On uniform approximation by splines

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1. Summary

Let $\pi : a = t_0 < t_1 < \cdots < t_n = b$ be a partition of the interval $I = [a, b]$, k an integer greater than one, and denote by S^k_π the set of all polynomial spline functions on $[a, b]$ of degree $k-1$ on π, i.e., with (interior) joints (or knots) at the points t_1, t_2, ..., t_{n-1}. This note is concerned with the behavior of

$$\operatorname{dist}(f, S^k_\pi) = \inf\{\|f - p\|_I : p \in S^k_\pi\},$$

as the mesh of π,

$$|\pi| = \max_i (t_{i+1} - t_i),$$

goes to zero. Here, f is an element of the real Banach space $C(I)$ with norm

$$\|g\|_I = \max\{|g(t)| : t \in I\}, \quad \text{for all } g \in C(I).$$

It is proved that, for all $f \in C(I)$,

$$\operatorname{dist}(f, S^k_\pi) = O(\omega(f; |\pi|)) \tag{1.1}$$

as $|\pi| \to 0$, where $\omega(f; \cdot)$ is the modulus of continuity of f. Hence, if $f \in C^{(r)}(I)$, then

$$\operatorname{dist}(f, S^k_\pi) = O(|\pi|^r \omega(f^{(r)}; |\pi|)), \tag{1.2}$$

for $0 \le r \le k-1$. In particular,

$$\operatorname{dist}(f, S^k_\pi) = O(|\pi|^k)$$

for $f \in C^{(k)}(I)$, or, more generally, for $f \in C^{(k-1)}(I)$, such, that $f^{(k-1)}$ satisfies a Lipschitz condition, a result proved earlier by different means [2]. These results are shown to be true even if I is permitted to become infinite and some of the knots are permitted to coalesce.

The argument is based on a "local" interpolation scheme P_π by splines, which is, in a way, a generalization of interpolation by broken lines, and which achieves the convergence rate (1.1). The linear projector (i.e., linear idempotent map) P_π can be shown to be bounded independently of π. Hence, the argument supplies the fact that any sequence $S^k_{\pi_n}$ with $\lim |\pi_n| = 0$ admits a corresponding uniformly bounded sequence P_{π_n} of linear projectors on $C(I)$ with $S^k_{\pi_n}$ the range of P_{π_n}, which converges strongly to the identity. Such sequences are important for the application of Galerkin's method and its generalizations to the approximate solution of functional equations (cf., e.g., [1]).

The following standard notation will be adhered to throughout. For T some set, $m(T)$ denotes the Banach space of all bounded real–valued functions on T, with norm

$$\|f\|_T = \sup_{t \in T} |f(t)|, \quad \text{for all } f \in m(T).$$

If T is a closed subset of the real line, $\mathbb{R}$, then $C(T)$ denotes the closed linear subspace of $m(T)$ consisting of all continuous (bounded) functions on T.

2. General Remarks

The arguments to follow derive from the following considerations.

Let X be a normed real linear space, $\{\phi_i\}_{i=1}^n$ a finite subset of X with S its linear span. A set $\{\lambda_i\}_{i=1}^n$ of linear functionals on X is said to be a *dual set for* $\{\phi_i\}_{i=1}^n$ if

$$\lambda_i \phi_j = \delta_{ij}, \quad i, j = 1, \ldots, n. \tag{2.1}$$

If $\{\phi_i\}_{i=1}^n$ has a dual set $\{\lambda_i\}_{i=1}^n$ consisting of continuous linear functionals, then the rule

$$Px = \sum_{i=1}^n (\lambda_i x)\phi_i, \quad \text{for all } x \in X, \tag{2.2}$$

defines a continuous linear projector P on X with range S. In fact, since $\{\phi_i\}_{i=1}^n$ is a finite set, there exists an A such that

$$\| \sum_{i=1}^n \alpha_i \phi_i \| \le A \|(\alpha_i)\|_\infty \quad \text{for all } (\alpha_i) \in \mathbb{R}^n. \tag{2.3}$$

Then, for all $x \in X$,

$$\|Px\| \le A \max_i |\lambda_i x| \le A \max_i \|\lambda_i\| \|x\|, \tag{2.4}$$

hence

$$\|P\| \le A \max_i \|\lambda_i\|. \tag{2.5}$$

The last inequality in (2.4) also shows that

$$(\max_i \|\lambda_i\|)^{-1} \|(\alpha_i)\|_\infty \le \| \sum_{i=1}^n \alpha_i \phi_i \| \quad \text{for all } (\alpha_i) \in \mathbb{R}^n, \tag{2.6}$$

since for $x = \sum \alpha_i \phi_i$, one has $\lambda_i x = \alpha_i$, $i = 1, \ldots, n$. This statement has the interesting converse:

Lemma 2.1. *Let X be a normed linear space, $\{\phi_i\}_{i=1}^n$ a subset of X. If there exists a $B > 0$ such that*

$$\|(\alpha_i)\|_\infty \le B \| \sum \alpha_i \phi_i \| \quad \text{for all } (\alpha_i) \in \mathbb{R}^n, \tag{2.7}$$

then $\{\phi_i\}$ has a dual set $\{\lambda_i\}$ of continuous linear functionals on X satisfying

$$\max_i \|\lambda_i\| \le B. \tag{2.8}$$

Proof: Let $1 \le i \le n$, and denote by S_i the linear span of $\{\phi_j : j = 1, \ldots, n;\ j \ne i\}$. By a corollary to the Hahn–Banach theorem, there exists a continuous linear functional, $\widehat{\lambda}$, on X such that $\widehat{\lambda}_i[S_i] = 0$, $\|\widehat{\lambda}_i\| = 1$, and $\widehat{\lambda}_i \phi_i = \operatorname{dist}(\phi_i, S_i)$. But

$$\begin{aligned}
\operatorname{dist}(\phi_i, S_i) &= \inf \big\{ \|\phi_i - \sum_{j \ne i} \alpha_j \phi_j \| : (\alpha_j) \in \mathbb{R}^n \big\} \\
&= \inf \big\{ \| \sum_{j=1}^n \alpha_j \phi_j \| : (\alpha_j) \in \mathbb{R}^n, \quad \alpha_i = 1 \big\} \\
&\ge \inf \{ B^{-1} \|(\alpha_j)\|_\infty : (\alpha_j) \in \mathbb{R}^n, \quad \alpha_i = 1 \} \ge B^{-1} > 0.
\end{aligned}$$

Hence, with $\lambda_i = (\widehat{\lambda}_i \phi_i)^{-1} \widehat{\lambda}_i$, $i = 1, \ldots, n$, $\{\lambda_i\}_{i=1}^n$ is a dual set for $\{\phi_i\}$ such that $\max_i \|\lambda_i\| \le B$. Q.E.D.

On combining Lemma 2.1 with (2.6), one gets

$$\inf \{ \| \sum \alpha_i \phi_i \| : \|(\alpha_i)\|_\infty = 1 \} = \min_i \operatorname{dist}(\phi_i, S_i). \tag{2.9}$$

Corollary. *Let* $\{\phi_i\}_{i=1}^n \subset X$*,* S_i *the linear span of* $\{\phi_j\}_{j\neq i}$*. If*

$$0 < \inf_i \operatorname{dist}(\phi_i, S_i), \tag{2.10}$$

then there exists a continuous linear projector P *on* X *with range the span* S *of* $\{\phi_i\}$ *such that*

$$\|P\| \leq \sup_{\|(\alpha_i)\|_\infty = 1} \|\sum \alpha_i\phi_i\| / \inf_{\|(\alpha_i)\|_\infty=1} \|\sum \alpha_i\phi_i\|. \tag{2.11}$$

Remark. The right–hand–side of (2.11) can be interpreted as the *condition number* of the basis $\{\phi_i\}$ for S. This leads to an interesting connection between the existence of linear projectors with range S of "small" norm and the existence of "well–conditioned" bases for S, which we will not pursue here further.

The finiteness of the set $\{\phi_i\}$ was not used in any essential way in the preceding discussion. The same arguments apply to a subset $\{\phi_i\}_{i\in\mathbb{Z}}$ of X, where $\mathbb{Z}$ denotes the integers, provided

$$\sum_{i\in\mathbb{Z}} \alpha_i\phi_i$$

can be interpreted in some reasonable way as an element of X for each $\alpha = (\alpha_i) \in m(\mathbb{Z})$, and, connected with this, one can ascertain the existence of a constant A such that

$$\|\sum_{i\in\mathbb{Z}} \alpha_i\phi_i\| \leq A\|\alpha\|_{\mathbb{Z}} \quad \text{for all } \alpha \in m(\mathbb{Z}).$$

3. Polynomial Splines on the Real Line

In order to circumvent certain (mostly notational) complications, and for its own interest, uniform approximation on the entire real line by splines is treated first.

A biinfinite real sequence $\pi = \{t_i\}_{i\in\mathbb{Z}}$ is called a *k–extended partition of the real line* $\mathbb{R}$ provided

$$\begin{aligned} t_i <& t_{i+k-1} \quad \text{for all } i \in \mathbb{Z}, \\ &\lim_{i\to\pm\infty} t_i = \pm\infty. \end{aligned} \tag{3.1}$$

Hence, if d_i denotes the frequency with which the number t_i occurs in π, then $d_i \leq k-1$ for all $i \in \mathbb{Z}$.

With π a k–extended partition of $\mathbb{R}$, $k \geq 2$, let S_π^k denote the set of all (polynomial) extended splines of degree $k-1$ on π, i.e., S_π^k consists of those real–valued functions on $\mathbb{R}$ which reduce to a polynomial of degree $\leq k-1$ on each of the intervals $[t_i, t_{i+1}]$, for all $i \in \mathbb{Z}$, and which have $k-1-d_i$ continuous derivatives in a neighborhood of t_i, for all $i \in \mathbb{Z}$. Further, define

$$B_\pi^k = S_\pi^k \cap C(\mathbb{R}), \tag{3.2}$$

the set of *bounded* splines of degree $k-1$ on π.

It is shown in [4; Theorem 5] that S_π^k is linearly isomorphic to $\mathbb{R}^{\mathbb{Z}}$, the isomorphism being

$$(\alpha_i) \mapsto \sum_{i\in\mathbb{Z}} \alpha_i M_i. \tag{3.3}$$

Here, with a slight change of notation as compared with [4],

$$M_i(t) = kg(t_i, t_{i+1}, \ldots, t_{i+k}; t) \tag{3.4}$$

is k times the k–th divided difference in s of the function

$$g(s;t) = (s-t)_+^{k-1} \tag{3.5}$$

on the points $t_i, \ldots, t_{i+k}$. Thus, if $t_i < t_{i+1} < \cdots < t_{i+k}$, then

$$M_i(t) = k \sum_{j=i}^{i+k} (t_j - t)_+^{k-1} / \prod_{\substack{m=i \\ m \neq j}}^{i+k} (t_j - t_m). \tag{3.6}$$

The basic properties of the M_i's all follow easily from the fact (already observed in [3]) that

$$f(t_i, \ldots, t_{i+k}) = \frac{1}{k!} \int_{-\infty}^{\infty} M_i(t) f^{(k)}(t)\, \mathrm{d}t \tag{3.7}$$

for all $f \in C^{(k)}$. It follows, in particular, that

$$M_i(t) \geq 0 \text{ with equality iff } t \notin (t_i, t_{i+k}), \tag{3.8}$$

and

$$\int_{-\infty}^{\infty} M_i(t)\, \mathrm{d}t = \int_{t_i}^{t_i+k} M_i(t)\, \mathrm{d}t = 1. \tag{3.9}$$

Note that (3.8) guarantees that $\sum_{i \in \mathbb{Z}} \alpha_i M_i(t)$ is well–defined at every $t \in \mathbb{R}$ for all $\alpha \in \mathbb{R}^{\mathbb{Z}}$, since, for $t \in [t_j, t_{j+1}]$,

$$\sum_{i \in \mathbb{Z}} \alpha_i M_i(t) = \sum_{i=j+1-k}^{j} \alpha_i M_i(t).$$

For the purposes of this note it is more convenient to work with the spline functions

$$\begin{aligned} \phi_i(t) &= \frac{t_{i+k} - t_i}{k} M_i(t) \\ &= g(t_{i+1}, \ldots, t_{i+k}; t) - g(t_i, \ldots, t_{i+k-1}; t), \end{aligned} \tag{3.10}$$

since this normalization gives

$$\sum_{i \in \mathbb{Z}} \phi_i(t) \equiv 1. \tag{3.11}$$

To prove (3.11), observe that

$$g(t_j, \ldots, t_{j+k-1}; t) = \begin{cases} 0, & t \geq t_{j+k-1}, \\ 1, & t \leq t_j, \end{cases}$$

since, in either case, $g(t_j, \ldots, t_{i+k-1}; t)$ is the $(k-1)$st divided difference of a polynomial in s of degree $\leq k-1$, this polynomial being $p(s) \equiv 0$ when $t \geq t_{j+k-1}$, and $p(s) \equiv (s-t)^{k-1}$ when $t \leq t_j$. Therefore, for $t \in [t_j, t_{j+1}]$,

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \phi_i(t) &= \sum_{i=j+1-k}^{j} [g(t_{i+1}, \ldots, t_{i+k}; t) - g(t_i, \ldots, t_{i+k-1}; t)] \\ &= g(t_{j+1}, \ldots, t_{j+k}; t) - g(t_{j+1-k}, \ldots, t_j; t) \\ &= 1. \end{aligned}$$

For later reference, various properties of the ϕ_i's are collected in the following

Lemma 3.1. *Let π be a k–extended partition of $\mathbb{R}$, and let $\phi_i(t)$ be defined on $\mathbb{R}$ by (3.10), for all $i \in \mathbb{Z}$. Then*

(i) $0 \leq \phi_i(t) \leq 1$ *for all* $t \in \mathbb{R}$ *and all* $i \in \mathbb{Z}$;
(ii) $\phi_i(t) = 0$ iff $t \notin (t_i, t_{i+k})$, *for all* $i \in \mathbb{Z}$;
(iii) $\sum_{i \in \mathbb{Z}} \phi_i(t) \equiv 1$;
(iv) $\| \sum_{i \in \mathbb{Z}} \alpha_i \phi_i \|_{\mathbb{R}} \leq \|\alpha\|_{\mathbb{Z}}$ *for all* $\alpha \in m(\mathbb{Z})$;

(v) *if* $\{\pi^{(n)}\}_{n=1}^{\infty}$ *is a sequence of* k*–extended partitions for* $\mathbb{R}$ *such that*

$$\lim_{n\to\infty} t_j^{(n)} = t_j, \quad j = i, \ldots, i+k,$$

then the corresponding sequence $\{\phi_i^{(n)}\}_{n=1}^{\infty}$ *converges uniformly to* ϕ_i*, i.e.,*

$$\lim_{n\to\infty} \|\phi_i^{(n)} - \phi_i\|_{\mathbb{R}} = 0. \tag{3.12}$$

Proof: (i) and (ii) follow from the corresponding statement (3.8) for the M_i's and from (iii); (iv) is a consequence of (i) and (iii). This leaves (v).

Since $\phi_i^{(n)}(t) = 0$ for $t \notin (t_i^{(n)}, t_{i+k}^{(n)})$, and $\lim_{n\to\infty} t_j^{(n)} = t_j$ for all $j \in \mathbb{Z}$, it is sufficient to prove

$$\lim_{n\to\infty} \|\phi_i^{(n)} - \phi_i\|_I = 0 \tag{3.13}$$

for some finite interval I containing $[t_i, t_{i+k}]$ in its interior. Now, since $g(s;t) = (s-t)_+^{k-1}$, g and its first $k-2$ partial derivatives with respect to s are jointly continuous in s and t uniformly on $I \times I$. The $(k-1)$st divided difference

$$g(s_i, \ldots, s_k; t)$$

is, therefore, jointly continuous in $s_i, \ldots, s_k$, t uniformly on

$$\{(s_1, \ldots, s_k) \in I \times \cdots \times I : s_1 \le s_k - \delta, \quad s_1 \le s_2 \cdots \le s_k\} \times I,$$

for each $\delta > 0$. But this implies (3.13), since $\phi_i(t)$ is the difference of two $(k-1)$st divided differences of $g(s;t)$ in s, and the $\pi^{(n)}$ and π are k–extended partitions and $\lim_{n\to\infty} t_j^{(n)} = t_j$, $j = i, \ldots, i+k$. Q.E.D.

The main result of this section is the following

Theorem 3.1. *Let* $k \ge 2$*, let* $\pi = \{t_i\}_{i\in\mathbb{Z}}$ *be a* k*–extended partition, and let* ϕ_i *be defined as in (3.10), for all* $i \in \mathbb{Z}$*. Then there exists a positive constant* D_k *depending on* k *but not on* π*, such that*

$$D_k^{-1} \le \inf_{i\in\mathbb{Z}} \operatorname{dist}_i(\phi_i, S_i), \tag{3.14}$$

where

$$\operatorname{dist}_i(\phi_i, S_i) := \inf\{\|\phi_i - \sum_{j\ne i} \alpha_j \phi_j\|_i : \alpha \in m(\mathbb{Z})\}, \tag{3.15}$$

and the seminorm $\|\cdot\|_i$ *is given by*

$$\|f\|_i = \max\{|f(t)| : t_{i+1} \le t \le t_{i+k-1}\}, \quad \text{for all } f \in C(\mathbb{R}). \tag{3.16}$$

Remark. In the light of Section 2, this theorem implies the existence of a dual set $\{\lambda_i\}_{i\in\mathbb{Z}}$ for $\{\phi_i\}$, such that

$$|\lambda_i f| \le D_k \|f\|_i \le D_k \|f\|_{\mathbb{R}} \quad \text{for all } f \in C(\mathbb{R}). \tag{3.17}$$

The linear projector P_π on $C(\mathbb{R})$, given by the rule

$$P_\pi f = \sum_{i\in\mathbb{Z}} (\lambda_i f)\phi_i \quad \text{for all } f \in C(\mathbb{R}), \tag{3.18}$$

has then B_π^k as its range, and satisfies $\|P_\pi\| \le D_k$. Moreover, since, by (3.17), each λ_i has its support in the interval $[t_{i+1}, t_{i+k-1}]$, one obtains the pointwise error bound

$$\begin{gathered}|f(s) - (P_\pi f)(s)| \le D_k \max\{|f(s) - f(t) : t \in [t_{i-k+2}, t_{i+k-1}]\}, \\ \text{for all } s \in [t_i, t_{i+1}], \text{ all } i \in \mathbb{Z}, \text{ and all } f \in C(\mathbb{R}),\end{gathered} \tag{3.19}$$

for the "local" interpolation scheme P_π.

Proof of Theorem 3.1. It is sufficient to prove the theorem for a strictly increasing partition π. For, if π is not strictly increasing, then one can find a sequence $\{\pi^{(n)}\}_{n=1}^{\infty}$ of strictly increasing partitions such that

$$\lim_{n\to\infty} t_j^{(n)} = t_j \quad \text{for all } j \in \mathbb{Z}.$$

By Lemma 3.1(v), one has then

$$\lim_{n\to\infty} \|\phi_j^{(n)} - \phi_j\|_{\mathbb{R}} = 0$$

for the corresponding sequence $\{\phi_j^{(n)}\}_{n=1}^{\infty}$, for all $j \in \mathbb{Z}$. Since on the finite interval $[t_{i+1}, t_{i+k-1}]$, all but finitely many of the $\phi_j^{(n)}$ vanish, one has

$$\lim_{n\to\infty} \| \sum_{j\in\mathbb{Z}} \alpha_j \phi_j^{(n)} - \sum_{j\in\mathbb{Z}} \alpha_j \phi_j \|_i = 0 \quad \text{for all } \alpha \in m(\mathbb{Z}).$$

Hence, for all $\alpha \in m(\mathbb{Z})$ and all $i \in \mathbb{Z}$,

$$\operatorname{dist}_i(\phi_i^{(n)}, S_i^{(n)}) \le \|\phi_i^{(n)} - \sum_{j\ne i} \alpha_j \phi_j^{(n)}\|_i \xrightarrow[n\to\infty]{} \|\phi_i - \sum_{j\ne i} \alpha_j \phi_j\|_i.$$

Therefore, for all $i \in \mathbb{Z}$,

$$\overline{\lim_{n\to\infty}} \operatorname{dist}_i(\phi_i^{(n)}, S_i^{(n)}) \le \operatorname{dist}_i(\phi_i, S_i). \tag{3.20}$$

Hence, once a positive constant D_k has been shown to exist such that for every strictly increasing partition

$$D_k^{-1} \le \inf_i \operatorname{dist}_i(\phi_i, S_i),$$

then, by (3.20), this inequality holds also with the same constant for every k–extended partition.

Hence, assume π to be strictly increasing, and let $i \in \mathbb{Z}$. For $k = 2$, there is little to prove. For, then

$$\|f\|_i = |f(t_{i+1})|,$$

while by Lemma 3.1,

$$\phi_j(t_{i+1}) = \delta_{ij} \quad \text{for all } j \in \mathbb{Z}.$$

Thus, $\operatorname{dist}_i(\phi_i, S_i) = 1$, and $D_2 = 1$ will do.

Assume, therefore, also, $k \ge 3$. Since $\sum_{j\in\mathbb{Z}} \phi_j = 1$, one has

$$\inf_{\alpha\in m(\mathbb{Z})} \|\phi_i - \sum_{j\ne i} \alpha_j \phi_j\|_i = \inf_{\alpha\in m(\mathbb{Z})} \|1 - \sum_{j\ne i} \alpha_j \phi_j\|_i \; .$$

Further, if $f(t) = 1 - \sum_{j\ne i} \alpha_j \phi_j(t)$, and $i+1 \le r < i+k-1$, then, for suitable $\beta_i, \ldots, \beta_{k-1}$, one has

$$f(t) = 1 + \sum_{j=1}^{k-1} \beta_j (t - t_{i+j})^{k-1}, \quad \text{for all } t \in [t_r, t_{r+1}].$$

To see this, observe that, by (3.10) and (3.6),

$$\phi_j(t) = (t_{j+k} - t_j) \sum_{m=j}^{j+k} (t_m - t)_+^{k-1} / \omega'(t_m),$$

where

$$\omega(t) = \prod_{m=j}^{j+k} (t - t_m).$$

But, since $(s-t)_+^{k-1} + (-1)^{k-1}(t-s)_+^{k-1} \equiv (s-t)^{k-1}$, one has also

$$\begin{aligned}\phi_j(t) &= (t_{j+k} - t_j) g(t_j, \ldots, t_{j+k}; t) \\ &= (-1)^k (t_{j+k} - t_j) g(t; t_j, \ldots, t_{j+k}) \\ &= (-1)^k (t_{j+k} - t_j) \sum_{m=j}^{j+k} (t - t_m)_+^{k-1} / \omega'(t_m).\end{aligned}$$

Hence, if $j < i$, then, on $[t_r, t_{r+1}]$, $\phi_j(t)$ can be written as a linear combination of the functions $(t - t_{r+1})^{k-1}, \ldots, (t - t_{j+k})^{k-1}$, while if $j > i$, then, on $[t_r, t_{r+1}]$, $\phi_j(t)$ can be written as a linear combination of the functions $(t - t_j)^{k-1}, \ldots, (t - t_r)^{k-1}$.

It follows that, for $i + 1 \le r < i + k - 1$,

$$\operatorname{dist}_i(\phi_i, S_i) \ge \inf_{\beta \in \mathbb{R}^{k-1}} \|1 + \sum_{j=1}^{k-1} \beta_j (t - t_{i+j})^{k-1}\|_{[t_r, t_{r+1}]} \, . \tag{3.21}$$

In particular, choose r such that also

$$t_{j+1} - t_j \le t_{r+1} - t_r, \quad \text{for } j = i+1, \ldots, i+k-2.$$

Then, since the right–hand–side of (3,21) is invariant under a change of scale and origin in $\mathbb{R}$, the proof of the theorem is complete, once the following lemma is proved:

Lemma 3.2. *Let $I = [-1, 1], n \ge 2$. There exists a positive constant C_n depending only on n, such that*

$$C_n^{-1} \le \|1 + \sum_{j=1}^{n} \beta_j (t - s_j)^n\|_I$$

whenever $(\beta_j) \in \mathbb{R}^n$ and

$$s_1 < s_2 < \cdots < s_m = -1, \quad 1 = s_{m+1} < \cdots < s_n, \tag{3.22}$$

$$s_{j+1} - s_j \le 2, \quad \text{for } j = 1, \ldots, n-1. \tag{3.23}$$

Proof: The argument is based on the fact that

$$|\gamma| = \inf_{\beta \in \mathbb{R}^n} \|1 + \sum_{j=1}^{n} \beta_j (t - s_j)^n\|_I \tag{3.24}$$

can be expressed in terms on the s_j's. Explicitly, one has

$$\gamma^{-1} = \sum_{i=0}^{n} \sigma_i \gamma_{n-i} / \binom{n}{i}, \tag{3.25}$$

where the σ_i's are the elementary symmetric functions in the s_j's, i.e.,

$$\prod_{j=1}^{n} (t + s_j) \equiv \sum_{i=0}^{n} \sigma_i t^i. \tag{3.26}$$

Further, the γ_i's are given by

$$T_n(t) \equiv \sum_{i=0}^{n} \gamma_i t^i, \tag{3.27}$$

where T_n is the Chebyshev polynomial of degree n.

It follows that γ^{-1} is linear in each of the s_j, hence [original text:] for some constant c_n depending only on n, one has

$$|\gamma^{-1}| \le c_n \max_j |s_j|.$$

But then, with (3.23),

$$|\gamma| \ge [c_n(2n+1)]^{-1},$$

so that $C_n = c_n(2n+1)$ will do. [replaced jan73 by the following:] continuous at all points $(s_i)_{i=1}^n$ of $\mathbb{R}^n$. $|\gamma^{-1}|$ is therefore bounded by some constant C_n^{-1} on the bounded subset of $\mathbb{R}^n$ described by (3.22), (3.23).

It remains to prove (3.25). To this end, observe that the functions

$$h_0(t) \equiv 1, \quad h_j(t) \equiv (t - s_j)^n, \quad j = 1, \ldots, n,$$

form a basis for the linear space $\mathbb{P}_n$ of all polynomials of degree $\le n$. To see this, note that the relation

$$\beta_0 + \sum_{j=1}^{n} \beta_j (t - s_j)^n \equiv \sum_{i=0}^{n} \widehat{\gamma}_i t^i \tag{3.28}$$

is equivalent to

$$\beta_0 \delta_{ni} + \sum_{j=1}^{n} \beta_j (-s_j)^i = \widehat{\gamma}_{n-i} / \binom{n}{i}, \quad i = 0, \ldots, n, \tag{3.29}$$

as one can easily see by comparing the coefficients of like powers of t in (3.28).

On setting $t = -s_j$ in (3.26), one finds

$$\sum_{i=0}^{n} \sigma_i (-s_j)^i = 0, \quad j = 1, \ldots, n,$$

hence

$$\begin{aligned} \sum_{i=0}^{n} \sigma_i \widehat{\gamma}_{n-i} / \binom{n}{i} &= \sigma_n \beta_0 + \sum_{i=0}^{n} \sigma_i \sum_{j=1}^{n} \beta_j (-s_j)^i \\ &= \beta_0 + \sum_{j=1}^{n} \beta_j \sum_{i=0}^{n} \sigma_i (-s_j)^i = \beta_0, \end{aligned} \tag{3.30}$$

showing that (3.28) may be solved for β_0. As for β_j, $j \ge 1$, note that the first n equations in (3.29) involve only β_j, $j \ge 1$, and may be solved for these, since their coefficient matrix is the Vandermonde matrix on the distinct points $-s_j$, $j = 1, \ldots, n$, and hence nonsingular. This shows that the set $\{h_j : j = 0, \ldots, n\}$ is generating for $\mathbb{P}_n$, hence a basis.

With this, $\{h_j(t) : j = 1, \ldots, n\}$ is easily seen to be a Chebyshev set on I.[2] For, assume by way of contradiction that

$$f(t) \equiv \sum_{j=1}^{n} \beta_j h_j(t)$$

vanishes at the points r_i, $i = 1, \ldots, n$, with

$$-1 \le r_1 < \cdots < r_n \le 1, \tag{3.31}$$

[2] For the definition and basic properties of Chebyshev sets, cf., e.g., [5, Chap.3]

while not all of the β_i's are zero. Then, since by the above, $\{h_j(t) : j = 1, \ldots, n\}$ is linearly independent on I, $f(t)$ is not identically zero. It is, therefore, no loss to assume that

$$\sum_{j=1}^{n} \beta_j h_j(t) \equiv \prod_{i=1}^{n} (t - r_i) \equiv \sum_{i=0}^{n} \widehat{\gamma}_i t^i,$$

which implies, with (3.28) and (3.30), that

$$\sum_{i=0}^{n} \sigma_i \widehat{\gamma}_{n-i} / \binom{n}{i} = 0. \tag{3.32}$$

But this is impossible. For

$$\sum_{i=0}^{n} \sigma_i \widehat{\gamma}_{n-i} / \binom{n}{i} = (n!)^{-1} \sum_{\tau} \prod_{i=1}^{n} (s_i - r_{\tau(i)}),$$

where the summation on the right is taken over all permutations τ of degree n. Because of (3.22) and (3.31), all terms in that sum are seen to have the same sign and, since $n \geq 2$ and the r_i's are distinct, not all terms are zero. Hence

$$\sum_{i=0}^{n} \sigma_i \widehat{\gamma}_{n-i} / \binom{n}{i} \neq 0,$$

contradicting (3.32).

It follows that if $e(t) \equiv 1 + \sum_{j=1}^{n} \beta_j (t - s_j)^n$ is the error in the best approximation $-\sum_{j=1}^{n} \beta_j h_j$ to h_0 with respect to the norm $\| \cdot \|_I$ then $e(t)$ must alternate at least $n+1$ times on I. Since $e \in \mathbb{P}_n$, e is, therefore, necessarily of the form

$$e(t) \equiv \gamma T_n(t),$$

and (3.25) follows from (3.28) and (3.30). Q.E.D.

Corollary 1. *The linear map Φ given by*

$$\Phi\alpha = \sum_{i \in \mathbb{Z}} \alpha_i \phi_i, \quad \text{for all } \alpha \in m(\mathbb{Z}),$$

is a linear homeomorphism from $m(\mathbb{Z})$ to its range. Hence, its range coincides with B_π^k, and B_π^k is a closed linear subspace of $C(\mathbb{R})$.

Proof: Let $\alpha \in m(\mathbb{Z})$. Then, for all $i \in \mathbb{Z}$ such that $\alpha_i \neq 0$, one has

$$\begin{aligned} \| \sum_{j \in \mathbb{Z}} \alpha_j \phi_j \|_{\mathbb{R}} &= |\alpha_i| \| \phi_i - \sum_{j \neq i} (-\alpha_j / \alpha_i) \phi_j \|_{\mathbb{R}} \\ &\geq |\alpha_i| \operatorname{dist}_i(\phi_i, S_i) \geq |\alpha_i| D_k^{-1}. \end{aligned} \tag{3.33}$$

Hence

$$\|\Phi\alpha\|_{\mathbb{R}} \geq \|\alpha\|_{\mathbb{Z}} D_k^{-1} \quad \text{for all } \alpha \in m(\mathbb{Z}), \tag{3.34}$$

showing that Φ is bounded below, hence boundedly invertible on its range. Since also, by Lemma 3.1,

$$\|\Phi\alpha\|_{\mathbb{R}} \leq \|\alpha\|_{\mathbb{Z}} \quad \text{for all } \alpha \in m(\mathbb{Z}), \tag{3.35}$$

the first assertion follows.

By (3.35), the range of Φ is contained in $C(\mathbb{R})$, hence in B_π^k. Further, by [4; Theorem 5], each $p \in S_\pi^k$ is of the form

$$p = \sum_{i \in \mathbb{Z}} \alpha_i \phi_i, \quad \text{for some } \alpha \in \mathbb{R}^{\mathbb{Z}}.$$

But then, by (3.33), $p \in B_\pi^k$ implies $\alpha \in m(\mathbb{Z})$, or, p is contained in the range of Φ. It follows that the range of Φ coincides with B_π^k, hence, in particular, that B_π^k is closed. Q.E.D.

Corollary 2. *There exists a linear projector P_π on $C(\mathbb{R})$ with range B^k_π such that*
(i) $\|P_\pi\| \le D_k$;
(ii) $|f(s) - (P_\pi f)(s)| \le D_k \max\{|f(s) - f(t)| : t_{i-k+2} \le t \le t_{i+k-1}\}$,
for all $s \in [t_i, t_{i+1}]$, *all* $i \in \mathbb{Z}$, *and all* $f \in C(\mathbb{R})$.

Proof: Let $i \in \mathbb{Z}$. By Theorem 3.1,

$$\operatorname{dist}_i(\phi_i, S_i) = \inf\bigl\{\|\phi_i - \sum_{j\ne i} \alpha_j\phi_j\|_i : \alpha \in m(\mathbb{Z})\bigr\} \ge D_k^{-1} > 0.$$

Hence, by a corollary to the Hahn–Banach theorem, there exists a linear functional λ_i on $C(\mathbb{R})$ such that

$$\begin{aligned} &\lambda_i\phi_j = \delta_{ij} \quad \text{for all } j \in \mathbb{Z}, \\ &|\lambda_i f| \le D_k\|f\|_i \quad \text{for all } f \in C(\mathbb{R}). \end{aligned} \tag{3.36}$$

With this, the rule

$$P_\pi f = \sum_{i\in\mathbb{Z}} (\lambda_i f)\phi_i, \quad \text{for all } f \in C(\mathbb{R}), \tag{3.37}$$

defines a linear projector on $C(\mathbb{R})$ whose range is B^k_π, by Corollary 1. Further, its norm is $\le D_k$, since

$$\|P_\pi f\|_{\mathbb{R}} \le \sup_i |\lambda_i f| \le \sup_i D_k\|f\|_i \le D_k\|f\|_{\mathbb{R}}.$$

To prove (ii), let $f \in C(\mathbb{R})$, $s \in \mathbb{R}$. Then

$$\begin{aligned} f(s) - (Pf)(s) &= f(s) - \sum_{j\in\mathbb{Z}} (\lambda_j f)\phi_j(s) \\ &= \sum_{j\in\mathbb{Z}} \lambda_j(f(s)\cdot 1 - f)\phi_j(s), \end{aligned}$$

since $1 = \sum_{j\in\mathbb{Z}} \phi_j \in B^k_\pi$, therefore $1 = P_\pi(1) = \sum_{j\in\mathbb{Z}} \lambda_j(1)\phi_j$. Hence, for $i \in \mathbb{Z}$, $s \in [t_i, t_{i+1}]$,

$$\begin{aligned} |f(s) - (Pf)(s)| &= \bigl|\sum_{j=i+1-k}^{i} \lambda_j(f(s)\cdot 1 - f)\phi_j(s)\bigr| \\ &\le \max\{|\lambda_j(f(s)\cdot 1 - f)| : i+1-k \le j \le i\} \\ &\le D_k \max\{\|f(s)\cdot 1 - f\|_j : i+1-k \le j \le i\} \\ &= D_k \max\{|f(s) - f(t)| : t_{i+2-k} \le t \le t_{i-1+k}\}, \end{aligned}$$

using (3.36) and the definition (3.16) of $\|\cdot\|_j$. Q.E.D.

4. Spline Approximation on a Finite Interval

The interpolation scheme $P_{\widehat\pi}$ introduced in the previous section for a k–extended partition $\widehat\pi = \{t_i\}_{i\in\mathbb{Z}}$ of $\mathbb{R}$ is "local" in the sense that, on $[t_0, t_n]$, $P_{\widehat\pi} f$ depends only on the values of f in the interval $[t_{2-k}, t_{n-2+k}]$; this follows directly from (ii) of Corollary 2. In particular, if $\widehat\pi$ is such that

$$t_{2-k} = t_{3-k} = \cdots = t_0 = a, \quad b = t_n = t_{n+1} = \cdots t_{n+k-2},$$

then $P_{\widehat\pi} f$ on $I = [a, b]$ depends only on the values of f on I. Hence, by the simple device of restricting attention to the interval I, $P_{\widehat\pi}$ becomes a linear projector P_π on $C(I)$ with range the set of extended polynomial splines S^k_π of degree $k-1$ on the restriction

$$\pi : a = t_0 < t_1 \le t_2 \le \cdots \le t_{n-1} < t_n = b$$

of $\widehat{\pi}$ to I. Since the bounds for $P_{\widehat{\pi}}$ derived in the previous section are also valid for P_π, one obtains, finally, the results announced in the introduction.

To make these statements precise, define for $I = [a, b]$ the restriction map from $C(\mathbb{R})$ to $C(I)$ by the rule

$$(R_I x)(t) = x(t), \quad \text{for all } t \in I,\ x \in C(\mathbb{R}). \tag{4.1}$$

R_I is a norm–reducing linear map, having the extension map E_I,

$$(E_I x)(t) = \begin{cases} x(a),\ t < a, \\ x(t),\ a \le t \le b, \\ x(b),\ b < t, \end{cases} \quad \text{for all } x \in C(I), \tag{4.2}$$

as a norm–preserving right inverse.

Call $\pi = \{t_i\}_{i=0}^n$ a $k-extended\ partition\ for\ I$, provided

$$\begin{aligned} &a = t_0 < t_1 \le \cdots \le t_{n-1} < t_n = b \\ &t_i < t_{i+k-1} \quad \text{for all } i. \end{aligned} \tag{4.3}$$

As before, let d_i denote the frequency with which the number t_i appears in π. Then define the set S_π^k of all polynomial extended splines of degree $k-1$ on π as the set of all real–valued functions on I which, on each of the intervals $[t_i, t_{i+1}]$, $i = 0, \ldots, n-1$, reduce to a polynomial of degree $\le k-1$, and have $k-1-d_i$ continuous derivatives in a neighborhood of t_i, $i = 1, \ldots, n-1$.

Lemma 4.1. *Let $I = [a, b]$ be some finite interval, $\pi = \{t_i\}_{i=0}^n$ a k–extended partition for I, and extend π in any way whatsoever to a k–extended partition $\widehat{\pi} = \{t_i\}_{i \in \mathbb{Z}}$ of $\mathbb{R}$, subject only to the restriction*

$$t_j = \begin{cases} a, & -k+2 \le j \le 0, \\ b, & n \le j \le n+k-2. \end{cases} \tag{4.4}$$

If $\widehat{P}$ is a linear projector on $C(\mathbb{R})$ with range $B_{\widehat{\pi}}^k$, then

$$P = R_I \widehat{P} E_I \tag{4.5}$$

is a linear projector on $C(I)$ with range S_π^k, satisfying $\|P\| \le \|\widehat{P}\|$.

Proof: Since the numbers t_0, t_n each appear in $\widehat{\pi}$ $k-1$ times, every $p \in B_\pi^k$ need only be continuous at t_0 and t_n. It follows that E_I maps S_π^k into $B_{\widehat{\pi}}^k$. Hence, as $\widehat{P}$ is the identity on its range, $B_{\widehat{\pi}}^k$, it follows that, for $p \in S_\pi^k$,

$$Pp = (R_I \widehat{P} E_I)p = R_I \widehat{P}(E_I p) = R_I(E_I p) = (R_I E_I)p = p,$$

or, P is the identity on S_π^k. But, since R_I maps the range $B_{\widehat{\pi}}^k$ of $\widehat{P}$ to S_π^k, the range of P must be contained in S_π^k. Hence, the range of P is S_π^k, and P is the identity on its range, i.e., P is a linear projector. Finally

$$\|P\| \le \|R_I\| \|\widehat{P}\| \|E_I\| = \|\widehat{P}\|. \qquad Q.E.D.$$

In particular, $P_\pi = R_I P_{\widehat{\pi}} E_I$ is a linear projector on $C(I)$ with range S_π^k, where $P_{\widehat{\pi}}$ is as described in Corollary 2 to Theorem 3.1. This gives

Theorem 4.1. *There exists a positive constant D_k depending only on k, with the property: For all k–extended partitions π of $I = [a, b]$, there exists a linear projector P_π on $C(I)$ with range S_π^k such that*

(i) $\|P_\pi\| \le D_k$

(ii) $|f(s) - (P_\pi f)(s)| \le D_k \max\{|f(s) - f(t)| : t \in [t_{i-k+2}, t_{i+k-1}]\}$, *for all $s \in [t_i, t_{i+1}]$ and all $f \in C(I)$, where $t_j = a, j \le 0,\ t_j = b, j \ge n$.*

Proof: Since $R_I E_I$ is the identity, one has, with $P_\pi = R_I P_{\widehat{\pi}} E_I$,

$$f(s) - (P_\pi f)(s) = (E_I f)(s) - [P_{\widehat{\pi}}(E_I f)](s), \quad \text{all } s \in [a, b];$$

hence, (ii) follows from Corollary 2 to Theorem 3.1.

Corollary 1. *For all $f \in C(I)$,*

$$\|f - P_\pi f\|_I \le D_k(k-1)\omega(f;|\pi|).$$

Proof: This is a consequence of (ii) of the preceding theorem.

Denote P_π by P_π^k, to emphasize dependence on k.

Corollary 2. *The preceding estimate can be improved for smooth f:*

(i) $\|f - P_\pi^k f\|_I \le \widehat{D}_k \widehat{D}_{k-1} \ldots \widehat{D}_{k-r} |\pi|^r \omega(f^{(r)};|\pi|)$,
for all $f \in C^{(r)}(I), r = 1, \ldots, k-1$,
with $\widehat{D}_k = D_k(k-1)$ *for* $k \ge 2$, *and* $\widehat{D}_1 = \frac{1}{2}$.

Hence,

(ii) $\|f - P_\pi^k f\|_I = O(|\pi|^k)$ *for all* $f \in \mathrm{Lip}_1^{(k-1)}(I)$,

where, as usual, $\mathrm{Lip}_1^{(k-1)}(I)$ *consists of all* $f \in C^{(k-1)}(I)$ *with* $f^{(k-1)}$ *satisfying a Lipschitz condition (with exponent 1) on* I.

Proof by induction on k. Consider $k = 2$. Then P_π^k is broken line interpolation, i.e.,

$$(P_\pi^2 f)(t) = f(t_i)\frac{t_{i+1} - t}{t_{i+1} - t_i} + f(t_{i+1})\frac{t - t_i}{t_{i+1} - t_i}, \quad t \in [t_i, t_{i+1}].$$

Assume, without loss in generality, that $t - t_i \le \frac{1}{2}(t_{i+1} - t_i)$. Then, with $f \in C^{(1)}(I)$,

$$\begin{aligned} f(t) - (P_\pi^2 f)(t) &= \int_{t_i}^{t} f'(s)dds - \frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}(t - t_i) \\ &= (f'(\eta) - f'(\xi))(t - t_i), \end{aligned}$$

for some η, $\xi \in (t_i, t_{i+1})$, from which (i) follows for this case.

As for the general case, observe that

$$S_\pi^{k-1} = \{p' : p \in S_\pi^k\},$$

unless π contains points repeated $k-1$ times, in which case, neither side is defined. But as π is a k–extended partition on I, I may be subdivided into finitely many subintervals $I_i = [a_i, a_{i+1}]$, $i = 1, \ldots, r$, with $a = a_i < a_2 < \cdots < a_{r+1} = b$, such that $\{a_i : i = 2, \ldots, r\}$ coincides with the set of points in π which are repeated $k-1$ times. If π_i denotes the restriction of π to I_i, then π_i is a $(k-1)$–extended partition of I_i, and Lemma 4.1 shows that

$$P_{\pi_i}^k = R_{I_i} P_{\widehat{\pi}} E_{I_i} = R_{I_i} P_\pi^k E_{I_i},$$

hence

$$f(t) - (P_\pi^k f)(t) = f(t) - (P_{\pi_i}^k f)(t), \quad \text{for all } t \in I_i.$$

It is, therefore, sufficient to prove the Corollary under the assumption that π is a $(k-1)$–extended partition, in which case

$$S_\pi^{k-1} = \{p' : p \in S_\pi^k\}.$$

Assume the corollary proved for $k-1$. One has for all $g \in S_\pi^k$,

$$\|f - P_\pi^k f\|_I = \|(f-g) - P_\pi^k(f-g)\|_I \le \widehat{D}_k \omega(f-g;|\pi|) \le \widehat{D}_k |\pi| \|f' - g'\|_I.$$

Hence, as $S_\pi^{k-1} = \{g' : g \in S_\pi^k\}$, one gets

$$\|f - P_\pi^k f\|_I \le \widehat{D}_k |\pi| \operatorname{dist}(f', S_\pi^{k-1}).$$

But as

$$\operatorname{dist}(f', S_\pi^{k-1}) \le \|f' - P_\pi^{k-1} f'\|_I,$$

all statements of the corollary for k follow from their assumed correctness for $k-1$. Q.E.D.

Remark. The statement in [2] to the effect that "$f \in \mathrm{Lip}_1^{(k-1)}(I)$" in (ii) of the preceding corollary can be weakened to "$f \in C^{(k-1)}(I)$ and $f^{(k-1)}$ is of bounded variation" is incorrect, as an examination of the simple case $k = 2$ quickly shows. The converse of (ii) will be considered in a subsequent note.

5. Remarks on Estimating D_k

As has just been pointed out, P_π^2 is broken line interpolation, i.e., the linear functionals λ_i are just point functionals,

$$\lambda_i f = f(t_{i+1}) \quad \text{for all } i.$$

For the case $k = 3$ of approximation by parabolic splines one may choose

$$\lambda_i f = -\frac{1}{2}\Big[f(t_{i+1}) - 4f\Big(\frac{t_{i+1}+t_{i+2}}{2}\Big) + f(t_{i+2})\Big],$$

giving

$$\|\lambda_i\| \le D_3 = 3 \quad \text{for all } i,$$

with strict inequality iff $t_{i+1} = t_{i+2}$.

Already for $k = 4$, the λ_i's become quite complicated, if one insists on choosing them as linear combinations of point functionals.

In view of Theorem 3.1 and Lemma 3.2, λ_i may be constructed in general as follows. Choose $r = r(i)$ such that $J_r = [t_r, t_{r+1}]$ is a largest among the intervals J_j, $j = i+1, \ldots, i+k-2$. Let

$$t_r = s_0 < s_1 < \cdots < s_{k-1} = t_{r+1}$$

be the extremal points of the Chebyshev polynomial $\tilde{T}_{k-1}$ of degree $k-1$ adjusted to the interval J_r. Define

$$C(\alpha_1, \ldots, \alpha_{k-1}) = \det((\alpha_m - t_{i+n})^{k-1})_{n,m=1}^{k-1},$$

and set

$$\widehat{\lambda}_i f = \sum_{m=0}^{k-1} (-1)^m \beta_m f(s_m) \quad \text{for all } f,$$

$$\beta_m = C(s_0, s_1, \ldots, s_{m-1}, s_{m+1}, \ldots, s_{k-1}), \quad m = 0, \ldots, k-1.$$

Then

$$\widehat{\lambda}_i (t - t_j)^{k-1} = 0, \quad j = i+1, \ldots, i+k-1,$$

hence

$$\widehat{\lambda}_i \phi_j = 0, \quad j \ne i.$$

Therefore, with

$$\lambda_i = \widehat{\lambda}_i / \widehat{\lambda}_i(1),$$

one has

$$\inf \|\phi_i - \sum_{j\ne i} \gamma_j \phi_j\|_{J_r} \ge \|\lambda_i\|^{-1}.$$

The argument in Lemma 3.2 merely shows that $\|\lambda_i\|$ can be computed as

$$\|\lambda_i\| = |\lambda_i \tilde{T}_{k-1}|.$$

This is so since $C(\alpha_1, \ldots, \alpha_{k-1})$ is a continuous function of the α_i's and is, by the argument in Lemma 3.2, not zero for $t_r \le \alpha_1 < \alpha_2 \cdots < \alpha_{k-1} \le t_{r+1}$. The β_m, $m = 0, \ldots, k-1$, are therefore all of one sign. Hence, as $\tilde{T}_{k-1}$ alternates on the points s_m, $m = 0, \ldots, k-1$, one has

$$|\lambda_i \tilde{T}_{k-1}| = \sum_{m=0}^{k-1} |\beta_m| / |\widehat{\lambda}_i(1)| = \|\lambda_i\|.$$

One computes D_4 to be ≤ 15 and $D_5 \le 100$. But it should be clear on examining closely the arguments in this note that the linear projectors P_π^k are probably far from being minimal in norm for the range S_π^k. The chief reason for this is the fact that the distance of ϕ_i from the linear span of the remaining ϕ_j's was measured only on some "small" interval rather than with respect to the norm on $C(I)$.

References

[1] C. de Boor, The method of projections as applied to the numerical solution of two point boundary value problems using cubic splines, Dissertation, Univ. Michigan (Ann Arbor MI), 1966.
[2] C. de Boor, On local spline approximation by moments, J. Math. Mech. 17 (1968) 729–735.
[3] H. B. Curry, I. J. Schoenberg, On spline distributions and their limits: The Polya distribution functions, Bull. Amer. Math. Soc. 53 (1947) 1114.
[4] H. B. Curry, I. J. Schoenberg, On Pólya frequency functions IV: the fundamental spline functions and their limits, J. Analyse Math. 17 (1966) 71–107.
[5] J. R. Rice, The Approximation of Functions, Vol. I, Addison-Wesley, Reading, Mass., 1964.

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