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Splines as linear combinations of B-splines. A Survey

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This paper is intended to serve as a postscript to the fundamental 1966 paper by Curry and Schoenberg on B-splines. It is also intended to promote the point of view that B-splines are truly *basic* splines: B-splines express the essentially local, but not completely local, character of splines; certain facts about splines take on their most striking form when put into B-spline terms, and many theorems about splines are most easily proved with the aid of B-splines; the computational determination of a specific spline from some information about it is usually facilitated when B-splines are used in its construction.

1. Introduction

The layout of the survey is as follows. After a short discussion of cardinal B-splines, i.e., of B-splines on a uniform knot sequence, in Section 2, B-splines for an arbitrary knot sequence are introduced in Section 3 and shown to be a basis for certain spaces of piecewise polynomial functions. Various simple properties of B-splines are listed in Section 4, and the relationship between a spline and its coordinates with respect to a B-spline basis is explored in Section 5. This leads naturally into the discussion of local spline approximation schemes, in Section 6. Results concerning existence and uniqueness of interpolating splines and the related total positivity and variation diminishing properties of B-splines are presented in Section 7. Section 8 describes the connection between splines and certain "best" interpolation schemes. Finally, Section 9 is devoted to generalized B-splines and ends with a new definition of polynomial B-splines in many variables due to I. J. Schoenberg.

No claim of completeness is made, and the author would be grateful to hear of any omissions.

The following notation is used throughout the paper, usually without further explanation:

$\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ the set of real numbers, and A^B the set of functions on B into A. Thus, $\mathbb{R}^{\mathbb{Z}}$ is the set of real bi-infinite sequences.

$m(B)$ is the linear space of bounded real functions on B, normed by $\|f\|_{\infty,B} := \sup_{x\in B}|f(x)|$. For $1 \le p \le \infty$, $\mathbb{L}_p(I)$ denotes the space of (equivalence classes of) functions f on the interval I for which $\|f\|_p := \|f\|_{p,I} := (\int_I |f|^p)^{1/p} < \infty$. $C^k(I)$ is the space of k times continuously differentiable functions on I. $\mathbb{L}_p^k(I)$ is the subspace of those $f \in C^{k-1}(I)$ whose $(k-1)$st derivative is absolutely continuous and whose k^{th} derivative is in $\mathbb{L}_p(I)$. $M^k(I)$ is the subspace of $C^{k-2}(I)$ whose elements have an absolutely continuous $(k-2)^{\text{nd}}$ derivative and a $(k-1)^{\text{st}}$ derivative of bounded variation. Finally, $\ell_p(\mathbb{Z}) := \{\alpha \in \mathbb{R}^{\mathbb{Z}} : \|\alpha\|_p := (\sum_i |\alpha_i|^p)^{1/p} < \infty\}$.

$\mathbb{P}_k$ denotes the linear space of all polynomials of order k (or, degree $< k$) with real coefficients. For a strictly increasing sequence $\boldsymbol{\xi} := (\xi_i)$, $\mathbb{P}_{k,\boldsymbol{\xi}}$ denotes the linear space of all piecewise polynomial (or, **pp**) functions of order k on $I := [\inf \xi_i, \sup \xi_i]$ with breakpoint sequence $\boldsymbol{\xi}$. Explicitly, $f \in \mathbb{P}_{k,\boldsymbol{\xi}} \iff f|_{(\xi_i,\xi_{i+1})} \in \mathbb{P}_k|_{(\xi_i,\xi_{i+1})}$, all i. In addition, $f \in \mathbb{P}_{k,\boldsymbol{\xi}}$ is taken to have two values at ξ_i, i.e., the values $f(\xi_i^-)$ and $f(\xi_i^+)$. If the reader finds it necessary to think of f as a single-valued function, he should choose some rule $f(\xi_i) := \alpha f(\xi_i^-) + (1-\alpha)f(\xi_i^+)$ (e.g., $\alpha = 1/2$) and stick with it.

If $\mathbf{v} = (v_i)$ is a sequence of nonnegative integers corresponding to $\boldsymbol{\xi}$, then $\mathbb{P}_{k,\boldsymbol{\xi},\mathbf{v}}$ denotes the linear subspace of $\mathbb{P}_{k,\boldsymbol{\xi}}$ consisting of those $f \in P_{k,\boldsymbol{\xi}}$ for which

$$\text{jump}_{\xi_i} f^{(v)} = 0 \quad \text{for} \quad v < v_i, \quad \text{all} \quad i.$$

The v–th derivative of f is also denoted by $D^v f$ as well as by $f^{(v)}$. $[\tau_0,\ldots,\tau_k]f$ stands for the k–th divided difference of f at the points $\tau_0,\ldots,\tau_k$. In particular, $[\tau_0]f = f(\tau_0)$.

$\text{const}_{\alpha,\ldots,\omega}$ denotes a constant which may depend on the quantities $\alpha,\ldots,\omega$.

2. Cardinal splines

B-splines made their first appearance in Schoenberg's 1946 paper on the approximation of equidistant data by analytic functions. There is no doubt that B-splines appear in earlier literature. They play a prominent role already in Favard's work [35], and Schoenberg has always maintained that they were already known to Laplace (see [70, p. 68]). But it is in Schoenberg's paper that they were thought important enough to be given a name, "basic k^{th}–order spline curves". Since this is the same paper in which Schoenberg introduces splines, I happily conclude that B-splines were there at the very beginning.

Schoenberg introduces the B-spline, *neé* basic spline curve, *alias* spline frequency function [29] *alias* fundamental spline function [71, 30]

$$M_k(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin u/2}{u/2}\right)^k e^{iux}du \tag{2.1}$$

and then observes that

$$M_k(x) = k[-k/2,\ 1-k/2,\dots,k/2]\,(\cdot - x)_+^{k-1}, \tag{2.2}$$

i.e., $M_k(x)$ is k times the k–th divided difference in y at the $k+1$ points $j-k/2,\ j=0,\dots,k$, of the function $(y-x)_+^{k-1} := (\max\{0,y-x\})^{k-1}$. These formulae show that M_k is the k–th convolution power of the characteristic function of the interval $[-1/2,1/2]$,

$$\begin{aligned} M_1(x) &= \begin{cases} 1, & \text{for } x\in(-1/2,1/2) \\ 0, & \text{for } x\notin[-1/2,1/2] \end{cases}, \\ M_k(x) &= (M_i * M_j)(x) = \int_{-\infty}^{\infty} M_i(x-y)M_j(y)\,dy \quad \text{for} \quad i+j=k. \end{aligned} \tag{2.3}$$

Therefore, — and this is why Laplace must have known B-splines — M_k is the density distribution of the error committed in the sum of k independent real random variables if each variable is replaced by its nearest integer value [70, p. 76].

It is easily seen from (2.2) or (2.3) that

$$M_k \in \mathbb{P}_{k,\mathbb{Z}+k/2}\cap C^{k-2} =: \quad \text{set of "spline curves of order } k\text{''}$$

as Schoenberg calls them. The subject matter of the paper [70] is the study of approximations of the form

$$Af := \sum_{n\in\mathbb{Z}} f(n)\,L(\cdot - n),$$

and the B-splines come in because they offer a convenient way of expressing, and thereby analyzing, the various pp "basic" functions L considered in the paper.

In the 60's, Schoenberg's results were rediscovered and considerably extended by those engaged in studying the mathematical aspects of the finite element method (see Aubin [1,2], Babuška [3], Bramble and Hilbert [19], Fix and Strang [38] and Strang and Fix [80], Di Guglielmo [33], and others). When restricted to the one-dimensional setting of Schoenberg's paper, these people are seen to consider approximation processes of the form

$$Af := \sum_{n\in\mathbb{Z}} (\lambda f(\cdot + n))\,L(\cdot - n)$$

for some convenient basic function L, e.g., $L = M_k$, and some linear functional $\lambda\in C^*(\mathbb{R})$, and to study the convergence behavior of

$$A_h := S_{1/h}AS_h, \qquad \text{with} \quad (S_\alpha f)(x) := f(\alpha x),$$

as $h\to 0$. The results of this study are nicely summarized by Link [57].

Schoenberg himself developed a particular aspect of this '46 paper, *viz* Cardinal spline interpolation, in considerable detail in a sequence of seven papers which appeared in the late 60's and early 70's. These papers have become the basis for his beautiful monograph [76] on cardinal spline interpolation. Readers interested in the properties and use of B-splines on *uniform* knot sequences are urged to consult that monograph.

3. B-splines defined

It was apparently Schoenberg's colleague H. B. Curry who observed that the formulation (2.2) of M_k as a k–th order difference generalizes naturally to a k–th order divided difference on arbitrary points,

$$M_{i,k}(x) \;:=\; k[t_i,\ldots,t_{i+k}]\,(\cdot - x)_+^{k-1}. \tag{3.1}$$

The resulting paper [30], though written in 1946 (see [29]), was finally published in 1966. The function $M_{i,k}$ is easily seen to be a pp function of order k with breakpoints $t_i,\ldots,t_{i+k}$, and with smoothness across each breakpoint t_j which depends on its multiplicity, i.e., on the frequency with which the number t_j occurs in the sequence $t_i,\ldots,t_{i+k}$. Further, one readily sees that

$$M_{i,k}(x) \ge 0 \quad \text{with equality if} \quad x \notin (t_i, t_{i+k}) \tag{3.2}$$

in case $t_i \le \cdots \le t_{i+k}$.

Now let $\mathbf{t} := (t_i)_{-\infty}^{\infty}$ be nondecreasing, with

$$t_{-\infty} \;:=\; \inf\, t_i, \quad t_\infty \;:=\; \sup\, t_i,$$

and let $(M_{i,k})_{-\infty}^{\infty}$ be the corresponding B-spline sequence. Then, the prescription

$$\Big(\sum_i \alpha_i M_{i,k}\Big)(x) \;:=\; \sum_i \alpha_i M_{i,k}(x), \quad \text{i.e., pointwise,}$$

makes sense for all $x \in \mathbb{R}$ and all $\alpha \in \mathbb{R}^{\mathbb{Z}}$ since, by (3.2), at most k of the terms in the second sum are nonzero for any given x.

In a later publication [73] (but see already Curry's review [28] of [70]), Schoenberg gave these functions $M_{i,k}$ the name *basic spline*, or *B-spline*, for the following reason.

Theorem 3.1 [30]. *If $\mathbf{t} := (t_i)_{-\infty}^{\infty}$ is nondecreasing, with $t_i < t_{i+k}$ and $d_i := \operatorname{card}\{j : t_j = t_i\}$, all i, then the corresponding sequence $(M_{i,k})_{-\infty}^{\infty}$ of B-splines is a basis for the linear space $\mathcal{S}_{k,\mathbf{t}}$ of all functions f on $\mathbb{R}$ which vanish off $(t_{-\infty}, t_\infty)$ and which, on $(t_{-\infty}, t_\infty)$, satisfy*

$$f|_{(t_i,t_{i+1})} \in \mathbb{P}_k|_{(t_i,t_{i+1})}, \quad \operatorname{jump}_{t_i} f^{(r)} = 0 \; for\; r < k - d_i, \quad all \quad i,$$

in the sense that the map $\mathbb{R}^{\mathbb{Z}} \to \mathcal{S}_{k,\mathbf{t}} : \alpha \mapsto \sum_i \alpha_i M_{i,k}$ is one-one and onto.

This theorem motivates the definition

$$\mathcal{S}_{k,\mathbf{t}} := \Big\{ \sum_i \alpha_i M_{i,k} : \alpha_i \in \mathbb{R}, \quad \text{all} \quad i \Big\}$$

for *arbitrary* nondecreasing $\mathbf{t}$, bi-infinite or not, with the sum taken over all i for which $(t_i,\ldots,t_{i+k})$ is a segment of $\mathbf{t}$. In particular,

$$\text{if} \quad \mathbf{t} = (t_i)_1^{n+k}, \quad \text{then} \quad \mathcal{S}_{k,\mathbf{t}} = \Big\{ \sum_{i=1}^{n} \alpha_i M_{i,k} : \alpha \in \mathbb{R}^n \Big\}\ .$$

Further, we will call $\mathcal{S}_{k,\mathbf{t}}$ the collection of (polynomial) **splines of order k with knot sequence $\mathbf{t}$**.

Corollary (Construction of a B-spline basis for $\mathbb{P}_{k,\boldsymbol{\xi},\mathbf{v}}$). *Let $\boldsymbol{\xi} := (\xi_i)_1^{p+1}$ be strictly increasing, $\mathbf{v} := (v_i)_1^{p+1}$ be a corresponding sequence of integers in $[0,k]$ with $v_1 = v_{p+1} = 0$, and let $\mathbb{P}_{k,\boldsymbol{\xi},\mathbf{v}}$ be the space of pp functions of order k on $[\xi_1, \xi_{p+1}]$ with breakpoints $\xi_2,\ldots,\xi_p$ and continuous v–th derivative at ξ_i for $v < v_i$, all i. If*

$$\mathbf{t} \;:=\; (t_i)_1^{n+k} = (\underbrace{\xi_1,\ldots,\xi_1}_{v_1 = k}, \underbrace{\xi_2,\ldots,\xi_2}_{v_2}, \ldots, \underbrace{\xi_{p+1},\ldots,\xi_{p+1}}_{v_{p+1} = k},$$

then $n = k + \sum_2^p (k - v_i)$*, and the sequence* $(M_{i,k})_1^n$ *of B-splines (restricted to* $[\xi_1, \xi_{p+1}]$*) of order* k *for the knot sequence* $\mathbf{t}$ *is a basis for* $\mathbb{P}_{k,\boldsymbol{\xi},\mathbf{v}}$.

For k even, $k = 2m$, it is customary to single out the subspace S of so-called "natural" splines in $\mathbb{P}_{k,\boldsymbol{\xi},\mathbf{v}}$. This subspace consists of those f in $\mathbb{P}_{k,\boldsymbol{\xi},\mathbf{v}}$ for which $f|_{(\xi_1,\xi_2)}$ and $f|_{(\xi_p,\xi_{p+1})}$ are both of degree $< m$ (see Section 8 below (8.8)). Greville [44] has described the following B-spline like basis for S,

$$\widehat{M}_{m+1,k}, \ldots, \widehat{M}_{k,k}, M_{k+1,k}, \ldots, M_{n-k,k}, \widehat{M}_{n-k+1,k}, \ldots, \widehat{M}_{n-m,k}$$

with the special functions $\widehat{M}_{i,k}$ defined as follows:

$$\begin{aligned} \widehat{M}_{i,k}(x) \; &:= k[t_{k+1}, \ldots, t_{i+k}]\,(\cdot - x)_+^{k-1} \quad \text{for } i \le k \\ \widehat{M}_{n-i,k}(x) \; &:= (-1)^k k[t_{n-i}, \ldots, t_n]\,(x - \cdot)_+^{k-1} \quad \text{for } i < k. \end{aligned} \tag{3.3}$$

For a different generalization of M_k to a "B-spline" with multiple knots (which are otherwise uniformly spaced), see Schoenberg and Sharma [77] and Lecture 5 of Schoenberg's monograph [76]. Certain technical assumptions made by them in their construction have recently been removed by Lee [56].

4. Simple properties of the B-spline

In this section, we list some simple properties of the B-spline, some of which are enlarged upon in subsequent sections. The definition of $M_{i,k}$ as a divided difference together with Taylor's formula with integral remainder readily imply that, for $t_i < t_{i+k}$,

$$[t_i, \ldots, t_{i+k}]\, f = \int M_{i,k}(s)\, f^{(k)}(s)\, ds/k!\,, \quad \text{all} \quad f \in \mathbb{L}_1^k[t_i, t_{i+k}]. \tag{4.1}$$

In particular,

$$\int M_{i,k}(s)\, ds = 1. \tag{4.2}$$

This shows that, on $[t_i, t_{i+k}]$,

$$\varphi(x) \; := \int_{-\infty}^{x} M_{i,k}(s)\, ds \tag{4.3}$$

is a spline of order $k+1$ with knots $t_i, \ldots, t_{i+k}$ and rises strictly monotonely from a value of 0 at t_i (and to the left of t_i) to a value of 1 at t_{i+k} (and to the right of t_{i+k}). This function is therefore useful in constructing piecewise monotone spline interpolants as is done in Passow [66], but without having to resort to multiple knots as he does. One obtains his construction as a special case by letting half the t_j's equal t_i and the other half equal t_{i+k}. Use of φ also produces a very quick proof that splines have property SAIN with respect to interpolation at a given set of points and the uniform norm (see Chui, Rozema, Smith, and Ward [24], who use (4.3) in the form (4.11)). Because of its local and monotone character, φ has also been instrumental in DeVore's successful investigation [32] of the order of approximation to smooth monotone functions by monotone splines.

It seems more convenient in computations to use the **normalized** B-spline

$$\begin{aligned} N_{i,k}(x) \; &:= \big([t_{i+1}, \ldots, t_{i+k}] - [t_i, \ldots, t_{i+k-1}]\big)\,(\cdot - x)_+^{k-1} \\ &= (t_{i+k} - t_i) M_{i,k}(x)/k, \end{aligned} \tag{4.4}$$

since it insures (see (5.8) below) that

$$\sum_{i=1}^{n} N_{i,k} = 1 \quad \text{on} \quad [t_k, t_{n+1}]. \tag{4.5}$$

Note that then

$$N_{i,k}^{(1)} = M_{i,k-1} - M_{i+1,k-1} = \frac{k-1}{t_{i+k-1}-t_i}\, N_{i,k-1} \;-\; \frac{k-1}{t_{i+k}-t_{i+1}}\, N_{i+1,k-1}. \tag{4.6}$$

If one follows [8] and applies Leibniz' formula

$$[t_i,\ldots,t_j]\,(fg) = \sum_{r=i}^{j} [t_i,\ldots,t_r]\,f\,[t_r,\ldots,t_j]\,g \tag{4.7}$$

for the divided difference of a product to

$$(\,\cdot\, - x)_+^j = (\,\cdot\, - x)_+^{j-1}(\,\cdot\, - x)$$

and notes that all divided differences of $(\,\cdot\, - x)$ of order > 1 vanish, then one obtains the recurrence relation

$$[t_i,\ldots,t_{i+k}]\,(\,\cdot\, - x)_+^j = \left(\frac{x-t_i}{t_{i+k}-t_i}\,[t_i,\ldots,t_{i+k-1}] + \frac{t_{i+k}-x}{t_{i+k}-t_i}\,[t_{i+1},\ldots,t_{i+k}] \right) (\,\cdot\, - x)_+^{j-1}\,, \tag{4.8}$$

which in turn implies that

$$\frac{k-j-1}{k-1}\, N_{i,k}^{(j)}(x) = \frac{x-t_i}{t_{i+k-1}-t_i}\, N_{i,k-1}^{(j)}(x) + \frac{t_{i+k}-x}{t_{i+k}-t_{i+1}}\, N_{i+1,k-1}^{(j)}(x). \tag{4.9}$$

For $j = 0$, this recurrence was found by the author [8] and by L. Mansfield, and by Cox [27] who proved it by a different argument and for distinct knots only, and gave a backward error analysis in that case for the evaluation algorithm based on the recurrence. The recurrence provides a scheme for the stable evaluation of B-splines since, on the interval (t_i, t_{i+k}) of interest, i.e., on the support of $N_{i,k}$, both weights in (4.9) are positive. This observation also allows us to establish, by induction on k, that

$$N_{i,k} > 0 \quad \text{on} \quad (t_i, t_{i+k}). \tag{4.10}$$

Similar recurrence relations for the integral of a B-spline have been given by Gaffney [39], and for the integral of products of B-splines by Lyche, Schumaker, and the author [17]. In this connection, we note that

$$\int_{-\infty}^{x} M_{i,k}(s)\,ds = \sum_{j=i}^{i+r} N_{j,k+1}(x) \quad \text{for} \quad x \le t_{i+r+1}. \tag{4.11}$$

B-splines are convenient for relating splines with multiple knots to splines with simple knots and vice versa (e.g., [7], Rice [68], Burchard [23], also the paper by P. Smith in these proceedings), since a B-spline *is a continuous function of its knots*, within reason. Specifically, writing

$$N_{t_i,\ldots,t_{i+k}} \;:=\; N_{i,k}$$

to stress the dependence of $N_{i,k}$ on its knots $t_i,\ldots,t_{i+k}$, *the map*

$$(\tau_j)_0^k \mapsto N_{\tau_0,\ldots,\tau_k}$$

is continuous as a map from $\{\tau \in \mathbb{R}^{k+1} | \tau_0 \le \cdots \le \tau_k,\ \tau_j < \tau_{j+k-1}\}$ *to* $C(\mathbb{R})$; *it is also continuous as a map from* $\{\tau \in \mathbb{R}^{k+1} | \tau_0 \le \cdots \le \tau_k,\ \tau_0 < \tau_k\}$ *to* $\mathbb{L}_p(\mathbb{R})$ *for every* $1 \le p < \infty$.

The precise behavior of $N_{i,k}$ near the boundary of its support can be read off directly from its definition as a divided difference. Since

$$(y-x)_+^{k-1} - (-)^k\,(x-y)_+^{k-1} = (y-x)^{k-1},$$

and the k–th divided difference of a polynomial of order k vanishes, one can write $N_{i,k}$ also in the form

$$N_{i,k}(x) = (-)^k \left([t_{i+1},\ldots,t_{i+k}] - [t_i,\ldots,t_{i+k-1}]\right)(x-\cdot)_+^{k-1}\,. \tag{4.12}$$

From this, one infers at once that, e.g., for x near t_i,

$$N_{i,k}(x) = (x-t_i)_+^{k-r}\prod_{j=1}^{k-r}\frac{k-j}{t_{i+k-j}-t_i} + O\big((x-t_i)_+^{k-r+1}\big) \tag{4.13}$$

if $t_i = t_{i+r-1} < t_{i+r}$, hence

$$N_{j,k}(x) = \begin{matrix} O\Big((x-t_i)N_{i,k}(x)\Big) & \text{for } j>i \quad \text{as} \quad x\to t_i \\ O\Big((x-t_{i+k})N_{i,k}(x)\Big) & \text{for } j<i \quad \text{as} \quad x\to t_{i+k}. \end{matrix} \tag{4.14}$$

If $t_i < \cdots < t_{i+k}$, then $N_{i,k}$ has a zero of order $k-1$ at t_i by (4.13) and also a $k-1$ fold zero at t_{i+k} by symmetry. This implies that

$$0 = N_{i,k}^{(j-r)}(t_{j+k}) = \int_{t_i}^{t_{i+k}} (t_{i+k}-s)^{r-1}N_{i,k}^{(j)}(s)\,ds/(r-1)!\,, \quad r=1,\ldots,j; \quad j=1,\ldots,k-1, \tag{4.15}$$

showing that $N_{i,k}^{(j)}$ *is orthogonal to* $\mathbb{P}_j$ *on* $[t_i, t_{i+k}]$, $j=1,\ldots,k-1$. (This fact was pointed out to me in 1973 by H. G. Burchard.)

5. The B-spline series

In this section the relationship

$$\sum_i \alpha_i N_i \leftrightarrow \alpha \tag{5.1}$$

between a spline and the sequence of its B-spline coefficients (with respect to the normalized B-splines) is discussed. Further aspects of this relationship will be mentioned in subsequent sections. From here on, we suppress the subscript k in $N_{i,k}$ and $M_{i,k}$ except when necessary. Also, we restrict the knot sequence $\mathbf{t}$ to be bi-infinite in order to avoid (mostly notational) complications. This is no essential restriction since any spline can always be extended to a spline with a bi-infinite knot sequence merely by adding to its expansion appropriate B-splines with zero coefficients.

A B-spline series may be differentiated by differencing the coefficients. Precisely, repeated application of (4.6) gives

$$\Big(\sum_i \alpha_i N_{i,k}\Big)^{(j)} = \sum_i \alpha_i^{(j)} N_{i,k-j} \tag{5.2a}$$

with

$$\alpha_i^{(j)} := \begin{cases} \alpha_i & , \quad j=0 \\ \dfrac{\alpha_i^{(j-1)} - \alpha_{i-1}^{(j-1)}}{(t_{i+k-j}-t_i)/(k-j)} & , \quad j>0 \end{cases} \tag{5.2b}$$

The recurrence relation (4.9) (with $j=0$) allows one to express a B-spline series as a series of lower order, but with polynomial coefficients. Precisely,

$$\sum_i \alpha_i N_{i,k}(x) = \sum_i \alpha_i^{[j]}(x) N_{i,k-j}(x) \tag{5.3a}$$

with

$$\text{(5.3}b\text{)} \qquad \alpha_i^{[j]}(x) \;:=\; \begin{cases} \alpha_i & , \quad j=0 \\ \dfrac{(x-t_i)\alpha_i^{[j-1]}(x)+(t_{i+k-j}-x)\alpha_{i-1}^{[j-1]}(x)}{t_{i+k-j}-t_i} & , \quad j>0\,. \end{cases}$$

In particular, $\alpha_i^{[k-1]}$ is a polynomial of degree $< k$ which agrees with $\sum_i \alpha_i N_{i,k}$ on $[t_i^+, t_{i+1}^-]$. Hence, (5.3) can be used to evaluate $\sum_i \alpha_i N_{i,k}$ at $x \in [t_i^+, t_{i+1}^-]$ by repeated formation of averages, starting with the k numbers $\alpha_{i-k+1}, \ldots, \alpha_i$ (see the first algorithm in [8]).

The quasi-interpolant of Fix and the author [16] provides an ofttimes convenient means for computing the B-spline coefficients of a given spline. The quasi-interpolant makes use of the linear functional λ_i given the rule

$$\text{(5.4)} \qquad \lambda_i f \;:=\; \lambda_{\tau_i,\psi_{i,k}} f \;:=\; \sum_{j<k} (-)^{k-1-j} \psi_{i,k}^{(k-1-j)}(\tau_i)\, f^{(j)}(\tau_i).$$

Here,

$$\psi_{i,k}(x) \;:=\; (t_{i+1}-x)\cdot \ldots \cdot (t_{i+k-1}-x)/(k-1)!$$

and τ_i is an *arbitrary* point in (t_i, t_{i+k}). Then, as one verifies directly [16],

$$\text{(5.5)} \qquad \lambda_i N_j = \delta_{i,j}, \quad \text{all} \quad j.$$

Since λ_i has support at a point only, it follows that $\lambda_i\bigl(\sum_j \alpha_j N_j\bigr) = \alpha_i$. †

The usefulness of this functional was demonstrated in [9]. For instance, it provides a quick proof of Theorem 3.1 and its corollary. As another instance, it provides a quick proof of the fact due to Curry and Schoenberg [30] that *B-splines are splines of minimal support*: If $f \in \mathcal{S}_{k,\mathbf{t}}$ has its support in (t_r, t_{r+s}) and $s < k$, then, for each i, one can choose τ_i in $(t_i, t_{i+k})\backslash(t_r, t_{r+s})$, hence then $\lambda_i f = 0$, all i, i.e., $f = 0$.

More generally, one obtains

Lemma 5.1. *If $t_i < t_{i+k}$, all i, then* $\overline{\operatorname{supp}(\sum_i \alpha_i N_i)} = \overline{\cup_{\alpha_i \neq 0} \operatorname{supp} N_i}$.

In order to compute the coefficients of specific splines, we observe that, *for f, $\psi \in \mathbb{P}_k$, $\alpha(\tau) := \lambda_{\tau,\psi} f$ is constant as a function of τ*, as is clear from the fact that $\alpha'(\tau) = \psi(\tau) f^{(k)}(\tau) - (-)^k \psi^{(k)}(\tau) f(\tau)$. Hence, with $\tau = y$, we get that

$$\lambda_i (y - \cdot\,)^{k-1} = \lambda_{y,\psi_{i,k}} (y - \cdot\,)^{k-1} = \psi_{i,k}(y)\,(-)^{k-1}(k-1)!\,.$$

This shows that

$$\text{(5.6)} \qquad (y-x)^{k-1} = \sum_i (y-t_{i+1})\cdots(y-t_{i+k-1}) N_{i,k}(x),$$

which is **Marsden's identity** [61]. More generally,

$$\lambda_i (y - \cdot\,)^{k-p}/(k-p)! \;= (-)^{p-1} \psi_{i,k}^{(p-1)}(y)\,(-)^{k-p},$$

so

$$\text{(5.7)} \qquad (y-x)^{k-p}/(k-p)! \;= (-)^{k-1} \sum_i \psi_{i,k}^{(p-1)}(y) N_{i,k}(x),$$

† added 1978: For a uniform knot sequence $\mathbb{Z}$ and $\tau_i = t_i^* := (t_{i+1} + \cdots + t_{i+k-1})/(k-1)$, (5.4)–(5.5) reduces to (24), (28) in I. J. Schoenberg's "Cardinal interpolation and spline functions. II", J. Approx. Theory **6** (1972), 404–420.

and, in particular, with $p = k$,

$$1 = \sum_i N_{i,k}. \tag{5.8}$$

Of course, all these identities hold on $(t_{-\infty}, t_\infty)$ only. One obtains similarly that

$$(y-x)_+^{k-1} = \sum_i (y - t_{i+1})_+ \cdots (y - t_{i+k-1})_+ N_{i,k}(x), \quad \text{for} \quad y \in \mathbf{t}. \tag{5.9}$$

For the uniform knot sequence $\mathbf{t} = \mathbb{Z}$ and for $k = 4$, one can find (5.6) and (5.9) already in Schoenberg [70].

Identities (5.6) and (5.9) illustrate a point to be made repeatedly in this survey, *viz* how closely a spline function is modelled by its B-spline coefficients. To elaborate on this point a little, note that, with $\boldsymbol{\tau} := (\tau_i)_0^k$ any subsequence of $\mathbf{t}$, (5.9) implies that

$$k[\tau_0, \ldots, \tau_k]\,(\,\cdot - x)_+^{k-1} = \sum_i \alpha_{\boldsymbol{\tau}}(i) M_{i,k}(x) \tag{5.10a}$$

where

$$\alpha_{\boldsymbol{\tau}}(i) \;:=\; (t_{i+k} - t_i)\,[\tau_0, \ldots, \tau_k]\,(\,\cdot - t_{i+1})_+ \cdot \ldots \cdot (\,\cdot - t_{i+k-1})_+ \;\geq\; 0 \tag{5.10b}$$

This supplies the formula

$$[\tau_0, \ldots, \tau_k] = \sum_i \alpha_{\boldsymbol{\tau}}(i)\,[t_i, \ldots, t_{i+k}] \tag{5.11}$$

for the k–th divided difference at some points in terms of the k–th divided differences at the points of a refinement of those points, with the coefficients nonnegative. The existence of such a formula with nonnegative weights $\alpha_{\boldsymbol{\tau}}$ was already known to Favard [35]. The formula is clearly a discrete analog of (4.1), and $\alpha_{\boldsymbol{\tau}}$ deserves to be called a **discrete B-spline with knots $\boldsymbol{\tau}$**. Indeed, $\alpha_{\boldsymbol{\tau}}$ has been called just that by Schumaker [79] in the special case when $\mathbf{t}$ is uniform, $t_i = t_0 + ih$, all i. In that case, if $f \in \mathcal{S}_{k,\mathbf{t}}$ has only the active knots $t_{i_0}, \ldots, t_{i_r}$ and $f = \sum_i \alpha_f(i) N_{i,k}$, then α_f is a discrete spline of order k with knots $i_0, \ldots, i_r$ in the sense of Mangasarian and Schumaker [60]. This means that, for each j, $\alpha_f(i)$ is a polynomial of order k in i on $i_j - k < i < i_{j+1}$. It should be said, though, that Mangasarian and Schumaker did not view discrete splines in this light as B-spline coefficients of continuous splines. They arrived at discrete splines as the solution of certain discrete minimization problems.

The size of the i–th B-spline coefficient of a spline is closely tied (at least for moderate k) to the size of that spline "nearby", i.e., on (t_i, t_{i+k}), as can be proved [9] with the aid of the linear functional (5.4). Slightly more refined arguments produce the following explicit result.

Theorem 5.1 [13]. *Let D_k be the smallest number with the property that for every $\mathbf{t}$, every i, and every $a < b$ with*

$$t_i \leq a \leq t_{i+1}, \quad t_{i+k-1} \leq b \leq t_{i+k},$$

there exists $h_i \in \mathbb{L}_\infty$ such that

$$\operatorname{supp}\, h_i \subseteq [a,b], \quad \|h_i\|_\infty \leq D_k/(b-a), \quad \int h_i N_j = \delta_{ij}, \quad \textit{all} \quad j. \tag{5.12}$$

Then $(\pi/2)^k/2 \leq D_k \leq 2k\;\, 9^{k-1}$.

Numerical evidence presented in [13] strongly suggests that actually $D_k \sim 2^k$.

The theorem implies that

$$|\alpha_i|(t_{i+k} - t_i)^{1/p} \leq D_k \|\sum_i \alpha_i N_i\|_{p,[t_i,t_{i+k}]}, \quad 1 \leq p \leq \infty, \tag{5.13}$$

which leads to

Theorem 5.2 [9] †. *Let E be the diagonal matrix $\lceil \ldots, (t_{i+k} - t_i)/k, \ldots \rfloor$. Then*

$$D_k^{-1} \|E^{1/p}\alpha\|_p \le \|\sum_i \alpha_i N_i\|_p \le \|E^{1/p}\alpha\|_p, \quad \text{all } \alpha \in \mathbb{R}^{\mathbb{Z}}, \quad 1 \le p \le \infty.$$

In particular, $\sum_i \alpha_i N_i \in \mathbb{L}_p$ if and only if $E^{1/p}\alpha \in \ell_p(\mathbb{Z})$.

The proof of the upper bound for $\|\sum_i \alpha_i N_i\|_p$ makes use of the fact that the N_i's are nonnegative and sum up to 1 while, by (4.2) and (4.4), $\int N_{i,k} = (t_{i+k} - t_i)/k$.

Corollary 1 [9]. *Let $N_{i,k,p} := (k/(t_{i+k} - t_i))^{1/p} N_{i,k}$. For $1 \le p < \infty$, $(N_{i,k,p})$ is a Schauder basis for $\mathcal{S}_{k,\mathbf{t}} \cap \mathbb{L}_p(\mathbb{R})$.*

We note the estimates

$$k^{1/p}/k \le \|N_{i,k,p}\|_p \le 1. \tag{5.14}$$

Corollary 2 [12]. *Let $\mathbf{t}$ be finite, infinite, or bi-infinite, let $G := (\int N_{i,k,2} N_{j,k,2})$, and let $G^{-1} = (\alpha_{ij})$. Then G^{-1} decays exponentially away from the diagonal. Explicitly,*

$$|\alpha_{ij}| \le \text{const } q^{|i-j|}$$

with $q = (1 - D_k^{-2})^{1/(2k-2)} \in (0, 1)$ and $\text{const} = D_k^3/q^{k-1}$ both depending only on k and not on $\mathbf{t}$.

This corollary was proved earlier for a finite *uniform* $\mathbf{t}$ by Domsta [34], and then used by Ciesielski and Domsta [26] in the construction of a basis for $C^{k-2}[0,1]^d$ which is, at the same time, also a basis for $\mathbb{L}_p^{k-2}[0,1]^d$ for $1 \le p < \infty$. The corollary was used in [12] for a somewhat related purpose, *viz* in order to show that least-squares approximation from $\mathcal{S}_{k,\mathbf{t}}$, considered as a map on $\mathbb{L}_p$, can be bounded in terms of the global mesh ratio

$$M_{\mathbf{t}} := \sup_{i,j} (t_{i+k} - t_i)/(t_{j+k} - t_j).$$

Corollary 3 [7], [13]. *Let $m\mathcal{S}_{k,\mathbf{t}} := \mathcal{S}_{k,\mathbf{t}} \cap m(\mathbb{R})$ be the subspace of bounded splines of order k with knot sequence $\mathbf{t}$. Then the rule $\alpha \mapsto \sum_i \alpha_i N_i$ maps $\ell_\infty(\mathbb{Z})$ onto $m\mathcal{S}_{k,\mathbf{t}}$. Further, with $\varphi : \ell_\infty(\mathbb{Z}) \to m\mathcal{S}_{k,\mathbf{t}} : \alpha \mapsto \sum_i \alpha_i N_i$, the condition (number) $\text{cond}_{k,\mathbf{t}} := \|\varphi\| \, \|\varphi^{-1}\|$ of the basis (N_i) for $m\mathcal{S}_{k,\mathbf{t}}$ is bounded by D_k independent of $\mathbf{t}$.*

Since $(D_1, D_2, D_3, D_4, \ldots) \le (1, \ 2.5, \ 5.3, \ 10.1, \ldots)$, this shows the B-spline basis to be well conditioned, independent of $\mathbf{t}$, for "small" k.

Finally, for another illustration of the fact that B-spline coefficients "model" the function they represent, observe that, for the particular choice

$$\tau_i = \tau_i^* := (t_{i+1} + \cdots + t_{i+k-1})/(k-1), \tag{5.15}$$

the coefficient of $f^{(1)}(\tau_i)$ in (5.4) vanishes. Then

$$\lambda_i f = f(\tau_i^*) + b_i$$

with

$$\begin{aligned} |b_i| &= \Big| \sum_{j=2}^{k-1} (-)^{k-1-j} \psi_{ik}^{(k-1-j)}(\tau_i^*) f^{(j)}(\tau_i^*) \Big| \\ &\le \text{const}_k \, (\max \, \Delta t_r)^2 \max_{2 \le j < k} \|f^{(j)}\|_\infty. \end{aligned}$$

Therefore, if, e.g., f is a fixed spline with $\|f^{(j)}\|_\infty < \infty$ for $2 \le j < k$, and we write f as a linear combination of B-splines on a knot sequence $\mathbf{t}$ which refines the knot sequence for f, then the resulting B-spline coefficient sequence α for f satisfies

$$\alpha_i = f(\tau_i^*) + O\big(\max \, (\Delta t_r)^2\big).$$

† added 1978: For a uniform $\mathbf{t}$, this theorem can be found in I. J. Schoenberg, *loc.cit.* which was the original inspiration for that part of [9].

6. Local spline approximation

Because of their local support, B-splines have been instrumental in the construction of local spline interpolation and approximation schemes. In such a scheme, the approximation is taken in the form

$$Af := \sum_i (\mu_i f) N_i \tag{6.1}$$

with μ_i a linear functional with support in $\operatorname{supp} N_i = (t_i, t_{i+k})$. Since then $(Af)|_{(t_j,t_{j+1})}$ depends only on $f|_{(t_{j+1-k},t_{j+k})}$, such an approximation scheme is capable of reflecting, and taking advantage of, the local behavior of f.

Lemma 6.1. *If A reproduces $\mathbb{P}_k$ on $(t_{-\infty}, t_\infty)$, then*

$$\|f - Af\|_{\infty,(t_j,t_{j+1})} \le \big(\sup_i \|\mu_i\|\big) \operatorname{dist}_{\infty,(t_{j+1-k},t_{j+k})}(f, \mathbb{P}_k).$$

The condition that A reproduce $\mathbb{P}_k$ is certainly satisfied in case A is a projector. This will happen iff (μ_i) is dual to (N_i), i.e., $\mu_i N_j = \delta_{ij}$, all i, j. In such a case, Af interpolates f at (μ_i) in the sense that $\mu_i A f = \mu_i f$, all i. A linear functional μ_i satisfying

$$\operatorname{supp} \mu_i \subseteq \operatorname{supp} N_i = (t_i, t_{i+k}), \quad \mu_i N_j = \delta_{ij}, \quad \text{all} \quad j, \tag{6.2}$$

seems to have been constructed for the first time in [5], for the purpose of demonstrating the linear independence over an interval of all B-splines which do not vanish identically on that interval. Since then, such linear functionals have been constructed in various ways and for a variety of jobs. A summary and detailed discussion is given in [13].

The first local spline interpolation scheme seems to have been Birkhoff's local spline approximation by moments [4]. A corrected and extended version can be found in [6]. The scheme was not given in the form (6.1). It was therefore somewhat of a surprise to find that local spline approximation by moments is a special case of the quasi-interpolant of Fix and the author [16], i.e., of the form (6.1) with $\mu_i = \lambda_i$ given by (5.4) with $\tau_i = t_{i+k/2}$, all i.

The quasi-interpolant approximates well to f *and* its first $k-1$ derivatives, but requires values of f and of its derivatives for its construction. An earlier scheme [7] constructs μ_i involving only function evaluations, and satisfying even

$$\operatorname{supp} \mu_i \subseteq (t_{i+1}, t_{i+k-1}), \quad \mu_i N_j = \delta_{ij}, \quad \text{all} \quad j, \tag{6.3}$$

and so that $\sup_i \|\mu_i\| < \infty$. This is possible since it can be shown that

$$\begin{aligned} D_{k,\infty} &:= \sup_{\mathbf{t}} \sup_i \inf \Big\{ \|\mu\| : \mu \in C^*[t_{i+1}, t_{i+k-1}], \quad \mu N_j = \delta_{ij}, \quad \text{all} \quad j \Big\} \\ &= \sup_{\mathbf{t}} \sup_i \; 1/\operatorname{dist}_{\infty,[t_{i+1},t_{i+k-1}]} \Big(N_i, \; \operatorname{span}(N_j)_{j \ne i} \Big) \end{aligned} \tag{6.4}$$

is finite. In fact, it follows from Theorem 5.1 that $D_{k,\infty} \le D_k < \infty$. Therefore, one finds that, for this scheme,

$$\|f - Af\|_{\infty,(t_j,t_{j+1})} \le D_{k,\infty} \operatorname{dist}_{\infty,(t_{j+2-k},t_{j+k-1})}(f, \mathbb{P}_k). \tag{6.5}$$

But it is not clear how well the derivatives of Af approximate those of f. Also, A is not applicable to arbitrary $f \in \mathbb{L}_p$.

The latter objection can be overcome by choosing μ_i of the form

$$\mu_i f = \int f h_i,$$

with $h_i \in \mathbb{L}_\infty[t_i, t_{i+k}]$ chosen as in Theorem 5.1 to satisfy (5.12). The resulting linear projector P,

$$Pf := \sum_i \Big(\int f h_i\Big) N_i, \tag{6.6}$$

is local and is bounded as a map on $\mathbb{L}_p$ by D_k for each $p \in [1, \infty]$ and independently of $\mathbf{t}$ [11]. But, in order to obtain also good approximations to derivatives (regardless of $\mathbf{t}$, i.e., without recourse to Markov's inequality), Lyche and Schumaker [59] found it necessary to give up the condition that Af interpolate f and to revert to the weaker condition that A merely reproduce $\mathbb{P}_k$. Such local approximation schemes have been further investigated by Demko [31].

An important local spline approximation scheme (which only reproduces $\mathbb{P}_2$) is Schoenberg's variation diminishing spline approximation. It will be discussed in the next section.

The use of local spline approximation schemes for gauging accurately the degree of approximation by splines is further pursued in DeVore's contribution to these proceedings.

We close this section with the remark that the dual to the linear projector P in (6.6), i.e., the linear projector P' given by

$$P'g := \sum_i \Big(\int f N_i\Big) h_i, \tag{6.7}$$

is helpful in settling two questions of "smooth" interpolation. The first, raised originally by Schoenberg [74] and partially answered by Golomb [42], concerns the existence of $g \in \mathbb{L}_p^k(\mathbb{R})$ which satisfies $g(t_i) = \alpha_i$, all i, for a given $\alpha \in \mathbb{R}^{\mathbb{Z}}$ and a given $\mathbf{t} = (t_i)$ taken strictly increasing for simplicity. Let $[t_i, \ldots, t_{i+k}]\alpha$ be the k–th divided difference of the data at $t_i, \ldots, t_{i+k}$ and recall the diagonal matrix $E := \lceil \ldots, (t_{i+k} - t_i)/k, \ldots \rfloor$ of the preceding section. Then it is easily seen that having $E^{1/p}([t_i, \ldots, t_{i+k}])$ in ℓ_p is a necessary condition for the existence of such a g. To see that this is also a sufficient condition, observe [13] that the function g, given by the conditions that $g(t_i) = \alpha_i, \quad i = 1, \ldots, k$ and that

$$g^{(k)} = (k-1)! \sum_i \Big([t_i, \ldots, t_{i+k}]\alpha\Big)(t_{i+k} - t_i) h_i, \tag{6.8}$$

is in $\mathbb{L}_p^k$ by Theorem 5.1 in case $E^{1/p}([t_i, \ldots, t_{i+k}]) \in \ell_p$, and agrees with α at $\mathbf{t}$ since, by (4.2), it has the same k–th divided differences at the points of $\mathbf{t}$ as does α.

The particular interpolant g to the given data α at $\mathbf{t}$ just constructed has the property that, on $[t_j, t_{j+1}]$, at most k of the h_i in (6.8) are not zero, while, by Theorem 5.1, $\|h_i\|_\infty (t_{i+k} - t_i) \le D_k$, all i. This proves [13] that, for given $\mathbf{t}$ and given α, there exists $g \in \mathbb{L}_\infty^k$ so that $g|_{\mathbf{t}} = \alpha$ and, for all $t_j < t_{j+1}$,

$$\|g^{(k)}\|_{\infty,[t_j,t_{j+1}]} \le \text{const} \max_{[t_j,t_{j+1}] \subseteq [t_i,t_{i+k}]} k!\, |[t_i, \ldots, t_{i+k}]\alpha|$$

for some const $\le D_k$. This answers a question by H.-O. Kreiss as to the existence and the size of such a const.

7. Total positivity and the variation diminishing properties of B-splines

The strict positivity of $N_{i,k}$ on (t_i, t_{i+k}) (see (4.10)) is a particular instance of the Schoenberg-Whitney theorem and the variation diminishing properties of B-splines, the subject of this section. A thorough discussion of these matters in the more general context of Chebyshev splines can be found in Chapter 10 of Karlin's book on total positivity [47].

Throughout this section, the knot sequence is taken to be finite,

$$\mathbf{t} = (t_i)_1^{n+k}, \quad \text{nondecreasing with } t_i < t_{i+k}, \quad \text{all} \quad i,$$

and $(N_i)_1^n$ is the corresponding sequence of B-splines of order k. $\mathcal{S}_{k,\mathbf{t}}$ has then dimension n. We consider spline interpolation at points $\tau_1 < \cdots < \tau_n$. This amounts to finding, for given f, $\alpha \in \mathbb{R}^n$ so that

$$\sum_{j=1}^n \alpha_j N_j(\tau_i) = f(\tau_i), \quad i = 1, \ldots, n. \tag{7.1}$$

The question of existence and uniqueness of such an interpolant was settled some time ago.

Theorem 7.1 (Schoenberg-Whitney [78]). *Let*

$$S := \left\{ \sum_{j=1}^{k} \alpha_j x^{j-1} + \sum_{j=k+1}^{n} \alpha_j (x - t_j)_+^{k-1} : \alpha \in \mathbb{R}^n \right\} \tag{7.2}$$

with $t_{k+1} < \cdots < t_n$. If $\tau_1 < \cdots < \tau_n$, then S contains, for arbitrary f, an s such that $s(\tau_i) = f(\tau_i)$, $i = 1, \ldots, n$ iff $\tau_{i-k} < t_i < \tau_i$, $i = k+1, \ldots, n$.

In this connection, it is interesting to note the following theorem published with an elegant proof in 1939, and pointed out to me by Allan Pinkus.

Theorem (Krein and Finkelstein [55]). *Let G be a Green's function for the k–th order linear differential operator*

$$L = \sum_{j=0}^{k} p_j D^j$$

with $p_j \in C[a,b]$, all j, and p_k never zero on $[a,b]$. Specifically, assume that G is of the form

$$G(x,y) = \begin{cases} \displaystyle\sum_{j=1}^{p} \varphi_j(x)\psi_j(y) & \text{for} \quad x > y, \\[2ex] \displaystyle\sum_{j=1}^{q} \widehat{\varphi}_j(x)\widehat{\psi}_j(y) & \text{for} \quad x < y, \end{cases}$$

with both $(\varphi_j)_1^p$ and $(\widehat{\varphi}_j)_1^q$ linearly independent and in $\ker L$. If

$$\det\, G \begin{pmatrix} x_1, \ldots, x_r \\ y_1, \ldots, y_r \end{pmatrix} \geq 0$$

for all nondecreasing $(x_i)_1^r$ and $(y_i)_1^r$, then

$$\det\, G \begin{pmatrix} x_1, \ldots, x_r \\ y_1, \ldots, y_r \end{pmatrix} > 0$$

for an increasing $(x_i)_1^r, (y_i)_1^r$ if and only if $x_{i-p} < y_i$, $i = p+1, \ldots, r$, and $y_i < x_{i+q}$, $i = 1, \ldots, r-q$.

Since S, as defined in (7.2), agrees with $\mathcal{S}_{k,\mathbf{t}}$ on $[t_k, t_{n+1}]$, it is possible to translate Theorem 7.1 into a statement involving B-splines provided we make the assumption that

$$\tau_1, \ldots, \tau_n \in [t_k, t_{n+1}]. \tag{7.3}$$

It is also possible to prove directly

Theorem 7.2. *If $\tau_1 < \cdots < \tau_n$, then $(N_j(\tau_i))_1^n$ is invertible if and only if $\tau_i \in \operatorname{supp} N_i$, i.e., $N_i(\tau_i) \neq 0$, all i.*

In other words, $(N_j(\tau_i))$ is invertible iff its diagonal is invertible. Burchard [21, Chap. III, 2(3)] and Karlin [47, Chap. 10, Lemma 4.1] both prove Theorem 7.2 explicitly in terms of B-splines, with simple knots, but, on the other hand, more generally for Chebyshev splines.

Karlin and Ziegler [53] remove the restriction in Theorem 7.2 to simple knots. They also allow for repeated or osculatory interpolation and consider Chebyshev splines rather than just polynomial splines. Straightforward translation of their result to B-splines would require assumption (7.3).

We will now quit belaboring this minor point and state the theorem directly in terms of B-splines.

Theorem 7.3 (Karlin-Ziegler [53] extension of Schoenberg-Whitney). *Let $\tau_1 \le \cdots \le \tau_n$ be such that*

$$\tau_{i+1} = \cdots = \tau_{i+r} = t_{j+1} = \cdots = t_{j+s} \quad \textit{implies} \quad r+s \le k,$$

and define linear functionals $(\mu_i)_1^n$ by the rule

$$\mu_i f \ := \ f^{(j)}(\tau_i) \quad \textit{with} \quad j \ := \ \max\{r : \tau_{i-r} = \tau_i\}.$$

Then $(\mu_i N_j)$ is invertible if and only if $N_i(\tau_i) \ne 0$, $i = 1, \ldots, n$.

A simple proof of this theorem, using only elementary properties of B-splines and Rolle's theorem, can be found in [15].

Theorem 7.3 states conditions under which it is possible to interpolate by linear combinations of *all* B-splines for a given knot sequence. A careful study of Karlin's proof [47] of the total positivity of $(N_j(\tau_i))$ reveals the fact that Theorem 7.3 remains valid if we replace the sequence (N_j) by one of its subsequences.

Theorem 7.4 [15]. *Under the same assumptions as those of Theorem 7.3, and for any subsequence $(q_1, \ldots, q_m)$ of $(1, \ldots, n)$, $\det(\mu_i N_{q_j})_{i,j=1}^m \ge 0$ with equality iff, for some i, $N_{q_i}(\tau_i) = 0$.*

This theorem implies at once the total positivity of $(N_j(\tau_i))$.

Theorem 7.5 (Karlin [47]). *Let $\tau_1 \le \cdots \le \tau_n$. Then $(N_j(\tau_i))$ is totally positive, i.e., all its minors are nonnegative.*

Karlin [47, p. 563] states that this theorem was communicated to him by Schoenberg.

Corollary. *(N_i) is a weak Descartes system, i.e., any subsequence $(N_{q_i})_1^m$ of $(N_i)_1^n$ is a weak Chebyshev system.*

The total positivity of $(N_j(\tau_i))$ provides bounds on the effect of rounding errors when solving (7.1) by Gauss elimination *without* pivoting which are smaller than those obtainable for general matrices even when using pivoting [18]. This means that it is reasonable to solve the *banded* system (7.1) without pivoting with the attendant savings in storage and program complexity.

The total positivity of $(N_j(\tau_i))$ is used in an essential way by Karlin and Pinkus [51] in their extension to splines and to higher derivatives of earlier results by C. Davis and Videnski concerning the existence of a polynomial of degree n on $[0, 1]$ with a prescribed sequence of $n+1$ extrema.

The total positivity of $(N_j(\tau_i))$ leads to one of the more striking spline approximation schemes, Schoenberg's variation diminishing spline approximation, which has found much use in computer-aided design (see e.g., Riesenfeld [69]). We recall some notation. A real-valued function f on some subset D of $\mathbb{R}$ has *at least m strong sign changes if f alternates (in sign) on some $(\tau_i)_0^m$* in D, i.e., if

$$f(\tau_0) \ne 0 \ \text{and, in case}\ m > 0, \quad f(\tau_{i-1})\, f(\tau_i) < 0 \quad \text{for} \quad i = 1, \ldots, m,$$

for some nondecreasing sequence $(\tau_i)_0^m$ in D. It is customary to denote by

$$S^-(f)$$

the total number of strong sign changes of f on its domain. It is well known (e.g., Theorem 5.1.4 of [47]) that, for a totally positive matrix A and any vector α,

$$S^-(A\alpha) \le S^-(\alpha),$$

i.e., a totally positive matrix transformation is variation diminishing. Since $(N_j(\tau_i))$ is totally positive, it follows that the linear map V_τ, given for some nondecreasing τ by

$$V_\tau f \ := \ \sum_{j=1}^n f(\tau_j)\, N_j, \quad \text{all} \quad f, \tag{7.4}$$

is variation diminishing, i.e., $S^-(V_\tau f) \le S^-(f)$. Recall now from Marsden's identity (see (5.6) and (5.7)) that, for any straight line p and any τ with $\tau_i \in (t_i, t_{i+k})$, all i,

$$p = \sum_{j=1}^{n} \left\{ p(\tau_j) + p'(\tau_j) \Big[\sum_{r=1}^{k-1} t_{j+r} - (k-1)\tau_j \Big] / (k-1) \right\} \quad N_j$$

on $[t_k, t_{n+1}]$. Therefore, with the particular choice

$$\tau_j^* \ := \ (t_{j+1} + \cdots + t_{j+k-1})/(k-1), \quad j = 1, \ldots, n, \tag{7.5}$$

mentioned already in (5.15), $V_{\tau *}$ reproduces $\mathbb{P}_2$ on $[t_k, t_{n+1}]$, and we have

$$S^-(V_{\tau *} f - p) \le S^-(f - p) \ \text{ on } [t_k, t_{n+1}], \quad \text{all } p \in \mathbb{P}_2, \quad \text{all } f. \tag{7.6}$$

The resulting approximation $V_{\tau *} f$ to f is **Schoenberg's variation diminishing spline approximation**, introduced by Schoenberg in [73] and further discussed in Marsden and Schoenberg [63].

We note the following result due to Marsden [62]: Write $V_{\tau *,k}$ to stress dependence on k, and restrict $\mathbf{t}$ so that $t_i = \cdots = t_k = 0$ and $t_{n+1} = \cdots = t_{n+k} = 1$. Then

$$V_{\tau *,k} \to 1 \ \textit{pointwise on } C[0,1] \ \text{ iff } \ \max_i \Delta t_i / k \to 0, \tag{7.7}$$

as Marsden shows with the aid of the Bohman-Korovkin theorem concerning strong convergence of positive operators to the identity on $C[0,1]$.

It is possible to refine the proof that $S^-(A\alpha) \le S^-(\alpha)$ for a totally positive matrix A for the particular choice $A = (N_j(\tau_i))$ so as to obtain the following theorem.

Theorem 7.6 [15]. *If $f := \sum_{j=1}^n \alpha_j N_j$ alternates on $(\tau_i)_0^m$, then*

$$f(\tau_i) \alpha_{q_i} N_{q_i}(\tau_i) > 0, \quad i = 0, \ldots, m,$$

for some subsequence q of $(1, \ldots, n)$.

Theorem 7.6 illustrates the point made earlier that B-spline coefficients "model" the function they represent. A spline cannot change sign at a point without its B-spline sequence also changing sign "nearby".

As a specific application of this theorem, consider the spline $N_i^{(j)}$ which, by (5.2), is the linear combination of $j+1$ B-splines (of order $k-j$), hence cannot have more than j strong sign changes, by Theorem 7.6. On the other hand, if $N_i^{(j-1)}$ is continuous, hence absolutely continuous, then $N_i^{(j)}$ is orthogonal to $\mathbb{P}_j$ on $[t_i, t_{i+k}]$, by (4.15), therefore must have at least j strong sign changes.

Corollary [30]. *B-splines are* **bell-shaped**. *Precisely, if $N_i^{(j-1)}$ is continuous for some $j < k$, then $N_i^{(j)}$ has exactly j zeros in (t_i, t_{i+k}), all simple, i.e., there exists $(\xi_r)_0^{j+1}$ with $t_i = \xi_0 < \cdots < \xi_{j+1} = t_{i+k}$ so that $(-)^r N_i^{(j)} > 0$ on (ξ_r, ξ_{r+1}), $r = 0, \ldots, j$.*

Finally, we record the relationship between B-splines and Pólya frequency functions discovered by Curry and Schoenberg [30]. By definition, a Pólya frequency distribution is any distribution function F (i.e., any function of the form $F(x) = \int_{-\infty}^x f(s)\,ds$ with f nonnegative and $F(\infty) = 1$) whose bilateral Laplace transform is of the form

$$\int_{-\infty}^{\infty} e^{-sx}\, dF(x) = 1/\psi(s)$$

with

$$\psi(s) = e^{-\gamma s^2 + \delta s} \prod_1^\infty (1 + \delta_v s)\, e^{-\delta_v s}$$

for some $\gamma \ge 0$, δ real, and $(\delta_v) \in \ell_2$. If $\psi(s) = e^{\delta s}$, then dF has its entire unit mass located at $x = \delta$. If $\psi(s) \ne e^{\delta s}$, then

$$\int_{-\infty}^{\infty} e^{-sx} \Lambda(x)\, dx = 1/\psi(s)$$

with Λ a Pólya frequency function, i.e., a nonnegative integrable function on $\mathbb{R}$ (normalized to have $\int \Lambda = 1$) for which the kernel

$$K(x,y) := \Lambda(x-y)$$

is totally positive of all orders.

Call F_k a spline distribution function of order k if F_k has a B-spline of order k as its density, i.e., if

$$F_k(x) = k\int_{-\infty}^{x} [\tau_0,\ldots,\tau_k]\,(\cdot - s)_+^{k-1}\,ds$$

for some $\tau_0 \le \cdots \le \tau_k$ with $\tau_0 < \tau_k$. Note that $F_k(x) = 0$ for $x \le \tau_0$ and $F_k(x) = 1$ for $x \ge \tau_k$, by (4.2). Further, say that F_k converges to a distribution function F in case $\lim_{k\to\infty} F_k(x) = F(x)$ for all points x at which F is continuous.

Theorem 7.6 [30]. *The distribution function F is a Pólya frequency distribution iff F is the limit of a sequence (F_k) of spline distributions, with F_k of order k, all k.*

8. "Best" interpolation

In this section, I finally discuss an aspect of splines which many consider to be the primary characteristic of splines, *viz* the fact that *splines are solutions to interesting variational problems*. This property of splines is closely related to the fact that the B-spline $M_{i,k}$ represents a k–th order divided difference. As mentioned already in (4.1), if $a \le t_i < t_{i+k} \le b$, then

$$[t_i,\ldots,t_{i+k}]\,f = \int_a^b M_{i,k}(s)\;f^{(k)}\,(ds)/k! \tag{8.1}$$

for every $f \in M^k[a,b] := \{f \in C_p^{k-2}[a,b] : f^{(k-2)} \text{ abs.const.}, f^{(k-1)} \in BV\}$.

Details for the material in this section can be found in [14] and its references.

Consider the problem of minimizing $\|f^{(k)}\|_p$ over

$$F_p := F_p(\boldsymbol{\tau},\alpha,k,[a,b]) := \left\{f \in \mathbb{L}_p^k[a,b] \;:\; f|_{\boldsymbol{\tau}} = \alpha\right\} \tag{8.2}$$

for given $\boldsymbol{\tau} := (\tau_i)_1^n$ in $[a,b]$, nondecreasing with $\tau_i < t_{i+k}$, all i, and given $\alpha \in \mathbb{R}^n$, with $[a,b]$ finite, positive $k \le n$ and $p \in [1,\infty]$. Here, $f|_{\boldsymbol{\tau}}$ is the sequence $(f_i)_1^n$ given by the rule

$$f_1 := f^{(j)}\,(\tau_i), \quad \text{with} \quad j := \max\{r : \tau_{i-r} = \tau_i\}.$$

F_p is not empty. It contains, e.g., exactly one polynomial of degree $< n$. Therefore,

$$F_p = \left\{f \in \mathbb{L}_p^k[a,b] : f|_{\boldsymbol{\tau}} = f_\alpha|_{\boldsymbol{\tau}}\right\}$$

for some fixed $f_\alpha \in F_p$. Favard [35] already knew and used the fact that

$$\inf_{f\in F_p} \|f^{(k)}\|_p = \inf_{g\in G_p} \|g\|_p$$

with

$$G_p := \left\{f^{(k)} : f \in F_p\right\} = \left\{g \in \mathbb{L}_p : \int M_{i,k}g = k!\,[\tau_i,\ldots,\tau_{i+k}]f_\alpha, \quad i = 1,\ldots,n-k\right\}. \tag{8.2$'$}$$

Let now $1 < p \le \infty$ and $1/p + 1/q = 1$. Then, following Krein [54], we recognize that minimization of $\|g_p\|$ over G_p can be viewed, dually, as the construction of an extension $\lambda \in \mathbb{L}_p = \mathbb{L}_q^*$ of minimal norm to all of $\mathbb{L}_q[a,b]$ of the linear functional λ_α, given on $\mathcal{S}_{k,\boldsymbol{\tau}} = \text{span}\,(M_{i,k})_1^{n-k} \subseteq \mathbb{L}_q[a,b]$ by

$$\lambda_\alpha : \mathcal{S}_{k,\boldsymbol{\tau}} \to \mathbb{R} : \sum_i \beta_i M_{i,k} \mapsto \sum_i \beta_i k!\,[\tau_i,\ldots,\tau_{i+k}]\,f_\alpha.$$

This is so since G_p, as a subset of $\mathbb{L}_q^*$, coincides with the set of all extensions of λ_α. Therefore

$$\inf_{f\in F_p} \|f^{(k)}\|_p = \min\Big\{\|\lambda\| : \lambda \in \mathbb{L}_q^*, \quad \lambda|_{\mathcal{S}_{k,\boldsymbol{\tau}}} = \lambda_\alpha\Big\} = \|\lambda_\alpha\|, \tag{8.3}$$

by the Hahn-Banach theorem, settling existence of a minimal f in F_p as well. Further, a minimal f must agree with f_α at $\tau_1, \ldots, \tau_k$ while its k–th derivative satifies

$$\int_a^b f^{(k)}(s)\psi(s)\,ds = \|f^{(k)}\|_p\|\psi\|_q \tag{8.4}$$

for any $\mathbb{L}_q$–**extremal** ψ of λ_α, i.e., for any ψ with

$$\psi \in \mathcal{S}_{k,\boldsymbol{\tau}} \quad \text{and} \quad \|\psi\|_q = 1 \quad \text{and} \quad \lambda_\alpha\psi = \|\lambda_\alpha\|. \tag{8.5}$$

If $\lambda_\alpha = 0$, then there is a polynomial of order k in F_p and it is the unique minimizer for all p. Otherwise $\lambda_\alpha \neq 0$. But then, for $1 < q < \infty$, λ_α has exactly one extremal and the equality (8.4) in Hölder's inequality then forces $f^{(k)}$ to satisfy

$$f^{(k)} = \|\lambda_\alpha\|\ |\psi|^{q-1} \text{ signum } \psi. \tag{8.6}$$

It follows that $\|f^{(k)}\|_p$ is uniquely minimized on F_p, and the minimizer is the unique element $\widehat{f}_p$ of the nonlinear family

$$\Big\{f \in \mathbb{L}_1^k[a,b] : f^{(k)} = |\psi|^{q-1}\text{signum } \psi \text{ for some } \psi \in \mathcal{S}_{k,\boldsymbol{\tau}}\Big\}, \tag{8.7}$$

for which $\widehat{f}_p|_{\boldsymbol{\tau}} = f_\alpha|_{\boldsymbol{\tau}}$. Such functions have been called $\mathbb{L}_p$–**splines** by Golomb [42] who was apparently the first to describe their structure.

For $p = 2$, the family (8.7) is linear and consists of all $f \in \mathbb{L}_1^k$ with $f^{(k)} \in \mathcal{S}_{k,\boldsymbol{\tau}}$. To describe the corresponding minimizer, let $\mathbf{t}$ be the extension of $\boldsymbol{\tau}$ to a nondecreasing sequence having both a and b occurring exactly $2k$ times. Then the minimizer in F_2 is the unique $\widehat{f}_2$ in $\mathcal{S}_{2k,\mathbf{t}}$ which, in addition to the condition $\widehat{f}_2|_{\boldsymbol{\tau}} = f_\alpha|_{\boldsymbol{\tau}}$, also satisfies

$$(\tau_i - a)\,\widehat{f}_2^{(2k-i)}(a^+) = (b - \tau_{n+1-i})\,\widehat{f}_2^{(2k-i)}(b^-) = 0, \quad i = 1, \ldots, k. \tag{8.8}$$

The minimizer has been called by Schoenberg [73] the **natural** spline interpolant, of order $2k$ with interior knots $\tau_1, \ldots, \tau_n$ for f_α in case $a < \tau_1$ and $\tau_n < b$, in which case all the constraints (8.8) on $\widehat{f}_2$ are active. At the other extreme, when none of the constraints (8.8) on f_2 is active, i.e., when $a = \tau_1 = \cdots = \tau_k$ and $\tau_{n-k+1} = \cdots = \tau_n = b$, the minimizer has been called by Schoenberg (see, e.g., Lecture 7 of [76]) the **complete** spline interpolant for f_α of order $2k$ with interior knots $\tau_{k+1}, \ldots, \tau_{n-k}$. The word "spline" itself was chosen by Schoenberg [70] because in the case $k = 2$ the resulting interpolating cubic spline approximates (for small slopes) the position of a mechanical or draftman's spline forced to go through the given data points. This connection between $(2k-1)$st degree spline interpolation at knots and least-squares approximation to the k–th derivative has remained for many the major reason for using splines.

For $p = \infty$, (8.4) fails to pin down the minimizer uniquely since it only implies that

$$f^{(k)} = \|\lambda_\alpha\|\,\text{signum } \psi \text{ off } N_\psi := \Big\{x \in [a,b] : \psi(x) = 0\Big\} \tag{8.9}$$

for every $\mathbb{L}_1$–extremal ψ of λ_α. Of course, if N_ψ has measure zero, then it follows that the minimizer $\widehat{f}$ is unique and its k–th derivative is absolutely constant, with $< n-k$ break points, by Theorem 7.5, since ψ is a nontrivial linear combination of $n-k$ B-splines. In the language of Glaeser [40,41], $\widehat{f}$ is a **perfect spline of degree** k, i.e., a pp function of order $k+1$ in C^{k-1} with absolutely constant k–th derivative.

Whether or not N_ψ has zero measure, supp $\psi = [a,b]\backslash N_\psi$ must contain the support of some B-spline of order k for the knot sequence $\boldsymbol{\tau}$, by Lemma 5.1, i.e., some interval (τ_i, τ_{i+k}) on which then, by (8.9), all minimizers must agree. This is the "core interval of uniqueness" of Fisher and Jerome [36]. In particular, the minimizer is uniquely determined in case $n = k+1$. It is also uniquely determined in case $n = 2k$ and

$$a = \tau_1 = \cdots = \tau_k, \quad \tau_{k+1} = \cdots = \tau_{2k} = b,$$

as was found by Glaeser [40,41], since now $\mathcal{S}_{k,\boldsymbol{\tau}} = \mathbb{P}_k|_{[a,b]}$. For the specific data $f_\alpha(x) := \int_a^x (s-a)^{k-1}(b-s)^{k-1}ds$, Louboutin [58] (see also Schoenberg [75,76]) found $\widehat{f}$ explicitly in this case: $f_\alpha^{(k)}$ is evidently orthogonal to $\mathbb{P}_{k-1} \subseteq \mathcal{S}_{k,\boldsymbol{\tau}}$ on $[a,b]$, therefore $\widehat{f}^{(k)}$ must be a step function with $< k$ jumps and orthogonal to $\mathbb{P}_{k-1}$ on $[a,b]$. But, since $\mathbb{P}_{k-1}$ is a Chebyshev system, this pins down signum $\widehat{f}^{(k)}$ uniquely up to multiplication by a sign $\sigma \in \{-1,1\}$,

$$\text{signum} \quad \widehat{f}^{(k)} = \sigma \quad \text{signum} \quad C_k^{(1)}$$

with $C_k(x) = (-)^{k-1}C_k(2\frac{x-a}{b-a} - 1)$ and C_k the Chebyshev polynomial of degree k. It follows that $f^{(1)}$ is a B-spline of order k with simple knots at the $k+1$ extrema of C_k on $[a,b]$ (see (4.15)). But, in general, there will be several distinct minimizers. Karlin [48] was the first to see that among these has to be at least one perfect spline $\widehat{f}$ of degree k with $< n-k$ interior knots. Its derivative $\widehat{f}^{(k)}$ can be constructed [10] as a limit point of the net $(g_\varepsilon)_{\varepsilon>0}$, with g_ε the unique minimizer of $\| \ \|_\infty$ in

$$G_{\infty,\varepsilon} \ := \ \Big\{ g \in \mathbb{L}_\infty[a,b] : \int_a^b \varphi g = \int_a^b \varphi \, f_\alpha^{(k)}, \quad \text{all} \quad \varphi \in S_\varepsilon \Big\}$$

where

$$S_\varepsilon \ := \ K_\varepsilon(\mathcal{S}_{k,\boldsymbol{\tau}}), \quad (K_\varepsilon\varphi)(x) \ := \ \int_{-\infty}^{\infty} \exp(-(y-x)^2/(2\varepsilon^2)\,\varphi(y)\,dy/(\varepsilon\sqrt{2\pi}).$$

The minimizer g_ε is in fact uniquely determined, absolutely constant and has $< n-k$ jumps, since the total positivity of $(N_{j,k}(\sigma_i))$ for increasing $\boldsymbol{\sigma}$ (see Theorem 7.5) implies [47] that $(K_\varepsilon N_{j,k}(\sigma_i))$ is strictly totally positive for strictly increasing $\boldsymbol{\sigma}$; therefore any nonzero element ψ of S_ε vanishes on $< n-k$ points. Finally, Favard [35] constructed a minimizer $\widehat{f}$ which is a spline of degree k with $< n-k$ interior knots, all simple, with the additional property that, for any $f \in F_\infty$, $|f^{(k)}| \le |\widehat{f}^{(k)}|$ implies that $f = \widehat{f}$. This minimality of "Favard's solution" is further underlined by the fact that it is, for any $r \in [1,\infty)$, the $\mathbb{L}_r^k$–limit of $\widehat{f}_p$ as $p \to \infty$ [25].

For $p = 1$, matters are least satisfactory since $\mathbb{L}_p$ now fails to be the dual for $\mathbb{L}_q$. Therefore, although (8.3) still holds for this case, it may happen that none of the norm preserving extensions of λ_α to all of $\mathbb{L}_\infty$ is representable as integration against an $\mathbb{L}_1$–function, in which case the infimum over F_1 is not attained. In this situation, one may be satisfied to follow the lead of Fisher and Jerome [37] and consider the slightly different problem of minimizing

$$\|f^{(k)}\| \ := \ \text{Var} \ f^{(k-1)}$$

over

$$F_1 \ := \ \Big\{ f \in M^k[a,b] : f|_{\boldsymbol{\tau}} = f_\alpha|_{\boldsymbol{\tau}} \Big\}$$

instead, which always has solutions. If $\tau_i < \tau_{i+k-1}$, all i, then among these solutions is a spline of order k with $\le n-k$ interior knots, all simple.

We close this section with yet another B-spline property, this one connected with perfect splines, optimal recovery (alias best class estimators) and $\mathbb{L}_1$–approximation by splines.

Lemma 8.1 (Micchelli [64]). *If $\boldsymbol{\tau} = (\tau_i)_1^n$ is nondecreasing in (a,b) with $n > k$, then there exists (up to multiplication by some $\sigma \in \{-1,1\}$) exactly one sign function h with $\le n-k$ jumps which is orthogonal to $\mathcal{S}_{k,\boldsymbol{\tau}}$ on $[a,b]$. If $a = \xi_0 < \cdots < \xi_{r+1} = b$, and, for this h, $(-)^i h = 1$ on (ξ_i, ξ_{i+1}), $i = 0, \ldots, r$, then $r = n-k$ and $\xi_i \in (\tau_i, \tau_{i+k})$, $i = 1, \ldots, r$.*

Micchelli's lemma is not entirely unrelated to the following fact about B-splines useful, e.g., in the characterization of best $\mathbb{L}_p$– approximations by splines.

Lemma 8.2. *If $\mathbf{t} = (t_i)_1^{n+k}$ is nondecreasing, in $[a,b]$, with $t_i < t_{i+k}$, all i, and $f \in \mathbb{L}_1[a,b]$ is orthogonal to $\mathcal{S}_{k,t}$ on $[a,b]$, then there exists $\boldsymbol{\xi} = (\xi_i)_1^{n+1}$ strictly increasing in $[a,b]$ with $t_i \le \xi_i \le t_{i+k-1}$ (any equality holding iff $t_i = t_{i+k-1}$), $i = 1,\ldots,n+1$, so that f is also orthogonal to $\mathcal{S}_{1,\boldsymbol{\xi}}$.*

Indeed, since, for appropriately chosen $p \in \mathbb{P}_k$, the function $F := p + \int_a^b (\,\cdot\, - y)_+^{k-1} f(y)\,dy/(k-1)!$ vanishes at $\mathbf{t}$ (counting multiplicities) by assumption, and F is in $C^{k-1}[a,b]$, Rolle's Theorem proves the existence of strictly increasing $(\xi_i)_1^{n+1}$ in $[a,b]$ with $t_i \le \xi_i \le t_{i+k-1}$, all i, at which $F^{(k-1)} = \mathrm{const} + \int_a^b (\,\cdot\, - y)_+^0 f(y)\,dy$ vanishes, which proves the lemma. In particular, if f is continuous, then it must vanish at the n points of some strictly increasing sequence $(\mu_i)_1^n$ with $t_i < \mu_i < t_{i+k}$, all i.

9. Generalizations

The trend started by Schoenberg [71] and Greville [43] toward ever more generalized splines continues unabated but has failed to bring with it a corresponding wealth of generalized B-splines. Schoenberg [71] actually described trigonometric B-splines and later, Burchard [21] and Karlin [47] independently constructed *Chebyshevian* B-splines with the aid of Popoviciu's [67] generalization of the divided difference notion. Yet another account can be found in Marsden's thesis, eventually published in [61], in which the generalization of Schoenberg's variation diminishing spline approximation for Chebyshev splines is given, but without a proof of its variation diminishing character. Such a scheme had already been described and proven to be variation diminishing by Karlin and Karon [49], and their assertion in [50] that Marsden's B-splines are essentially different from Karlin's is incorrect.

Here are some of the details of the construction.

Let Pf be the polynomial of degree $< k$ which agrees with f at the distinct points $\tau_1,\ldots,\tau_k$. If $\varphi_j(x) = x^{j-1}$, all j, then

$$f - Pf = \det\begin{pmatrix}\tau_1,\ldots,\tau_k, & \cdot\\ \varphi_1,\ldots,\varphi_k, & f\end{pmatrix} \Big/ \det\begin{pmatrix}\tau_1,\ldots,\tau_k\\ \varphi_1,\ldots,\varphi_k\end{pmatrix}. \tag{9.1}$$

Therefore, since $[\tau_1,\ldots,\tau_k,x]\,f$ is the leading coefficient in the polynomial of degree $\le k$ which agrees with f at $\tau_1,\ldots,\tau_k,x$, we have

$$f - Pf = \big([\tau_1,\ldots,\tau_k,\,\cdot\,]\,f\big)\,(\varphi_{k+1} - P\varphi_{k+1}) \tag{9.2}$$

with

$$\begin{aligned}[\tau_1,\ldots,\tau_k,\,\cdot\,]\,f &= (f-Pf)/(\varphi_{k+1} - P\varphi_{k+1})\\ &= \det\begin{pmatrix}\tau_1,\ldots,\tau_k, & \cdot\\ \varphi_1,\ldots,\varphi_k, & f\end{pmatrix} \Big/ \det\begin{pmatrix}\tau_1,\ldots,\tau_k, & \cdot\\ \varphi_1,\ldots,\varphi_{k+1}\end{pmatrix}.\end{aligned}$$

If now, more generally, $(\varphi_j)_1^{k+1}$ is a Chebyshev system (on some interval I), then $\det\binom{\tau_1,\ldots,\tau_{k+1}}{\varphi_1,\ldots,\varphi_{k+1}} \ne 0$ for distinct $\tau_1,\ldots,\tau_{k+1}$ in I and the following definition makes sense: *The k–th divided difference of f at the distinct points $\tau_1,\ldots,\tau_{k+1}$ in I with respect to the sequence $\varphi := (\varphi_j)_1^{k+1}$ is* [67]

$$[\tau_1,\ldots,\tau_{k+1}]_\varphi\, f \;:=\; \det\begin{pmatrix}\tau_1,\ldots,\tau_{k+1}\\ \varphi_1,\ldots,\varphi_k,\ f\end{pmatrix} \Big/ \det\begin{pmatrix}\tau_1,\ldots,\tau_{k+1}\\ \varphi_1,\ldots,\varphi_{k+1}\end{pmatrix}. \tag{9.3}$$

Then, with Pf denoting, more generally, the unique element in $\mathrm{span}\,(\varphi_j)_1^k$ which agrees with f at $\tau_1,\ldots,\tau_{k+1}$, we have

$$f - Pf = \big([\tau_1,\ldots,\tau_k,\,\cdot\,]_\varphi f\big)\,(\varphi_{k+1} - P\varphi_{k+1})$$

which is the formal analog of (9.2). The definition shows the generalized divided difference (9.3) to be a symmetric function of the τ_i's. The definition even allows for some confluence among the τ_i's provided the φ_j's are sufficiently smooth and one defines (for nondecreasing $\boldsymbol{\tau}$)

$$\det\begin{pmatrix}\tau_1,\ldots,\tau_{k+1}\\ \varphi_1,\ldots,\varphi_{k+1}\end{pmatrix} \;:=\; \det\begin{pmatrix}\mu_1,\ldots,\mu_{k+1}\\ \varphi_1,\ldots,\varphi_{k+1}\end{pmatrix} \;= \det\,(\mu_i\varphi_j)$$

with $\mu_i f := f^{(j)}(\tau_i)$ and $j := \max\{r : \tau_{i-r} = \tau_i\}$, in the manner of Theorem 7.3. More detail about these generalized divided differences are provided by Popoviciu [67], and see also Mühlbach [65].

Assume that, in addition, $(\varphi_j)_1^k$ spans the kernel of a k–th order linear ordinary differential operator

$$L^* := D^k + \sum_{j<k} a_j D^j \tag{9.4}$$

with $a_j \in C^j(I)$, all j, so that the formal adjoint

$$L := (-)^k D^k + \sum_{j<k} (-)^j D^j (a_j \,\cdot\,) = (-)^k \Big(D^k + \sum_{j<k} b_j D^j \Big) \tag{9.5}$$

is an operator of the same kind. Green's function $G(x,y)$ for the initial value problem $L^* f = g$, $f^{(j)}(a) = 0$, $j = 0, \ldots, k-1$, can then be constructed as

$$G(x,y) = (x-y)_+^0 \sum_{j=1}^k \varphi_j(x)\psi_j(y) \tag{9.6a}$$

with $(\psi_j)_1^k$ the basis for ker L adjunct to $(\varphi_i)_1^k$, i.e.,

$$\sum_{j=1}^k \varphi_j^{(i-1)}(x)\psi_j(x) = \delta_{ik}, \quad i = 1, \ldots, k, \quad x \in I. \tag{9.6b}$$

With $\mathbf{t} = (t_i)_1^{n+k}$ nondecreasing and $t_i < t_{i+k}$, all i, the function

$$M_{i,L}(y) := [t_i, \ldots, t_{i+k}]_\varphi \, G(\,\cdot\,, y) \tag{9.7}$$

is then piecewise in ker L with breakpoints $t_i, \ldots, t_{i+k}$, and in C^{k-2} in case $t_i < \cdots < t_{i+k}$. In the language of Greville [43], $M_{i,L}$ is a generalized spline function with respect to ker L. Coincidences among the t_i's reduce the smoothness of $M_{i,L}$ across t_j in the usual way. Further,

$$M_{i,L} \quad \text{vanishes off} \quad (t_i, t_{i+k}) \tag{9.8}$$

since, for $y > t_{i+k}$, $G(\,\cdot\,, y)|_{(t_i, t_{i+k})} = 0$ while, for $y < t_i$, $G(\,\cdot\,, y)|_{(t_i, t_{i+k})} \in \ker L^*$ by (9.6). One also has the analog

$$[t_i, \ldots, t_{i+k}]_\varphi \, f = \int_{t_i}^{t_{i+k}} M_{i,L}(y) L^* f(y) \, dy. \tag{9.9}$$

If, in addition, $(\varphi_i)_1^{k+1}$ is an extended complete Chebyshev (or, ECT) system, then Burchard [21] and Karlin [47] have shown the analog of the Schoenberg-Whitney theorem 7.1 that, *for strictly increasing* $\mathbf{t}$ *and strictly increasing* $\boldsymbol{\tau} = (\tau_i)_1^n$, $\det(M_{j,L}(\tau_i)) \geq 0$ *with strict inequality iff* $M_{i,L}(\tau_i) \neq 0$, *all* i. Further, Karlin [47] showed that $(M_{j,L}(\tau_i))$ *is totally positive* in this case, as was mentioned earlier. Few facts beyond these are known for Chebyshev B-splines. While the analog of Marsden's identity (5.7) can be found in [61], the analog of the linear functional (5.4) has not been described, although that should be fairly easy. More importantly for computations, a recurrence relation like (4.9) has been searched for in vain so far.

It is actually quite unnecessary to assume that $(\varphi_j)_1^k$ is a Chebyshev system in order to construct L–splines (in the sense of Greville) of local support. Continue to assume that $(\varphi_j)_1^k$ is a basis for the kernel of the differential operator L^* of (9.4) with L of (9.5) its adjoint and G the Green's function given by (9.6). If $\mathbf{t} = (t_i)_1^n$ is strictly increasing, then, for each i, the span of $([t_j])_{j=i}^{i+k}$ contains a nontrivial $\mu \perp \ker L^*$ since ker L^* has dimension k. But then

$$M_{\mu,L}(x) := \mu \, G(\,\cdot\,, x) \tag{9.10}$$

defines an L–spline with knots $t_i, \ldots, t_{i+k}$ and support in (t_i, t_{i+k}). Clearly, $M_{\mu,L}$ represents μ with respect to the pairing $\langle f, g\rangle := \int f L^* g$. If now $(\varphi_j)_1^k$ fails to be a Chebyshev system on $[t_i, t_{i+k}]$, then there exists a nontrivial μ in the span of $([t_j])_i^{i+r}$ and orthogonal to ker L^* for some $r < k$, i.e., the corresponding $M_{\mu,L}$ has even smaller support. More explicitly, let $(\mu_i)_1^{n+2k-2}$ be the sequence

$$[t_1]D^{k-1}, \ldots, [t_1]D,\ [t_1],\ [t_2], \ldots, [t_n],\ [t_n]D, \ldots, [t_n]D^{k-1}$$

of linear functionals and, for each i, let v_i be the linear functional of the form $v_i = \mu_i + \sum_{i+1}^{i+r} \beta_j \mu_j$ which is orthogonal to ker L^*, with r as small as possible. The corresponding sequence $(M_{v_i,L})_1^{n+k-2}$ of basic L–splines is then a basis for the space of all L–splines on $[t_1, t_n]$ with simple interior knots $t_2, \ldots, t_{n-1}$.

A construction like this was used by Jerome [45] under the additional assumption that, for each i, $t_{i+k} - t_i$ is small enough so that $(\varphi_j)_1^k$ is a Chebyshev system on $[t_i, t_{i+k}]$. Earlier, Jerome and Schumaker [46] had used such considerations in connection with Lg–splines, i.e., when the linear functionals (μ_i) above are, more generally, of the form $\mu_i = \sum_1^k \alpha_{ij}[t_i]D^{j-1}$. Related developments of great generality can be found in Brown [20].

We close this section with yet another B-spline property discovered by Curry and Schoenberg [30].

Lemma 9.1 [30]. *Let $M_{0,k}$ be the B-spline defined by (3.1), and let σ be any k–simplex in $\mathbb{R}^k$ of unit volume with vertices $v^{(i)}$, $i = 0, \ldots, k$ and so that $v_1^{(i)} = t_i$, $i = 0, \ldots, k$. Then, for all x,*

$$M_{0,k}(x) = |\sigma \cap \{v \in \mathbb{R}^k : v_1 = x\}|,$$

i.e., $M_{0,k}(x)$ gives the $(k-1)$–dimensional volume of the intersection of the simplex σ with the hyperplane in $\mathbb{R}^k$ which intersects the v_1–axis at $v_1 = x$ and is orthogonal to it.

In a letter [72] to P. Davis, Schoenberg recalls the Hermite-Genocchi formula

$$[z_0, \ldots, z_k]\, f = \int \underset{\tau_n}{\cdots} \int f^{(k)}\,(v_0 z_0 + v_1 z_1 + \cdots + v_k z_k)\; dv_1 \cdots dv_k \tag{9.11}$$

with $v_0 = 1 - v_1 - \cdots - v_k$ and where the integration is to be carried out over the complex

$$\tau_n : v_1 \geq 0, \ldots, v_k \geq 0, \quad \sum_1^k v_i \leq 1,$$

and points out that Lemma 9.1 follows from this on comparison with (4.1). Schoenberg further recalls that the Hermite-Genocchi formula remains valid if $z_0, \ldots, z_k$ are points in the complex plane not all on one line and if f is a complex-valued function regular in the convex hull $\prod$ of $z_0, \ldots, z_k$. The formula (4.1) now becomes

$$[z_0, \ldots, z_k]\, f = \int_{\prod} M(x, y; z_0, \ldots, z_k)\, f^{(k)}\,(x, y)\; dxdy/k!\ . \tag{9.12}$$

At the point $z = (x, y)$, $M(x, y; z_0, \ldots, z_k)$ is therefore the $(k-2)$ –dimensional volume of the intersection of the plane $\{v \in \mathbb{R}^k : v_1 = x,\ v_2 = y\}$ with a simplex of unit volume whose i–th vertex $v^{(i)}$ satisfies $(v_1^{(i)}, v_2^{(i)}) = z_i$. In particular, M is positive on $\prod$ and zero off $\prod$ and is a spline of order $k-1$ along any straight line, with knots only at the points where such a line intersects a segment $[z_i, z_j]$. Schoenberg's letter even contains a drawing of such a B-spline in two variables for $k = 4$.

This suggests the following definition.

Definition. *Let σ be a nontrivial simplex in $\mathbb{R}^{s+k}$. On $\mathbb{R}^s$, define the B-spline of order k from σ by*

$$M_{k,\sigma}(x_1, \ldots, x_s) \ :=\ |\sigma \cap \{v \in \mathbb{R}^{s+k} : v_i = x_i, \quad i = 1, \ldots, s\}| \quad all \quad x \in \mathbb{R}^s.$$

Then $M_{k,\sigma}$ is unimodal, nonnegative, piecewise polynomial of total order k, and in C^{k-1} in general. Its support is the projection of σ onto $\mathbb{R}^s$, i.e., the convex hull of the projections $((v_j^{(i)})_{j=1}^s)_{i=0}^k$ of the vertices of σ to $\mathbb{R}^s$.

At this point, I have no idea how useful these B-splines might be, even only for the writing of papers. It is easy to visualize how such B-splines can be made to give a partition of unity: One takes some suitable convex set C in $\mathbb{R}^k$ of unit volume and then subdivides the cylinder $\mathbb{R}^s \times C$ in $\mathbb{R}^{s+k}$ into nontrivial simplices. The corresponding B-splines will then add up to one. But it is unlikely that these B-splines will become very useful unless one finds some means of evaluating them such as a recurrence relation like (4.9).

In any event, I think these B-splines are very beautiful.

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