^{1/2} \\
&\le \frac{1}{|c_\phi|}2^{(m-1/2)k}\left(\sum_{|\alpha|=m} c_\alpha\,\|D^\alpha T_\Omega f\|^2_{B^{1/2}_{2,\infty}}\right)^{1/2} \\
&\le \mathrm{const}(\Omega, m)2^{(m-1/2)k}\,\|f\|_{B^{m+1/2}_{2,1}}, \quad \text{by Theorem 6.1.}
\end{aligned}$$

Hence $\|\mu\|_{B^{-m+1/2}_{2,\infty}} = \sup_{k\in\mathbb{N}_0} 2^{(-m+1/2)k}\,\|\widehat{\mu}\|_{L_2(A_k)} \le \mathrm{const}(\Omega, m)\,\|f\|_{B^{m+1/2}_{2,1}}$. $\square$

REFERENCES

Ad. Adams, R.A. (1975), *Sobolev Spaces*, Academic Press, New York.

Ag. Agmon, S. (1965), *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, New York.

Bej. Bejancu A. 1998, *Local accuracy for radial basis function interpolation on finite uniform grids*, manuscript.

Bu1. Buhmann M.D. 1990, *Multivariate cardinal interpolation with radial basis functions*, Constr. Approx. **8**, 225–255.

Bu2. Buhmann M.D. 1993, *New developments in the theory of radial basis function interpolation*, Multivariate Approximation: From CAGD to Wavelets (K. Jetter, F.I. Utreras, eds.), World Scientific, Singapore, pp. 35–75.

BDL. Buhmann, M.D., N. Dyn and D. Levin, *On quasi-interpolation by radial basis functions with scattered centres*, Constr. Approx. **11** (1995), 239–254.

Du1. Duchon, J. (1977), *Splines minimizing rotation-invariant seminorms in Sobolev spaces*, Constructive Theory of Functions of Several Variables, Lecture Notes in Mathematics 571 (W. Schempp, K. Zeller, eds.), Springer-Verlag, Berlin, pp. 85–100.

Du2. Duchon, J., *Sur l'erreur d'interpolation des fonctions de plusieur variables par les D^m-splines*, RAIRO Analyse Numerique **12** (1978), 325–334.

DR. Dyn, N. and A. Ron, *Radial basis function approximation: from gridded centres to scattered centres*, Proc. London Math Soc (3) **71** (1995), 76–108.

GS. Gelfand, I. M. and G. E. Shilov, *Generalized Functions*, vol. 1, Academic Press, 1964.

JL. Jia R.-Q., & Lei J., *Approximation by Multiinteger Translates of Functions Having Global Support*, J. Approx. Theory **72** (1993), 2–23.

J1. Johnson, M.J., *A bound on the approximation order of surface splines*, Constr. Approx. **14** (1998), 429–438.

J2. Johnson, M.J., *An improved order of approximation for thin-plate spline interpolation in the unit disk*, Numer. Math. (to appear).

J3. Johnson, M.J., *On the error in surface spline interpolation of a compactly supported function*, manuscript.

MN. Madych, W.R., and S.A. Nelson, *Bounds on multivariate polynomials and exponential error estimates for multiquadric interpolation*, J. Approx. Th. **70** (1992), 94–114.

Pe. Peetre, J., *New Thoughts on Besov Spaces*, Math. Dept. Duke Univ., Durham, NC, 1976.

P1. Powell M.J.D., *The theory of radial basis function approximation in 1990*, Advances in Numerical Analysis II: Wavelets, Subdivision, and Radial Functions (W.A. Light, ed.), Oxford University Press, Oxford, 1992, pp. 105–210.

P2. Powell M.J.D., *The uniform convergence of thin plate spline interpolation in two dimensions*, Numer. Math. **68** (1994), 107–128.

S1. Schaback R., *Error estimates and condition numbers for radial basis function interpolation*, Adv. Comp. Math. **3** (1995), 251–264.

S2. Schaback R., *Improved error bounds for radial basis function interpolation*, Math. Comp. (to appear).

Tr1. Triebel, H., *Theory of function spaces II*, Birkhäuser, 1992.

Tr2. Triebel, H., *Fractals and spectra*, Birkhäuser, 1997.

WS. Wu Z., & Schaback R., *Local error estimates for radial basis function interpolation of scattered data*, IMA J. Numer. Anal. **13** (1993), 13–27.