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On the construction of multivariate (pre)wavelets

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Abstract: A new approach for the construction of wavelets and prewavelets on $\mathbb{R}^d$ from multiresolution is presented. The method uses only properties of shift-invariant spaces and orthogonal projectors from $L_2(\mathbb{R}^d)$ onto these spaces, and requires neither decay nor stability of the scaling function. Furthermore, this approach allows a simple derivation of previous, as well as new, constructions of wavelets, and leads to a complete resolution of questions concerning the nature of the intersection and the union of a scale of spaces to be used in a multiresolution.

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1. Introduction

We present a new approach for the construction of wavelets and prewavelets on $\mathbb{R}^d$ from multiresolution. Our method, which is based on our earlier work [BDR] and [BDR1], uses only properties of shift-invariant spaces and orthogonal projectors from $L_2(\mathbb{R}^d)$ onto these spaces, and requires neither decay nor stability of the scaling function. Furthermore, this approach allows us to derive in a simple way previous constructions of wavelets, as well as new constructions, and to settle completely certain basic questions about multiresolution.

A univariate function $\psi \in L_2(\mathbb{R})$ is called an **orthogonal wavelet** if its normalized, translated dilates $\psi_{j,k} := 2^{k/2}\psi(2^k \cdot -j)$, $j,k \in \mathbb{Z}$, form an orthonormal basis for $L_2(\mathbb{R})$. In other words, this system is complete and satisfies the orthogonality conditions

$$\int_{\mathbb{R}} \psi_{j,k}\overline{\psi}_{j',k'} = \delta(j-j')\delta(k-k'), \quad j,k,j',k' \in \mathbb{Z}, \tag{1.1}$$

with δ the delta function on $\mathbb{Z}$. The concept of **prewavelet** is somewhat more general in that it requires (1.1) to hold only when $k \neq k'$ and hence the functions there are not assumed to be orthogonal at a fixed dyadic level k. In particular, $\psi(\cdot - j)$, $j \in \mathbb{Z}$, are not necessarily orthogonal, and, instead, one assumes that $\big(\psi(\cdot - j)\big)_{j\in\mathbb{Z}}$ forms a stable basis for $L_2(\mathbb{R})$ (see the end of this section and §2 for the definition of stability).

On $\mathbb{R}^d$, wavelet and prewavelet bases are generated by the translation and dilation of the elements of a set Ψ of $2^d - 1$ functions from $L_2(\mathbb{R}^d)$. We shall say that they are an **orthogonal wavelet set** if $\{\psi_{j,k} := 2^{kd/2}\psi(2^k \cdot -j) :\ \psi \in \Psi, j \in \mathbb{Z}^d, k \in \mathbb{Z}\}$ is an orthonormal basis for $L_2(\mathbb{R}^d)$. Analogously, the set Ψ is a **prewavelet set** if $\{\psi_{j,k} :\ \psi \in \Psi, j \in \mathbb{Z}^d, k \in \mathbb{Z}\}$ is a stable basis for $L_2(\mathbb{R}^d)$, and in addition we have orthogonality between levels:

$$\int_{\mathbb{R}^d} \psi_{j,k}\overline{\phi}_{j',k'} = 0, \quad k \neq k', \quad j,j' \in \mathbb{Z}^d, \quad \psi,\phi \in \Psi. \tag{1.2}$$

The construction of orthogonal wavelets has a rich history described in the monograph of Meyer [Me] and the article of Daubechies [D1]. Prewavelets have been stressed only in recent years beginning with Battle [B]. The paper of Jia and Micchelli [JM] discusses their brief history. Most methods used for the construction of wavelets are based on the notion of multiresolution as introduced by Mallat [Ma] and Meyer (see [Me]). Multiresolution, which we now describe, will also form the starting point for our constructions.

We say that a space $\mathcal{S}$ of functions defined on $\mathbb{R}^d$ is **shift-invariant** if, for each $s \in \mathcal{S}$, the **shifts**, $s(\cdot - j)$, $j \in \mathbb{Z}^d$, of s are also in $\mathcal{S}$. More generally, we say that $\mathcal{S}$ is h-shift-invariant if it is closed under $h\mathbb{Z}^d$-translations. All shift-invariant spaces considered in this paper are assumed to be closed subspaces of $L_2(\mathbb{R}^d)$. Important examples of shift-invariant spaces are those generated by a finite set $\Phi := \{\phi_1, \ldots, \phi_n\}$ of functions from $L_2(\mathbb{R}^d)$. For such Φ, we define

$$\mathcal{S}^k(\Phi)$$

to be the $L_2(\mathbb{R}^d)$-closure of the finite linear combinations of the 2^{-k}-shifts of the functions from Φ. We write

$$\mathcal{S}(\Phi) := \mathcal{S}^0(\Phi).$$

In case Φ consists of a single element ϕ, we write $\mathcal{S}^k(\phi)$ (instead of $\mathcal{S}^k(\{\phi\})$) and we say that $\mathcal{S}(\phi)$ is a **principal** shift-invariant space.

Now suppose that we hold in hand a sequence of spaces $\{\mathcal{S}^k\}_{k\in\mathbb{Z}}$, with $\mathcal{S}^k$ a 2^{-k}-shift-invariant space for each $k \in \mathbb{Z}$. We shall say that $\{\mathcal{S}^k\}$ forms a **multiresolution** if the following conditions are satisfied:

$$\begin{aligned} &\text{(i)} \quad \mathcal{S}^k \subset \mathcal{S}^{k+1},\ k \in \mathbb{Z};\\ &\text{(ii)} \quad \overline{\cup \mathcal{S}^k} = L_2(\mathbb{R}^d);\\ &\text{(iii)} \quad \cap\, \mathcal{S}^k = \{0\}. \end{aligned} \tag{1.3}$$

In the usual definition of multiresolution analysis as proposed by Mallat [Ma], it is also assumed that (a) $\mathcal{S}^k$ is the 2^k-dilate of some fixed principal shift-invariant space $\mathcal{S}(\phi)$, and that (b) the shifts of ϕ form an orthonormal family. We do not assume these conditions in our definition (1.3) in order that we can discuss more general situations that are covered by the techniques of this paper. However, for the remainder of this introduction, in order to keep the discussion simple, we shall assume (a), i.e., that $\mathcal{S}^k$ is of the form

$$\mathcal{S}^k = \{s(2^k\cdot) : \ s \in \mathcal{S}(\phi)\},$$

for some principal shift-invariant space $\mathcal{S}(\phi)$. Equivalently, each $\mathcal{S}^k$ is generated by the 2^{-k}-shifts of the dilated function $\phi(2^k\cdot)$. In this case, the condition (1.3)(i) is already implied by

$$\mathcal{S}^0 \subset \mathcal{S}^1. \tag{1.4}$$

Previous constructions of wavelets assume that ϕ has $L_2(\mathbb{R}^d)$-stable shifts, a notion which we now introduce. For a collection $F \subset L_2(\mathbb{R}^d)$, we say that F is **a stable basis** (for the space it generates) if there exist positive constants $C_1(F)$ and $C_2(F)$ such that, for any finitely supported $a := \big(a_f\big)_{f\in F}$,

$$C_1(F)\|a\|_{\ell_2(F)} \le \|\sum_{f\in F} a_f f\| \le C_2(F)\|a\|_{\ell_2(F)}. \tag{1.5}$$

In the context of wavelets, the family F is taken to be the $2^{-k}\mathbb{Z}^d$-shifts of some finite set Φ, with the totality of shifts taken over all $k \in \mathbb{Z}$ or for some fixed k. Discussions of the stability question, including earlier references, can be found in [JM] (also for norms other than the 2-norm) and in [BDR1]. Because the finitely supported sequences are dense in $\ell_2(\mathbb{Z}^d)$, the L_2-stability of the shifts of a function ϕ implies that the map

$$\ell_2(\mathbb{Z}^d) \to \mathcal{S}(\phi) : a \mapsto \sum_{j\in\mathbb{Z}^d} \phi(\cdot - j)a(j)$$

is well-defined and induces a Hilbert space isomorphism between $\ell_2(\mathbb{Z}^d)$ and $\mathcal{S}(\phi)$.

Thus, under the stability assumption, (1.4) is equivalent to having a **refinement equation**

$$\phi(x) = \sum_{j\in\mathbb{Z}^d} \phi(2x - j)a(j) \tag{1.6}$$

hold for some sequence $a \in \ell_2(\mathbb{Z}^d)$, or, equivalently, to have

$$\widehat{\phi} = A\widehat{\phi}(\cdot/2)$$

for some 4π-periodic, locally square-integrable function A, called the **(refinement) mask**.

Given a shift-invariant space $\mathcal{S} := \mathcal{S}^0$ whose 2^k-dilates $\mathcal{S}^k$, $k \in \mathbb{Z}$, satisfy (1.3)(i), we define the **wavelet space** W as the orthogonal complement of $\mathcal{S}^0$ in $\mathcal{S}^1$:

$$W := \mathcal{S}^1 \ominus \mathcal{S}^0.$$

It follows that $W^k := \mathcal{S}^{k+1} \ominus \mathcal{S}^k$, $k \in \mathbb{Z}$, is the 2^k-dilate of W. The spaces W^k, $k \in \mathbb{Z}$, are mutually orthogonal.

One can equivalently define W by projections. If $P := P_{\mathcal{S}}$ is the orthogonal projector from $L_2(\mathbb{R}^d)$ onto $\mathcal{S}$, then $W = \{s - Ps : \ s \in \mathcal{S}^1\}$. If (1.3)(ii) and (iii) are also satisfied, then one has the orthogonal decomposition

$$L_2(\mathbb{R}^d) = \bigoplus_{k\in\mathbb{Z}} W^k \tag{1.7}$$

since, for each $f \in L_2(\mathrm{I\!R}^d)$, we have

$$f = \sum_{k\in\mathbb{Z}} (P_k f - P_{k-1} f), \quad \text{with} \quad P_k f - P_{k-1} f \in W^{k-1},\ k \in \mathbb{Z},$$

and P_k the $L_2(\mathrm{I\!R}^d)$ projector onto $\mathcal{S}^k$ (which is obtained from P by dilation). Indeed, the condition (1.3)(ii) implies that $\lim_{k\to\infty} P_k f = f$, and (1.3)(iii) implies that $\lim_{k\to-\infty} P_k f = 0$. Note that $P_k - P_{k-1}$ is the orthogonal projector of $L_2(\mathrm{I\!R}^d)$ onto W^{k-1}.

One obtains wavelets and prewavelets from multiresolution analysis by finding generators for the space W. For example, in the univariate case, Mallat [Ma] begins with a function $\phi \in L_2(\mathrm{I\!R})$ which has orthonormal shifts and satisfies (1.3) (with $\mathcal{S}^k := \mathcal{S}^k(\phi)$) and shows that W is a principal shift-invariant space $W = \mathcal{S}(\psi)$ with ψ an orthogonal wavelet. One can also apply the Mallat construction to a function ϕ whose shifts are only $L_2(\mathrm{I\!R})$-stable by first orthonormalizing these shifts.

Unfortunately, if ϕ is of compact support (and its shifts are not orthonormal) then the orthogonal wavelet ψ will generally not have compact support. This motivated the study of prewavelets. We obtain prewavelets ψ by finding generators of W whose shifts form an $L_2(\mathrm{I\!R})$-stable basis for W (but not necessarily an orthonormal system). Chui and Wang [CW] and Micchelli [Mi] have shown in the *univariate* case that if ϕ has compact support and $L_2(\mathrm{I\!R})$-stable shifts and (1.3) is satisfied (again with $\mathcal{S}^k := \mathcal{S}^k(\phi)$) then there is a *compactly supported* prewavelet ψ which generates W. Chui and Wang even characterize the $\psi \in W$ of minimal support (in a sense to be made clear in §5) which generates W. We shall give a simple derivation of (a slightly stronger version of) these facts in §5.

In the multivariate case, the construction of orthogonal wavelet and prewavelet sets is by far more involved. Micchelli [Mi] and Jia and Micchelli [JM] have studied multiresolution in the case the function ϕ has $L_2(\mathrm{I\!R}^d)$-stable shifts and satisfies two regularity conditions. The first of these is that the periodization

$$|\phi|^\circ := \sum_{j\in\mathbb{Z}^d} |\phi(\cdot - j)| \tag{1.8}$$

of $|\phi|$ is in $L_2(\mathrm{T\!T}^d)$. (Note that this requirement is satisfied if ϕ has suitable decay at ∞). Secondly, they require that ϕ satisfies the refinement equation (1.6) with the coefficient sequence a in $\ell_1(\mathbb{Z}^d)$.

In contrast with the present literature, we only need to assume here that the function ϕ is in $L_2(\mathrm{I\!R}^d)$, satisfies the refinement condition (1.3)(i), (with $\mathcal{S}^0 := \mathcal{S}(\phi)$, and $\mathcal{S}^k$ the 2^k-dilate of $\mathcal{S}^0$) and that its Fourier transform $\widehat{\phi}$ satisfies

$$\operatorname{supp} \widehat{\phi} = \mathrm{I\!R}^d. \tag{1.9}$$

Here and later, the support of an $L_2(\mathrm{I\!R}^d)$-function f is defined only modulo a null-set as $\{x : f(x) \neq 0\}$ and the Fourier transform of a function $f \in L_1(\mathrm{I\!R}^d)$ is defined by

$$\widehat{f}(y) := \int_{\mathrm{I\!R}^d} e_{-y} f, \tag{1.10}$$

where, here and throughout,

$$e_\theta : x \mapsto e^{ix\cdot\theta}$$

is the complex exponential with frequency $\theta \in \mathrm{I\!R}^d$. (We shall use without further mention basic facts from Fourier analysis including the fact that the Fourier transform has an extension to $L_2(\mathrm{I\!R}^d)$.) In particular, our analysis applies whenever ϕ has compact support since then $\widehat{\phi}$ is analytic and its zero set is of measure zero (unless $\phi = 0$). We note that we shall not need to assume that ϕ has $L_2(\mathrm{I\!R}^d)$-stable shifts, nor impose any decay conditions, nor any conditions on the refinement coefficients a. In fact, we do not even need to assume the refinement condition in the form (1.6), only in the original form (1.3)(i).

Under the above assumptions, we show in this paper that the conditions (1.3)(ii) and (1.3)(iii) of multiresolution automatically hold. Further, our derivation of (1.3)(iii) from (1.3)(i) does not make use of (1.9). We even provide a characterization of property (1.3)(ii) for the case when (1.9) fails to hold. We also show that (1.3)(ii) and (iii) automatically hold whenever ϕ is of compact support. Details can be found in §4. Previous results on the matter (cf. e.g., [JM] and [Sö]) were derived under the stability assumption and under suitable decay conditions.

The main goal of multiresolution is to construct a set of Ψ of 2^d-1 functions which generate the wavelet space W (i.e., $W = \mathcal{S}(\Psi)$) and have other prescribed properties. We shall index the elements in Ψ by the set $V' := V \setminus \{0\}$, with V the set of vertices of the cube $[0\,\mathrm{d}1/2]^d$. A major advantage of our approach is that it is almost trivial to find generating sets Ψ for W. Once one such set Ψ is found, we can then find (all) other generating sets by simple operations on the Fourier transforms of the elements of Ψ.

Two particularly interesting generating sets which are obtained by our construction are discussed in §3. First of all, we show that (1.9) implies that W always possesses a generating set Ψ which provides an orthonormal basis for W, i.e., an orthogonal wavelet set. Secondly, under slightly more restrictive assumptions on ϕ, we show that there is a function $w \in L_2(\mathrm{I\!R}^d)$ whose half-shifts $w(\cdot + v)$, $v \in V'$, form a generating set for W. Special cases of this latter result have been proved in [MRU] and [Mi1], see also [LM].

A more delicate problem is the construction of multivariate wavelets and prewavelets which have compact support. If the function ϕ of multiresolution has compact support, it is quite easy to find generating sets Ψ for W whose elements are compactly supported. On the other hand, if ϕ has $L_2(\mathrm{I\!R}^d)$-stable shifts, we would like the shifts of the functions in Ψ to form an $L_2(\mathrm{I\!R}^d)$-stable basis for W. While it has been shown by Meyer [Me; Chapter III, §6], and Jia and Micchelli [JM1] that such generating sets always exist, their proofs are not constructive. On the other hand, several authors, including Riemenschneider and Shen [RS1], Chui, Stöckler, and Ward [CSW], Lorentz and Madych [LM], and Stöckler [Sö], have given constructions of prewavelet sets Ψ under various conditions on ϕ and in some cases with restrictions on the space dimension d. We shall discuss this question in §7 where we shall use our characterizations of the wavelet space W to recover and slightly improve some of these constructions.

A particularly interesting application of wavelet constructions is to functions ϕ which are B-splines or box splines. In this regard, we obtain the compactly supported univariate spline prewavelets of Chui and Wang [CW1] and derive various orthogonal wavelets and prewavelets obtained from box splines.

As we have already noted, our construction of wavelets and prewavelets is based on our earlier results on the structure of shift-invariant spaces. We use two facts repeatedly. The first is an exact description of finitely generated shift-invariant spaces. For example, we have shown in [BDR] that the principal shift-invariant space $\mathcal{S}(\phi)$ is described by its Fourier transforms:

$$\widehat{\mathcal{S}(\phi)} = \{\tau\widehat{\phi} \in L_2(\mathrm{I\!R}^d) : \ \tau \text{ is } 2\pi\text{-periodic}\}. \tag{1.11}$$

Here and later, for a set of functions F, we define $\widehat{F} := \{\widehat{f} : \ f \in F\}$ to be the set of its Fourier transforms. A similar characterization holds for a finitely generated space (see §2).

In the case that ϕ has $L_2(\mathrm{I\!R}^d)$-stable shifts, (1.11) is well-known, and the functions τ must be in $L_2(\mathrm{T\!T}^d)$, with

$$\mathrm{T\!T}^d$$

the d-dimensional torus, i.e., the cube $[-\pi\,\mathrm{d}\pi]^d$ with the usual identification of its boundary points. Assuming $\operatorname{supp}\widehat{\phi} = \mathrm{I\!R}^d$, we have shown in [BDR1] that there is always a function ϕ_* which generates $\mathcal{S}(\phi)$, (i.e. $\mathcal{S}(\phi_*) = \mathcal{S}(\phi)$), whose shifts are $L_2(\mathrm{I\!R}^d)$-stable; in fact they can be taken to be orthonormal.

The second result which we frequently employ is the explicit formula (2.11) of the next section for the orthogonal projector $P = P_\phi$ from $L_2(\mathrm{I\!R}^d)$ onto the principal shift-invariant space $\mathcal{S}(\phi)$.

With these results in mind, our construction proceeds as follows. We show that if ϕ is in $L_2(\mathbb{R}^d)$ with $\operatorname{supp}\widehat{\phi} = \mathbb{R}^d$ and if the space sequence $\big(\mathcal{S}^k(\phi)\big)_{k\in\mathbb{Z}}$ satisfies (1.3)(i), then we can give an alternate description for $\mathcal{S}^1$:

$$\mathcal{S}^1 = \mathcal{S}(\Phi) \tag{1.12}$$

where $\Phi := \big(\phi(\cdot + v)\big)_{v\in V}$. It follows that

$$w_v := \phi(\cdot + v) - P_\phi(\phi(\cdot + v)), \quad v \in V',$$

are a set of generators for W. We then use our characterization of finitely generated shift-invariant spaces to obtain other generators with more favorable properties. Because of our description of the projector P_ϕ, all these generators are described in a concrete way in terms of their Fourier transforms. In this way, we are able to construct an orthonormal basis for the wavelet space without using either the refinement equation or the mask, and further, without making any assumption on the stability or orthogonality of the shifts of ϕ. Under further assumptions (e.g., when ϕ is compactly supported), generators of the wavelet space that can be written as a finite linear combinations of $\phi(2\cdot -j)$, $j \in \mathbb{Z}^d$, are obtained.

As already alluded to in our definition (1.3), we shall actually work in the more general setting of **non-stationary wavelets** in this paper, which means that our spaces $\mathcal{S}^k$, while still being assumed to be generated by the 2^{-k}-shifts of some function ϕ_k, will not be assumed to be the dilate of $\mathcal{S}^0$ or of any other space in the sequence $\{\mathcal{S}^j\}$. It turns out that this generalization can be handled at no additional cost and leads to interesting bases for $L_2(\mathbb{R}^d)$. For example, in §6, we discuss such an example based on exponential splines, and in §8 we discuss their multivariate analog, the exponential box splines.

An outline of the present paper is as follows. In §2, we review and extend results from our earlier work which will be needed in the sequel. In §3, we describe generating sets and bases for wavelet spaces W. In §4, we analyze conditions (1.3)(ii) and (iii) of multiresolution. In §5, we apply our constructions to derive univariate wavelets and prewavelets with various desirable properties. In §6, we discuss exponential B-splines as wavelets. In §7, we consider the construction of wavelets in the multivariate case. We conclude with a brief discussion in §8 of exponential box splines as wavelets, and describe stable bases for their associated wavelet spaces.

2. Shift-invariant spaces

Our analysis will be based on the structure of shift-invariant spaces given in our earlier work [BDR] and [BDR1]. In this section, we review some of these facts and develop them somewhat further in directions pertinent to the construction of wavelets. We have already mentioned in (1.11) a characterization of $\mathcal{S} = \mathcal{S}(\phi)$ in terms of Fourier transforms. A similar characterization of the space $\mathcal{S}^k(\phi)$, generated by the 2^{-k}-shifts of ϕ, easily follows from (1.11) by dilation:

$$\widehat{\mathcal{S}^k(\phi)} = \{\tau\widehat{\phi} \in L_2(\mathbb{R}^d) : \ \tau \text{ is } 2^{k+1}\pi\text{−periodic}\}. \tag{2.1}$$

In the context of the principal shift-invariant space $\mathcal{S}(\phi)$, it will be important to know whether some given function $f \in \mathcal{S}(\phi)$ generates this space, i.e., whether $\mathcal{S}(f) = \mathcal{S}(\phi)$. With the aid of (1.11), we obtain the following simple answer [BDR1] to this problem:

Corollary 2.2. *Let $\mathcal{S}(\phi)$ be a principal shift-invariant space, and let $f \in \mathcal{S}(\phi)$. Then f generates $\mathcal{S}(\phi)$ if and only if* $\operatorname{supp} \widehat{f} \supset \operatorname{supp} \widehat{\phi}$.

Proof: If $\phi \in \mathcal{S}(f)$, there exists, by (1.11), a 2π-periodic τ such that $\widehat{\phi} = \tau \widehat{f}$, and hence $\operatorname{supp} \widehat{f} \supset \operatorname{supp} \widehat{\phi}$.

For the converse, we assume that $\operatorname{supp} \widehat{f} \supset \operatorname{supp} \widehat{\phi}$ and want to show that $\mathcal{S}(f) = \mathcal{S}(\phi)$. Since we assume that $f \in \mathcal{S}(\phi)$, then, by (1.11), there exists τ such that $\widehat{f} = \tau \widehat{\phi}$ a.e. Defining τ' to be $1/\tau$ on $\operatorname{supp} \tau$ and 0 elsewhere, we obtain that a.e. $\widehat{\phi} = \tau' \widehat{f}$ on $\operatorname{supp} \tau$, but since $\operatorname{supp} \tau \supset \operatorname{supp} \widehat{f} \supset \operatorname{supp} \widehat{\phi}$ (the last inclusion by assumption), the equality holds everywhere. By (1.11), we conclude that $\phi \in \mathcal{S}(f)$, and hence $\mathcal{S}(f) = \mathcal{S}(\phi)$. □

The above description of principal shift-invariant spaces in terms of their Fourier transforms can be generalized to the finitely generated shift-invariant space $\mathcal{S}(\Phi)$ as follows (cf. Theorem 1.7 of [BDR1]):

$$\widehat{\mathcal{S}(\Phi)} = \{\Sigma_{\phi \in \Phi} \tau_\phi \widehat{\phi} \in L_2(\mathbb{R}^d) : \ \tau_\phi \text{ is } 2\pi\text{-periodic}, \ \phi \in \Phi\}. \tag{2.3}$$

From the description (1.11) of the principal shift-invariant space $\mathcal{S}(\phi)$, we see that the Fourier transform of a function $s \in \mathcal{S}(\phi)$ is determined by its values on $\mathbb{T}^d$ (at least when $\operatorname{supp} \widehat{\phi} = \mathbb{R}^d$). It is possible to factor out this redundancy with the aid of the **bracket product** which is defined for $f, g \in L_2(\mathbb{R}^d)$ by the formula:

$$[f, g] := (f\overline{g})^\circ = \sum_{\beta \in 2\pi \mathbb{Z}^d} f(\cdot + \beta) \overline{g}(\cdot + \beta). \tag{2.4}$$

Note that $[f, g]$ is in $L_1(\mathbb{T}^d)$ whenever $f, g \in L_2(\mathbb{R}^d)$. Also, by the Cauchy-Schwarz inequality,

$$|[f, g]|^2 \ \leq \ [f, f][g, g], \tag{2.5}$$

with the right side finite a.e. The importance of (2.4) lies in part in the following formula, valid for $f, g \in L_2(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} f(x - j)\overline{g}(x)\, dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e_{-j} \widehat{f} \overline{\widehat{g}} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e_{-j} [\widehat{f}, \widehat{g}], \quad j \in \mathbb{Z}^d, \tag{2.6}$$

which shows that the inner product of $f(\cdot - j)$ with g is the j-th Fourier coefficient of $[\widehat{f}, \widehat{g}]$.

It is easy to derive the following three elementary properties of the bracket product. The first two follow from (2.6), while the third one follows directly from the definition (2.4).

Lemma 2.7. *If $f, g \in L_2(\mathbb{R}^d)$, then the shifts of f are orthogonal to the shifts of g if and only if $[\widehat{f}, \widehat{g}] = 0$.*

Lemma 2.8. *If $f, g \in L_2(\mathbb{R}^d)$ are compactly supported, then $[\widehat{f}, \widehat{g}]$ is a trigonometric polynomial.*

Lemma 2.9. *If $f, g \in L_2(\mathbb{R}^d)$ and τ has period 2π, then $[\tau \widehat{f}, \widehat{g}] = \tau [\widehat{f}, \widehat{g}] = [\widehat{f}, \overline{\tau} \widehat{g}]$.*

The bracket product also appears naturally in the computation of the norms of elements $s \in \mathcal{S}(\phi)$. By (1.11), $\widehat{s} = \tau \widehat{\phi}$ and

$$(2\pi)^{d/2} \|s\|_{L_2(\mathbb{R}^d)} = \|\widehat{s}\|_{L_2(\mathbb{R}^d)} = \|\tau [\widehat{\phi}, \widehat{\phi}]^{1/2}\|_{L_2(\mathbb{T}^d)}. \tag{2.10}$$

There is a simple description for the orthogonal projector $P := P_\phi$ from $L_2(\mathbb{R}^d)$ onto the principal shift-invariant space $\mathcal{S}(\phi)$. For each $f \in L_2(\mathbb{R}^d)$, $P_\phi f$ is the best $L_2(\mathbb{R}^d)$-approximation to f from $\mathcal{S}(\phi)$ and is characterized by the orthogonality of the error $f - P_\phi f$ to $\mathcal{S}(\phi)$. It was shown in [BDR] that P_ϕ is described by

$$\widehat{P_\phi f} = \frac{[\widehat{f}, \widehat{\phi}]}{[\widehat{\phi}, \widehat{\phi}]} \widehat{\phi} \tag{2.11}$$

where we use the convention (throughout this paper) that 0 times any extended number is 0; in particular $0/0$ is defined to be 0. (We note that, by the definition of the bracket product, $\widehat{\phi}$ vanishes whenever $[\widehat{\phi}, \widehat{\phi}]$ does.) There is a similar formula (which we shall not need) in the case $\mathcal{S}$ is finitely generated (see [BDR1]).

There are several interesting points to be made about bracket products and the projector P_ϕ. First of all, from (2.6), it follows that ϕ has orthonormal shifts if and only if $[\widehat{\phi}, \widehat{\phi}] = 1$ a.e. on $\mathbb{T}^d$. In this case, formula (2.11) is the (Fourier transform of the) usual one for projecting onto $\mathcal{S}(\phi)$. In the case that ϕ does not have orthonormal shifts, but $[\widehat{\phi}, \widehat{\phi}] \neq 0$ a.e., the function ϕ_* with Fourier transform

$$\widehat{\phi_*} := \frac{\widehat{\phi}}{[\widehat{\phi}, \widehat{\phi}]^{1/2}}, \tag{2.12}$$

is in $\mathcal{S}(\phi)$, has orthonormal shifts, and generates $\mathcal{S}(\phi)$, i.e., $\mathcal{S}(\phi_*) = \mathcal{S}(\phi)$.

Since the square root of the bracket product appears very frequently, we introduce the following notation:

$$\widetilde{\phi} := [\widehat{\phi}, \widehat{\phi}]^{1/2} = \Big(\sum_{\beta \in 2\pi \mathbb{Z}^d} |\widehat{\phi}(\cdot + \beta)|^2 \Big)^{1/2}, \tag{2.13}$$

with the sum taken, offhand, pointwise, hence defined for any ϕ on $\mathbb{R}^d$ if we allow it to take the value ∞. From (2.6), we conclude that the map $L_2(\mathbb{R}^d) \to L_2(\mathbb{T}^d) : \widehat{\phi} \mapsto \widetilde{\phi}$ is non-linear, norm-preserving, and onto:

Lemma 2.14. *The function ϕ is in $L_2(\mathbb{R}^d)$ if and only if $\widetilde{\phi} \in L_2(\mathbb{T}^d)$. Moreover, $\|\widehat{\phi}\|_{L_2(\mathbb{R}^d)} = \|\widetilde{\phi}\|_{L_2(\mathbb{T}^d)}$.*

Turning back to the orthogonal projection, we can write it in the form

$$\widehat{P_\phi f} = [\widehat{f}, \widehat{\mu}]\widehat{\phi}, \quad \widehat{\mu} := \frac{\widehat{\phi}}{\widetilde{\phi}^2}, \tag{2.15}$$

and check that $\widetilde{\mu}\widetilde{\phi} = 1$ on $\operatorname{supp} \widetilde{\phi}$, and therefore, by the above lemma, $\widehat{\mu} \in L_2(\mathbb{R}^d)$ if and only if $1/\widetilde{\phi} \in L_2(\operatorname{supp} \widetilde{\phi})$. In such a case, by (2.5), $[\widehat{f}, \widehat{\mu}] \in L_1(\mathbb{T}^d)$, and we can formally write the orthogonal projection $P_\phi f$ in the form

$$P_\phi f = \sum_{j \in \mathbb{Z}^d} \phi(\cdot + j) [\widehat{f}, \widehat{\mu}]^\vee(j), \tag{2.16}$$

with $[\widehat{f}, \widehat{\mu}]^\vee(j)$ the jth Fourier coefficient of $[\widehat{f}, \widehat{\mu}]$.

A special case of the above occurs when ϕ has $L_2(\mathbb{R}^d)$-stable shifts, i.e., when $\big(\phi(\cdot + j)\big)_{j \in \mathbb{Z}^d}$ forms a stable basis for $\mathcal{S}(\phi)$. As will be explained below, this stability is equivalent to having $C_1 \leq \widetilde{\phi} \leq C_2$ a.e. on $\mathbb{T}^d$ for constants $C_1, C_2 > 0$. In this case, the formal expansion (2.16) converges (unconditionally) and hence the orthogonal projection takes the explicit form

$$P_\phi f = \sum_{j \in \mathbb{Z}^d} \phi(\cdot + j) \mu_f(j), \quad \mu_f(j) := \int_{\mathbb{R}^d} f(x) \overline{\mu(x + j)}\, dx. \tag{2.17}$$

A similar analysis applies to the structure of the spaces $\mathcal{S}^k(\phi)$ generated by the 2^{-k}-shifts of the function ϕ. We shall only need the case $\mathcal{S}^1(\phi)$ in the sequel. In this case, the bracket product is replaced by the **double-bracket product**

$$[\![f,g]\!] := \sum_{\beta\in 4\pi\mathbb{Z}^d} f(\cdot+\beta)\overline{g}(\cdot+\beta), \tag{2.18}$$

which is a 4π-periodic function. The role of $\widetilde{f}$ is played by

$$\overset{\approx}{f} := \Big(\sum_{\beta\in 4\pi\mathbb{Z}^d} |\widehat{f}(\cdot+\beta)|^2\Big)^{1/2}, \quad f\in L_2(\mathbb{R}^d). \tag{2.19}$$

Note that $\overset{\approx}{f}$ is a non-negative 4π-periodic function, and

$$\widetilde{f}(x)^2 = \sum_{\beta\in 4\pi V} \overset{\approx}{f}(x+\beta)^2, \tag{2.20}$$

with V the vertices of the cube $[0\,\mathrm{d}1/2]^d$.

In particular, $\overset{\approx}{\phi}$ characterizes stability and orthogonality of the *half*-shifts of ϕ. Namely, the orthogonality of the half-shifts is equivalent to $\overset{\approx}{\phi} = 2^{-d/2}$ a.e., and the stability is equivalent to the boundedness a.e. of $\overset{\approx}{\phi}$ and $1/\overset{\approx}{\phi}$.

We next describe in more detail the structure of the finitely generated space $\mathcal{S} = \mathcal{S}(\Phi)$; a more complete discussion can be found in [BDR1]. First, for $s\in\mathcal{S}$, the representation (2.3) for $\widehat{s}$ is *local* in the sense that we can independently assign the values $\tau_\phi(x)$ for $x\in\mathbb{T}^d$ and $\phi\in\Phi$. The choice of $\tau_\phi(x)$ determines the value of $\widehat{s}$ at all points in $x+2\pi\mathbb{Z}^d$. This means that the structure of $\mathcal{S}$ is determined by the vectors $(\widehat{\phi}(x+\beta))_{\beta\in 2\pi\mathbb{Z}^d}$, $\phi\in\Phi$, $x\in\mathbb{T}^d$. For example, for any fixed x, these vectors are linearly independent if and only if the associated **Gramian matrix**

$$G(\Phi) := \Big([\widehat{\phi},\widehat{\psi}]\Big)_{\phi,\psi\in\Phi} \tag{2.21}$$

has non-zero determinant at x. In particular, if $\det G(\Phi)=0$ on a set of positive measure in $\mathbb{T}^d$, then the representation (2.3) is not unique.

We say that the set of generators Φ provides a **basis** for $\mathcal{S}$ (that is, their shifts are a basis) if the representation (2.3) is unique for each $s\in\mathcal{S}$, or, equivalently, if $\det G(\Phi)$ is nonzero a.e. All bases for $\mathcal{S}$ have the same number of elements. We note, [BDR1], that not every finitely generated shift-invariant space $\mathcal{S}$ contains a basis. We also note that $G(\Phi)$ is a non-negative definite matrix, hence Φ is a basis if and only if $\det G(\Phi)>0$ a.e. on $\mathbb{T}^d$.

We further say that a set of generators Φ provides an L_2-**stable basis** for $\mathcal{S}(\Phi)$ if each $s\in\mathcal{S}$ has a unique representation

$$s = \sum_{\phi\in\Phi}\sum_{j\in\mathbb{Z}^d} \phi(\cdot-j)c_{j,\phi}(s) \tag{2.22}$$

and the coefficients satisfy

$$C_1\sum_{\phi\in\Phi}\sum_{j\in\mathbb{Z}^d}|c_{j,\phi}(s)|^2 \le \|s\|^2_{L_2(\mathbb{R}^d)} \le C_2\sum_{\phi\in\Phi}\sum_{j\in\mathbb{Z}^d}|c_{j,\phi}(s)|^2 \tag{2.23}$$

with absolute constants $C_1, C_2 > 0$. Any L_2-stable basis is obviously a basis. One easily checks that the present definition coincides with the (seemingly weaker) one given in the introduction.

We recall from [BDR1] that the finite generating set Φ for $\mathcal{S}$ is an L_2-stable basis for $\mathcal{S}$ if and only if, for some matrix norm $\|\cdot\|$ (and hence all matrix norms),

$$\|G(\Phi)\|,\ \|G(\Phi)^{-1}\| \in L_\infty(\mathbb{T}^d). \tag{2.24}$$

In particular, this is the case only if

$$C_1 \le \det G(\Phi) \le C_2, \quad \text{a.e. on } \mathbb{T}^d, \tag{2.25}$$

for absolute constants $C_1, C_2 > 0$.

In the case that the entries of $G(\Phi)$ are continuous, the 'a.e.' in (2.25) can be removed, and more importantly, (2.25) becomes *equivalent* to the L_2-stability of Φ. Furthermore, in this case the right inequality of (2.25) trivially holds, and thus, due to the continuity of $\det G(\Phi)$, stability is equivalent to the condition

$$\det G(\Phi)(x) > 0, \quad \text{all } x \in \mathbb{T}^d.$$

This latter characterization of stability was obtained by Jia and Micchelli [JM] under slightly stronger assumptions (which imply the continuity assumption).

With the notions of basis and L_2-stable basis in hand, the following theorem shows how from one Φ which provides a (stable) basis for $\mathcal{S}$ we can obtain other sets with the same property. In this theorem, $\mathrm{T} := (\tau_{\psi,\phi})_{\psi\in\Psi,\phi\in\Phi}$ denotes a square matrix whose entries are 2π-periodic measurable functions.

Theorem 2.26. *Let Φ provide a basis for $\mathcal{S} = \mathcal{S}(\Phi)$. For any set Ψ of functions from $\mathcal{S}(\Phi)$, we have:*

(i) Ψ provides a basis for $\mathcal{S}$ if and only if $\widehat{\Psi} = \mathrm{T}\widehat{\Phi}$ for some T which is nonsingular a.e.;

(ii) Ψ provides a basis for $\mathcal{S}$ if and only if it generates $\mathcal{S}$ and $\#\Psi = \#\Phi$;

(iii) Ψ provides a basis for $\mathcal{S}$ if and only if $\#\Psi = \#\Phi$ and $\det G(\Psi) \neq 0$ a.e.

(iv) Ψ provides an L_2-stable basis for $\mathcal{S}$ if Φ does and $\widehat{\Psi} = \mathrm{T}\widehat{\Phi}$ with $\|\mathrm{T}\|, \|\mathrm{T}^{-1}\|$ in $L_\infty(\mathbb{T}^d)$.

The above is easily proved by noticing the effect of the transformation T on the Gramian:

$$G((\mathrm{T}\widehat{\Phi})^\vee) = \mathrm{T}G(\Phi)\mathrm{T}^*, \tag{2.27}$$

with T^* the conjugate transpose of T; see [BDR1; Cor.3.31] for more details.

A special case of the above occurs when T is a diagonal matrix. In this case, Ψ provides a new basis if and only if $\Psi \subset L_2(\mathbb{R}^d)$ and the 2π-periodic functions $\{\tau_{\psi,\psi}\}$ are different from zero a.e. on $\mathbb{T}^d$. Also, Ψ is an L_2-stable basis if Φ is, and the $\tau_{\psi,\psi}$ and their reciprocals are in $L_\infty(\mathbb{T}^d)$.

If Φ provides a basis for $\mathcal{S} := \mathcal{S}(\Phi)$, then Φ can be orthonormalized by a Gram-Schmidt orthogonalization. We summarize this fact in the following theorem whose proof is left to the reader (and can be found in [BDR1]).

Theorem 2.28. *Let Φ provide a basis for $\mathcal{S} := \mathcal{S}(\Phi)$.*

(i) Then, there is a set Φ^* of generators for $\mathcal{S}$ that provides an orthonormal basis for $\mathcal{S}$.

(ii) If the functions in Φ have compact support, then there is a set $\Phi^* = \{\phi_1^*, \ldots, \phi_n^*\}$ of compactly supported functions which give the orthogonal decomposition:

$$\mathcal{S} = \mathcal{S}(\phi_1^*) \oplus \cdots \oplus \mathcal{S}(\phi_n^*). \tag{2.29}$$

The half-shift-invariant space $\mathcal{S}^1(\phi)$, generated by ϕ, is identical with $\mathcal{S}(\Phi)$, $\Phi := \big(\phi(\cdot + v)\big)_{v\in V}$, and V the vertices of the cube $[0\,\mathrm{d}1/2]^d$, as before. Clearly, orthonormality or L_2-stability of the half-shifts of ϕ is the same as orthonormality or L_2-stability of the full shifts of Φ. Thus, there must be a relation between $G(\Phi)$ and $\overset{\approx}{\phi}$ which we shall now derive.

Given a 4π-periodic function f, the functions

$$Q_v(f) := \sum_{\mu\in 4\pi V} e_v(\cdot + \mu) f(\cdot + \mu), \quad v \in \mathbb{Z}^d/2 \tag{2.30}$$

are 2π-periodic. If Λ has been obtained by choosing exactly one point from each of the cosets $v + \mathbb{Z}^d$, $v \in V$, then

$$f = \sum_{v\in\Lambda} e_{-v} Q_v(f)/2^d \tag{2.31}$$

is a decomposition of f into its 2π-periodic components $Q_v(f)/2^d$, $v \in \Lambda$.

Since $\mathbb{Z}^d/2$ is the disjoint sum $V + \mathbb{Z}^d$, and e_v is 4π-periodic for $v \in \mathbb{Z}^d/2$, we find that, for $v, u \in V$,

$$\begin{aligned}[e_v\widehat{\phi}, e_u\widehat{\phi}] &= \sum_{\beta\in 2\pi\mathbb{Z}^d} e_v(\cdot+\beta) e_{-u}(\cdot+\beta) |\widehat{\phi}(\cdot+\beta)|^2 \\ &= \sum_{\mu\in 4\pi V} e_{v-u}(\cdot+\mu) [\![\widehat{\phi}, \widehat{\phi}]\!](\cdot+\mu)^2 = Q_{v-u}(\overset{\approx}{\phi}{}^2).\end{aligned} \tag{2.32}$$

With this, we can easily compute the eigenvalues and eigenvectors of $G(\Phi)$.

Lemma 2.33. *For each $\mu \in 4\pi V$ and $x \in \mathbb{T}^d$, the number $2^d \overset{\approx}{\phi}(x+\mu)^2$ is an eigenvalue of $G(\Phi)(x)$, with eigenvector $a_\mu := \big(e_v(x+\mu)\big)_{v\in V}$.*

Proof: In view of (2.32), the v-th component of $G(\Phi)a_\mu$ is

$$\sum_{u\in V} Q_{v-u}(\overset{\approx}{\phi}{}^2) e_u(\cdot+\mu) = e_v(\cdot+\mu) \sum_{u\in V} e_{u-v}(\cdot+\mu) Q_{v-u}(\overset{\approx}{\phi}{}^2) = 2^d e_v(\cdot+\mu) \overset{\approx}{\phi}(\cdot+\mu)^2$$

where, in the last equality, we used (2.31) as well as the 2π-periodicity of $Q_{v-u}(\overset{\approx}{\phi}{}^2)$. □

Corollary 2.34. *If $\operatorname{supp}\widehat{\phi} = \mathbb{R}^d$, then the set $\Phi := \big(\phi(\cdot + v)\big)_{v\in V}$ is a basis for $\mathcal{S}(\Phi)$.*

Proof: From Lemma 2.33, $\det G(\Phi) = \text{const} \prod_{\mu\in 4\pi V} \overset{\approx}{\phi}(\cdot+\mu)^2$. From our assumption on the support of $\widehat{\phi}$, it follows that $\overset{\approx}{\phi}$, hence $\det G(\Phi)$, does not vanish a.e., hence Φ is a basis for $\mathcal{S}(\Phi)$. □

3. Generators for the wavelet space

We give in this section various descriptions for wavelet spaces and their generators and bases. As mentioned earlier, we develop this analysis in the following more general framework than usually considered in the multiresolution construction of wavelets: We suppose that ϕ and η are functions in $L_2(\mathbb{R}^d)$ with the property

$$\mathcal{S}(\phi) \subset \mathcal{S}^1(\eta). \tag{3.1}$$

As before, $\mathcal{S}(\phi)$ denotes the principal shift-invariant space generated by ϕ, and $\mathcal{S}^1(\eta)$ denotes the space generated by the half-shifts of η. Thus, $S^1(\eta)$ is also the 2-dilate of $\mathcal{S}(\eta(\cdot/2))$.

We define the **wavelet space** W as the orthogonal complement of $\mathcal{S}(\phi)$ in $\mathcal{S}^1(\eta)$:

$$W := \mathcal{S}^1(\eta) \ominus \mathcal{S}(\phi).$$

The purpose of this section is to find generating sets and bases for W. The analysis of this section also applies to the other wavelet spaces

$$W^k := \mathcal{S}^{k+1} \ominus \mathcal{S}^k \tag{3.2}$$

(which might be generated by some other L_2-functions) after a suitable dilation. In the case of **stationary decompositions** usually considered in wavelet constructions, we have $\eta = \phi(2\cdot)$, and $\mathcal{S}^1(\eta)$ is the 2-dilate of $\mathcal{S}(\phi)$. However, (3.1) is much more general since we can begin with any $\eta \in L_2(\mathbb{R}^d)$ and take for ϕ any element in $\mathcal{S}^1(\eta)$. Indeed, $\mathcal{S}(\phi) \subset \mathcal{S}^1(\eta)$, because $\mathcal{S}^1(\eta)$ is invariant under shifting by half-integers, *a fortiori* by integers.

Since $\mathcal{S}(\phi) \subset \mathcal{S}^1(\eta)$, we have

$$\widehat{\phi} = A\widehat{\eta} \tag{3.3}$$

for some 4π-periodic function A. We call A the **refinement mask**. In effect, we are are extending the concept of refinability here since we are not assuming that ϕ can be written in the form

$$\phi = \sum_{j \in \mathbb{Z}^d/2} \eta(\cdot - j)a(j) \tag{3.4}$$

for some sequence a (with convergence in the sense of $L_2(\mathbb{R}^d)$).

We do assume that

$$\operatorname{supp} \widehat{\eta} = \operatorname{supp} \widehat{\phi}. \tag{3.5}$$

Although this assumption is not essential for our wavelet constructions (e.g., a referee has pointed out that many of our results hold under the weaker assumption that $\widetilde{\phi}$ and $\widetilde{\widetilde{\eta}}$ are both positive a.e.), it significantly simplifies the underlying analysis. In case ϕ and η are compactly supported, as is the case in almost all wavelet constructions, (3.5) automatically holds, since then $\operatorname{supp} \widehat{\phi} = \operatorname{supp} \widehat{\eta} = \mathbb{R}^d$.

We shall now describe our first set of generators for W; other generating sets and bases for W will be obtained from this set by using the transformations T described in Theorem 2.26. We let (as earlier) V denote the vertices of the cube $[0\,\mathrm{d}1/2]^d$. We continue to use the abbreviation $V' := V\backslash\{0\}$.

It is clear that the space $\mathcal{S}^1(\eta)$ is generated by the half-shifts $\big(\eta(\cdot + v)\big)_{v \in V}$ of η, and this is, indeed, the usual starting point for most of the wavelet constructions now in the literature. However, we will show below that, because of (3.5), $\mathcal{S}^1(\eta) = \mathcal{S}^1(\phi)$, and therefore $\mathcal{S}^1(\eta) = \mathcal{S}(\Phi)$ with

$$\Phi := \big(\phi_v := \phi(\cdot + v)\big)_{v \in V}. \tag{3.6}$$

The generating set Φ is attractive since the generator ϕ of $\mathcal{S}(\phi)$ is one of its elements.

Theorem 3.7. *If $\mathcal{S}(\phi) \subset \mathcal{S}^1(\eta)$ and $\operatorname{supp} \widehat{\phi} = \operatorname{supp} \widehat{\eta}$, then*

$$\mathcal{S}^1(\eta) \ = \mathcal{S}^1(\phi) = \ \mathcal{S}(\Phi) \tag{3.8}$$

with $\Phi := \big(\phi(\cdot + v)\big)_{v \in V}$.

Proof: The second equality is clear. As for the first equality, let $g := \eta(\cdot/2)$ and $f := \phi(\cdot/2)$. Since $\phi \in \mathcal{S}^1(\eta)$, we have $f \in \mathcal{S}(g)$. By assumption, $\operatorname{supp} \widehat{g} = \operatorname{supp} \widehat{f}$. Therefore, by Corollary 2.2, $\mathcal{S}(f) = \mathcal{S}(g)$. Our claim then follows from the fact that $\mathcal{S}^1(\phi)$ and $\mathcal{S}^1(\eta)$ are the 2-dilates of $\mathcal{S}(f)$ and $\mathcal{S}(g)$ respectively. □

It is very simple to find elements of W. If $f \in \mathcal{S}^1(\eta)$, then since P_ϕ is the orthogonal projector onto $\mathcal{S}(\phi)$, the error $f - P_\phi f$ is in W. If we choose $2^d - 1$ such functions f in an appropriate way, we obtain a basis for W. Most wavelet constructions begin with the functions $f = \eta(\cdot + v)$, $v \in V$. However, there are too many of these functions and one of them must somehow be eliminated (destroying symmetry). Our last theorem asserts that $\mathcal{S}^1(\eta)$ is also generated by $\big(\phi(\cdot + v)\big)_{v\in V}$. Starting with the $\big(\phi_v := \phi(\cdot + v)\big)_{v\in V}$ gives the set

$$\mathcal{W} := \big(w_v := \phi_v - P_\phi \phi_v\big)_{v\in V'}.$$

It is easy to see that $\mathcal{W}$ is a generating set for W, i.e., $W = \mathcal{S}(\mathcal{W})$. Indeed, since $P_\phi\phi_v$ is in $\mathcal{S}(\phi)$, ϕ_v must be in $\mathcal{S}(\phi) \oplus \mathcal{S}(w_v)$. Therefore, $\{\phi\} \cup \mathcal{W}$ generates $\mathcal{S}^1(\eta)$. It follows that $\mathcal{W}$ generates W.

From (2.11), we obtain a simple description of the Fourier transform of the $\widehat{w_v}$:

$$\widehat{w_v} := \widehat{\phi_v} - \widehat{P_\phi \phi_v} = \widehat{\phi_v} - \frac{[\widehat{\phi_v}, \widehat{\phi}]}{[\widehat{\phi}, \widehat{\phi}]}\widehat{\phi}, \quad v \in V' := V\backslash\{0\}. \tag{3.9}$$

Theorem 3.10. *If $S(\phi) \subset S^1(\eta)$ and $\operatorname{supp}\widehat{\phi} = \operatorname{supp}\widehat{\eta}$, then $W := \mathcal{S}_1 \ominus \mathcal{S}_0$ is a finitely generated shift-invariant space and $\mathcal{W} := \big(w_v\big)_{v\in V'}$ (defined as in (3.9)) is a generating set for W:*

$$W \;=\; \mathcal{S}(\mathcal{W}). \tag{3.11}$$

If $\operatorname{supp}\widehat{\phi} = \mathbb{R}^d$, then $\mathcal{W}$ provides a basis for W.

Proof: We have already shown that $\mathcal{W}$ is a generating set for W. If $\operatorname{supp}\widehat{\phi} = \mathbb{R}^d$, then by Corollary 2.34, the set Φ provides a basis for $\mathcal{S}(\Phi)$. We have shown that $\Phi_* := \{\phi\} \cup \mathcal{W}$ is another generating set for $\mathcal{S}(\Phi)$. Since $\#\Phi_* = \#\Phi$, Theorem 2.26(ii) asserts that Φ_* also provides a basis for $\mathcal{S}(\Phi)$. Hence, $\det G(\Phi_*)$ is nonzero a.e. From the orthogonality between W and $\mathcal{S}(\phi)$, we find that $\det G(\Phi_*) = \widetilde{\phi}^2 \det G(\mathcal{W})$. Therefore, $\det G(\mathcal{W})$ is also nonzero a.e., thus $\mathcal{W}$ provides a basis for W. □

The set $\mathcal{W}$ is our first set of generators for W. We shall find several others in the following sections. The idea is simple. We transform $\mathcal{W}$ to a new set of generators by using one of the matrices T whose entries are 2π-periodic functions. The intent is to choose T in such a way that the new set $(\mathrm{T}\widehat{\mathcal{W}})^\vee$ of generators has more favorable properties. Our next result illustrates this procedure and provides a set of generators which are compactly supported functions whenever ϕ is, – a property the generators in $\mathcal{W}$ lack.

Theorem 3.12. *Assume that $\mathcal{S}(\phi) \subset \mathcal{S}^1(\eta)$, $\operatorname{supp}\widehat{\phi} = \operatorname{supp}\widehat{\eta} = \mathbb{R}^d$, and that $[\widehat{\phi}, \widehat{\phi}]$ (or equivalently $\widetilde{\phi}$) is bounded. Then the $2^d - 1$ functions*

$$\mathcal{W}_c := \big(([\widehat{\phi}, \widehat{\phi}]\widehat{\phi}_v - [\widehat{\phi}_v, \widehat{\phi}]\widehat{\phi})^\vee\big)_{v\in V'}$$

provide a basis for the wavelet space W. If ϕ has compact support, then the functions in $\mathcal{W}_c$ are also of compact support.

Proof: Since $\widehat{\mathcal{W}_c}$ is obtained from $\widehat{\mathcal{W}}$ by multiplying by the 2π-periodic scalar matrix $[\widehat{\phi}, \widehat{\phi}]I$, which is assumed here to be bounded, we conclude that $\widehat{\mathcal{W}_c} \subset L_2(\mathbb{R}^d)$. Furthermore, $\operatorname{supp}\widehat{\phi} = \mathbb{R}^d$ implies, by Theorem 3.10, that $\mathcal{W}$ provides a basis. Hence, by Theorem 2.26, $\mathcal{W}_c$ provides a basis for W as well.

It remains to show that the functions in $\mathcal{W}_c$ are compactly supported whenever ϕ is. By Lemma 2.8, $[\widehat{\phi}_v, \widehat{\phi}]$ is a trigonometric polynomial. Therefore, the inverse transform of $[\widehat{\phi}_v, \widehat{\phi}]\widehat{\phi}$ is a finite linear combination of the shifts of ϕ, hence is compactly supported since ϕ is. The same argument shows that the inverse transform of $[\widehat{\phi}, \widehat{\phi}]\widehat{\phi}_v$ is also compactly supported, and thus, indeed, the functions in $\mathcal{W}_c$ are compactly supported. □

The next theorem shows that W always has a set of generators consisting of orthogonal wavelets.

Theorem 3.13. *Let $\mathcal{S}(\phi) \subset \mathcal{S}^1(\eta)$ and $\operatorname{supp}\widehat{\phi} = \operatorname{supp}\widehat{\eta} = \mathrm{I\!R}^d$. Then:*
(i) there is a set of generators Ψ for W which provides an orthonormal basis for W;
(ii) if in addition ϕ has compact support, then there is a subset $\Psi = \left(\psi_v\right)_{v\in V'}$ of compactly supported functions from W which provides a basis for W and satisfies

$$\mathcal{S}(\psi_v) \perp \mathcal{S}(\psi_u),\ v \neq u.$$

Proof: (i): By Theorem 3.10, $\mathcal{W}$ is a basis for W and therefore we need only apply Theorem 2.28(i). (ii): By Theorem 3.12, the functions in $\mathcal{W}_c$ provide a basis for W and are of compact support whenever ϕ is. Therefore, we need only apply Theorem 2.28(ii). □

We shall next discuss conditions under which the half-shifts $w(\cdot + v)$, $v \in V'$, of a function $w \in L_2(\mathrm{I\!R}^d)$ provide a basis for W. Clearly, w must be an element of $\mathcal{S}^1(\eta)$.

Theorem 3.14. *Let $\mathcal{S}(\phi) \subset \mathcal{S}^1(\eta)$ and $\operatorname{supp}\widehat{\phi} = \operatorname{supp}\widehat{\eta} = \mathrm{I\!R}^d$. Then:*
(i) If $w \in \mathcal{S}^1(\eta)$, then the functions $w(\cdot + v)$, $v \in V'$, are all in W if and only if $[\![\widehat{w}, \widehat{\phi}]\!]$ is 2π-periodic.
(ii) If w is a generator for $\mathcal{S}^1(\eta)$ and $[\![\widehat{w}, \widehat{\phi}]\!]$ is 2π-periodic, then the functions $\left(w(\cdot + v)\right)_{v\in V'}$ provide a basis for W.

Proof: (i): Since $w \in L_2(\mathrm{I\!R}^d)$, the function $[\![\widehat{w}, \widehat{\phi}]\!]$ is in $L_1([-2\pi \,\mathrm{d} 2\pi]^d)$. Proceeding as in (2.6),

$$\int_{\mathrm{I\!R}^d} w(x-j)\overline{\phi}(x)\,dx = \frac{1}{(2\pi)^d}\int_{\mathrm{I\!R}^d} e_{-j}\widehat{w}\overline{\widehat{\phi}} = \frac{1}{(2\pi)^d}\int_{[-2\pi \,\mathrm{d}2\pi]^d} e_{-j}[\![\widehat{w}, \widehat{\phi}]\!], \quad j \in \mathbb{Z}^d/2. \tag{3.15}$$

Now, the functions $w(\cdot + v)$, $v \in V'$, are all in W if and only if the inner products appearing in (3.15) are zero whenever $j \in (\mathbb{Z}^d/2) \setminus \mathbb{Z}^d$, i.e., if and only if $[\![\widehat{w}, \widehat{\phi}]\!]$ has period 2π.

(ii): From the facts that the half-shifts of w generate $\mathcal{S}^1(\eta)$, and $\operatorname{supp}\widehat{\eta} = \mathrm{I\!R}^d$, we easily conclude that $\operatorname{supp}\widehat{w} = \mathrm{I\!R}^d$, and therefore, by Corollary 2.34, $\mathcal{W}_* := \left(w(\cdot+v)\right)_{v\in V}$ provides a basis for $\mathcal{S}^1(\eta)$. Equivalently, $G(\mathcal{W}_*)$ is non-zero a.e. on $\mathrm{T\!T}^d$. It follows that the Gramian of any subset of $\mathcal{W}_*$, and in particular the subset $\mathcal{W}_*\backslash\{w\}$, is non-zero a.e. as well, while, by (i), this latter set lies in W, since we also assume that $[\![\widehat{w}, \widehat{\phi}]\!]$ is 2π-periodic. Thus, we have found $2^d - 1$ functions in W (viz., the functions in $\mathcal{W}_*\backslash\{w\}$) whose Gramian is non-zero a.e., while Theorem 3.10 asserts that W contains a basis of cardinality $2^d - 1$. Thus, Theorem 2.26(iii) ensures that $\mathcal{W}_*\backslash\{w\}$ is basis for W. □

We give some examples of functions w which satisfy the assumptions of Theorem 3.14. Since any function $w \in W$ is in $\mathcal{S}^1(\eta)$, we must have $\widehat{w} = \tau\widehat{\eta}$ for some 4π-periodic τ, and so

$$[\![\widehat{w}, \widehat{\phi}]\!] = [\![\tau\widehat{\eta}, \widehat{\phi}]\!] = \tau[\![\widehat{\eta}, \widehat{\phi}]\!]. \tag{3.16}$$

We would like the function in (3.16) to be of period 2π. One obvious choice is to take $\tau = 1/[\![\widehat{\eta}, \widehat{\phi}]\!]$ which gives the function w_0 with Fourier transform

$$\widehat{w}_0 := \widehat{\eta}/[\![\widehat{\eta}, \widehat{\phi}]\!] = \widehat{\phi}/[\![\widehat{\phi}, \widehat{\phi}]\!]. \tag{3.17}$$

The half-shifts $w_0(\cdot+v)$, $v \in V'$, will generate W provided $w_0 \in L_2(\mathrm{I\!R}^d)$. Note the intimate relation between the present w_0 and the "dual function" μ which was defined in (2.15). Indeed, w_0 is orthogonal to each $\phi(\cdot + j)$, $j \in \mathbb{Z}^d/2\backslash\{0\}$ since $[\![\widehat{w}_0, \widehat{\phi}]\!] = 1$.

In general, w_0 will not be in $L_2(\mathbb{R}^d)$ because of the division by $[\![\widehat{\phi}, \widehat{\phi}]\!] = \widetilde{\widetilde{\phi}}^2$. However, we can multiply $\widehat{w}_0$ by any 2π-periodic function and obtain the Fourier transform of other candidates. For example, multiplying by $\prod_{\mu\in 4\pi V} \widetilde{\widetilde{\phi}}(\cdot+\mu)^2$ gives the function w with Fourier transform

$$\widehat{w} := \widehat{\phi} \prod_{\mu\in 4\pi V'} \widetilde{\widetilde{\phi}}(\cdot+\mu)^2. \tag{3.18}$$

This function w also has the advantage of being of compact support whenever ϕ is. Since ϕ is a generator for $\mathcal{S}^1(\eta)$ and $\widetilde{\widetilde{\phi}}^2 > 0$ a.e. (because $\operatorname{supp} \widehat{\phi} = \mathbb{R}^d$), the function w of (3.18) is a generator for $\mathcal{S}^1(\eta)$ (because its support is $\mathbb{R}^d$). Applying (ii) of Theorem 3.14, we obtain the following Corollary.

Corollary 3.19. *Let $\mathcal{S}(\phi) \subset \mathcal{S}^1(\eta)$ and $\operatorname{supp}\widehat{\phi} = \operatorname{supp}\widehat{\eta} = \mathbb{R}^d$. If the function w, defined by (3.18), is in $L_2(\mathbb{R}^d)$, then the set $\{w(\cdot+v) : v \in V'\}$ provides a basis for W. If ϕ has compact support, then so does w.*

We note that the above w is in L_2 whenever $\widetilde{\widetilde{\phi}}$ or $\widetilde{\phi}$ is bounded. For example, this is the case whenever ϕ has compact support.

We have shown so far that it is easy to obtain generating sets for W with various properties. They can be chosen to provide a basis or an orthogonal basis for W. They can be chosen to be the shifts of one function w and to have compact support if ϕ has. There is one important problem we have not yet discussed, and that is how to find an $L_2(\mathbb{R}^d)$-stable basis for W consisting of compactly supported functions. It is easy to see that the generating set $\mathcal{W}_c$ will have this property if the half shifts of ϕ are $L_2(\mathbb{R}^d)$-stable (this assumption is not realistic in the stationary case, but can be satisfied in other situations, cf. §8). We discuss this problem in §5 (in the univariate case) and in §7 (for the multivariate case). But, first, we examine in the next section the other two conditions in (1.3), i.e., (1.3)(ii,iii).

4. Multiresolution

In this section, we analyze conditions (1.3)(ii) and (iii). Our setting is as follows. We have for each $k \in \mathbb{Z}$ a function $\phi_k \in L_2(\mathbb{R}^d)$ and the space $\mathcal{S}^k := \mathcal{S}^k(\phi_k)$ generated by its 2^{-k}-shifts. Alternatively, $\mathcal{S}^k$ is the 2^k-dilate of the principal shift-invariant space $\mathcal{S}(\phi_k(2^{-k}\cdot))$. We noted in (2.1) that the functions $s \in \mathcal{S}^k$ are characterized by the representation

$$\widehat{s} = \tau\widehat{\phi_k}, \quad \tau \text{ of period } 2^{k+1}\pi. \tag{4.1}$$

We first study condition (1.3)(ii). It should be noted that we have completely characterized in [BDR] density properties of (arbitrary) shift-invariant subspaces of $L_2(\mathbb{R}^d)$. However, the present setting is so simple that it does not require any of this general machinery.

We begin with the following lemma:

Lemma 4.2. *Let $\mathcal{S}^k$, $k \in \mathbb{Z}$, be a nested sequence. Then $\overline{\cup\mathcal{S}^k}$ is a closed translation-invariant subspace of $L_2(\mathbb{R}^d)$.*

Proof: Let $X := \cup\mathcal{S}^k$. Then $\overline{X}$ is certainly closed. Now, let $f \in X$. Since $\mathcal{S}^k \subset \mathcal{S}^{k+1}$, $f \in \mathcal{S}^k$ for all k sufficiently large. Since $\mathcal{S}^k$ is 2^{-k}-shift-invariant, $f_t := f(\cdot+t)$ is in X for any $t = 2^{-k}j$, $j \in \mathbb{Z}^d$, which means that f_t is in X for all dyadic $t = 2^{-k}j$, $j \in \mathbb{Z}^d$, $k \in \mathbb{Z}$. Since translation is a continuous operation in $L_2(\mathbb{R}^d)$, we obtain that f_t is in $\overline{X}$ for all $t \in \mathbb{R}^d$. Moreover, if $g \in \overline{X}$, and $f \in X$, then $\|g_t - f_t\| = \|g - f\|$. Approximating g by functions $f \in X$ shows that $g_t \in \overline{X}$. ☐

It is well known (see, e.g., [Ru; pp.203–206]) that a closed translation-invariant subspace X of $L_2(\mathbb{R}^d)$ is characterized by its Fourier transforms. Precisely, $\widehat{X} = L_2(\Omega)$ for some measurable set Ω (called the **spectrum of** X).

Theorem 4.3. *Let $\left(\mathcal{S}^k := \mathcal{S}^k(\phi_k)\right)_{k\in\mathbb{Z}}$ be a nested sequence. Then $\overline{\cup\mathcal{S}^k} = L_2(\mathbb{R}^d)$ if and only if $\Omega_0 := \cup \operatorname{supp}\widehat{\phi_k} = \mathbb{R}^d$ (modulo a null-set).*

Proof: Let $X := \overline{\cup\mathcal{S}^k}$. From the above remarks on translation-invariant spaces, $\widehat{X} = L_2(\Omega)$ for some measurable set $\Omega \subset \mathbb{R}^d$. We have $X = L_2(\mathbb{R}^d)$ if and only if $\Omega = \mathbb{R}^d$ modulo a null-set. We verify that $\Omega = \Omega_0$ modulo a null-set which will complete the proof. Since each ϕ_k is in X, we must have $\operatorname{supp}\widehat{\phi_k} \subset \Omega$ modulo a null-set, and so, $\Omega_0 \subset \Omega$ modulo a null-set. Now suppose that $\Omega\backslash\Omega_0$ contains a set Ω_1 of positive measure. From (4.1), each element in $\mathcal{S}^k$, $k \in \mathbb{Z}$, has Fourier transform which vanishes on Ω_1. Hence, each element in $\cup\mathcal{S}^k$ has Fourier transform which vanishes on Ω_1. Taking the closure, we see that each element in X has Fourier transform which vanishes on Ω_1. This is absurd since $\widehat{X}$ contains $L_2(\Omega_1)$. □

The role of (1.3)(ii) in multiresolution analysis is to guarantee that $\lim_{k\to\infty} P_k f = f$ for each $f \in L_2(\mathbb{R}^d)$.

Corollary 4.4. *Let $\left(\mathcal{S}^k := \mathcal{S}^k(\phi_k)\right)_{k\in\mathbb{Z}}$ be a nested sequence, and $\cup_{k\in\mathbb{Z}} \operatorname{supp}\widehat{\phi_k} = \mathbb{R}^d$. Then, the orthogonal projectors P_k from $L_2(\mathbb{R}^d)$ onto $\mathcal{S}^k$ satisfy $\lim_{k\to\infty} P_k f = f$ for all $f \in L_2(\mathbb{R}^d)$.*

Proof: Since $\mathcal{S}^k \subset \mathcal{S}^{k+1}$, Theorem 4.3 says that $\|f - P_k f\| = \operatorname{dist}(f, \mathcal{S}^k) \to 0$. □

We next consider in more detail the stationary case $\mathcal{S}^k = \mathcal{S}^k(\phi(2^k\cdot))$, $k \in \mathbb{Z}$, i.e., the case when $\mathcal{S}^k$ is the 2^k-dilate of $\mathcal{S}^0 = \mathcal{S}(\phi)$, which is the usual situation treated in multiresolution. The following is a very simple sufficient condition for (1.3)(ii), in the event of a stationary multiresolution.

Theorem 4.5. *Let ϕ be an $L_2(\mathbb{R}^d)$-function, and for each $k \in \mathbb{Z}$, let $\mathcal{S}^k$ be the 2^k-dilate of $\mathcal{S}(\phi)$. Assume that $\left(\mathcal{S}^k\right)_k$ is nested. Then, $\left(\mathcal{S}^k\right)_k$ satisfies (1.3)(ii) if $\widehat{\phi}$ is non-zero a.e. in some neighborhood of the origin.*

Proof: Here $\phi_k = \phi(2^k\cdot)$, and therefore $\widehat{\phi}_k = c_k\widehat{\phi}(\cdot/2^k)$. Thus if $\widehat{\phi}$ is non-zero a.e. on Ω, then $\widehat{\phi}_k$ is non-zero a.e. on $2^k\Omega$. Now, if Ω is some neighborhood of the origin, we obtain that $\cup_k \operatorname{supp}\widehat{\phi}_k = \mathbb{R}^d$, since $\cup_k 2^k\Omega = \mathbb{R}^d$. By Theorem 4.3, (1.3)(ii) holds. □

Of course, (1.3)(ii) can also hold when $\widehat{\phi}$ vanishes at every neighborhood of the origin on a set of positive measure. For example, this is the case if $d = 2$ and $\operatorname{supp}\widehat{\phi} = \{x \in \mathbb{R}^2 :\ x_2^2 \le |x_1|\}$.

Special cases of Theorem 4.5 have been established by other authors. For example, in the univariate case and under certain restrictions on the smoothness and decay of ϕ, Mallat [Ma] showed that whenever ϕ has orthonormal shifts, assumption (1.3)(i) implies that $\overline{\cup\mathcal{S}^k} = L_2(\mathbb{R}^d)$. Recently, this was generalized to the multivariate case by Jia and Micchelli [JM] who replaced orthonormality by $L_2(\mathbb{R}^d)$-stability and replaced Mallat's other conditions by the requirements that $\phi^\circ \in L_2(\mathbb{T}^d)$, and that ϕ satisfy the refinement equation (1.6) for a sequence a from $\ell_1(\mathbb{Z}^d)$. In both of these examples, the conditions used imply that $\widehat{\phi}$ is continuous and $\widehat{\phi}(0) \ne 0$, hence these results indeed follow from Theorem 4.5. Note that, by the same token, Theorem 4.5 certainly applies whenever $\phi \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ and $\widehat{\phi}(0) \ne 0$.

We return to the general case of the spaces $\mathcal{S}^k = \mathcal{S}^k(\phi_k)$ introduced at the beginning of this section, in order to discuss condition (1.3)(iii). We shall need the notion of Lebesgue points: recall that if f is locally in $L_1(\mathbb{R}^d)$, a point $x \in \mathbb{R}^d$ is said to be a **Lebesgue point** of f if

$$\lim_{|Q|\to 0} \frac{1}{|Q|}\int_Q f(u)\,du = f(x),$$

with the limit taken over cubes Q which contain x. For each locally integrable f, almost every point is a Lebesgue point (see e.g. [BS; p. 121]). We shall need the following simple lemma (which certainly is known).

Lemma 4.6. *If Ω is a measurable subset of $\mathbb{R}^d$ and $\alpha \neq 0$ is a fixed real constant such that for each dyadic $t \in \mathbb{R}^d$, we have $\Omega + \alpha t = \Omega$ modulo a null-set, then $\Omega = \mathbb{R}^d$ or $\Omega = \emptyset$, modulo a null-set. Moreover, if f is any measurable function on $\mathbb{R}^d$ which, for each dyadic t, satisfies $f(\cdot + \alpha t) = f$ a.e., then $f = \mathrm{const}$ a.e.*

Proof: By replacing f (respectively, Ω) by $f(\alpha\cdot)$ (respectively, $\alpha^{-1}\Omega$), we can assume that $\alpha = 1$. Let $a \in \mathbb{R}^d$ be a Lebesgue point of $\chi := \chi_\Omega$. Then with $Q_\delta := [-\delta/2 \,\mathrm{d}\, \delta/2]^d$, we have

$$\lim_{\delta \to 0} \delta^{-d} \int_{Q_\delta} \chi(x + a)\, dx = \chi(a). \tag{4.7}$$

Now, for any dyadic number t, we have $\chi(x + t) = \chi(x)$, a.e. in x. Hence, for any set Q of finite measure,

$$\int_Q \chi(x + a + t)\, dx = \int_Q \chi(x + a)\, dx. \tag{4.8}$$

If $y \in \mathbb{R}^d$ is any other Lebesgue point of χ, then using the density of the dyadic points, we can, for each $\delta > 0$, find a t_δ such that $y \in a + t_\delta + Q_\delta$. Using (4.8), we find

$$\chi(y) = \lim_{\delta \to 0} \delta^{-d} \int_{Q_\delta} \chi(x + a + t_\delta)\, dx = \lim_{\delta \to 0} \delta^{-d} \int_{Q_\delta} \chi(x + a)\, dx = \chi(a).$$

Hence, χ is constant a.e. and our result follows.

If the function f is as described in the lemma's statement, then for each $y \in \mathbb{R}$, the set $\Omega := \{x : f(x) \leq y\}$ satisfies $\Omega + t = \Omega$, modulo a null-set, for each dyadic t. Hence $\Omega = \mathbb{R}^d$ or $\Omega = \emptyset$, modulo a null-set, and it follows that f is a constant a.e. If f is complex-valued, this argument can be applied to both its real and its imaginary parts. $\square$

Note that the only property of the set of dyadic points used in the proof of this lemma is their density.

Theorem 4.9. *Given the sequence $\mathcal{S}^k := \mathcal{S}^k(\phi_k)$, $k \in \mathbb{Z}$, set $Y := \cap_k \mathcal{S}^k$. Then Y is a linear subspace of $L_2(\mathbb{R}^d)$ of dimension ≤ 1.*

Note that the nestedness assumption (1.3)(i) is *not* made here. Further, as the proof below makes clear, the result remains valid even when $\{\mathcal{S}^k\}_k$ is replaced by a subsequence $\{\mathcal{S}^{k_j}\}_j$, provided that $\lim_{j \to -\infty} k_j = -\infty$.

Proof: Assuming $Y \neq \{0\}$, we will show that $\dim Y = 1$.

Let f, g be two functions in Y, hence

$$\widehat{f} = \tau_k \widehat{\phi}_k, \qquad \widehat{g} = \eta_k \widehat{\phi}_k, \qquad \tau_k, \eta_k \ \ 2^{k+1}\pi\text{-periodic}, \ \ k \in \mathbb{Z}. \tag{4.10}$$

We prove that, necessarily, $\operatorname{supp} \widehat{f} = \operatorname{supp} \widehat{g}$. For this, consider

$$A := \operatorname{supp} \widehat{g} \cap (\mathbb{R}^d \backslash \operatorname{supp} \widehat{f}).$$

Since $\widehat{g}(x) \neq 0$ on A, it follows from (4.10) that $\widehat{\phi}_k(x) \neq 0$ a.e. on A for all k, and hence, since $\widehat{f}(x) = 0$ on A, we must have $\tau_k(x) = 0$ a.e. on A for all k. Since each τ_k is $2^{k+1}\pi$-periodic, we conclude (by taking $k \to -\infty$ in the first equality of (4.10)) that $\widehat{f} = 0$ on the set $A + D$, with $D := \{j 2^k \pi : \ j \in \mathbb{Z}^d, \ k \in \mathbb{Z}\}$. Since $A + D$ is invariant under dyadic shifts, Lemma 4.6 shows that either $A + D = \mathbb{R}^d$ or $A + D$ is a null-set. Since $\widehat{f}$ is non-trivial and vanishes on $A + D$, we conclude that $A + D$ is a null-set, hence so is A. By symmetry, it follows that $\operatorname{supp} \widehat{f} = \operatorname{supp} \widehat{g}$.

Hence, to complete the proof, it is sufficient to show that the function

$$F : \mathbb{R}^d \to \mathbb{C} : x \mapsto \begin{cases} \widehat{g}(x)/\widehat{f}(x), & \widehat{f}(x) \neq 0; \\ 0, & \text{otherwise,} \end{cases}$$

equals $c\chi_{\operatorname{supp}\widehat{f}}$ for some constant c. Since $\widehat{f}$ is not trivial, F takes on nonzero values. In particular, for given $\varepsilon > 0$, we can choose a set $A \subset \mathbb{R}^d$ of positive measure such that $0 \notin F(A)$, and $F(A)$ is contained in a disc of diameter ε. Let $x \in A$ and $t = 2^{k+1}\pi j$ with $k \in \mathbb{Z}$ and $j \in \mathbb{Z}^d$. Then $\widehat{f}(x) \neq 0$. If also $\widehat{f}(x+t) \neq 0$, then

$$F(x) = \widehat{g}(x)/\widehat{f}(x) = \eta_k(x)/\tau_k(x) = \eta_k(x+t)/\tau_k(x+t) = \widehat{g}(x+t)/\widehat{f}(x+t) = F(x+t),$$

while $F(x+t) = 0$ otherwise. It follows that $F(A+D) \subset F(A) \cup \{0\}$, while by Lemma 4.6 (and the fact that A, hence $A+D$, is not null), $A+D = \mathbb{R}^d$. We conclude that F assumes its non-zero values in a disc of arbitrary small diameter, and hence must be constant on its support. □

It can indeed happen that Y has dimension 1. For example if $\phi_k = \phi$ for each k (not to be confused with the stationary case: $\phi_k = \phi(2^k\cdot)$, all k), then ϕ is obviously in Y. Other, less trivial, examples are also possible (see §6). In passing, we note the following immediate consequence of Corollary 2.2 (and its scaled versions):

Proposition 4.11. *Let $f \in \cap_{k\in\mathbb{Z}}\mathcal{S}^k(\phi_k)$. Then f generates all the spaces $\mathcal{S}^k$ if and only if $\operatorname{supp}\widehat{f} = \operatorname{supp}\widehat{\phi}_k$, all k. In particular, the spaces $\big(\mathcal{S}^k(\phi_k)\big)_k$ are generated all by a single function only if $\operatorname{supp}\phi_k = \operatorname{supp}\phi_{k'}$ for all $k, k' \in \mathbb{Z}$.*

We also note that, for any nested sequence $X_k \subset X_{k+1}$ of closed subspaces of a Hilbert space, the corresponding orthogonal projectors $P_k := P_{X_k}$ converge strongly to the orthogonal projector $P_{-\infty}$ onto $X_{-\infty} := \cap_k X_k$ as $k \to -\infty$ (hence converge strongly to the orthogonal projector P_∞ onto $X_\infty := \overline{\cup_k X_k}$ as $k \to \infty$). Therefore, in particular:

Theorem 4.12. *Let $\mathcal{S}^k := \mathcal{S}^k(\phi_k)$, $k \in \mathbb{Z}$, be a nested sequence, and let $Y = \cap_{k\in\mathbb{Z}}\mathcal{S}^k$ be the (one- or zero-dimensional) space of Theorem 4.9. Then $\lim_{k\to-\infty} P_k f = P_Y f$, for all $f \in L_2(\mathbb{R}^d)$.*

Proof: Here, for completeness, is a proof which only uses the fact that $X_k := \mathcal{S}^k$ is a nested sequence of closed linear subspaces of a Hilbert space, with $Y := \cap_k X_k$.

To show that $\lim_{k\to-\infty} P_k f = P_Y f$, it is sufficient to show that $P_k f \to g$ weakly implies that $g = P_Y f$ and $g = \lim_{k\to-\infty} P_k f$. For, it implies that $P_Y f$ is the only limit point of $(P_k f)_k$, and, further, implies that $(P_k f)_k$ has limit points since, being bounded, it has weak limit points.

So, let g be the weak limit of $\big(P_k f\big)_k$. Since every X_j is closed and convex, hence weakly closed, and contains every $P_k f$ with $k \le j$, it must contain g; therefore, $g \in Y$. On the other hand, $x_k := f - P_k f$ is perpendicular to X_k, hence to Y, therefore, so is the weak limit, $x := f - g$. In short, $g = P_Y f$. Since Y is the intersection of the nested sequence $\big(X_k\big)_k$, it follows that

$$\lim_{k\to-\infty} \|x_k\| = \sup_k \operatorname{dist}(f, X_k) \le \operatorname{dist}(f, Y) = \|x\|.$$

Because of the weak convergence we also have $\langle x_k, x\rangle \to \langle x, x\rangle$, and therefore

$$0 \le \|P_k f - g\|^2 = \|x - x_k\|^2 = \|x\|^2 - 2\operatorname{Re}\langle x, x_k\rangle + \|x_k\|^2 \quad \to \quad -\|x\|^2 + \lim_k \|x_k\|^2 \le 0$$

as k runs to $-\infty$. In other words, $\lim_k P_k f = g = P_Y f$. □

Since $Y \subset \mathcal{S}^k$, it is orthogonal to each of the wavelet spaces $W^k := \mathcal{S}^{k+1} \ominus \mathcal{S}^k$, $k \in \mathbb{Z}$. Therefore, applying Corollary 4.4 and Theorem 4.12 , we obtain the following orthogonal decomposition of $L_2(\mathbb{R}^d)$.

Corollary 4.13. *Let $\mathcal{S}^k := \mathcal{S}^k(\phi_k)$, $k \in \mathbb{Z}$, be a nested sequence, and let $\Omega_0 := \cup \operatorname{supp} \widehat{\phi_k} = \mathbb{R}^d$ (modulo a null-set). If Y is the (one- or zero-dimensional) subspace of Theorem 4.9, then*

$$L_2(\mathbb{R}^d) = Y \oplus \bigoplus_{k \in \mathbb{Z}} W^k.$$

The significance of the last corollary is the following. Let ψ_Y be any nontrivial element of Y with $\|\psi_Y\| = 1$. If for each $k \in \mathbb{Z}$, the set Ψ_k provides an orthonormal basis for the wavelet space W^k, then the totality of functions ψ_Y and $\psi(\cdot - j2^{-k})$, $j \in \mathbb{Z}^d$, $\psi \in \Psi_k$, $k \in \mathbb{Z}$, is an orthonormal basis for $L_2(\mathbb{R}^d)$. Thus, even when Y is nontrivial, multiresolution produces a basis for $L_2(\mathbb{R}^d)$. Similarly, we obtain an L_2-stable basis whenever the Ψ_k provide an L_2-stable basis for W^k whose stability constants are independent of $k \in \mathbb{Z}$.

In the stationary case, i.e., the case when $\phi_k = \phi_0(2^k \cdot)$, the following corollary shows that Y is necessarily trivial.

Corollary 4.14. *For $\phi \in L_2(\mathbb{R}^d)$, define $\mathcal{S}^k := \mathcal{S}^k(\phi(2^k \cdot))$, $k \in \mathbb{Z}$. Then $\cap_{k \in \mathbb{Z}} \mathcal{S}^k = \{0\}$.*

Note that, as in Theorem 4.9, the nestedness condition (1.3)(i) is not required, hence is not assumed.

Proof: We suppose that f is a non-trivial function in $L_2(\mathbb{R}^d)$ which is in each of the spaces $\mathcal{S}^k$ and derive a contradiction. By the assumptions here, each $\mathcal{S}^k$ is the 2-dilate of $\mathcal{S}^{k-1}$, and hence $\cap_k \mathcal{S}^k$ is invariant under dilation by 2. On the other hand, by Theorem 4.9, this space is at most one-dimensional. Therefore, if $f \in \cap_k \mathcal{S}^k$, then there exists some λ such that

$$f(2\cdot) = \lambda f, \quad \text{a.e. on } \mathbb{R}^d. \tag{4.15}$$

It is now easy to show that this is impossible for $f \in L_2(\mathbb{R}^d) \backslash \{0\}$. Indeed, for each $C > 0$, the sets $F_k := \{x : 2^k \leq |x| < 2^{k+1} \text{ and } |f(x)| > C|\lambda|^k\}$ satisfy

$$F_k = 2F_{k-1} \quad \text{and} \quad \operatorname{meas}(F_k) = 2^d \operatorname{meas}(F_{k-1}), \quad \text{for all } k.$$

If f is not the zero function, then for some $C > 0$, $\operatorname{meas}(F_0) \neq 0$. From (4.15), $|f(x)| \geq C|\lambda|^k$ for $x \in 2^k F_0$. Therefore,

$$\int_{\mathbb{R}^d} |f|^2 \geq C^2 \operatorname{meas}(F_0) \sum_{k \in \mathbb{Z}} (2^d |\lambda|^2)^k,$$

which shows that f is not in $L_2(\mathbb{R}^d)$ because the series diverges. □

The importance of Corollary 4.14 is that, in the stationary case, it is not necessary to assume property (1.3)(iii). Moreover, in the case that ϕ has compact support, condition (1.3) (ii) is already implied by (1.3)(i). We have therefore the following corollary.

Corollary 4.16. *If, for the compactly supported function ϕ in $L_2(\mathbb{R}^d)$, the sequence $\mathcal{S}^k := \mathcal{S}^k(\phi(2^k \cdot))$, $k \in \mathbb{Z}$, is nested, i.e., satisfies (1.3)(i), then the conditions (1.3)(ii,iii) are automatically satisfied and we have the orthogonal decomposition*

$$L_2(\mathbb{R}^d) = \bigoplus_{k \in \mathbb{Z}} W^k$$

with $W := \mathcal{S}^1 \ominus \mathcal{S}^0$ the wavelet space and W^k its 2^k-dilate, $k \in \mathbb{Z}$.

5. Univariate wavelets and prewavelets

After showing in the last section that conditions (ii) and (iii) of (1.3) hold in quite a general setting, we now turn our attention back to wavelet constructions. We start with a separate discussion of the univariate case, since this case is significantly simpler than its multivariate counterpart.

As in §3, we are only interested in studying one of the wavelet spaces, namely, $W := \mathcal{S}^1 \ominus \mathcal{S}^0$. The other wavelet spaces, $W^k := \mathcal{S}^{k+1} \ominus \mathcal{S}^k$, are obtained by identical methods, and furthermore, in the stationary case each of the wavelet spaces is obtained from W by dilation.

We work in the same setting as in §3: We assume that $\phi, \eta \in L_2(\mathbb{R})$ satisfy

$$\mathcal{S}(\phi) \subset \mathcal{S}^1(\eta) \tag{5.1}$$

and

$$\operatorname{supp} \widehat{\phi} = \operatorname{supp} \widehat{\eta} = \mathbb{R}. \tag{5.2}$$

As before, we remind the reader that this last assumption is always satisfied when ϕ and η are compactly supported.

We have seen in §3 that W is a principal shift-invariant space and there is always a generator w_* for W whose shifts form an orthonormal system. However, in general we can say nothing about the support of w_*, or the decay of $w_*(x)$ as $|x| \to \infty$. In this section, we want to go further and find other generators for W with favorable decay properties. In particular, in the case $\eta = \phi(2\cdot)$ usually studied in wavelet constructions, we shall recover various generators for W given by other authors.

Our starting point is the function w of Theorem 3.10, i.e., the function whose Fourier transform is

$$\widehat{w} := e_{1/2}\widehat{\phi} - \frac{[e_{1/2}\widehat{\phi}, \widehat{\phi}]}{[\widehat{\phi}, \widehat{\phi}]}\widehat{\phi}. \tag{5.3}$$

We know from Theorem 3.10 that w provides a basis for W.

We wish to express the generators that follow in terms of η. For this we shall use the refinement relation (1.6):

$$\widehat{\phi} = A\widehat{\eta} \tag{5.4}$$

with A a 4π-periodic function.

Theorem 5.5. *Let*

$$\widehat{\psi} := 2e_{-1/2}\overline{A}(\cdot + 2\pi)\widetilde{\widetilde{\eta}}(\cdot + 2\pi)^2\widehat{\eta} = 2e_{-1/2}[\![\widehat{\eta}, \widehat{\phi}]\!](\cdot + 2\pi)\widehat{\eta}. \tag{5.6}$$

If $\widehat{\psi} \in L_2(\mathbb{R})$, then its inverse transform ψ is a generator for the wavelet space. Moreover, ψ has orthonormal (respectively, stable) shifts if the shifts of ϕ and the half-shifts of η are orthonormal (respectively, stable).

Proof: We already know that the function w of (5.3) is a generator for W. We will show that $\widehat{w}/\widehat{\psi}$ is 2π-periodic. Since (5.2) implies that $\operatorname{supp} \widehat{\psi} = \mathbb{R}$, this will prove, by (1.11), that $w \in \mathcal{S}(\psi)$ and hence ψ generates W.

Since $e_{1/2}(\cdot + 2\pi) = -e_{1/2}$, and $e_{1/2}$ is 4π-periodic, we see that $[e_{1/2}\widehat{\phi}, \widehat{\phi}] = e_{1/2}(\widetilde{\widetilde{\phi}}^2 - \widetilde{\widetilde{\phi}}(\cdot + 2\pi)^2)$, while $[\widehat{\phi}, \widehat{\phi}] = \widetilde{\widetilde{\phi}}^2 + \widetilde{\widetilde{\phi}}(\cdot + 2\pi)^2$. Substituting this into (5.3), we obtain that

$$\widehat{w} = \frac{2\widetilde{\widetilde{\phi}}(\cdot + 2\pi)^2}{\widetilde{\phi}^2} e_{1/2}\widehat{\phi}.$$

Since $\widehat{\phi} = A\widehat{\eta}$, and A is 4π-periodic, $\widetilde{\widetilde{\phi}} = |A|\widetilde{\widetilde{\eta}}$. Therefore, we see that

$$\tau := \widehat{w}/\widehat{\psi} = (\widetilde{\phi})^{-2} A\, A(\cdot + 2\pi) e_1.$$

Since A is 4π-periodic and $\widetilde{\phi}$ and e_1 are 2π-periodic, we conclude that, indeed, the ratio $\widehat{w}/\widehat{\psi}$ is 2π-periodic and hence ψ generates W.

To prove the rest of the theorem, we first compute $\widetilde{\psi}$ as follows:

$$\begin{aligned}
\widetilde{\psi}^2 &= 4\left\{|A(\cdot+2\pi)|^2 \widetilde{\widetilde{\eta}}(\cdot+2\pi)^4 \widetilde{\widetilde{\eta}}^2 + |A|^2 \widetilde{\widetilde{\eta}}^4 \widetilde{\widetilde{\eta}}(\cdot+2\pi)^2\right\} \\
&= 4\widetilde{\widetilde{\eta}}(\cdot+2\pi)^2 \widetilde{\widetilde{\eta}}^2 \left\{|A(\cdot+2\pi)|^2 \widetilde{\widetilde{\eta}}(\cdot+2\pi)^2 + |A|^2 \widetilde{\widetilde{\eta}}^2\right\} \\
&= 4\widetilde{\widetilde{\eta}}(\cdot+2\pi)^2 \widetilde{\widetilde{\eta}}^2 \left\{\widetilde{\widetilde{\phi}}(\cdot+2\pi)^2 + \widetilde{\widetilde{\phi}}^2\right\} = 4\widetilde{\widetilde{\eta}}(\cdot+2\pi)^2 \widetilde{\widetilde{\eta}}^2 \widetilde{\phi}^2.
\end{aligned} \tag{5.7}$$

If ϕ has orthonormal shifts and η has orthonormal half-shifts, then $\widetilde{\phi}^2 = 1$ a.e., and $\widetilde{\widetilde{\eta}}^2 = 1/2$ a.e. We conclude that $\widetilde{\psi}^2 = 1$ a.e., and hence ψ has orthonormal shifts. Similarly, if ϕ has $L_2(\mathbb{R})$-stable shifts and η has $L_2(\mathbb{R})$-stable half-shifts, then the functions $\tilde{\phi}$ and $\widetilde{\widetilde{\eta}}$ and their reciprocals are bounded. It follows that $\widetilde{\psi}$ has the same property and hence the shifts of ψ are $L_2(\mathbb{R})$-stable. □

Remark 5.8. *It also follows from (5.7) that the stability constants $C_1(\psi), C_2(\psi) > 0$ for ψ, i.e., the positive constants in the inequality $C_1(\psi) \le \widetilde{\psi} \le C_2(\psi)$ a.e., can be chosen as*

$$C_j(\psi) = 2C_j(\phi)C_j(\eta)^2, \quad j = 1, 2,$$

where $C_j(\phi)$ and $C_j(\eta)$, $j = 1, 2$, are the stability constants associated with $\widetilde{\phi}$ and $\widetilde{\widetilde{\eta}}$ respectively.

Note that when $\eta = \sqrt{2}\phi(2\cdot)$, the orthogonality assumption or the stability assumption on the half-shifts of η is equivalent to the corresponding assumption on the shifts of ϕ. Further, in the orthogonal case, $\widetilde{\widetilde{\eta}}(\cdot + 2\pi)^2 = 1/2$, hence the formula (5.6) is reduced to

$$\widehat{\psi} = e_{-1/2}\overline{A}(\cdot + 2\pi)\widehat{\eta}, \tag{5.9}$$

which gives the usual wavelet obtained by multiresolution. Note also that the theorem incidentally proves that $\widehat{\psi} \in L_2(\mathbb{R})$, hence ψ is in $L_2(\mathbb{R})$ whenever η has $L_2(\mathbb{R})$-stable half-shifts.

Mallat has proved the orthonormal part of the above theorem (for $\eta := \sqrt{2}\phi(2\cdot)$) without the assumption (5.2), but with additional hypotheses on the decay and smoothness of ϕ. Several authors have used Mallat's approach to construct orthonormal wavelets, including Daubechies [D] in her celebrated construction of wavelets w of compact support and arbitrary high orders of differentiability. However, the difficult part of the Daubechies construction is to show the existence of compactly supported functions ϕ which satisfy (5.4), have arbitrarily high orders of differentiability, and have shifts which are orthonormal.

As an example, if ϕ is the B-spline $\phi = N(\cdot|0, \ldots, r)$ of order r with knots at $0, \ldots, r$, then $\mathcal{S}(\phi)$ is the space of all cardinal splines of order r which are in $L_2(\mathbb{R})$. The function ψ is then the spline wavelet of Battle and Lemarie (see [B]).

The prewavelet part of Theorem 5.5 has been proved by Micchelli in [Mi], but under different hypotheses. He does not assume (5.2), but assumes that ϕ satisfies the refinement equation (1.6) with coefficients $a \in \ell_1(\mathbb{Z})$. Similar ideas have been employed by Chui and Wang [CW], [CW1]. In particular, when ϕ is the cardinal B-spline, the prewavelet ψ of Theorem 5.5 is their compactly supported spline wavelet (except for an integer shift).

The remainder of this section will be devoted to the important case when the functions ϕ and η are *compactly supported*. We shall be interested in finding functions w from W which have minimal support. In the context of compact support, one encounters the notion of *linear independence*: We say that the shifts of the compactly supported ϕ are **linearly independent** if, for each sequence c, the sum $\sum_{j\in\mathbb{Z}} \phi(\cdot - j)c(j)$ is identically zero if and only if $c(j) = 0$ for all $j \in \mathbb{Z}$ (note that for each $x \in \mathbb{R}$, the series has only a finite number of nonzero terms and hence converges pointwise). We remark that linear independence of the shifts of ϕ implies that these shifts are $L_2(\mathbb{R})$-stable, (cf. [JM]).

In what follows, we denote by $\operatorname{diam}\Omega$ the length of the smallest interval containing the subset Ω of $\mathbb{R}$. With the aid of [R2], the following result on linearly independent generators was proved in [BDR1].

Result 5.10. *Let $\mathcal{S}$ be a univariate principal shift-invariant space which is generated by a compactly supported function. Then, there exists a compactly supported $\phi \in \mathcal{S}$ that satisfies all of the following conditions:*

(a) The shifts of ϕ are linearly independent.

(b) Every compactly supported $f \in \mathcal{S}$ can be written as a finite linear combination of the shifts of ϕ.

(c) $\operatorname{diam}\operatorname{supp}\phi \le \operatorname{diam}\operatorname{supp} f$*, for every $f \in \mathcal{S}$.*

Furthermore, up to a shift and a scalar multiplication, ϕ is characterized by any of these three properties.

Corollary 5.11. *If ϕ has compact support, then the wavelet space W has a compactly supported generator whose shifts are linearly independent. This generator enjoys all the properties of ϕ in Result 5.10.*

Proof: By the case $d = 1$ in Theorem 3.12, W is principal and has a compactly supported generator. It is therefore enough to apply Result 5.10. □

In view of the attractive properties of a linearly independent generator, it is desirable to find a constructive method to find the linearly independent generator of W. For this, we shall assume (without loss of generality in view of Result 5.10) that the generator η for $\mathcal{S}^1(\eta)$ has linearly independent half-shifts. In view of Result 5.10, any compactly supported function in $\mathcal{S}^1(\eta)$ has Fourier transform $\tau\widehat{\eta}$ with τ a 4π-periodic trigonometric polynomial. We are interested in the properties of τ that characterize linear independence of the shifts of $(\tau\widehat{\eta})^\vee$.

If τ is a non-trivial 4π-periodic trigonometric polynomial, then $\tau = e_{m/2}\sum_{j=0}^{n}\alpha(j)e_{j/2}$ with $\alpha(0)\alpha(n) \ne 0$ and $m \in \mathbb{Z}$. We call n the **modified degree** of τ and write $\operatorname{mdeg}\tau := n$. From this it easily follows that if f and g are compactly supported and $\widehat{g} = \tau\widehat{f}$ for some 4π-periodic trigonometric polynomial, then

$$\operatorname{diam}\operatorname{supp} g = \operatorname{diam}\operatorname{supp} f + (\operatorname{mdeg}\tau)/2. \tag{5.12}$$

If τ and ζ are two 4π-periodic trigonometric polynomials, we say that ζ divides τ if τ/ζ is also a 4π-periodic trigonometric polynomial. With this, we have the following characterization of the linearly independent generators of W.

Proposition 5.13. *Assume η is compactly supported and has linearly independent half-shifts. Let w be any compactly supported generator of the wavelet space W. Then w is the linearly independent generator of the wavelet space (and thus enjoys all the properties of the ϕ in Result 5.10) if and only if the 4π-periodic trigonometric polynomial τ in the representation $\widehat{w} = \tau\widehat{\eta}$ is not divisible by a non-constant 2π-periodic trigonometric polynomial.*

Proof: By Result 5.10, every compactly supported $w \in W \subset \mathcal{S}^1(\eta)$ can be expressed in the form $\widehat{w} = \tau\widehat{\eta}$ for some 4π-periodic trigonometric polynomial τ. If $\tau = \lambda\zeta$, where λ is a 2π- and ζ is 4π-periodic trigonometric polynomial, then, by Corollary 2.2, $w' := (\zeta\widehat{\eta})^\vee$ is also a compactly supported generator for W.

By (5.12),

$$\operatorname{diam}\operatorname{supp} w = \operatorname{diam}\operatorname{supp}\eta + (\operatorname{mdeg}\tau)/2, \quad \operatorname{diam}\operatorname{supp} w' = \operatorname{diam}\operatorname{supp}\eta + (\operatorname{mdeg}\zeta)/2,$$

hence $\operatorname{diam}\operatorname{supp} w' \leq \operatorname{diam}\operatorname{supp} w$, with equality if and only if $\operatorname{mdeg}\zeta = \operatorname{mdeg}\tau$, i.e., if and only if $\operatorname{mdeg}\lambda = 0$. Our claim follows then from the fact (cf. Result 5.10) that the linearly independent generator is characterized by the minimality of its support. $\square$

In view of the last result, the search for the linearly independent generator of W can be carried out as follows: assuming η has linearly independent half-shifts, we find some particular compactly supported $w \in W$, and write $\widehat{w} = \tau\widehat{\eta}$. Then τ is necessarily a trigonometric polynomial. Factoring $\tau = \lambda\zeta$ where λ is a 2π-periodic factor of maximal degree, ψ_* defined by $\widehat{\psi}_* = \zeta\widehat{\eta}$ is the linearly independent generator of W.

Corollary 5.14. *Assume that ϕ and η are compactly supported and the half-shifts of η are linearly independent, and that $\widehat{\phi} =: A\widehat{\eta}$. Then the linearly independent generator ψ_* for the wavelet space $W = \mathcal{S}^1(\eta) \ominus \mathcal{S}(\phi)$ is given by*

$$\widehat{\psi}_* := \zeta\widehat{\eta},$$

where $\zeta := \tau/\lambda$, and λ is a 2π-periodic trigonometric polynomial of maximal degree that divides

$$\tau := e_{-1/2}\overline{A}(\cdot + 2\pi)\widetilde{\widetilde{\eta}}(\cdot + 2\pi)^2 = e_{-1/2}[\![\widehat{\eta}, \widehat{\phi}]\!](\cdot + 2\pi).$$

Proof: From Theorem 5.5, we know that $\psi := (\tau\widehat{\eta})^\vee$ generates W. Thus the claim will follow from the argument preceding this corollary, as soon as we show that τ indeed is a trigonometric polynomial.

The function $\widetilde{\widetilde{\eta}}^2$ is a trigonometric polynomial by the analogue of Lemma 2.8, since it is the 4π-periodization of $|\widehat{\eta}|^2$ for the compactly supported η. The mask A is also a trigonometric polynomial by Result 5.10, since η and ϕ are compactly supported and η is a linearly independent generator of $\mathcal{S}^1(\eta)$. $\square$

Result 5.10 tells us that the search for a linearly independent generator is, necessarily, the same as the search for a minimally supported generator in the sense that we are minimizing $\operatorname{diam}\operatorname{supp} w$ among all generators w. Chui and Wang, [CW], considered a slightly different notion of minimality: they were interested in finding a generator w for W which can be expressed in the form $\widehat{w} = \tau\widehat{\eta}$, with τ a trigonometric polynomial of minimal degree (they assume that the refinement mask $A = \widehat{\phi}/\widehat{\eta}$ is a polynomial, to guarantee the existence of such τ). Thus, while we minimize $\operatorname{diam}\operatorname{supp} w$ over all possible generators w, Chui and Wang minimize $\operatorname{diam}\operatorname{supp} w$ only over those w which can be written as a finite linear combination of the half-shifts of η. However, because of Result 5.10, the two notions coincide if we assume (as we do) that the half-shifts of η are linearly independent, and, furthermore, as is proved by Jia and Wang in [JW], this assumption holds in the stationary case in case ϕ has stable shifts and the mask has no 2π-periodic polynomial factor. In any event, with straightforward modifications, the arguments used in Proposition 5.13 and Corollary 5.14 can be applied to show that the same characterization holds for the "minimal w" in the [CW] sense.

Chui and Wang stated their results in terms of the symmetric zeros of the trigonometric polynomials involved. Let us pause for a moment to see how symmetric zeros enter into the characterizations provided above. If τ is a 4π-periodic trigonometric polynomial, then, up to some exponential factor, we can write $\tau = p(e_{1/2})$ for some algebraic polynomial p with $\deg p = \operatorname{mdeg}\tau$. But, for any algebraic polynomial q, $q(e_{1/2})$ is 2π-periodic if and only if it can be written as an algebraic polynomial in $e_1 = {e_{1/2}}^2$, i.e., if and only if q involves only even powers, or, what is the same, if and only if all the zeros of q occur in symmetric pairs. Thus the quotient τ/λ in Corollary 5.14 can be equivalently characterized by the lack of symmetric zeros in p/q.

If we take for ϕ a cardinal B-spline and for η its 2-dilate, then the half-shifts of η are linearly independent. In this case, the spline wavelet ψ of Chui and Wang (given by Theorem 5.5) is the minimally supported wavelet for W guaranteed by Corollary 5.11 because the function τ of Corollary 5.14 is known to have no 2π-periodic polynomial factor. It thus follows that ψ has linearly independent shifts.

6. An example of non-stationary decompositions: exponential B-splines

We have carried out the analysis in this paper without making the assumption that η is the 2-dilate of ϕ. The reason for this is two-fold: first, the assumption $\eta = \phi(2\cdot)$ does not simplify either the idea or the details of our approach. Second, and more importantly, there are various interesting examples where the "finer" function η is not obtained from ϕ by dilation. This is the case for example for exponential B-splines, exponential box splines, and various radial basis functions. In this section, we briefly discuss what seems to be the simplest example in this direction: the exponential B-splines.

The exponential B-spline $N_\lambda := N_\lambda(\cdot|0, ..., n)$ is a generalization of the (polynomial) B-spline $N(\cdot|0, ..., n)$. It can be defined by its Fourier transform as follows. Let λ be a parameter vector $(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$. Then

$$\widehat{N}_\lambda(y) = \prod_{m=1}^{n} \frac{e^{\lambda_m - iy} - 1}{(\lambda_m - iy)}.$$

The polynomial B-spline corresponds to the choice $\lambda = 0$. *Splines in tension* correspond to the choice $n = 4$, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -\lambda_4$.

In general, N_λ is $(n-2)-$times continuously differentiable and is supported on $[0\,\mathrm{d}n]$. On each interval $[j\,\mathrm{d}j+1]$, N_λ coincides with a function in the kernel K_λ of the differential operator $\mathcal{D} := \prod_{m=1}^{n}(D - \lambda_m)$. The shifts of N_λ are linearly independent if and only if

$$\lambda_m - \lambda_j \notin 2\pi i \mathbb{Z} \backslash 0, \quad \text{all } m, j. \tag{6.1}$$

Furthermore, when (6.1) holds, every $f \in K_\lambda$ can be expressed as a linear combination of the shifts of N_λ.

With the above knowledge in hand, it should be clear that N_λ cannot be written in terms of its 2-dilate, unless $\lambda = 0$: upon dilating N_λ we obtain a function which is piecewise in $K_{2\lambda}$ and therefore every element of $\mathcal{S}^1(N_\lambda(2\cdot))$ is piecewise in $K_{2\lambda}$ while N_λ is piecewise in K_λ. Thus, the usual framework of multiresolution analysis cannot be applied to exponential B-splines.

On the other hand, from the Fourier transform of N_λ, we see that

$$\widehat{N}_\lambda(y) = \prod_{m=1}^{n} \frac{e^{\lambda_m/2 - iy/2} + 1}{2} \prod_{m=1}^{n} \frac{e^{\lambda_m/2 - iy/2} - 1}{(\lambda_m/2 - iy/2)}. \tag{6.2}$$

The second factor on the right hand side of (6.2) is recognized as $\widehat{N}_{\lambda/2}(\cdot/2)$, and thus

$$\widehat{N}_\lambda = A_{\lambda/2} \widehat{N}_{\lambda/2}(\cdot/2),$$

with A_λ the 4π-periodic trigonometric polynomial

$$A_\lambda(y) := \prod_{m=1}^{n} \frac{e^{\lambda_m - iy/2} + 1}{2}.$$

Note that $\widehat{N}_{\lambda/2}(\cdot/2)$ is the Fourier transform of $2N_{\lambda/2}(2\cdot)$ which is supported on $[0\,\mathrm{d}n/2]$, and is piecewise in K_λ (with breakpoints at the half-integers).

We fix a vector λ and define the spaces $\mathcal{S}^k := \mathcal{S}^k(\phi_k)$, $k \in \mathbb{Z}$, with $\phi_k := N_{\lambda/2^k}(2^k\cdot)$. The generators ϕ_k then satisfy the non-stationary refinement equations

$$\widehat{\phi}_k = 2A_{\lambda/2^{k+1}}(\cdot/2^k)\widehat{\phi}_{k+1}.$$

We observe that $2A_{\lambda/2^{k+1}}(\cdot/2^k)$ can be written as

$$2^{-n}\sum_{j=0}^{n}\sigma_j(e^{\lambda_1/2^k},...,e^{\lambda_n/2^k})e^{-ijy/2^{k+1}},$$

where $\sigma_j(t_1,...,t_n)$ is the homogeneous symmetric polynomial of degree j in $t_1,...,t_n$.

The scale of spaces $\mathcal{S}^k$, $k \in \mathbb{Z}$, clearly satisfies condition (1.3)(i) of multiresolution. Since $\operatorname{supp}\widehat{\phi_k} = \mathbb{R}$, (1.3) (ii) follows from Theorem 4.3. According to Theorem 4.9, the space $Y := \cap_k \mathcal{S}^k$ has dimension ≤ 1. The following theorem, which is a special case of Theorem 8.4, provides a complete description of this space.

Theorem 6.3. *Let $\{\mathcal{S}^k\}$ be a multiscale of spaces generated by exponential B-splines. Let $Y := \cap_{k\in\mathbb{Z}}\mathcal{S}^k$. Then Y is one-dimensional if and only if $\operatorname{Re}\lambda_j \neq 0$, $j = 1,...,n$. Otherwise, Y is trivial. In case Y is one-dimensional, it is spanned by the Green's function G (or more precisely the fundamental solution of the differential operator $\mathcal{D}$) whose Fourier transform is given by*

$$\widehat{G}(y) = \prod_{m=1}^{n}(\lambda_m - iy)^{-1}. \tag{6.4}$$

In this case, $\mathcal{S}^k = \mathcal{S}^k(G)$ for every k.

For convenience, we define from now on $W_{-\infty} := \cap_k \mathcal{S}^k$, and obtain in this fashion the decomposition

$$L_2(\mathbb{R}) = \bigoplus_{-\infty\leq k<+\infty} W_k,$$

valid for the wavelet decomposition based on any exponential B-spline.

An interesting and important problem in the context of non-stationary subdivisions is the stability question. Let ψ_k be the compactly supported wavelet function given by (the appropriate scaled version of) Theorem 5.5 for $\phi_k := N_{\lambda/2^k}(2^k\cdot)$. Then the wavelet space $W^k := \mathcal{S}^{k+1}\ominus\mathcal{S}^k$ is a principal 2^{-k}-shift-invariant space generated by ψ_k. The 2^{-k}-shifts of ϕ_k are linearly independent iff

$$\lambda_m - \lambda_j \notin 2^{k+1}\pi i\mathbb{Z}\backslash 0, \tag{6.5}$$

as can be easily concluded from (6.1) by rescaling. We see that for large enough k (say, $k > k_0 \geq -\infty$), ϕ_k is always a linearly independent generator, and in particular a stable generator of its space. Further, if $\lambda \in \mathbb{R}^n$, then the linear independence (and hence the stability) holds for all k. By Theorem 5.5, ψ_k provides a stable basis for the wavelet space W_k for every $k > k_0$. At the same time, the sum $\bigoplus_k W_k$ is orthogonal, *a fortiori* the sum $\bigoplus_{k>k_0} W_k$ is orthogonal. Nevertheless, these arguments *do not imply* that $\{\psi_k(\cdot - 2^{-k}j)\}_{k>k_0, j\in\mathbb{Z}^d}$ forms a stable basis for $\bigoplus_{k>k_0} W_k$, since one still needs to show that the stability constants associated with the basis $\{\psi_k(\cdot - 2^{-k}j)\}_{j\in\mathbb{Z}^d}$ of W_k can be chosen independently of $k \geq k_0$. This question does not arise in the stationary case, since then ψ_k is obtained by dilating ψ_0 and the stability constants do not change with k.

The main tool in this discussion of stability is the following consequence of Theorem 5.5 and Remark 5.8:

Corollary 6.6. *Let $\left(\mathcal{S}^k = \mathcal{S}^k(\phi_k)\right)_k$ be a nested sequence of spaces in $L_2(\mathbb{R})$, and, for $k \in \mathbb{Z}$, define $\psi_k := 2^{k/2}\psi(2^k\cdot)$, with ψ the wavelet generator of Theorem 5.5 corresponding to the choice $\phi := \phi_k(\cdot/2^k)$, $\eta := \phi_{k+1}(\cdot/2^k)$ in that theorem. Let $\eta_k := \phi_k(\cdot/2^k)$, $k \in \mathbb{Z}$, and $-\infty \le k_0 < k_1 \le \infty$. Then, the set*

$$\Psi := \{\psi_k(\cdot - 2^{-k}j)\}_{k_0<k<k_1, j\in\mathbb{Z}^d}$$

is an L_2-stable basis for the space $\bigoplus_{k_0<k<k_1} W_k$ (with $W_k := \mathcal{S}^{k+1} \ominus \mathcal{S}^k$) if and only if there exist positive constants C_1 and C_2 such that

$$\|\widetilde{\eta}_k\|_{L_\infty(\mathbb{T})} \le C_2 \quad \text{and} \quad \|1/\widetilde{\eta}_k\|_{L_\infty(\mathbb{T})} \le 1/C_1,$$

for every $k_0 < k < k_1$. Furthermore, the stability constants $C_j(\Psi)$ for the choice $F := \Psi$ in (1.5) then can be taken as C_j^3, $j = 1, 2$.

Proof: As explained in the paragraph preceding this corollary, we only need to check, for each k, the stability constants associated with the basis $\Psi_k := \left(\psi_k(\cdot - 2^{-k}j)\right)_{j\in\mathbb{Z}^d}$ for W_k. By Theorem 5.5 and Remark 5.8, these constants are determined by the constants associated with the sequence $\left(\phi_m(\cdot - 2^{-m}j)\right)_{j\in\mathbb{Z}^d}$, $m = k, k+1$. By scaling, these latter constants are observed to be identical with the constants associated with the sequences $\left(\eta_m(\cdot - j)\right)_{j\in\mathbb{Z}}$, $m = k, k+1$. With this, the bounds C_j^3, $j = 1, 2$, follow from Remark 5.8. □

Corollary 6.7. *Let $\left(\mathcal{S}^k(\phi_k)\right)_k$ be a nested sequence of exponential B-spline spaces, i.e., $\phi_k = N_{\lambda/2^k}(2^k\cdot)$ for some (fixed) $\lambda \in \mathbb{C}^n$. Let $k_0 > -\infty$ be chosen such that (6.5) holds for every $k \ge k_0$. Let ψ_0 be the generator of W_0 defined by Theorem 5.5, and let ψ_k be the analogous generator of W_k, $k \in \mathbb{Z}$. Let*

$$\Psi := \left(\psi_k(\cdot - 2^{-k}j)\right)_{k\ge k_0, j\in\mathbb{Z}^d}.$$

Then Ψ forms a stable basis for $\bigoplus_{k\ge k_0} W_k$.

Proof: We observe that $\eta_k := \phi_k(2^{-k}\cdot)$ is the function $N_{\lambda/2^k}$, and, by the assumption here, the shifts of each η_k form a stable basis for $\mathcal{S}(\eta_k)$. All the functions η_k, $k \in \mathbb{Z}$, are supported in $[0\,\mathrm{d}n]$ and they converge uniformly as $k \to \infty$ to the polynomial B-spline N_0. From this it easily follows the $\widetilde{\eta}_k$ converges uniformly, as $k \to \infty$, to $\widetilde{N}_0$. Thus, for sufficiently large k_1, and for every $k \ge k_1$,

$$\|\widetilde{\eta}_k\|_{L_\infty(\mathbb{T}^d)} \le \|\widetilde{N}_0\|_{L_\infty(\mathbb{T}^d)} + \varepsilon,$$

and

$$\|1/\widetilde{\eta}_k\|_{L_\infty(\mathbb{T}^d)} \le \|1/\widetilde{N}_0\|_{L_\infty(\mathbb{T}^d)} + \varepsilon.$$

It follows, thus, that $\sup_{k\ge k_0} \|\widetilde{\eta}_k\|_{L_\infty(\mathbb{T}^d)}$, and $\sup_{k\ge k_0} \|1/\widetilde{\eta}_k\|_{L_\infty(\mathbb{T}^d)}$ are finite, and our claim follows from Corollary 6.6. □

A more subtle analysis is required in the consideration of the stability of the full basis $\left(\psi_k(\cdot - 2^{-k}j)\right)_{k,j}$. We omit these details here.

7. Multivariate prewavelets

We have given in §3 various sets of generators for the wavelet space W. In particular, we have shown how to obtain generating sets which provide an L_2-stable basis or more generally an orthonormal basis for W. However, our constructions were lacking in the following sense: If η has compact support, then the elements in the generating sets which provide an L_2-stable basis need not be of compact support, nor can they be shown to decay at any rate. On the other hand, it has been proved by Meyer [Me; Chapter III, §6] (and also Jia and Micchelli [JM1]) that, under some general assumptions on the generator ϕ of $\mathcal{S}^0$ (e.g., ϕ is compactly supported and provides a stable basis for $\mathcal{S}^0$), there always are generating sets consisting of nicely decaying functions which provide an L_2-stable basis for W. However, the proofs of these facts are not constructive, hence leave open the question of how to obtain such generating sets explicitly. We shall not provide a solution to this problem in its entirety, but we shall build on previous constructions, of Lorentz and Madych [LM] and Riemenschneider and Shen [RS], which can be applied in certain special but important cases.

We assume throughout this section that ϕ and η are $L_2(\mathbb{R}^d)$-functions that satisfy

$$\mathcal{S}(\phi) \subset \mathcal{S}^1(\eta) \tag{7.1}$$

and

$$\operatorname{supp} \widehat{\phi} = \operatorname{supp} \widehat{\eta} = \mathbb{R}^d. \tag{7.2}$$

As before, we denote the refinement mask by A, i.e.,

$$\widehat{\phi} = A\widehat{\eta}.$$

The refinement mask plays a major role in the context of orthogonal wavelets (cf. (5.9)). But, as already observed in Theorem 5.5, the construction of prewavelets is based on the function

$$B := [\![\widehat{\eta}, \widehat{\phi}]\!] = \overline{A}\widetilde{\widehat{\eta}}^2, \tag{7.3}$$

and for that reason we assigned it the above special notation, B.

The derivations of generators and bases for W that were carried out in §3 involved only the function ϕ. In order to construct stable bases for W from ϕ that imitated the decay properties of ϕ, we would have to assume that ϕ has L_2-stable half-shifts, and this is a restrictive assumption, and applies only to non-stationary refinements (see the next section). Thus, we change our focus from ϕ to η, under the assumption that the half-shifts of the new generator, η, are stable. Indeed, it is the L_2-stability of η which will allow the construction of an L_2-stable basis for W.

We recall the operator Q_0 of (2.30).

Corollary 7.4. *Assume that ϕ and η satisfy (7.1) and (7.2). A necessary and sufficient condition that $w \in L_2(\mathbb{R}^d)$ be in W is that there is a 4π-periodic function τ such that*

$$\widehat{w} = \tau\widehat{\eta} \tag{7.5}$$

and

$$Q_0(\tau B) = \sum_{v \in 4\pi V} (\tau B)(\cdot + v) = 0. \tag{7.6}$$

If η has compact support, then a sufficient condition that w have compact support is that τ is a trigonometric polynomial (of period 4π). Moreover, this last property characterizes the compactly supported elements of W, whenever η has linearly independent half-shifts.

Proof: The first equality in (7.6) is merely the definition of Q_0. As for the second, since $W \subset \mathcal{S}^1(\eta)$, any function in W has Fourier transform of the form (7.5). Since, for any functions $f, g \in L_2(\mathrm{IR}^d)$, $[f, g] = Q_0([\![f, g]\!])$, we conclude that

$$[\widehat{w}, \widehat{\phi}] = Q_0([\![\tau\widehat{\eta}, \widehat{\phi}]\!]) = Q_0(\tau B).$$

Since $w \in W$ if and only if $[\widehat{w}, \widehat{\phi}] = 0$, the main claim of this corollary follows.

If τ is a polynomial and η is compactly supported, then $\tau\widehat{\eta}$ certainly is the Fourier transform of a compactly supported function. In case η has linearly independent half-shifts, Theorem 1.3 in [BR] implies that $\widehat{f} = \tau\widehat{\eta}$, with τ a trigonometric polynomial, whenever $f \in \mathcal{S}^1(\eta)$ is of compact support. $\square$

With Corollary 7.4 in mind, we would like to find a set $V_0 \subset V$ of cardinality $2^d - 1$ and 4π-periodic functions τ_v, $v \in V_0$, that satisfy $Q_0(\tau_v B) = 0$. Then, the functions $\tau_v\widehat{\eta}$, $v \in V_0$, are in $\widehat{W}$. Under certain conditions we can choose the τ_v, $v \in V_0$, so that the $w_v := (\tau_v\widehat{\eta})^\vee$, $v \in V_0$, provide an L_2-stable basis for W. We begin by generalizing a construction used by Lorentz and Madych [LM] (see also [JM] and [Sö]).

We can decompose the function $B = [\![\widehat{\eta}, \widehat{\phi}]\!]$ into its 2π-periodic components, as in (2.31):

$$B = \sum_{v \in V} e_{-v} B_v, \quad B_v := Q_v(B)/2^d. \tag{7.7}$$

If η and ϕ are of compact support, then (by the half-shift analog of Lemma 2.8), $B = [\![\widehat{\eta}, \widehat{\phi}]\!]$ is a 4π-periodic polynomial. In such a case, the functions B_v are 2π-periodic polynomials.

For our first construction, we assume that B is bounded, and that, for some $v_0 \in V$, B_{v_0} is bounded away from zero a.e., and set $V_0 := V \setminus \{v_0\}$. These requirements are fulfilled, for example, for $v_0 = 0$, hence $V_0 = V'$, whenever η has L_2-stable half-shifts, and, further, the 4π-periodic refinement mask A is *real, nonnegative* and continuous, with no 2π-periodic zeros.

Theorem 7.8. *Assume that ϕ and η satisfy (7.1) and (7.2), and let B_v be as in (7.7). Let $v_0 \in V$, $V_0 := V \backslash \{v_0\}$. Then the functions*

$$\tau_v := e_{v_0} B_v - e_v B_{v_0}, \quad v \in V_0, \tag{7.9}$$

satisfy $Q_0(\tau_v B) = 0$, hence the functions w_v, $v \in V_0$, with Fourier transform $\widehat{w}_v := \tau_v\widehat{\eta}$, are in W, provided that $\widehat{w}_v \in L_2(\mathrm{IR}^d)$. If η and ϕ have compact support, then the w_v, $v \in V_0$, have compact support as well. If η has L_2-stable half-shifts, and both B and $1/B_{v_0}$ are essentially bounded on $\mathrm{T\!T}^d$, then $\big(w_v\big)_{v \in V_0}$ provides an L_2-stable basis for W.

Proof: Let $v, u \in V$. Since B_v is 2π-periodic,

$$Q_0(e_u B_v B) = B_v Q_0(e_u B) = B_v Q_u(B) = B_v B_u 2^d.$$

Application of this equality, once with $v = v,\ u = v_0$ and then with the opposite choice, proves that

$$2^{-d} Q_0(\tau_v B) = B_v B_{v_0} - B_{v_0} B_v = 0.$$

Hence (7.6) is satisfied and the functions w_v, $v \in V_0$, are in W.

If η and ϕ have compact support, then, by the half-shift analog of Lemma 2.8, $B = [\![\eta, \phi]\!]$ is a trigonometric polynomial, hence so is each B_v and each τ_v. This implies that each w_v is compactly supported.

To show the L_2-stability of the $\big(w_v\big)_{v\in V_0}$, we consider the matrix $\mathrm{T} := \big(\tau_{v,u}\big)_{v,u\in V}$ with diagonal elements $\tau_{v,v} := -B_{v_0}$, $v \in V$, and with off-diagonal elements $\tau_{v,v_0} = B_v$, $v \in V_0$, and with all other entries zero. We observe that $\mathrm{T}\big(e_v\widehat{\eta}\big)_{v\in V}$ coincides with $\big(\widehat{w}_v\big)_{v\in V_0}$ in all the V_0-entries, and therefore, for proving the desired stability it suffices to show that the shifts of the inverse transforms of $\mathrm{T}\big(e_v\widehat{\eta}\big)_{v\in V}$ are stable. Recall that we are assuming that the half-shifts of η are stable, or, equivalently, that the full-shifts of $\big(\eta(\cdot+v)\big)_{v\in V}$ are stable. Thus, by (iv) of Theorem 2.26, it remains to show that $\|\mathrm{T}\|$ and $\|\mathrm{T}^{-1}\|$ are essentially bounded on $\mathbb{T}^d$. Since we assume that B is bounded, so is each component B_v, hence T has all entries bounded. On the other hand, $|\det \mathrm{T}| = |B_{v_0}|^{2^d}$ hence by our assumption, is bounded away from 0. This implies that both $\|\mathrm{T}\|$ and $\|\mathrm{T}^{-1}\|$ are bounded a.e. □

We note that the boundedness assumption on B is automatically satisfied whenever the full-shifts of η and ϕ are stable, since $|B| = \widetilde{\widetilde{\eta}}\widetilde{\widetilde{\phi}}$, with each factor on the right being bounded because of the stability assumption.

The other assumptions of Theorem 7.8 are also met in many instances. The most important example is recorded in the following corollary, which also admits straightforward extensions to the non-compact support case.

Corollary 7.10. *Let ϕ and η be two compactly supported functions with the half-shifts of η and the full-shifts of ϕ being L_2-stable. Suppose that (7.1) and (7.2) are satisfied, and that the refinement mask A is a trigonometric polynomial (continuity of A would suffice, as well). Then $\phi_1 := \phi * \overline{\phi(-\cdot)}$ and $\eta_1 := \eta * \overline{\eta(-\cdot)}$ satisfy the conditions of Theorem 7.8 for $v_0 = 0$, and hence the sequence $\big(w_v\big)_{v\in V'}$, of (compactly supported) functions defined there (with respect to ϕ_1 and η_1) forms a stable basis for the corresponding wavelet space.*

Proof: We observe that $\widehat{\eta}_1 = |\widehat{\eta}|^2$ and $\widehat{\phi}_1 = |\widehat{\phi}|^2$, and hence $\widehat{\phi}_1 = |A|^2\widehat{\eta}_1$. Because of the compact support of η_1 and ϕ_1, the non-negative function

$$B := [\![\widehat{\eta}_1, \widehat{\phi}_1]\!] = |A|^2\widetilde{\widetilde{\eta}}_1{}^2$$

is a trigonometric polynomial (Lemma 2.8), hence bounded (alternatively, it is bounded because of the stability assumptions on η and ϕ). Since the half-shifts of η are stable, $\widehat{\eta}$ possesses no 4π- periodic zero, hence neither does $\widehat{\eta}_1 = |\widehat{\eta}|^2$. Consequently, $\widetilde{\widetilde{\eta}}_1$ vanishes nowhere. Also, because of the stability of the shifts of ϕ, A has no 2π-periodic zeros (since such zeros would be inherited by $\widehat{\phi}$, hence by $\widetilde{\phi}$). This means that B is a non-negative 4π-periodic function without any 2π-periodic zeros. Consequently, B_0 is a strictly positive (2π-periodic) trigonometric polynomial. Now apply Theorem 7.8. □

We next describe a general procedure for constructing functions τ which satisfy (7.6). The vertices V form a group under addition modulo one. If J is one of its subgroups, then the distinct cosets $v + J$ form a partition of V. We let $R \subset V$ be a set of representers for these distinct cosets. A partition $R = R_0 \cup R_1$ into disjoint sets gives the sets $K_j := \cup_{v\in R_j}(v + J)$, $j = 0, 1$, which are a partition of V. Note that if e_v, $v \in V$, is an exponential which is not constant on $4\pi J$, then $\sum_{\nu\in 4\pi J} e_v(\nu) = 0$.

Theorem 7.11. *Assume that ϕ and η satisfy (7.1) and (7.2), and define, as before, $B := [\![\widehat{\eta}, \widehat{\phi}]\!]$. Let J be any subgroup of the group V, let v be any element of V for which e_v is non-constant on $4\pi J$, and let K be any union of cosets (in V) of J which contains 0. Then the function $w_{v,K}$, with Fourier transform*

$$\widehat{w}_{v,K} := e_v\widehat{\eta} \prod_{\alpha\in 4\pi K\backslash 0} B(\cdot+\alpha),$$

is in W provided it is in $L_2(\mathbb{R}^d)$. Moreover, if η and ϕ have compact support, then w is also of compact support.

Proof: In case η and ϕ are compactly supported, $B = [\![\widehat{\eta}, \widehat{\phi}]\!]$ is a trigonometric polynomial, and hence each $w_{v,K}$ (which is then a well-defined L_2-function) is compactly supported.

To prove the main claim of this theorem, it is enough, in view of Corollary 7.4, to show that

$$\tau := \prod_{\alpha \in 4\pi K} B(\cdot + \alpha)$$

satisfies

$$\sum_{\mu \in 4\pi V} e_v(\cdot + \mu)\tau(\cdot + \mu) = 0.$$

Since $\tau(\cdot + \nu) = \tau$, $\nu \in 4\pi J$, we can write this last sum as

$$\sum_{r \in 4\pi R} \sum_{\nu \in 4\pi J} e_v(\cdot + r + \nu)\tau(\cdot + r + \nu) = \sum_{r \in 4\pi R} e_v(\cdot + r)\tau(\cdot + r) \sum_{\nu \in 4\pi J} e_v(\nu).$$

The last sum is 0 because e_v is not constant on $4\pi J$. □

The choice $K = V$ in the last theorem shows that all the $(\mathbb{Z}^d/2)\backslash\mathbb{Z}^d$-translates of the function w, defined by

$$\widehat{w} := \widehat{\eta} \prod_{\alpha \in 4\pi V'} B(\cdot + \alpha),$$

are in W, provided that $\widehat{w} \in L_2$. It is then easy to prove that the V'-shifts of w provide a basis for the wavelet space. There is a close relation between the function w here and the generator w of W of Corollary 3.19, only that there we used $\widetilde{\widetilde{\phi}}^2 = [\![\widehat{\phi}, \widehat{\phi}]\!]$, while here we use the function $B = [\![\widehat{\eta}, \widehat{\phi}]\!]$. It follows, for example, that if ϕ and η are compactly supported and the refinement mask A is a polynomial, w here enjoys a smaller support than w of Corollary 3.19. However, unless the half-shifts of ϕ are stable, neither of these generators is expected to provide a *stable* basis for W.

The simplest instance of Theorem 7.11 occurs when $J := \{0, \alpha\}$ is a group of order 2. We obtain the following extension of Theorem 5.5 to the multivariate setting. Here, as before, B is defined as $[\![\widehat{\eta}, \widehat{\phi}]\!] = \overline{A}\widetilde{\widetilde{\eta}}^2$.

Corollary 7.12. *Assume that ϕ and η satisfy (7.1) and (7.2). If $v \in V'$ and $\alpha \in 4\pi V'$ satisfy $e_v(\alpha) = -1$, then the function w with Fourier transform*

$$\widehat{w} = e_v\, B(\cdot + \alpha)\widehat{\eta}$$

is in W provided it is in $L_2(\mathbb{R}^d)$. Moreover, if η and ϕ are of compact support, then w is also of compact support.

In some instances, it is possible to find an L_2-stable basis from among the functions of Corollary 7.12, as is shown in the following theorem of Riemenschneider and Shen [RS] (see also [JM], [RS1] and [CSW]):

Theorem 7.13. *Assume that ϕ and η satisfy (7.1) and (7.2) and that $B := [\![\widehat{\eta}, \widehat{\phi}]\!]$ is real-valued. Assume that ϕ has L_2-stable full-shifts and η has L_2-stable half-shifts. Assume further that there is a one-one mapping α from V' to $4\pi V'$ that satisfies the following two conditions:*
(a) $e_v(\alpha(v)) = -1$, for every $v \in V'$;
(b) $e_{v-u}(\alpha(v) - \alpha(u)) = -1$, for all $v, u \in V'$, unless $v = u$.
Then, the functions w_v, $v \in V'$, defined by their Fourier transforms

$$\widehat{w}_v := 2^d e_v\, B(\cdot + \alpha(v))\widehat{\eta}, \quad v \in V',$$

provide an L_2-stable basis for W. Furthermore, if the full-shifts of ϕ are orthonormal and the half-shifts of η are also orthonormal, then $\big(w_v\big)_{v\in V'}$ provides an orthonormal basis for W. If η and ϕ have compact support, then the functions w_v, $v \in V'$, are also of compact support.

Proof: It is easy to conclude from the stability assumption on η that each $\widehat{w}_v$, $v \in V'$, is in $L_2(\mathbb{R}^d)$, and hence each w_v is a well-defined L_2-function. From Corollary 7.12 and assumption (a), we conclude that each w_v is in W. This corollary also implies that w_v is compactly supported whenever η and ϕ are.

We introduce the functions w_v^*, $v \in V'$ with Fourier transform $\widehat{w}_v^* = \widehat{w}_v/\widetilde{\widetilde{\eta}}$. These functions are in $L_2(\mathbb{R}^d)$ because $\widetilde{\widetilde{\eta}}$ is bounded away from zero, thanks to the stability assumption on the half-shifts of η. We now compute the Gramian of these functions. First, we see that

$$2^{-2d}[\![\widehat{w}_v^*, \widehat{w}_u^*]\!] = e_{v-u}B(\cdot+\alpha(v))B(\cdot+\alpha(u))[\![\widehat{\eta},\widehat{\eta}]\!]/\widetilde{\widetilde{\eta}}^2 = e_{v-u}B(\cdot+\alpha(v))B(\cdot+\alpha(u)).$$

(Here, we wrote $B(\cdot+\alpha(u))$ is instead of $\overline{B}(\cdot+\alpha(u))$, since B is assumed to be real.) Therefore,

$$[\widehat{w}_v^*, \widehat{w}_u^*] = \sum_{\mu\in 4\pi V} [\![\widehat{w}_v^*, \widehat{w}_u^*]\!](\cdot+\mu) = 2^{2d}\sum_{\mu\in 4\pi V} e_{v-u}(\cdot+\mu)B(\cdot+\alpha(v)+\mu)B(\cdot+\alpha(u)+\mu). \tag{7.14}$$

For any $\mu \in 4\pi V$, the terms in (7.14) corresponding to μ and $\mu+\alpha(v)-\alpha(u)$ are negatives of one another because of our assumption (b), and the $(\alpha(v)-\alpha(u))$-periodicity of the term $B(\cdot+\alpha(v)+\mu)B(\cdot+\alpha(u)+\mu)$. Hence $[\widehat{w}_v^*, \widehat{w}_u^*] = 0$, $v \neq u$. On the other hand, for $v = u$,

$$[\widehat{w}_v^*, \widehat{w}_v^*] = 2^{2d}\sum_{\mu\in 4\pi V} B(\cdot+\alpha(v)+\mu)^2 = 2^{2d}\sum_{\mu\in 4\pi V} B(\cdot+\mu)^2 = 2^{2d}Q_0(B^2). \tag{7.15}$$

Since $B^2 = A^2\widetilde{\widetilde{\eta}}^4 = |A|^2\widetilde{\widetilde{\eta}}^4 = \widetilde{\widetilde{\phi}}^2\widetilde{\widetilde{\eta}}^2$ and $Q_0(\widetilde{\widetilde{\phi}}^2) = \widetilde{\phi}^2$, we have

$$Q_0(B^2)(x) \in [m \,\mathrm{d} M](\widetilde{\phi}(x))^2 \text{ a.e.},$$

with m and M the essential infimum, respectively, supremum of $\widetilde{\widetilde{\eta}}^2$. Since both m and M are positive and finite by the stability assumption on η, while $\widetilde{\phi}^2$ is essentially bounded away from 0 and infinity by the stability assumption on ϕ, we conclude that also $Q_0(B^2)$ is essentially bounded away from 0 and infinity. We thus conclude that the Gramian associated with $\big(w_v^*\big)_{v\in V'}$ is diagonal, with the diagonal entries bounded above and below by positive constants. On the other hand, $\widehat{\mathcal{W}} := \big(\widehat{w}_v\big)_{v\in V}$ is obtained from $\big(\widehat{w}_v^*\big)_{v\in V'}$ by multiplying by the scalar matrix $\mathrm{T} := \widetilde{\widetilde{\eta}}^2 I$. Again, the stability assumption on the half-shifts of η implies that $\widetilde{\widetilde{\eta}}$ and $1/\widetilde{\widetilde{\eta}}$ are bounded, hence that $\|\mathrm{T}\|$ as well as $\|\mathrm{T}^{-1}\|$ are bounded. Thus, from Theorem 2.26(iv), we conclude that the basis $\big(w_v\big)_{v\in V'}$ is stable.

Finally, when η has orthonormal half-shifts, $\widetilde{\widetilde{\eta}} = 2^{-d/2}$ a.e., and hence

$$2^{2d}Q_0(B^2) = 2^{2d}Q_0(\widetilde{\widetilde{\phi}}^2\widetilde{\widetilde{\eta}}^2) = 2^d Q_0(\widetilde{\widetilde{\phi}}^2) = 2^d\widetilde{\phi}^2.$$

If also ϕ has orthonormal full-shifts, $\widetilde{\phi} = 1$ and hence $2^{2d}Q_0(B^2) = 2^d$. Thus, (7.15) implies that $[\widehat{w}_v^*, \widehat{w}_v^*] = 2^d$, hence $[\widehat{w}_v, \widehat{w}_v] = \widetilde{\widetilde{\eta}}^2[\widehat{w}_v^*, \widehat{w}_v^*] = 2^{-d}2^d = 1$, and we conclude that $G(\mathcal{W}) = I$, or equivalently, that $\mathcal{W}$ is an orthonormal basis. □

We make the following additional remarks concerning Theorem 7.13. As Riemenschneider and Shen [RS] note, it is easy to construct mappings with properties (a) and (b) in the case $d = 1, 2, 3$. However, Riemenschneider and Shen also note that there are no such mappings when $d > 3$. On the other hand, there is some hope that turning to the more general elements of W given in Theorem 7.11, an analogue of Theorem 7.13 may be established in higher dimensions.

We have assumed in Theorem 7.13 that the function $B = [\![\widehat{\eta}, \widehat{\phi}]\!]$ is real. Since also $B = \overline{A}\widetilde{\widehat{\eta}}^2$, B is real if and only if the mask A is real. This is true for example if ϕ is real-valued and symmetric about the origin and $\eta = \phi(2\cdot)$. Moreover, the assumption that B (or A) is real can be somewhat weakened. For example, the proof given above supports the following claim:

Remark 7.16. *The construction detailed in Theorem 7.13 remains valid in case $B = e_j B'$ for some real B' and some $j \in \mathbb{Z}^d/2$.*

8. Box Splines

Box splines were introduced by the first two authors in [BD] and their exponential generalization (sometimes referred to as "exponential box splines") was introduced by the third author in [R1]. Box splines have become a main theme in Multivariate Spline Theory, and it is certainly beyond the scope of this section to provide a good account on box splines. We do not even attempt to provide an overview of box splines in the context of wavelet decompositions, because of the already rich literature on that matter. Thus, our only aim here is to illustrate the material detailed in previous sections via a discussion of this class of examples.

To define a box spline, we let Γ be a finite index set consisting of pairs of the form

$$\gamma = (x_\gamma, \lambda_\gamma), \quad x_\gamma \in \mathbb{R}^d \backslash 0, \ \lambda_\gamma \in \mathbb{C}.$$

The box spline $M := M_\lambda$ can then be defined via its Fourier transform as

$$\widehat{M}(y) = \prod_{\gamma \in \Gamma} \frac{e^{\lambda_\gamma - iy \cdot x_\gamma} - 1}{\lambda_\gamma - iy \cdot x_\gamma}. \tag{8.1}$$

The notation is indicative of the fact that we usually hold the **directions** $\left(x_\gamma\right)_{\gamma\in\Gamma}$ fixed, but may vary the parameters $\lambda := \left(\lambda_\gamma\right)_{\gamma\in\Gamma}$. Assuming that span $\left(x_\gamma\right)_{\gamma\in\Gamma} = \mathbb{R}^d$ (as we do throughout), the box spline is a compactly supported piecewise-exponential-polynomial function supported in the zonotope

$$Z_\Gamma := \{\sum_{\gamma\in\Gamma} t_\gamma x_\gamma : \ t_\gamma \in [0 \,\mathrm{d} 1]\}.$$

The polynomial case corresponds to the choice $\lambda = 0$. Exponential B-splines are obtained when $d = 1$ and $x_\gamma = 1$, all γ. Tensor splines are obtained whenever all the directions are standard unit vectors. The box spline is positive in the interior of Z_Γ whenever λ is real-valued.

We first observe that

$$\widehat{M}_\lambda(y) = A_{\lambda/2}(y/2)\widehat{M}_{\lambda/2}(y/2),$$

with

$$A_\lambda(y) := \prod_{\gamma\in\Gamma} \frac{e^{\lambda_\gamma - iy\cdot x_\gamma} + 1}{2}.$$

This suggests the choice

$$\widehat{\phi}_k := 2^{-kd}\widehat{M}_{\lambda/2^k}(\cdot/2^k),$$

since then $\widehat{\phi}_k = 2^d A_{\lambda/2^{k+1}}(\cdot/2^{k+1})\widehat{\phi}_{k+1}$. To ensure the fact that A_λ is a 2π-periodic polynomial, we assume that

$$x_\gamma \in \mathbb{Z}^d\backslash 0, \quad \text{all } \gamma \in \Gamma. \tag{8.2}$$

Assuming (8.2), we can define **the multiscale generated by the box spline** M_λ as

$$\mathcal{S}^k := \mathcal{S}^k(\phi_k), \quad \phi_k := M_{\lambda/2^k}(2^k\cdot). \tag{8.3}$$

As before, we use special notation for ϕ_0 and ϕ_1:

$$\phi := \phi_0, \quad \eta := \phi_1.$$

Note that this is a stationary multiscale if and only if $\lambda = 0$, i.e., if and only if M_λ is a polynomial box spline.

Since each ϕ_k is compactly supported, we know that property (1.3)(ii) is satisfied here. With regard to (1.3)(iii), we have the following extension of Theorem 6.3:

Theorem 8.4. *Let $\{\mathcal{S}^k\}$ be a multiscale of spaces generated by the box spline M_λ (as in (8.3)). Let $Y := \cap_{k\in\mathbb{Z}}\mathcal{S}^k$. Then Y is one-dimensional if and only if $\operatorname{Re}\lambda_\gamma \neq 0$, $\gamma \in \Gamma$. Otherwise, Y is trivial. In case Y is one-dimensional, it is spanned by the fundamental solution of the differential operator $\mathcal{D} := \prod_{\gamma\in\Gamma}(D_{x_\gamma} - \lambda_\gamma)$ (where D_x is the directional derivative in the x-direction) whose Fourier transform is given by*

$$\widehat{G}(y) = \prod_{\gamma\in\Gamma}(\lambda_\gamma - iy\cdot x_\gamma)^{-1}. \tag{8.5}$$

In this case, $\mathcal{S}^k = \mathcal{S}^k(G)$, for every k.

Proof: Let $\psi \in \cap_k\mathcal{S}^k$ be a non-zero function. Since $\psi \in \mathcal{S}^{-k}$, it is a linear combination of the 2^k-shifts of the box spline $\phi_k := M_{2^k\lambda}(\cdot/2^k)$. Since the ratio $\widehat{\phi}_k/\widehat{G}$ is (a trigonometric polynomial) of period $2^{k+1}\pi$, (2.1) implies that every function in $\mathcal{S}^k$ must have the form

$$\widehat{f} = \tau_k\widehat{G},$$

with τ_k $2^{k+1}\pi$-periodic. From the fact that $\operatorname{supp}\widehat{G} = \mathbb{R}^d$, we conclude that all τ_k agree a.e. with one measurable function τ, and this function is necessarily invariant under all 2π-dyadic shifts. Lemma 4.6 then implies that $\tau = \text{const}$, hence the Fourier transform of every function in the intersection is a scalar multiple of $\widehat{G}$. Therefore, this intersection is trivial if and only if it does not contain G, and otherwise it is spanned by G. Since the ratio $\widehat{\phi}_k/\widehat{G}$ is $2^{k+1}\pi$-periodic, then, again by (2.1), $G \in \mathcal{S}^k$ if and only if $G \in L_2(\mathbb{R}^d)$. Consequently, the proof of the theorem is reduced to the proof of the following claim:
"$G \in L_2(\mathbb{R}^d)$ if and only if $\operatorname{Re}\lambda_\gamma \neq 0$ for every $\gamma \in \Gamma$."

If $\operatorname{Re}\lambda_j \neq 0$ for every characteristic value λ_j, then we easily verify that, because X_Γ is of rank d, $\widehat{G}$ is in L_2, hence so is G. On the other hand, if, for some γ, $\operatorname{Re}\lambda_\gamma = 0$, then $\widehat{G}$ cannot lie in $L_2(\mathbb{R}^d)$, since it is not even in $L_2(\Omega)$ whenever the open set Ω contains points from the zero set of $y \mapsto \lambda_\gamma - iy\cdot x_\gamma$. □

Assuming (8.2), the shifts of M_λ are linearly independent only if Γ is **unimodular**, which means, by definition, that every $d \times d$ matrix whose rows are taken from the multiset $\bigl(x_\gamma\bigr)_{\gamma\in\Gamma}$ has determinant $-1, 0$ or 1. Further, if λ is real-valued, the unimodularity assumption is also *sufficient* for linear independence. For these reasons, we assume for the remainder of this section that Γ is unimodular and λ is real-valued.

We want now to consider the possible applications of the constructions proposed in the last section to box splines. It is hard to apply Theorem 7.8 directly, since it requires information on the function B, while the available information here is on the mask A. Nevertheless, if A has the form

$$A = \mathrm{const}_\lambda e_{-j} A',$$

where $j \in \mathbb{Z}^d/2$ and A' is non-negative, then we might choose $v \in V$ such that $j - v \in \mathbb{Z}^d$ to obtain

$$B_v := Q_v(B)/2^d = \mathrm{const} \sum_{\mu\in 4\pi V} e_v(\cdot+\mu)e_{-j}(\cdot+\mu)B'(\cdot+\mu) = e_{v-j}Q_0(B'),$$

where $B' := (1/\mathrm{const})e_j B$. Since $B' = A'\widetilde{\widetilde{\eta}}^2$, and A' is non-negative, so is B'. This, together with the stability assumption on the half-shifts of η and the shifts of ϕ, implies that $Q_0(B')$ does not vanish, hence $1/B_v$ is bounded, and we arrive at the following conclusion:

Corollary 8.6. *Let M be a box spline defined by a unimodular Γ with real parameter vector λ. Assume that Γ satisfies also the following "parity" condition: "Γ can be partitioned into pairs such that each pair (γ, γ') satisfies*

$$(x_\gamma, \lambda_\gamma) = \varepsilon(\gamma,\gamma')(x_{\gamma'}, -\lambda_{\gamma'}),$$

where $\varepsilon(\gamma,\gamma') \in \{\pm1\}$." Let $B := [\![\widehat{\phi}_1, \widehat{\phi}_0]\!]$, with ϕ_k defined as in (8.3). Then B_{v_0} vanishes nowhere on $\mathbb{T}^d$, where $v_0 \in V$ is determined by the condition

$$v_0 = \sum_{\gamma\in\Gamma} x_\gamma/4, \quad \bmod \mathbb{Z}^d.$$

Consequently, the construction detailed in Theorem 7.8 can be applied with respect to this v_0.

Proof: Since B here is a polynomial, it is clear that the functions $\widehat{w}_v$ defined in Theorem 7.8 are in L_2. Also, for the same reason, B is bounded. Thus, to apply Theorem 7.8, we, indeed, need only to prove the boundedness of $1/B_{v_0}$. In view of the remarks preceding this corollary, it suffices to show that the mask A in the equation $\widehat{\phi}_0 = 2^d A\widehat{\phi}_1$ is of the form

$$A = e_{-j}A',$$

with A' non-negative and $j - v_0 \in \mathbb{Z}^d$. Here,

$$A(y) = \prod_{\gamma\in\Gamma} \frac{e^{\lambda_\gamma/2 - ix_\gamma\cdot y/2}+1}{2}.$$

Let (γ,γ') be a pair in the partitioning of Γ. Then

$$\frac{e^{\lambda_\gamma/2 - ix_\gamma\cdot y/2}+1}{2}\;\frac{e^{\lambda_{\gamma'}/2 - ix_{\gamma'}\cdot y/2}+1}{2} =$$
$$\Bigl(\frac{1-\varepsilon(\gamma,\gamma')}{4}e^{\lambda_\gamma/2} + \frac{1+\varepsilon(\gamma,\gamma')}{4}e^{-ix_\gamma\cdot y/2}\Bigr)(\cosh(\lambda_\gamma/2)+\cos(x_\gamma\cdot y/2)).$$

The second factor above is non-negative. Multiplying the first factor over all pairs (γ,γ'), we obtain an expression of the form

$$\mathrm{const}_\lambda e_{-j},$$

where $j := \sum_{\varepsilon(\gamma,\gamma')=1} x_\gamma/2 = 1/4\sum_{\gamma\in\Gamma} x_\gamma \in \mathbb{Z}^d$, since each direction appears either as an x_γ or an $x_{\gamma'}$ and $x_{\gamma'} = x_\gamma$ if $\varepsilon(\gamma,\gamma') = 1$ while $x_{\gamma'} = -x_\gamma$ otherwise. ◻

It should be clear that, under the assumptions of the last corollary, Theorem 7.8 can be applied to obtain stable bases for all the wavelet spaces of the multiscale generated by the box spline M. Also, the assumption that λ is real is convenient but not essential. In general, to obtain a box spline that satisfies the above assumptions, one can start with any M that is defined by a unimodular Γ, and replace M by $M * \overline{M}(-\cdot)$. The box spline obtained in this way corresponds to the choice $v_0 = 0$ in the above corollary. The other variants can be obtained the shifting that box spline by $j \in V$.

If $d \leq 3$, we can also try to employ the construction detailed in Theorem 7.13. Here, given a unimodular Γ, we want the mask A to be of the form

$$A = e_j A'$$

for some real A'. In the polynomial case (i.e., when $\lambda = 0$), this assumption is always satisfied since then for $A := A_0$ we have

$$A(y) = e_j(y) \prod_{\gamma \in \Gamma} \cos(y \cdot x_\gamma),$$

with $j := -1/2 \sum_{\gamma\in\Gamma} x_\gamma$. This observation immediately extends to the case when $\lambda \in i\mathbb{R}^d$, but, however, does not extend to an arbitrary λ. On the other hand, if M is a box spline as in Corollary 8.6, and M' is a polynomial box spline (with a unimodular set of direction), then $M * M'$ satisfies the requirements of Theorem 7.13.

We mentioned previously that for non-stationary subdivisions one should not exclude the possibility that the half-shifts of ϕ are stable. Box splines provide an excellent illustration of this point. In order to check the stability of the half-shifts of ϕ, we consider, as before, the function $\overset{\approx}{\phi} = |A_{\lambda/2}(\cdot/2)|\overset{\approx}{\eta}$. By our assumptions, the half-shifts of η are linearly independent, hence stable, which means that $\overset{\approx}{\eta}$ is positive on $\mathbb{R}^d$. Therefore, the zeros of $\overset{\approx}{\phi}$ are identical with those of the mask

$$A_{\lambda/2}(y/2) = \prod_{\gamma\in\Gamma} \frac{(e^{\lambda_\gamma/2 - iy/2\cdot x_\gamma} + 1)}{2}.$$

We observe that the factor $e^{\lambda_\gamma/2 - iy/2\cdot x_\gamma} + 1$ has zeros in $\mathbb{R}^d$ if and only if $\lambda_\gamma = 0$ (recall that we are already assuming that λ_γ is real). Thus we obtain the following interesting result:

Corollary 8.7. *Let M_λ be the box spline given by a unimodular Γ and a real λ. Then $\overset{\approx}{M}_\lambda$ vanishes nowhere if and only if λ contains no zero entry. Consequently, this last condition is equivalent to the stability of the half-shifts of M_λ.*

Stronger results can be obtained by a finer analysis. It can be shown that, assuming only (8.2) (which is imbedded in the last corollary in the unimodularity assumption on Γ), the stability of the half-shifts of M_λ is equivalent to the existence of non-trivial functions in the intersection $\cap_k \mathcal{S}^k$, with $\left(\mathcal{S}^k\right)_k$ the multiscale generated by the box spline M_λ.

The stability of the half-shifts of M_λ leads to painless constructions of compactly supported stable bases for the wavelet space. Here is a sample result in this direction:

Proposition 8.8. *Assume that ϕ and η satisfy (7.1) and (7.2), and assume that the half-shifts of ϕ are stable. Let w be either the generator for the wavelet space introduced in Corollary 3.19 or in the paragraph after Theorem 7.11. Then w provides (i.e., the $(\mathbb{Z}^d/2)\backslash\mathbb{Z}^d$-translates of w form) a stable basis for W.*

The last result is less impressive than it might look at first. Indeed, considering the box spline multiscale and assuming, say, that λ is real and contains non-zero entries, we can easily find single compactly supported stable generators to each of the wavelet spaces associated with the multiscale $\left(\phi_k\right)_k$ generated by M. Still, as mentioned already in section 6, it is crucially important to understand the behavior of the stability constants as k varies, and in the case of box splines these constants deteriorate fast as k increases. This can be observed as follows: if we rescale each ϕ_k and ϕ_{k+1} by 2^{k+1}, and denote the functions obtained by η_k and η_{k+1} respectively, we obtain a refinement equation of the form

$$\widehat{\eta}_k = 2^d A_{\lambda/2^{k+1}} \widehat{\eta}_{k+1}.$$

Thus complex zeros of the kth order mask converge (exponentially) to the real domain, as k increases. One might attempt to choose very large initial entries for λ, yet the results of [DR] indicate that the asymptotic approximation properties of $\mathcal{S}^k(\phi_k)$ deteriorate exponentially with the growth of λ.

The above discussion demonstrates the difficulty of controlling the stability constants in case the wavelet constructions are based on the stability of the half-shifts of M_λ. On the other hand, the constructions that make use of the refinement equation (such as the one detailed in Corollary 8.6) require only the stability of the half-shifts of $M_{\lambda/2}(2\cdot)$. Using methods similar to those employed in §6, it can be shown that for such constructs the stability constants do not blow up as $k \to \infty$.

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