Zonotopal algebra

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Abstract

A wealth of geometric and combinatorial properties of a given linear endomorphism X of $\mathbb{R}^N$ is captured in the study of its associated zonotope $Z(X)$, and, by duality, its associated hyperplane arrangement $\mathcal{H}(X)$. This well-known line of study is particularly interesting in case $n := \operatorname{rank} X \ll N$. We enhance this study to an algebraic level, and associate X with three algebraic structures, referred herein as *external, central, and internal.* Each algebraic structure is given in terms of a pair of homogeneous polynomial ideals in n variables that are dual to each other: one encodes properties of the arrangement $\mathcal{H}(X)$, while the other encodes by duality properties of the zonotope $Z(X)$. The algebraic structures are defined purely in terms of the combinatorial structure of X, but are subsequently proved to be equally obtainable by applying suitable algebro-analytic operations to either of $Z(X)$ or $\mathcal{H}(X)$. The theory is universal in the sense that it requires no assumptions on the map X (the only exception being that the algebro-analytic operations on $Z(X)$ yield sought-for results only in case X is unimodular), and provides new tools that can be used in enumerative combinatorics, graph theory, representation theory, polytope geometry, and approximation theory.

1 Introduction

1.1 General

We are interested in combinatorial, geometric, algebraic and analytic properties of low rank linear endomorphisms X of $\mathbb{R}^N$. This setup is relevant in quite a few areas in mathematics from linear algebra to algebraic graph theory to semi-simple group representations to approximation theory (box splines), and underlies interesting connections among rather different mathematical problems.

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Consider X as a map from $\mathbb{R}^N$ to $\mathbb{R}^n$, and identify it with the columns of its matrix representation. Important geometric information about X is captured by the image

$$Z(X) := \{\sum_{x \in X} t_x x : \quad t \in [0,1]^X\}$$

of the unit cube $[0,1]^X$ under the action of X. The resulting polytope is known as a *zonotope*. Zonotopes exhibit special symmetries that general polytopes lack. Underlying those special features is the fact that their normal cone fan is linear, i.e., is a (central) hyperplane arrangement. The duality between zonotopes and hyperplane arrangements is rich, and includes intriguing connections between the different tilings of the zonotope into sub-zonotopes on the one hand, and the geometries obtained by translating the hyperplanes in the hyperplane arrangement on the other hand (see [35], [34], [43], [42], [49, Chapter 7], [8, Chapter 2], [39], [46]). While we briefly touch in Section 2 on these known connections, the focus of this paper is neither on the linear algebra surrounding the map X, nor on the geometry and combinatorics of the zonotope $Z(X)$ *per se.*

The theory of *zonotopal algebra* that is developed in the current article is algebraic. At its core one finds three pairs of zero-dimensional homogeneous polynomial ideals in n variables: an external pair $(\mathcal{I}_+(X), \mathcal{J}_+(X))$, a central pair $(\mathcal{I}(X), \mathcal{J}(X))$, and an internal pair $(\mathcal{I}_-(X), \mathcal{J}_-(X))$. The ideals within each pair are dual to each other; in particular, their Hilbert series are identical. To keep this introduction brief, we do not describe in depth the actual ingredients of the theory that is developed here. Instead, we present a number of results that capture the flavor of the general theory and its many potential applications.

The definition of the $\mathcal{I}$-ideals goes as follows. First, given $y \in \mathbb{R}^n$, let p_y be the linear form

$$p_y : \mathbb{R}^n \to \mathbb{R} \ : t \mapsto y \cdot t := \sum_{i=1}^{n} y(i)t(i).$$

Further, let

$$\mathcal{F}(X)$$

be the set of *facet hyperplanes* of X, viz., $H \in \mathcal{F}(X)$ if and only if H is a subspace of $\mathbb{R}^n$ of dimension $n-1$, and $\mathrm{span}(X \cap H) = H$. Finally, for any facet hyperplane, let η_H be the normal to H, and let $m(H)$ be the cardinality of the vectors in $X \backslash H$:

$$m(H) := m_X(H) := \#(X \backslash H).$$

The three $\mathcal{I}$-ideals are generated each by the polynomials

$$p_{\eta_H}^{m(H)+\epsilon}, \quad H \in \mathcal{F}(X).$$

The external ideal $\mathcal{I}_+(X)$ corresponds to the choice $\epsilon = 1$, the central ideal $\mathcal{I}(X)$ corresponds to the choice $\epsilon = 0$, while the internal ideal $\mathcal{I}_-(X)$ corresponds to the choice $\epsilon = -1$.

The Hilbert series of these three ideals are closely related to the external activity variable of the Tutte polynomial that is associated with X. We explain (and prove) this connection later. A more rudimentary result is as follows (see Section 2.1 for the definition of unimodularity). We denote by

$$\Pi := \mathbb{C}[t_1, \ldots, t_n]$$

the space of polynomials in n variables, and by

$$\Pi_k \quad (\Pi_k^0, \ \text{respectively})$$

the subspace of Π that contains all polynomials of degree $\le k$ (all homogeneous polynomials of exact degree k, respectively). Also, for any homogeneous ideal $I \subset \Pi$, we denote

$$\ker I := \{p \in \Pi : \ q(D)p = 0, \ \forall q \in I\} = \{p \in \Pi : \ q(D)p(0) = 0, \ \forall q \in I\}.$$

Our first result provides a very basic combinatorial connection between the $\mathcal{I}$-ideals on the one hand and the zonotope $Z(X)$, the integer points in it, as well as the integer points in its interior $\mathrm{int}(Z(X))$.

Proposition 1.1 *Let $X \subset \mathbb{R}^n$ be unimodular, and let $Z(X)$ be the associated zonotope. Then:*

(1) $\dim \ker \mathcal{I}_+(X) = \#(Z(X) \cap \mathbb{Z}^n)$.

(2) $\dim \ker \mathcal{I}(X) \ \ = \mathrm{vol}(Z(X))$.

(3) $\dim \ker \mathcal{I}_-(X) = \#(\mathrm{int}(Z(X)) \cap \mathbb{Z}^n)$.

Another related result is that the number of (unbounded) n-dimensional regions in $\mathcal{H}(X)$ equals $\dim \ker \mathcal{I}_+(X) - \dim \ker \mathcal{I}_-(X)$; this result holds for a general X. As a matter of fact, far deeper connections between the zonotope $Z(X)$ and the $\mathcal{I}(X)$-ideals are demonstrated in this paper: the $\mathcal{I}$-ideals can be derived, each, by applying suitable algebro-analytic operations to a suitably chosen subset of $Z(X) \cap \mathbb{Z}^n$.

An important highlight of the $\mathcal{I}$-ideals is that their associated kernels can be described cleanly and explicitly in terms of the columns of X.[1] Our second illustration moves in this direction and considers, for a given X, the possible use of the polynomials

$$p_Y := \prod_{y \in Y} p_y, \quad Y \subset X,$$

for the representation of the ring Π. For example, denote

$$\mathcal{P}_+(X) := \mathrm{span}\{p_Y : Y \in 2^X\}.$$

Proposition 1.2 *With $\mathcal{P}_+(X)$ as above,* $\Pi = \mathcal{P}_+(X) \oplus \mathcal{I}_+(X)$.

The above result follows directly from the fact that $\mathcal{P}_+(X)$ equals $\ker \mathcal{I}_+(X)$. Even more interesting decompositions are obtained when using the $\mathcal{J}$-ideals, since these ideals are generated by polynomials of the form p_Y, with Y a (multi)subset of X (or of a slightly augmented version of it). For example, one way to express the duality between $\mathcal{I}(X)$ and $\mathcal{J}(X)$ is via the direct sum decomposition (cf. §3)

$$\Pi = \mathcal{J}(X) \oplus \ker \mathcal{I}(X).$$

This decomposition corresponds to a decomposition of the power set 2^X: it will be shown that $\ker \mathcal{I}(X)$ is spanned by p_Y, $Y \in S(X)$, with $S(X)$ a suitable subset of 2^X. The ideal $\mathcal{J}(X)$ is generated by the remaining polynomials p_Y, $Y \in 2^X \backslash S(X)$.

[1]Unfortunately, there are no known simple representations for the kernels of the $\mathcal{J}$-ideals.

Special types of zonotopal algebras are intimately connected to group representations. The connection is particularly simple in the case of SL_{n+1}-representations, since in this case the underlying X is unimodular.[2] Fixing n, we let X^k, $k \geq 1$ be a k-fold multiset of the edge set[3] of a complete graph with $n+1$ vertices (see Example 2.1). A basic result, which applies to all finite-dimensional SL_{n+1} representations, is that the character (or more precisely the Fourier coefficients of the character) of the representation is piecewise polynomial (see, e.g., [47]), with the polynomial pieces all lying in the kernel of the ideal $\mathcal{J}(X^1)$. Here is a rather different result.

Example 1.3 *Fix $n \geq 2$ and a positive integer k, and let Γ_k be the irreducible SL_{n+1} representation of highest weight $(k, k, \ldots, k)$. Then there exists a unique polynomial $p \in \mathcal{P}_+(X^k)$ whose values on the spectrum of Γ_k determine the character of Γ_k: at each eigenvalue α, $p(\alpha)$ equals the multiplicity of the eigenvalue in Γ_k.*

This result follows directly from the theory of this paper, thanks to the fact that the convex hull of the spectrum of the above Γ_k is the zonotope $Z(X^k)$. However, the connection between zonotopal algebras and group representations extends beyond examples of this type, as the next example makes clear. In that result, X^k retains its meaning from the previous one. Note that, in general, the convex hull of the spectrum of an SL_3-representation is not a zonotope.

Example 1.4 *Let Γ be an irreducible SL_3 representation of highest weight $(k+j, k-j)$, for some integers $0 \leq j \leq k$. Then, with σ the spectrum of Γ, and for every $c \in \mathbb{C}^\sigma$ there exists a unique (bivariate) polynomial $p \in \mathcal{P}_+(X^k)$, such that:*

(i) $\deg p \leq 3k - j$, *and*

(ii) $p_{|\sigma} = c$.

Let us now illustrate connections with algebraic graph theory and with the notion of parking functions from combinatorics (see [37], [48], [32]). For simplicity, we present the connection for the edge set X of a complete graph X with $n+1$ vertices; similar results are valid for general graphical X. Note that here and elsewhere

$$\mathbb{Z}_+$$

stands for the non-negative integers (including 0).

Definition 1.5 *Set $V := \{1, \ldots, n\}$. For $r \in \mathbb{Z}_+^V$, and $v \in V$, set $V_{r,v} := \{v' \in V : \ r(v') \geq r(v)\}$. Then r is called:*

(i) An **external parking function** *if, for each $v \in V$, one of the following two conditions holds:*

$$\begin{array}{ll} \textit{Either} & \#V_{r,v} < n - r(v) + 1, \\ \textit{or} & \#V_{r,v} = n - r(v) + 1, \ \ \textit{and} \ \ r(v') = r(v) \ \textit{for} \ v' = \min V_{r,v}. \end{array}$$

(ii) An **internal parking function** *if, for each $v \in V$, one of the following two conditions holds:*

[2]Group representations are connected with a discrete version of zonotopal algebras, that are not discussed in this paper. In the unimodular case, however, the discrete version coincides with the continuous version, which is the version studied here.

[3]One needs also to choose correctly the basis for $\mathbb{R}^n$ in the definition of the edge set.

Either $\#V_{r,v} < n - r(v)$,

or $\#V_{r,v} = n - r(v)$, *and* $r(v') = r(v)$, *for some* $v' \neq \max V_{r,v}$.

Parking functions define a monomial set in Π whose monomial complement spans a monomial ideal. This monomial ideal "monomizes" a corresponding zonotopal ideal, and the above holds for every graphical X; this point was already made explicit in [37] (for the central zonotopal case). Here is a pertinent statement concerning the external case. We use here $R_+(X)$ to denote the set of external parking functions of X.

Example 1.6 *Let* X *be the edge set of a complete graph with* $n+1$ *vertices. Then there exists an injection* $T : R_+(X) \to 2^X$ *such that*

- *The polynomials* $\{p_{Tr} : r \in R_+(X)\}$ *form a basis for* $\mathcal{P}_+(X)$.
- *For each* $r \in R_+(X)$, *the monomial* t^r *appears (with non-zero coefficient) in the monomial expansion of* p_{Tr}.

In particular, $\deg p_{Tr} = \sum_{v \in V} r(v)$, *for every* $r \in R_+(X)$.

Since parking functions are well known to be connected with other combinatorial aspects of graphs, such as the number of inversions in its spanning trees, [45], results as the above draw connections with graph theory beyond parking functions *per se.* We study connections of this type in [30].

We now move in a completely different direction, and point out connections between zonotopal spaces and special types of multivariate polynomial interpolation problems. Connections of this type are at the core of zonotopal algebras, were fully developed before for the central case, and are well explained in the body of this paper. Here is one illustration (cf. Section 4).

Proposition 1.7 *Denote* $\mathcal{Z}_+(X) := Z(X) \cap \mathbb{Z}^n$. *Then the restriction map*

$$f \mapsto f_{|\mathcal{Z}_+(X)}$$

is a bijection between $\mathcal{P}_+(X)$ *and* $\mathbb{C}^{\mathcal{Z}_+(X)}$, *provided that* X *is unimodular.*

Our final example is about connections of the theory developed here with approximation theory. We recall that a (polynomial) *box spline* M_X (with $X \subset \mathbb{R}^n$ the given multiset) is a smooth piecewise polynomial function in n variables supported on the zonotope $Z(X)$. It can be defined as the convolution product of the measures M_x, $x \in X$, with the mass of each M_x uniformly distributed on the line connecting 0 to x. One of the early key problems in box spline theory was to understand the properties of the polynomial space

$$\mathcal{D}(X)$$

(which is defined and reviewed in Section 3 here, and which is known to be) spanned by the polynomials in the local structure of M_X. The "mere" attempt to understand the *dimension* of that space spawned an industry of techniques for estimating the dimension of joint kernels of differential and other operators (see [41] and references therein). We present below a potential box spline application of our results that is of a different flavor.

Conjecture 1.8 *Let X be unimodular, and let*

$$\mathcal{Z}_-(X)$$

be the set of integer points in the interior of $Z(X)$. Let f be any function defined on $\mathcal{Z}_-(X)$. Then, there exists a unique polynomial $p \in \ker \mathcal{I}_-(X)$ such that $p(D)M_X$ equals f on $\mathcal{Z}_-(X)$.

This conjecture follows (albeit in a somewhat non-trivial way) from Conjecture 6.1[4], but may be true even if the latter is disproved.

1.2 Historical context

Special zonotopal algebras (viz. spaces of the type $\mathcal{D}(X)$ for special maps X) appear implicitly in Weyl's character formulæ, and the connection is valid for representations of all semi-simple Lie algebras, [47]. Zonotopal spaces associated with general maps X (viz. the spaces $\mathcal{D}(X)$) made their debut in [15]. The dimension formula for $\mathcal{D}(X)$ was established in [11] (continuous version) and in [12] (discrete version). This result was extended to non-matroidal structures by multiple authors and in multiple ways (see [41]). Our approach here, in Section 3, bypasses these developments, but uses in an essential way methods for bounding dimensions of such spaces *from below* [6, 19]. The dual space $\mathcal{P}(X)$ was introduced independently in [1] and in [24], with the latter containing the details concerning the construction of the homogeneous basis for $\mathcal{P}(X)$ (Section 3.2). The identification of $\mathcal{I}(X)$ as the annihilating ideal of $\mathcal{P}(X)$ is found in [17]. A chapter in [16] is devoted to the study of these and other related algebraic aspects of box spline theory. Newer treatments of the central case are presented in [22] and the book [23], where several aspects of the central algebra are re-explored and extended, including its relations with modules over the Weyl algebra and D-modules, as well as with toric arrangements and their cohomology. A nice connection between the space $\mathcal{D}(X)$ and the cohomology of toric hyperKähler varieties is described in [28] via the so-called Volume Polynomials shown to span $\mathcal{D}(X)$ as a D-module (for subsequent developments, see [27], [29]).

Our interest in extending the theory of zonotopal spaces beyond the central pair was stimulated by discussions we had in the mid 1990's with Nira Dyn and Uli Reif, concerning the possibility of a result *á la* Conjecture 1.8, and was enhanced by discussions we had a few years later with Frank Sottile, who pointed out to us connections between our external theory and the work of [38]. Our delay in publishing this theory was primarily caused by inherent difficulties we encountered in the internal study due to the absence of a "canonical" basis for $\ker \mathcal{I}_-(X)$. We believe that the theory as presented here alleviates ramifications of this hurdle to the extent possible.

As we alluded to above, the novelty of this paper lies exclusively in the theory of the internal and external algebra, whose foundations we develop here, as well as in pointing out various connections of this theory with other fields – most notably, enumerative combinatorics and representation theory (see Sections 4 and 5, also Section 1.1). It should be mentioned that our second task is by no means completed in this paper, due to the multitude and richness of those connections. A description of combinatorial connections alone is a subject of our forthcoming paper [30], currently in preparation.

We hope that this paper will offer a new perspective and new tools to researchers working in algebra, analysis and combinatorics, along with a glimpse into exciting developments yet to come.

[4] We note that Conjecture 6.1 implies that point evaluation is well defined on $p(D)M_X$.

1.3 Layout of this article

The paper is organized as follows. Section 2 contains background results that will be used in the rest of the paper. This section is subdivided into five subsections: Section 2.1 is devoted to linear algebra and matroid theory, Section 2.2 to hyperplane arrangements, Section 2.3 to zonotopes, Section 2.4 to polynomial interpolation and Section 2.5 to polynomial ideals and their kernels.

The bulk of the paper is in Sections 3, 4 and 5. Those three sections are made parallel to each other, with two subsections in each, the first containing main theory and the second discussing grading, the Hilbert series, and homogeneous bases for the polynomial spaces in question. While the material in Section 3 is known, we feel it is crucial to present it in this way here, for the rest of the paper to be much more easily understandable, as well as for the streamlined approach itself. The paper ends with Section 6 containing a few additional remarks and conjectures.

2 Preliminary results

2.1 Linear algebra

Consider a finite multiset $X \subset \mathbb{R}^n \backslash \{0\}$ of full rank n and of size $\#X$. At times, we will associate X with some full ordering. In this case, we may consider the vectors in X to comprise the columns of an n-by-$\#X$ matrix, which we will still denote by X. Such a multiset (or a matrix) X gives rise to a *linear matroid* (see, e.g., [36]) via the standard convention that the independent sets of the matroid are exactly the linearly independent subsets of the columns of X.

We now single out three sub-collections of the power set 2^X that will play a crucial role in this paper. The reader may notice right away that all three are defined in purely matroidal terms. The first of the three is the set $\mathbb{B}(X)$ of all bases of X:

$$\mathbb{B} := \mathbb{B}(X) := \{B \subset X \ : \ B \text{ is a basis for } \mathbb{R}^n\}.$$

The second is the collection $\mathbb{I}(X)$ of all independent subsets of X:

$$\mathbb{I} := \mathbb{I}(X) := \{I \subset X, \ I \text{ is independent in } \mathbb{R}^n\}.$$

Note that the empty set is independent. The third is the set of internal bases, and is defined in the sequel.

Clearly, $\mathbb{B}(X) \subset \mathbb{I}(X)$, and the inclusion is proper. Nonetheless, it is convenient to consider the independent sets as full-rank bases, too. To this end, we choose a fixed basis B_0 of $\mathbb{R}^n$, and append B_0 to X:

$$X' := X \cup B_0.$$

We then impose some arbitrary, but fixed, ordering $\prec$ on B_0, and associate each $I \in \mathbb{I}(X)$ with $\mathrm{ex}(I) \in \mathbb{B}(X')$ which is the greedy completion of I to a basis, using the elements of B_0, i.e., $b \in \mathrm{ex}(I)$ if and only if $b \in I$ or else $b \in B_0$ and

$$b \notin \mathrm{span}\{I \cup \{b' \in B_0 : b' \prec b\}\}.$$

That creates a 1-1 map from $\mathbb{I}(X)$ into $\mathbb{B}(X')$. The range of this extension map is denoted by $\mathbb{B}_+(X)$:

$$\mathbb{B}_+(X) := \{\mathrm{ex}(I) \in \mathbb{B}(X') \ : \ I \in \mathbb{I}(X)\}.$$

We refer to the bases in $\mathbb{B}_+(X)$ as the *external bases* of X. Note that every basis of X is external directly from the definition, but not every external basis of X is a basis of X.

Next, we define the notion of an internal basis. To this end, we assume to be given an order $\prec$ on X. A vector $b \in B$ in a basis $B \in \mathbb{B}(X)$ is said to be *internally active* in B if b is the last element in $X \backslash H$, where $H := \operatorname{span}(B \backslash b)$:

$$b \succ x, \ \forall x \in X \backslash (H \cup b). \tag{1}$$

A basis B that contains *no* internally active vectors is said to be an *internal basis*. We denote

$$\mathbb{B}_-(X) := \{B \in \mathbb{B}(X) \ : \ B \text{ is an internal basis}\}.$$

It is obvious that the notion of an internal basis depends on the ordering. In fact, assuming that the last n vectors of X form a basis B_1, only the ordering within B_1 counts here, since, whatever $B \in \mathbb{B}(X)$ we choose, only the vectors in $B \cap B_1$ can be internally active in B. Thus

$$\mathbb{B}(X \backslash B_1) \subset \mathbb{B}_-(X) \subset \mathbb{B}(X) \subset \mathbb{B}_+(X) \subset \mathbb{B}(X \cup B_0).$$

We will see later that the number of internal bases is independent of the ordering of X.

We say that X is *unimodular* if $X \subset \mathbb{Z}^n$ and

$$\forall B \in \mathbb{B}(X), \ \operatorname{span}_{\mathbb{Z}} B = \mathbb{Z}^n \ (\iff \det(B) = \pm 1).$$

Example 2.1 [the edge set of a graph]. Let G be a connected undirected graph with $n+1$ vertices $V = \{v_i\}_{i=0}^n$. Let $e_0 := 0$. Let $(e_i)_{i=1}^n$ be a basis for $\mathbb{R}^n$. Identify an edge e_{ij} that connects the vertices v_i and v_j ($i < j$) with the vector $e_i - e_j \in \mathbb{R}^n$. With this identification, one chooses X to be the edge set of G. Note that the edge (multi)set X of a graph is always unimodular (assuming, say, that (e_i) is the standard basis. Otherwise, "unimodularity" here is with respect to the lattice spanned by the basis.) The corresponding matroid is called *graphical.* A particular interesting example is when G is chosen to be a *complete graph,* i.e., a graph in which every pair of vertices is connected by exactly one edge.

Remark. Although it is not obvious, there is a certain level of symmetry in the definition of external bases and internal ones. To demonstrate this point, let us assume that X is graphical, and let $B_1 := (e_i)_{i=1}^n$. Assuming that $B_1 \in \mathbb{B}(X)$ (which means that there is an edge between v_0 and each of the other vertices), we place B_1 last in X, and order its vectors according to the enumeration of the vertices ($e_i \prec e_j$ iff $i < j$). Using this order to define $\mathbb{B}_-(X)$, one finds that $B \in \mathbb{B}(X)$ is internal if $B \in \mathbb{B}(X \backslash B_1)$. Otherwise, $B \backslash B_1$ defines a partition $(V_0, \ldots, V_k)$, $v_0 \in V_0$, on the vertex set V, with the k vectors in $B \cap B_1$ connecting v_0 to each of $V_1, \ldots, V_k$. The basis B is then internal if and only if, for $i = 1, \ldots, k$, the edge in $B \cap B_1$ that connects v_0 and V_i is *not* connected to the *maximal* vertex of V_i.

Now, let us append another copy of B_1 to X. We call this copy B_0, and denote $X' := X \cup B_0$. We retain the order on B_0 as above and use this external copy B_0 to define $\mathbb{B}_+(X)$. We then need to determine what the greedy extension $\operatorname{ex}(I)$ of $I \in \mathbb{I}(X)$ should be. Again, each such I determines a partition $(V_0, \ldots, V_k)$ as before. The greedy extension is performed by connecting, for $i = 1, \ldots, k$, the vertex v_0 to the *minimal* vertex in V_i. □

2.2 Hyperplane arrangements

Recall from the introduction the definition

$$\Pi := \mathbb{C}[t_1, \cdots, t_n],$$

as well as the notations Π_k and Π_k^0. We first associate each direction $x \in X$ with a constant $\lambda_x \in \mathbb{R}$, and define an affine polynomial $p_{x,\lambda} \in \Pi$:

$$p_{x,\lambda} : \mathbb{R}^n \to \mathbb{R} : t \mapsto x \cdot t - \lambda_x.$$

The *X-hyperplane arrangement* $\mathcal{H}(X, \lambda)$ is determined by the zero sets of the above polynomials, viz., by

$$H_{x,\lambda} := \{t \in \mathbb{R}^n \ : \ p_{x,\lambda}(t) = 0\}, \quad x \in X.$$

We will usually assume that λ is chosen *generically,* i.e., so that the intersection of any collection of $n+1$ hyperplanes is empty. Note that different choices of λ may result in hyperplane arrangements with different geometries. Of particular interest are the following three geometric characteristics of the hyperplane arrangement:

1. $V(X, \lambda)$: the set of vertices

2. $CC(X, \lambda)$: the set of n-dimensional connected components

3. $BCC(X, \lambda)$: the set of n-dimensional bounded connected components

As a reader of this article should observe, the set $V(X, \lambda)$ is analyzed in Section 3; however, the set $CC(X, \lambda)$ appears nowhere in this paper past the current location. The reason is mainly technical: the tools that we introduce and employ allow us to study zero-dimensional sets. We bypass this limitation by associating $CC(X, \lambda)$ and $BCC(X, \lambda)$ with suitable supersets and/or subsets of $V(X, \lambda)$, and utilize to this end the notions of external and internal bases. It is thus worth mentioning the following known facts.

Result 2.2 *For any generic hyperplane arrangement determined by a multiset X,*

$$\begin{aligned} \#V(X, \lambda) &= \#\mathbb{B}(X), \\ \#CC(X, \lambda) &= \#\mathbb{B}_+(X) \ (= \#\mathbb{I}(X)), \\ \#BCC(X, \lambda) &= \#\mathbb{B}_-(X). \end{aligned}$$

The result shows in addition that the number of objects of each type is a geometric invariant of generic arrangements. In this connection, it is worthwhile to point out the relevance of the (univariate) *Ehrhart polynomial*[5]:

$$E_X(t) := \sum_{I \in \mathbb{I}(X)} t^{\#I}.$$

It is known that (see [5]) $E_X(1) = \#\mathbb{I}(X) = \#\mathbb{B}_+(X)$, and that $E_X(-1) = (-1)^n \#\mathbb{B}_-(X)$. The first equality is a triviality; the second one can be proved by induction on n. It shows that $\#\mathbb{B}_-(X)$, indeed, is independent of the order on X.

[5]Strictly speaking, this is the true Ehrhart polynomial of X only when X is unimodular (see, e.g., [44]).

2.3 Zonotopes

Let us now consider X as a map:

$$X : \mathbb{R}^X \to \mathbb{R}^n : t \mapsto \sum_{x \in X} t_x x.$$

Then the *zonotope of* X is defined as the image of the unit cube under this map

$$Z(X) := X([0,1]^X).$$

Assuming X to be unimodular, we have the following formulæ for the volume of $Z(X)$, the number of integer points in $Z(X)$, and the number of integer points in the interior of $Z(X)$, respectively:

1. $\mathrm{vol}(Z(X)) = \#\mathbb{B}(X)$,

2. $\#(Z(X) \cap \mathbb{Z}^n) = \#\mathbb{B}_+(X) = \#\mathbb{I}(X)$,

3. $\#(\mathrm{int}(Z(X)) \cap \mathbb{Z}^n) = \#\mathbb{B}_-(X)$.

Every zonotope $Z(X)$ is a disjoint (up to a nullset) union of the translated parallelepipeds [44, 16]:

$$t_B + Z(B), \quad B \in \mathbb{B}(X).$$

The translation $t_B \in \mathbb{R}^n$ equals $\sum_{x \in X(B)} x$, with $X(B)$ a suitable subset of $X \backslash B$. There are multiple ways of choosing these translations, hence there are multiple tilings of the zonotope. A canonical approach to obtaining a tiling is based on ordering X (in any way). Each such ordering corresponds to a different geometry on the hyperplane arrangement. In this duality, the vertices of the hyperplane arrangements are associated with the parallelepipeds that tile the zonotope, the bounded regions of the arrangement correspond to the vertices of the parallelepipeds that are interior to the zonotope, while the unbounded regions of the arrangement correspond to the vertices on the boundary of the zonotope. Thus, for example, the number of vertices of a connected region of the arrangement must agree with the number of parallelepipeds that intersect at the corresponding "lattice point" of the zonotope. This geometric duality is well known and is discussed, e.g., in [46, 45, 4, 5]. A reader who is interested in the above-mentioned geometric duality may wish to revisit the discussion here after reviewing the construction of a homogeneous basis for $\mathcal{P}(X)$ in Section 3.2.

2.4 The least map and polynomial interpolation

Given a power series f in n variables

$$f = f_0 + f_1 + f_2 + \cdots,$$

where f_j is a homogeneous polynomial of degree j, and define the *least map* via $f \mapsto f_\downarrow$ by

$$f_\downarrow := f_j, \quad f_j \neq 0, \;\; f_i = 0, \;\; \forall i < j.$$

In other words, $f_\downarrow$ is the first non-zero term in the above expansion of f. We adopt the convention that $0_\downarrow := 0$. For a collection F of functions analytic at the origin, we define

$$F_\downarrow := \mathrm{span}\{f_\downarrow : f \in F\}.$$

The least map plays an important role in polynomial interpolation, as was shown in [17, 18, 19]. Here are the details. A pointset $\sigma \subset \mathbb{C}^n$ is called *correct* for a polynomial space $P \subset \Pi$ if the restriction map $p \mapsto p|_\sigma$ is invertible (as a map from P to $\mathbb{C}^\sigma$) i.e., if interpolation from P at the points of σ is *correct* (the latter means that the interpolating polynomial exists and is unique).

With σ a finite subset of $\mathbb{R}^n$, consider the point evaluation functional

$$\delta_\alpha : \Pi \to \mathbb{C} \,:\, p \mapsto p(\alpha),$$

and define

$$\Lambda := \operatorname{span}\{\delta_\alpha \,:\, \alpha \in \sigma\}.$$

Then Λ is a subspace of the dual space Π' of Π. Given σ, the correctness of a space $F \subset \Pi$ for interpolation on σ is equivalent to the isomorphism

$$\Lambda|_F \cong F',$$

where F' is the dual space of F.

Now, associate $p \in \Pi$ with a differential operator:

$$p(D) := p\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_n}\right).$$

(In particular, if $p(t) = x \cdot t,\ x \in \mathbb{R}^n \backslash \{0\}$, then $p(D)$ is the directional derivative in the x-direction.) Then, given a polynomial p and a formal power series f, define their pairing $\langle p, f \rangle$ as

$$\langle p, f \rangle := (p(D)f)(0). \tag{2}$$

The functional δ_α is represented using this pairing by the exponential

$$e_\alpha : \mathbb{R}^n \to \mathbb{R} \,:\, t \mapsto e^{\alpha \cdot t}, \quad \textit{i.e.,} \qquad \langle p, e_\alpha \rangle = p(\alpha) = \delta_\alpha p.$$

Thus the space Λ is represented by the exponential space

$$\operatorname{Exp}(\sigma) := \operatorname{span}\{e_\alpha \,:\, \alpha \in \sigma\}.$$

Finally, we define

$$\Pi(\sigma) := \operatorname{span}\{f_\downarrow \,:\, f \in \operatorname{Exp}(\sigma)\}.$$

We now check that the dimension of $\Pi(\sigma)$ is exactly $\#\sigma$. Let $T_j : \operatorname{Exp}(\sigma)$ be the jth degree Taylor expansion; i.e., for $f \in \operatorname{Exp}(\sigma)$, $T_j f$ is the j-th degree Taylor expansion of f at 0. Note that $\deg(f_\downarrow) = j$ if and only if $f \in \ker T_{j-1} \backslash \ker T_j$. Thus, with T'_j the restriction of T_j to $\ker T_{j-1}$,

$$\dim \operatorname{span}\{f_\downarrow : f \in \operatorname{Exp}(\sigma),\ \deg(f_\downarrow) = j\} = \operatorname{rank} T'_j = \dim \ker T_{j-1} - \dim \ker T_j.$$

Summing from $j = 0$ to ∞ (where $T_{-1} := 0$), we obtain

$$\sum_{j=0}^{\infty} \Big(\dim(\ker T_{j-1}) - \dim(\ker T_j)\Big) = \dim(\ker T_{-1}) = \dim \operatorname{Exp}(\sigma) = \#\sigma.$$

Here we used the fact that every finite set of exponentials is linearly independent.

Now, for any analytic function $f \neq 0$,

$$\langle f_\downarrow, f \rangle \neq 0.$$

This means that there exists no $f \in \mathrm{Exp}(\sigma)\backslash\{0\}$ that satisfies

$$\langle p, f \rangle = 0, \quad \forall p \in \Pi(\sigma).$$

Thus, the spaces $\mathrm{Exp}(\sigma)$ and $\Pi(\sigma)$ serve as duals of each other. In summary:

Result 2.3 ([18]) *The map* $\mathrm{Exp}(\sigma) \to \Pi(\sigma)'$ *defined by* $f \mapsto \langle \cdot, f \rangle$ *is an isomorphism, and the set* σ *is correct for the space* $\Pi(\sigma)$. *In particular,* $\dim \Pi(\sigma) = \#\sigma$.

Now, given any non-zero polynomial $p \in \Pi$, let $p_\uparrow$ be the highest degree homogeneous polynomial in p, i.e., $p_\uparrow$ is homogeneous, and $\deg(p - p_\uparrow) < \deg p$. Likewise, define, for $F \subset \Pi$,

$$F_\uparrow := \mathrm{span}\{f_\uparrow \ : \ f \in F\}.$$

This defines the so-called *most map.* The following result describes the interaction of the least map $(\cdot)_\downarrow$ and the most map $(\cdot)_\uparrow$:

Result 2.4 ([18]) *Let* $p \in \Pi$ *be fixed. Then*

$$(p(D)\,(\mathrm{Exp}(\sigma)))_\downarrow \supset p_\uparrow(D)(\Pi(\sigma)). \tag{3}$$

Proof. Let p be a polynomial and let f be an analytic function. Set $p =: p_\uparrow + q$, then $\deg q < \deg p$. For $f \in \mathrm{Exp}(\sigma)$, set $f =: f_\downarrow + g$. Then we have

$$p(D)f = p_\uparrow(D)f_\downarrow + q(D)f_\downarrow + p_\uparrow(D)g + q(D)g.$$

Note that $p_\uparrow(D)f_\downarrow$ is the lowest order term in the right hand side. Hence $p_\uparrow(D)f_\downarrow = (p(D)f)_\downarrow$ unless $p_\uparrow(D)f_\downarrow = 0$. Applying this observation to an arbitrary function $f \in \mathrm{Exp}(\sigma)$, we conclude that (3) holds. □

This theorem can be used as follows: suppose that we have a homogeneous polynomial r, and we would like to understand the action of $r(D)$ on $\Pi(\sigma)$. Then it makes sense to find an inhomogeneous polynomial p such that (i) $p_\uparrow = r$, and (ii) p vanishes at as many points of σ as possible. Indeed, one easily verifies that

$$p(D)\mathrm{Exp}(\sigma) = \mathrm{Exp}(\sigma\backslash Z_p),$$

with Z_p the zero-set of p. In particular, $p(D)$ annihilates $\mathrm{Exp}(\sigma)$ iff p vanishes on σ. In that latter case, we obtain the following corollary from the previous result:

Corollary 2.5 ([18]) *If* p *vanishes on* σ, *then* $p_\uparrow$ *annihilates* $\Pi(\sigma)$.

Corollary 2.6 ([18]) *Let* $\sigma \subset \mathbb{R}^n$ *be finite, and let* P *be a homogeneous subspace of* Π. *Then* σ *is correct for* P *if the map*

$$p \mapsto \langle \cdot, p \rangle$$

is a bijection between P *and* $\Pi(\sigma)'$.[6]

[6] The converse of this result is true, too, once we assume in addition that $\dim(P \cap \Pi_j) \geq \dim(\Pi(\sigma) \cap \Pi_j)$, for all j.

Proof. We may assume that $\dim P = \#\sigma$, since otherwise the bijection cannot hold. Now, if $\dim P = \#\sigma$, and σ is not correct for P, then some $p \in P$ vanishes on σ, hence $p_\uparrow(D)$ annihilates $\Pi(\sigma)$, and, in particular, $p_\uparrow \perp \Pi(\sigma)$. Since P is homogeneous, $p_\uparrow \in P$, hence the bijection does not hold. □

These results can be used to prove dimension formulæ for polynomial spaces of interest, which we now introduce. Given a vector λ indexed by X, recall that we associate each $x \in X$ with an affine polynomial

$$p_{x,\lambda} : \mathbb{R}^n \to \mathbb{R} : t \mapsto x \cdot t - \lambda_x.$$

For simplicity, we denote the linear polynomial $p_{x,0}$ by p_x. For a multi-subset $Y \subset X$, define

$$p_{Y,\lambda} := \prod_{y \in Y} p_{y,\lambda}, \qquad p_Y := p_{Y,0}. \tag{4}$$

Let $V(X,\lambda)$ denote the vertex set of the corresponding X-hyperplane arrangement $\mathcal{H}(X,\lambda)$. Recall that $\#V(X,\lambda) = \#\mathbb{B}(X)$ for a generic λ. Moreover, if λ is generic, then there is a natural bijection $B \mapsto v_B$ between $\mathbb{B}(X)$ and $V(X,\lambda)$, where each $B \in \mathbb{B}(X)$ is mapped to the unique common zero v_B of $\{p_{y,\lambda} : \ y \in B\}$. This implies that each subset $\mathbb{B}'$ of $\mathbb{B}(X)$ is associated in a unique way with $V' \subset V(X,\lambda)$. We define

$$\mathcal{D}_{\mathbb{B}'}(X) := \{f \in \Pi \, : \, p_Y(D)f = 0, \quad \forall Y \subset X \ \text{ s.t. } \ Y \cap B \neq \emptyset, \ \forall B \in \mathbb{B}'\}.$$

Then we have the following results, [19]:

Theorem 2.7 *Let $X \subset \mathbb{R}^n \backslash \{0\}$ be a finite multiset of full rank n. Then, for any subset $\mathbb{B}'$ of $\mathbb{B}(X)$,*

$$\Pi(V') \subseteq \mathcal{D}_{\mathbb{B}'}(X),$$

with V' the vertices that correspond to $\mathbb{B}'$ in any generic X-hyperplane arrangement $\mathcal{H}(X,\lambda)$.

Corollary 2.8 *In the setting of Theorem 2.7,*

$$\dim \mathcal{D}_{\mathbb{B}'}(X) \geq \#\mathbb{B}'.$$

Proof of Theorem 2.7 and Corollary 2.8. Let λ be generic. Note that, for $Y \subset X$ and $B \in \mathbb{B}(X)$, $p_{Y,\lambda}(v_B) = 0$ if and only if $Y \cap B \neq \emptyset$. Now, set $V' := \{v_B : \ B \in \mathbb{B}'\}$, and let Y be a multi-subset of X such that

$$Y \cap B \neq \emptyset \quad \forall B \in \mathbb{B}'.$$

Then we conclude that $p_{Y,\lambda}$ vanishes on V'. Then, by Corollary 2.5, we have

$$(p_{Y,\lambda})_\uparrow(D)(\Pi(V')) = 0.$$

Since $(p_{Y,\lambda})_\uparrow = p_Y$, we conclude that

$$\Pi(V') \subseteq \mathcal{D}_{\mathbb{B}'}(X).$$

But,

$$\dim \Pi(V') = \#V' = \#\mathbb{B}'.$$

□

We will apply these results three times in this paper: once with respect to $\mathbb{B}' := \mathbb{B}(X)$, then with respect to $\mathbb{B}' := \mathbb{B}_+(X)$, and finally with respect to $\mathbb{B}' := \mathbb{B}_-(X)$. In all these cases, we will show that equality holds in Corollary 2.8, hence that $\Pi(V') = \mathcal{D}_{\mathbb{B}'}(X)$.

2.5 Polynomial ideals and their kernels

Here we state, for the convenience of the reader, some basic results from commutative algebra, which will be used in the rest of the paper. The majority of these results are standard and can be found in commutative algebra textbooks, e.g., [26] or [31]. The fact that kernels of polynomial ideals can be synthesized from finitely many localizations can be found in [33]; while this result is non-trivial, we will need only the result for the simpler special case of zero-dimensional ideals. Some of the actual presentation here follows [19].

Let I be an ideal in the ring $\Pi := \mathbb{C}[t_1, \cdots, t_n]$. If I is generated by a set $L \subset \Pi$, this will be denoted as $I = \mathrm{Ideal}(L)$. The *codimension* of I is the dimension of the quotient space Π/I or, equivalently, the dimension of its *annihilator*

$$\{\mu \in \Pi' \;:\; \mu f = 0 \quad \forall f \in I\}.$$

Since the dual space Π' can be realized as the space $\mathbb{C}[[t_1, \ldots, t_n]]$ of formal power series, the codimension of I is also the dimension of the orthogonal complement of I in $\mathbb{C}[[t_1, \ldots, t_n]]$ with respect to the pairing (2), i.e., $\langle f, g\rangle := (f(D)g)(0)$. The *variety* $\mathrm{Var}(I)$ of I

$$\mathrm{Var}(I) := \{\theta \in \mathbb{C}^n \;:\; p(\theta) = 0 \quad \forall p \in I\}$$

forms a subset of the annihilator of I, where evaluation at a point $\theta \in \mathrm{Var}(I)$ is realized by the exponential e_θ. The *kernel* of I is defined as

$$\ker I := \mathrm{span}\{e_\alpha p \;:\; \alpha \in \mathrm{Var}(I), \;\; p \in \Pi \quad \text{s.t.} \quad \langle e_\alpha p, q\rangle = 0, \quad \forall\, q \in I\}.$$

With that definition, the kernel $\ker I$ is *total* in the sense that

$$(\ker I) \perp := \{q \in \Pi \;:\; \langle k, q\rangle = 0, \quad \forall\, k \in \ker I\} = I.$$

An ideal I is called *zero-dimensional* if its variety is finite. In that case, each of the multiplicity spaces

$$(\ker I)_\alpha := \{p \in \Pi : e_\alpha p \in \ker I\}$$

is finite-dimensional, hence $\ker I$ is a finite-dimensional space of exponential polynomials:

$$\ker I := \sum_{\alpha \in \mathrm{Var}(I)} (\ker I)_\alpha.$$

Furthermore, we have then that

$$\dim(\ker I) = \dim \Pi/I,$$

hence, in particular, I is of finite codimension. Defining

$$(\ker I)_\downarrow := \mathrm{span}\{f_\downarrow : f \in \ker I\},$$

we have

Result 2.9 *If I is a zero-dimensional ideal, then*

$$\Pi = I \oplus (\ker I)_\downarrow.$$

The homogenization of the kernel of an ideal via the least map $(\cdot)_\downarrow$ is dual to the homogenization of the ideal itself via the most map $(\cdot)_\uparrow$. Here are the details. Given an ideal I, we define

$$I_\uparrow := \mathrm{span}\{p_\uparrow \ : \ p \in I\}.$$

$I_\uparrow$ is a *homogeneous* ideal, i.e., is generated by homogeneous polynomials. We have:

Result 2.10 *For a zero-dimensional ideal I, the following properties hold:*

1. $\mathrm{Var}(I_\uparrow) = \{0\}$.

2. $\ker(I_\uparrow)$ *is a finite-dimensional polynomial space.*

3. $\ker(I_\uparrow) = \{p \in \Pi : q(D)p = 0, \ q \in L\}$, *with $L \subset \Pi$ any set that generates $I_\uparrow$.*

4. $\ker(I_\uparrow) = (\ker I)_\downarrow$.

5. $(I_\uparrow) \oplus \ker(I_\uparrow) = \Pi$.

Finally, if I is 0-dimensional, and F is a polynomial space, then the relation $F + I = \Pi$ implies that $\dim F \geq \dim \Pi/I$ with equality iff the above sum is direct. Hence we have:

Result 2.11 *Let I be a zero-dimensional ideal, and let F be a linear subspace of* Π. *If*

$$F + I = \Pi, \tag{5}$$

then $\dim F \geq \dim \ker I$. *Moreover, if* $\dim F = \dim \ker I$, *then the sum (5) is direct.*

3 Central zonotopal algebra

3.1 Main results

We have mentioned the geometric duality between the hyperplane arrangement and the zonotope that are associated with the multiset X. The focus of this paper is on an algebraic counterpart of that duality. We discuss in the paper three pairs of finite-dimensional polynomial spaces, all of which can be alternatively described as kernels of certain zero-dimensional ideals. Each polynomial space will be shown to be a dual space of its pair-mate via the map $p \mapsto \langle p, \cdot \rangle$ where $\langle \cdot, \cdot \rangle$ is our pairing (2).

The first pair will be referred to as the *central pair* of X. The space $\mathcal{P}(X)$ below is the central space of the zonotope $Z(X)$, while the space $\mathcal{D}(X)$ is the central space of the hyperplane arrangement $\mathcal{H}(X, 0)$. The theory of this pair of polynomial spaces was developed in the 80s and 90s in [1, 2, 3, 6, 18, 19, 20, 21, 17, 10, 11, 13, 14, 9, 24, 25]. We have discussed some historical aspects of this theory in the Introduction, and will discuss the history of specific results in more detail towards the end of this section. The section contains a streamlined and abbreviated theory of the central pair. To keep this paper close to being self-contained, we provide most proofs.

The polynomials spaces $\mathcal{P}(X)$ and $\mathcal{D}(X)$ are best described in terms of a partition of the power set 2^X into two disjoint sets of long subsets $L(X) \subset 2^X$ and short subsets $S(X) = 2^X \backslash L(X)$:

$$\begin{aligned} L(X) &:= \{Y \subset X \ : \ Y \cap B \neq \emptyset, \quad \forall B \in \mathbb{B}(X)\}, \\ S(X) &:= \{Y \subset X \ : \ \mathrm{rank}(X \backslash Y) = n\}. \end{aligned}$$

Note that the elements of $S(X)$ are exactly the independent sets of the matroid dual to X, and those of $L(X)$ are its dependent sets, as the independence of a set Y in the dual matroid is equivalent to the set $X \setminus Y$ being of full rank in the original matroid. Using the notation (4) for polynomials p_Y, the spaces $\mathcal{P}(X)$ and $\mathcal{D}(X)$ are defined as follows:

$$\begin{aligned}\mathcal{D}(X) &:= \{f \in \Pi : p_Y(D)f = 0, \quad \forall Y \in L(X)\},\\ \mathcal{P}(X) &:= \operatorname{span}\{p_Y : Y \in S(X)\}.\end{aligned}$$

Immediately, Theorem 2.7 and Corollary 2.8 imply as a special case the following two statements:

Theorem 3.1 *Let $V(X,\lambda)$ denote the vertices of a generic X-hyperplane arrangement associated with X. Then $\Pi(V(X,\lambda)) \subseteq \mathcal{D}(X)$.* □

Corollary 3.2 $\dim \mathcal{D}(X) \geq \#\mathbb{B}(X)$. □

We now let $\mathcal{J}(X)$ denote the ideal generated by the long polynomials:

$$\mathcal{J}(X) := \operatorname{Ideal}\{p_Y : Y \in L(X)\}.$$

Then $\mathcal{D}(X)$ could also be defined as the polynomial kernel of $\mathcal{J}(X)$: $\mathcal{D}(X) = \Pi \cap \ker \mathcal{J}(X)$. It is easy to check that the only common zero of the long polynomials is 0, i.e., that $\operatorname{Var}(\mathcal{J}(X)) = \{0\}$. Since the ideal $\mathcal{J}(X)$ is homogeneous, this tells us (see Section 2.5) that the kernel of $\mathcal{J}(X)$ is finite-dimensional and is a subspace of Π. Thus, $\mathcal{D}(X)$ is precisely the kernel of $\mathcal{J}(X)$:

$$\ker \mathcal{J}(X) = \mathcal{D}(X) \subset \Pi. \tag{6}$$

Theorem 3.3 $\mathcal{P}(X) + \mathcal{J}(X) = \Pi$. □

Proof, [24]. Set $F := \mathcal{P}(X) + \mathcal{J}(X)$, and assume that F is proper in Π. Since F is homogeneous, its orthogonal complement $F\perp$ in Π (with respect to the pairing (2)) contains non-zero polynomials. We will show that $F\perp$ is D-invariant, i.e., closed under differentiations. This will imply that $F\perp$ must contain the constants, which is absurd, since $\mathcal{P}(X)$, hence F, contains the constants.

We need thus to prove that, for $p \in F\perp$ and $a \in \mathbb{R}^n$, $D_a p \perp F$. First, since

$$p \in F\perp \subset \mathcal{J}(X)\perp = \ker \mathcal{J}(X),$$

it follows that $D_a p \perp \mathcal{J}(X)$ (since $\mathcal{J}(X)$ is an ideal and the kernel of an ideal is always D-invariant). It remains to show that $D_a p \perp \mathcal{P}(X)$, i.e., that $D_a p \perp p_Y$, for every short Y. Fix such Y and choose $B \in \mathbb{B}(X \backslash Y)$. Since we can write $a = \sum_{b \in B} c(b) b$, it suffices to prove that $D_b p \perp p_Y$ (for every $b \in B$). Now,

$$\langle D_b p, p_Y \rangle = \langle p, p_b p_Y \rangle = \langle p, p_{Y \cup b} \rangle.$$

Since $Y \cup b \subset X$, and since F contains, by assumption, every polynomial $p_{Y'}$, $Y' \subset X$, it follows that $p_{Y \cup b} \in F$, hence $\langle p, p_{Y \cup b} \rangle = 0$. □

Corollary 3.4 $\dim \mathcal{P}(X) \geq \dim \mathcal{D}(X)$.

Proof. This inequality follows by applying Result 2.11 to the sum from Theorem 3.3 and by recalling the formula (6) that identifies $\ker \mathcal{J}(X)$ with $\mathcal{D}(X)$. $\square$

Next, consider the set of *facet hyperplanes* of X:

$$\mathcal{F}(X) := \{H \,:\, H \text{ is a subspace of } \mathbb{R}^n,\ \dim H = n-1,\ \operatorname{span}(X \cap H) = H\}. \tag{7}$$

Note that these hyperplanes are in fact parallel to the facets of the zonotope $Z(X)$, which explains this terminology. Given any facet hyperplane $H \in \mathcal{F}(X)$, let η_H be a non-zero normal to H: $\eta_H \perp H$. We also define

$$m(H) := m_X(H) := \#(X \backslash H). \tag{8}$$

Define

$$\mathcal{I}(X) := \operatorname{Ideal}\{p_{\eta_H}^{m(H)} \,:\, H \in \mathcal{F}(X)\},$$

where, as above, $p_x : t \mapsto x \cdot t$. Then we have the following theorem:

Theorem 3.5 *$\mathcal{P}(X) \subseteq \ker \mathcal{I}(X)$.*

Proof. We only need to check that

$$D_{\eta_H}^{m(H)}(p_Y) = 0 \quad \forall H \in \mathcal{F}(X) \quad \forall Y \in S(X),$$

i.e., that any generator of $\mathcal{I}(X)$ acting as a differential operator annihilates any generator of $\mathcal{P}(X)$. Indeed, note that, for a single vector $\xi \in X$, we get $D_{\eta_H}(p_\xi) = 0$ if and only if $\xi \perp \eta_H$ or, equivalently, $\xi \in H$. So, a polynomial of the form p_Y can survive $m(H)$ differentiations in the direction η_H only if $\#(Y \backslash H) \geq \#(X \backslash H)$, i.e., only if $\#(Y \backslash H) = \#(X \backslash H)$ since $Y \subseteq X$. The last equality is possible if and only if all vectors in $X \backslash Y$ belong to H. But Y is a short subset of X, so its complement is of full rank. Contradiction! Hence p_Y is annihilated by $D_{\eta_H}^{m(H)}$. $\square$

Corollary 3.6 *$\dim \mathcal{P}(X) \leq \dim \ker \mathcal{I}(X)$.* $\square$

Theorem 3.7 ([17]) *$\dim \ker \mathcal{I}(X) \leq \#\mathbb{B}(X)$.*

Sketch of proof. The proof is by induction on $\#X$ and n. Assuming that this statement is correct for X, we define $X' := X \cup \{\xi\}$, and consider for every facet hyperplane $H \in \mathcal{F}(X)$ the space $P_H := \mathcal{P}((X \cap H) \cup \{\xi\})$. If $\xi \in H$, then $P_H = 0$; otherwise, P_H has positive dimension. Note that each $B \in \mathbb{B}(X')$ lies either in $\mathbb{B}(X)$ (in case it does not contain ξ), or else in a unique $(X \cap H) \cup \{\xi\}$, $H \in \mathcal{F}(X)$. Therefore,

$$\dim \mathcal{P}(X) + \sum_{H \in \mathcal{F}(X)} \dim P_H = \#\mathbb{B}(X').$$

We then define a map T as follows:

$$T : \ker I(X') \to \times_{H \in \mathcal{F}(X)} P_H \quad : \quad f \mapsto (D_{\eta_H}^{m(H)} f)_{H \in \mathcal{F}(X)}.$$

The kernel of this map is, directly from the definition, $\ker I(X)$, hence, by induction, $\mathcal{P}(X)$. Our previous computation then shows that

$$\dim \ker I(X') \leq \dim \ker T + \dim \operatorname{ran} T = \#\mathbb{B}(X').$$

The only missing item in the argument is to show that the map T is well-defined, i.e., that $D_{\eta_H}^{m(H)} \ker I(X') \subset P_H$. This is trivially true in case $\xi \in H$. Proving the above for the case $\xi \in H$ is the hard part of the proof, which is omitted here. Some of these missing details are discussed as a part of the proof of Theorem 4.7 in the next section. See [17] for details. □

We are now in a position to prove the main theorem of this section.

Theorem 3.8

(1) $\dim \mathcal{P}(X) = \dim \mathcal{D}(X) = \#\mathbb{B}(X)$.

(2) The map $p \mapsto \langle p, \cdot \rangle$ *is a bijection between* $\mathcal{D}(X)$ *and* $\mathcal{P}(X)'$.

(3) $\mathcal{D}(X) = \Pi(V(X, \lambda)) = \ker \mathcal{J}(X)$.

(4) The point set $V(X, \lambda)$ *is correct for* $\mathcal{D}(X)$ *as well as for* $\mathcal{P}(X)$.

(5) $\mathcal{P}(X) = \ker \mathcal{I}(X)$.

(6) $\mathcal{P}(X) \oplus \mathcal{J}(X) = \Pi$.

Proof. Putting together the inequalities obtained in Corollaries 3.2, 3.4, 3.6 and in Theorem 3.7, we get

$$\#\mathbb{B}(X) \leq \dim \mathcal{D}(X) \leq \dim \mathcal{P}(X) \leq \dim \ker \mathcal{I}(X) \leq \#\mathbb{B}(X).$$

This shows that equalities must hold throughout. Invoking Theorems 3.1, 3.3 and 3.5, along with Result 2.3, Corollary 2.6 and Result 2.11, we obtain the remaining claims of this theorem. □

Remark. Let us assume that $X = B$, with B some basis for $\mathbb{R}^n$. Then $S(X) = \{\emptyset\}$, hence $\mathcal{P}(X) = \text{span}\{1\} = \Pi_0^0$. On the other hand, $\{b\} \in L(X)$, for every $b \in B$, and hence $\mathcal{J}(X)$ contains the linear polynomials $\{p_b : b \in B\}$. Thus, $\mathcal{J}(X)$ is the maximal ideal $\{p \in \Pi : p(0) = 0\}$, and the decomposition $\mathcal{P}(X) \bigoplus \mathcal{J}(X) = \Pi$ becomes obvious. There are other cases when this decomposition can be obtained directly: for example, when X is in general position. However, for a general X, this decomposition is non-trivial.

As a by-product of Theorem 3.8, we obtain an additional result that characterizes the least space obtained from integer points in the half-open half-closed zonotope in the case when X is unimodular, i.e., when all vectors in X have only integer components and every basis of X is invertible over $\mathbb{Z}$ (see Section 2.3). We recall that the zonotope $Z(X)$ is defined as the image $Z(X) := X([0,1]^X)$ of the unit cube $[0,1]^X$ under the map $X : \mathbb{R}^X \to \mathbb{R}^n$.

Thus assume that X is unimodular and consider its zonotope and associated hyperplane arrangements. In the context of hyperplane arrangements, a set of interest is the vertex set $V(X, \lambda)$ of the arrangement, whose precise geometry depends on the vector $\lambda \in \mathbb{C}^X$. For a generic λ, the vertex set $V(X, \lambda)$ is of *maximal* cardinality $\#\mathbb{B}(X)$. The dual vertex set, $\mathcal{Z}(X, t)$, is parameterized by $t \in \mathbb{R}^n$. As we will shortly see, for a generic t this set is of *minimal* cardinality $\#\mathbb{B}(X)$. Let us begin with a definition:

$$\mathcal{Z}(X, t) := \{\alpha \in \mathbb{Z}^n : \ t - \alpha \in Z(X)\}.$$

We consider t to be *generic* if it does not lie in any of the hyperplanes

$$\mathbb{Z}^n + H, \quad H \in \mathcal{F}(X).$$

If t is generic, then it is well known (see, e.g., [16]) that

$$\#\mathcal{Z}(X,t) = \mathrm{Vol}(Z(X)) = \#\mathbb{B}(X).$$

We will assume that t is fixed and generic and will occasionally denote $\mathcal{Z}(X,t)$ simply by $\mathcal{Z}(X)$.

Theorem 3.9 ([17]) *Let X be unimodular. Then $\Pi(\mathcal{Z}(X)) = \mathcal{P}(X) = \ker \mathcal{I}(X)$, hence $\mathcal{Z}(X)$ is correct for $\mathcal{P}(X)$ as well as for $\mathcal{D}(X)$.*

Proof. With $\mathcal{Z}(X) = \mathcal{Z}(X,t)$, we already know that $\#\mathcal{Z}(X) = \mathrm{Vol}(Z(X)) = \#\mathbb{B}(X)$. Recall that for a given $\sigma \subset \mathbb{R}^n$, we defined

$$\mathrm{Exp}(\sigma)_{\downarrow} =: \Pi(\sigma).$$

By Theorem 2.3, $\dim \Pi(\sigma) = \#\sigma$ for any set σ, so, for a unimodular X, we get

$$\dim \Pi(\mathcal{Z}(X)) = \#\mathcal{Z}(X) = \#\mathbb{B}(X).$$

Since both spaces $\Pi(\mathcal{Z}(X))$ and $\mathcal{P}(X)$ have the same dimension, we only need to prove that one is included in the other. We will show that $\Pi(\mathcal{Z}(X)) \subset \ker \mathcal{I}(X)$. To this end, we recall Corollary 2.5: if $f, g \in \Pi$ satisfy $f_{\uparrow} = g$, and if $f|_{\sigma} = 0$, then

$$g(D)(\Pi(\sigma)) = 0.$$

We choose g to be one of the generators of $\mathcal{I}(X)$, i.e.,

$$g : t \mapsto (\eta_H \cdot t)^{m(H)} = p_{\eta_H}^{m(H)}(t), \quad H \in \mathcal{F}(X).$$

We need to find f such that $f_{\uparrow} = g$ and f vanishes on $\mathcal{Z}(X)$. Once we manage to do so for every g as above, we are done. To this end, we fix $H \in \mathcal{F}(X)$, and will define f as (with $\eta := \eta_H$) $f := (p_\eta + c_1)(p_\eta + c_2)\cdots(p_\eta + c_{m(H)})$, with $(c_i)_i$ some constants. Obviously, for such f we always have that $f_{\uparrow} = g$. We need also to ensure that f vanishes on $\mathcal{Z}(X)$. In the argument below we assume for convenience that all vectors $X \backslash H =: \{y_1, \ldots, y_{m(H)}\}$ lie on one side of H. It is then straightforward to see that the zonotope $Z(X)$ lies between the hyperplane H, and the hyperplane

$$H' := H + \sum_{i=1}^{m(H)} y_i.$$

A simple consequence of the unimodularity is that there are exactly $m(H) - 1$ translates of H that lie properly between H and H' and contain integers. Precisely, these are the hyperplanes

$$H + \sum_{i=1}^{j} y_i =: H + c_j, \quad j = 1, \ldots, m(H) - 1.$$

Since t is generic, it does not lie on any of these hyperplanes, hence we may assume without loss that it lies between H and $H + y_1$. We conclude that

$$\mathcal{Z}(X,t) \subset \cup_{j=0}^{m(H)-1} (H + c_j), \quad c_0 := 0,$$

hence that the polynomial

$$f := \prod_{j=0}^{m(H)-1} (p_\eta + c_j)$$

vanishes on $\mathcal{Z}(X,t) = \mathcal{Z}(X)$. This proves that $\Pi(\mathcal{Z}(X)) = \mathcal{P}(X) = \ker \mathcal{I}(X)$, the second equality by Part (5) of Theorem 3.8.

The correctness of $\mathcal{Z}(X,t)$ for $\mathcal{P}(X)$ follows then from Result 2.3, while its correctness for $\mathcal{D}(X)$ follows from the duality between $\mathcal{P}(X)$ and $\mathcal{D}(X)$ proved in Part (2) of Theorem 3.8 and from Result 2.6. $\square$

Additional historical remarks. The space $\mathcal{D}(X)$ was introduced in [15]. The inequality $\dim \mathcal{D}(X) \geq \#\mathbb{B}(X)$ was first proved by Dahmen and Micchelli in [11] by induction on $\#X$ and on n. A non-inductive analytic argument is given in [6]. The equality $\dim \mathcal{D}(X) = \#\mathbb{B}(X)$ is also due to Dahmen and Micchelli [11]. They subsequently provided, in [13], a very elegant proof for the inequality $\dim \mathcal{D}(X) \leq \#\mathbb{B}(X)$, which uses the matroidal structure of X. In [24], the space $\mathcal{P}(X)$ is proved to be dual to every space of the form $\mathcal{D}(X,\lambda)$, with the definition of the latter obtained from the definition of $\mathcal{D}(X)$ by replacing each p_x by $p_{x,\lambda}$; here λ need not be generic. The space $\mathcal{D}(X,\lambda)$ plays an important role in the theory of *exponential box splines,* [40], but not in this paper.

3.2 Homogeneous basis and Hilbert series for $\mathcal{P}(X)$

Let Π_j^0 be the space of homogeneous polynomials of degree j (in n variables). Since both $\mathcal{P}(X)$ and $\mathcal{D}(X)$ are *graded* or *homogeneous,* i.e., are spanned by homogeneous polynomials and since the pairing (2) respects grading, the isomorphism shown in part (2) of Theorem 3.8 implies that, for every j,

$$\dim(\Pi_j^0 \cap \mathcal{D}(X)) = \dim(\Pi_j^0 \cap \mathcal{P}(X)).$$

We refer to the homogeneous dimensions of the space $\mathcal{P}(X)$ as the *central Hilbert series of X:*

$$h_X : \mathbb{N} \to \mathbb{Z}_+ \ : \ j \mapsto \dim(\Pi_j^0 \cap \mathcal{P}(X)).$$

Note that

$$\sum_j h_X(j) = \#\mathbb{B}(X).$$

The adjective "central" is chosen in anticipation of the introduction of two other Hilbert series that will be labeled "internal" and "external" respectively.

We focus now on building a homogeneous basis for $\mathcal{P}(X)$, which will enable us to compute the homogeneous dimensions $h_X(j)$ of $\mathcal{P}(X)$. We will see soon that h_X can be computed directly by studying the dependence/independence relations among the vectors in X.

Example 3.10 *Let*

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}.$$

Then $\#\mathbb{B}(X) = 16$, and

$$h_X = (1, 3, 6, 6).$$

$\square$

An algorithm for computing h_X. First, we impose an arbitrary order $\prec$ on X. Then we associate each $B \in \mathbb{B}(X)$ with the homogeneous polynomial $p_{X(B)}$, where

$$X(B) := \{y \in X \,:\, y \notin \operatorname{span}\{b \in B \,:\, b \preceq y\}\}. \tag{9}$$

Note that $X(B) \in S(X)$, since $B \subset (X \backslash X(B))$. Define

$$\operatorname{val} B := \# X(B).$$

Then

$$h_X(j) = \#\{B \in \mathbb{B}(X) : \operatorname{val} B = j\}.$$

This assertion follows from the stronger assertion in Theorem 3.12 below. The algorithm, incidentally, draws an intimate relation between the Tutte polynomial of X [7] and the central Hilbert function.

Example 3.11 *Let* $X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =: [x_1 \ x_2 \ x_3 \ x_4]$ *be an ordered multiset. Then*

$$X([x_1, x_3]) = \emptyset, \qquad X([x_2, x_4]) = \{x_1, \ x_3\}.$$

The algorithm easily produces the Hilbert series

$$h_X = (1, 2, 2, 0, 0, \ldots)$$

□

Theorem 3.12 ([24]) *The set*

$$\{Q_B := p_{X(B)} \,:\, B \in \mathbb{B}(X)\} \tag{10}$$

is a basis for $\mathcal{P}(X)$.

Proof. It is clear that $p_{X(B)} \in \mathcal{P}(X)$, since $X(B) \cap B = \emptyset$, i.e., $X(B) \in S(X)$. Since $\dim \mathcal{P}(X) = \#\mathbb{B}(X)$, it is sufficient to show that $\{Q_B \,:\, B \in \mathbb{B}(X)\}$ is linearly independent. We will prove this by induction on $\#X$, with the case $\#X = n$ being trivial.

Assume, thus, the set in (10) to be linearly independent. Let $X' = X \cup \{\xi\}$ where ξ is the last element in X'. The induction step requires us to show that, given $p \in \mathcal{P}(X)$ and $B_0 \in \mathbb{B}(X') \backslash \mathbb{B}(X)$, if, for some constants $(a(B))_B$,

$$p + \sum_{B \in \mathbb{B}(X') \backslash \mathbb{B}(X)} a(B)\, p_{X(B)} = 0,$$

then $a(B_0) = 0$. To this end, we define

$$B_0 =: B_0' \cup \{\xi\}, \quad H := \operatorname{span} B_0' \in \mathcal{F}(X).$$

Let η be a non-zero vector such that $\eta \perp H$. Then

$$0 = D_\eta^{m(H)} 0 = D_\eta^{m(H)} p + \sum_{B \in \mathbb{B}(X') \backslash \mathbb{B}(X)} a(B) D_\eta^{m(H)} p_{X(B)}, \tag{11}$$

where $m(H) := \#(X \backslash H)$. Since $D_\eta^{m(H)} \mathcal{P}(X) = 0$ by Theorem 3.5, we conclude that $D_\eta^{m(H)} p = 0$. Next, consider

$$D_\eta^{m(H)} p_{X(B)}, \quad B \in \mathbb{B}(X') \backslash \mathbb{B}(X).$$

We know that $\#(X' \backslash H) = m(H) + 1$. If $\#(B \cap H) < n - 1$, then $X' \backslash X(B)$ contains at least 2 vectors from $X' \backslash H$. So, $\#(X(B) \cap (X' \backslash H)) < m(H)$. Consequently,

$$D_\eta^{m(H)} p_{X(B)} = 0.$$

If $\#(B \cap H) = n - 1$, then $\#(X(B) \backslash H) = m(H)$ so that

$$D_\eta^{m(H)} p_{X(B)} = c(B)\, p_{X(B) \cap H},$$

for some non-zero coefficient $c(B)$. From equation (11), we get

$$0 = \sum_{B \in K} a(B) c(B)\, p_{X(B) \cap H},$$

where $K := \{B \in \mathbb{B}(X') \backslash \mathbb{B}(X) \ : \ \#(X(B) \backslash H) = m(H)\}$. Now, it is easily observed that, with $W := \xi \cup (X' \cap H)$, the polynomials

$$p_{X(B) \cap H}, \quad B \in K,$$

are exactly the polynomials in the homogeneous basis for $\mathcal{P}(W)$, with the order on W being the order induced from X. Our induction hypothesis implies (since $\#W < \#X'$) that those polynomials are independent, hence that $a(B_0) = 0$. $\square$

Remark. The construction provides us with the direct sum decomposition

$$\mathcal{P}(X \cup \{\xi\}) = \mathcal{P}(X) \bigoplus (\bigoplus_{\xi \notin H \in \mathcal{F}(X)} p_{X \backslash H} \mathcal{P}((X \cap H) \cup \{\xi\}).$$

The decomposition allows us to compute the Hilbert series $h_{X'}$ by summing the Hilbert series of the various summands:

$$h_{X \cup \xi}(j) = h_X(j) + \sum_{\xi \notin H \in \mathcal{F}(X)} m_X(H) + h_{X_H}(j),$$

with $X_H := (X \cap H) \cup \xi$. This means that we do not need to impose one fixed order on X: once ξ is known to be placed last, the order of the remaining elements can be chosen separately (hence independently) for each summand. This is consistent with the known invariance of the Tutte polynomial, [7].

4 External algebra

4.1 Main results

Recall that, in Section 2.1, we let $\mathbb{I}(X)$ denote the collection of all independent subsets of X and let $B_0 \subset \mathbb{R}^n$ be a fixed ordered basis. We also denoted

$$X' := X \cup B_0$$

and defined a bijection $\mathrm{ex} : \mathbb{I}(X) \to \mathbb{B}_+(X) \subset \mathbb{B}(X')$ to the set $\mathbb{B}_+(X)$ of all external bases of X. We now define

$$\begin{aligned} L_+(X) &:= \{Y \subset X' : Y \cap B \neq \emptyset, \quad B \in \mathbb{B}_+(X)\}, \\ \mathcal{P}_+(X) &:= \mathrm{span}\{p_Y : Y \subset X\}, \\ \mathcal{D}_+(X) &:= \{f \in \Pi : p_Y(D)f = 0, \ \forall Y \in L_+(X)\}. \end{aligned}$$

Our goal is to show that $\mathcal{D}_+(X)$ and $\mathcal{P}_+(X)$ are dual to each other, and to determine their annihilating ideals. The main result will be established in several steps, analogously to Section 3.1.

Let $V(X', \lambda)$ be the set of vertices of a generic X'-hyperplane arrangement. With a slight abuse of notation we denote by $V_+(X, \lambda)$ the subset of $V(X', \lambda)$ that corresponds to $\mathbb{B}_+(X) \subset \mathbb{B}(X')$. Applying Theorem 2.7 and its Corollary 2.8 to this case (i.e., with X there replaced by X' here, and $\mathbb{B}'$ there being our $\mathbb{B}_+(X)$) we obtain the following results.

Theorem 4.1 $\Pi(V_+(X, \lambda)) \subseteq \mathcal{D}_+(X)$. □

Corollary 4.2 $\dim \mathcal{D}_+(X) \geq \#\mathbb{B}_+(X)$. □

Define

$$\mathcal{J}_+(X) := \mathrm{Ideal}\{p_Y : Y \in L_+(X)\}.$$

Note that, (almost) directly from the definition of $\mathcal{D}_+(X)$, $\ker \mathcal{J}_+(X) = \mathcal{D}_+(X)$.

Theorem 4.3 $\mathcal{P}_+(X) + \mathcal{J}_+(X) = \Pi$.

Proof. We start with the fact that $\mathcal{P}(X) + \mathcal{J}(X) = \Pi$, which is established in Theorem 3.3. Since $\mathcal{P}_+(X) \supset \mathcal{P}(X)$, we conclude that

$$\mathcal{P}_+(X) + \mathcal{J}(X) = \Pi.$$

So, we need to prove that

$$\mathcal{J}(X) \subset \mathcal{P}_+(X) + \mathcal{J}_+(X).$$

Let $Y \in L(X)$, $f \in \Pi$. Since every polynomial in $\mathcal{J}(X)$ is a combination of polynomials of the form $p_Y f$, it suffices to prove that

$$p_Y f \in \mathcal{P}_+(X) + \mathcal{J}_+(X),$$

a claim that we prove by reverse induction on $\#Y$. Thus, assume that the claim is correct for every $Y' \in L(X)$ such that $\#Y' > \#Y$. Put $S := \mathrm{span}(X \backslash Y)$. Then $\dim S < n$, since Y is long. Let $I \subset X \backslash Y$ be a basis for S and $B := \mathrm{ex}(I)$. Since B is a basis for $\mathbb{R}^n$,

$$\mathrm{Ideal}\{p_b : b \in B\} + \Pi_0^0 = \Pi.$$

So, we can write f in the following form:

$$f = c_0 + \sum_{b \in B} p_b f_b, \quad f_b \in \Pi.$$

Consequently,

$$p_Y f = c_0 p_Y + \sum_{b \in B} p_{Y \cup \{b\}} f_b.$$

We claim that each term above belongs to $\mathcal{P}_+(X) + \mathcal{J}_+(X)$. Since $Y \subset X$, it is clear that $p_Y \in \mathcal{P}_+(X)$. Now, for $p_{Y\cup\{b\}}$, we have either $b \in I$ or $b \in B_0$.

Case I. If $b \in I \subset X$, then $Y' := Y \cup \{b\} \subset X$. By induction,

$$p_{Y\cup\{b\}} f_b \in \mathcal{P}_+(X) + \mathcal{J}_+(X).$$

Case II. Let $b \in B_0$. We show that

$$p_{Y\cup\{b\}} f_b \in \mathcal{J}_+(X),$$

and to this end it is enough to show that $Y \cup \{b\} \in L_+(X)$. Let $B' \in \mathbb{B}_+(X)$. If $Y \cap B' = \emptyset$, then $B' \cap X \subset X \backslash Y \subset S$. Hence $B' = \mathrm{ex}(I')$, for $I' \subset S$. Thus $\mathrm{span} I' \subset \mathrm{span} I$, and the definition of the extension map implies that in such case we always have that $B_0 \cap \mathrm{ex}(I) \subset B_0 \cap \mathrm{ex}(I')$. Consequently, $b \in B'$, and thus

$$(Y \cup \{b\}) \cap B' \neq \emptyset.$$

We conclude that $Y \cup \{b\} \in L_+(X)$, hence that, directly from the definition of $\mathcal{J}_+(X)$, $p_{Y\cup\{b\}} f_b \in \mathcal{J}_+(X)$; *a fortiori* $p_{Y\cup\{b\}} f_b \in \mathcal{P}_+(X) + \mathcal{J}_+(X)$. ∎

Remark. Note that the only property of the extension ex that was used is that once $\mathrm{span} I' \subset \mathrm{span} I$, then $B_0 \cap I \subset B_0 \cap I'$. It is probably easy to show that every extension of such type is a greedy extension with respect to some ordering of B_0. □

Invoking Result 2.11, we obtain

Corollary 4.4 $\dim \mathcal{P}_+(X) \geq \dim \mathcal{D}_+(X)$. ∎

The corollary implies that $\dim \mathcal{P}_+(X) \geq \#\mathbb{B}_+(X)$. This last estimate can be proved directly: Order X' so that B_0 is placed after X, and the internal order within B_0 is retained. Then follow the construction of a homogeneous basis for $\mathcal{P}(X')$ from Section 3.2. Observe that a polynomial in that basis is of the form $p_{X'(B)}$, $X'(B) \subset X'$, and that $X'(B)$ is then a subset of X if and only if $B \in \mathbb{B}_+(X)$. Thus

$$\{p_{X'(B)} : \ B \in \mathbb{B}_+(X)\} \subset \mathcal{P}_+(X),$$

and we get the desired bound from the linear independence of these polynomials. We will come back to this issue later, since the polynomials above form a *basis* for $\mathcal{P}_+(X)$, and we will use the cardinality of the sets $X'(B)$, $B \in \mathbb{B}_+(X)$, in order to provide an algorithm for computing the forthcoming external Hilbert series $h_{X,+}$ of X.

Recalling (7) and (8), we define

$$\mathcal{I}_+(X) := \mathrm{Ideal}\{p_{\eta_H}^{m(H)+1} \ : \ H \in \mathcal{F}(X), \ \eta_H \perp H\}.$$

Theorem 4.5 $\mathcal{P}_+(X) \subseteq \ker \mathcal{I}_+(X)$.

Proof. Given any $Y \subset X$ and any facet hyperplane H, we have that $D_{\eta_H}(p_Y) = p_{Y\cap H} D_{\eta_H} p_{Y\backslash H}$. The result then follows from the fact that $\#(Y \backslash H) \leq m(H)$. ∎

Corollary 4.6 $\dim P_+(X) \leq \dim \ker \mathcal{I}_+(X)$. ∎

Theorem 4.7 $\dim\ker\mathcal{I}_+(X) \le \#\mathbb{B}_+(X)$.

Before we embark on a proof of this theorem, we must make a few auxiliary statements first. For our next result, we will use the symbol

$$\mathcal{P}_N^0(S) := \operatorname{span}\{p_Y : Y \subset S, \#Y = N\}$$

to denote the space of homogeneous polynomials of degree N in the variables S, for any (possibly infinite) set S and any nonnegative integer N.

Proposition 4.8 *Let I be an ideal of Π and let S be a subspace of $\mathbb{R}^n$ of dimension $d \ge 2$. Let $S_1, \ldots, S_k$ be distinct subspaces of S, each of dimension $d-1$. Suppose that, for $n_1, \ldots, n_k \in \mathbb{N}$, the ideal I contains all homogeneous polynomials in variables S_i of degree n_i:*

$$\mathcal{P}_{n_i}^0(S_i) \subset I.$$

Then

$$\mathcal{P}_N^0(S) \subset I \quad \text{whenever} \quad (N+1)(k-1) \ge \sum_{i=1}^k n_i.$$

Proof. Pick $l \in S$. We need to show that the polynomial p_l^N lies in the ideal I. Choose a subspace $V \subset S$ of dimension 2 such that $l \in V$ but $V \not\subset S_i$ for all $i = 1, \ldots, k$. The dimension formula

$$\dim V + \dim S_i = \dim(V \cap S_i) + \dim(V + S_i)$$

immediately implies that $\dim(V \cap S_i) = 1$ for each $i = 1, \ldots, k$. So, for each i, there exists a nonzero vector $h_i \in V \cap S_i$. By the assumption of the Proposition,

$$p_{h_i}^{n_i} \in I, \qquad i = 1, \ldots, k.$$

Observe that

$$H_i := p_{h_i}^{n_i}\mathcal{P}_{N-n_i}^0(V) \subset I \cap \mathcal{P}_N^0(V), \qquad i = 1, \ldots, k.$$

We will now argue that $\sum_{i=1}^k H_i = \mathcal{P}_N^0(V)$. Indeed, if not, then there exists a nonzero polynomial $q \in \mathcal{P}_N^0(V)$ such that $\langle q, p\rangle = 0$ for all $p \in H_i$. This means

$$0 = \langle q, p_{h_i}^{n_i} p\rangle = \langle p(D)q, p_{h_i}^{n_i}\rangle \tag{12}$$

for any $p \in \mathcal{P}_{N-n_i}^0(V)$. But the last expression in (12) is, up to the factor $n_i!$, equal to $(p(D)q)(h_i)$. Our last setup can therefore be reformulated as a univariate problem: there exists a nonzero polynomial $q \in \mathcal{P}_N(\mathbb{R})$ and distinct points $h_i \in \mathbb{R}$, $i = 1, \ldots, k$ such that

$$(p(D)q)(h_i) = 0, \qquad i = 1, \ldots, k \quad \text{for all} \quad p \in \mathcal{P}_{N-n_i}(\mathbb{R}). \tag{13}$$

But this is a Vandermonde linear homogeneous system of $\sum_{i=1}^k (N - n_i + 1)$ equations in $N+1$ unknown coefficients of q, which has a nontrivial solution if and only if

$$\sum_{i=1}^k (N - n_i + 1) < N+1 \quad \text{iff} \quad Nk - \sum_{i=1}^k n_i + k < N+1 \quad \text{iff} \quad (N+1)(k-1) < \sum_{i=1}^k n_i,$$

contrary to the assumption of this Proposition. Thus $p_l^N \in I$ for all nonzero vectors $l \in S$, hence every homogeneous polynomial in S of degree N is in I and therefore $\mathcal{P}_N^0(S) \subset I$. □

Corollary 4.9 *Let $Y \subset X$, and let $0 \neq \eta \perp \operatorname{span} Y$. Then $p_\eta^{\#(X \setminus \operatorname{span} Y)+1} \in \mathcal{I}_+(X)$.*

Proof. We run the proof by induction on $n - \dim(\operatorname{span} Y)$. When Y spans a hyperplane, we have $p_\eta^{\#(X \setminus \operatorname{span} Y)+1} \in \mathcal{I}_+(X)$ by definition of $\mathcal{I}_+(X)$. When $\dim(\operatorname{span} Y) \leq n-2$, we denote $S := (\operatorname{span} Y)\perp$ and consider all possible ways to add one more vector to the set Y to increase the dimension of $\operatorname{span} Y$. Call the orthogonal complements of the spans of these sets S_1 through S_k. Note that addition of different vectors x to Y may produce the same subspace $\operatorname{span}(Y \cup \{x\})$ and therefore the same orthogonal complement. If that is the case, we list such a subspace S_i only once.

With $\eta \in S$ and S_1 through S_k subspaces of S of codimension 1, we are now in the setting of Proposition 4.8, so may conclude that

$$p_\eta^N \in \mathcal{I}_+(X)$$

whenever $(N+1)(k-1) \geq \sum_{i=1}^k n_i$, where $n_i = \#(X \setminus (S_i \perp)) + 1$ by the inductive hypothesis. Note that the count $\#(X \setminus (S_i \perp))$ performed for all i accounts for every vector of $(X \setminus (S\perp)) = X \setminus \operatorname{span} Y$ exactly $k-1$ times, hence

$$\begin{aligned}
\sum_{i=1}^k \#(X \setminus (S_i \perp)) &= (k-1) \cdot \#(X \setminus \operatorname{span} Y), \quad \text{hence} \\
(k-1)(N+1) &\geq (k-1) \cdot \#(X \setminus \operatorname{span} Y) + k, \quad \text{or, equivalently,} \\
N &\geq \#(X \setminus \operatorname{span} Y) + 1/(k-1).
\end{aligned}$$

The last inequality is satisfied whenever $N \geq \#(X \setminus \operatorname{span} Y) + 1$, so we are done. $\square$

We are now in a position to give a proof of Theorem 4.7.

Proof of Theorem 4.7. Step I. We append to X an auxiliary basis B_0, and obtain $X' := X \cup B_0$. We choose B_0 to be in general position with respect to X. Note that

$$\mathcal{I}(X') \subset \mathcal{I}_+(X).$$

Indeed, let $H \in \mathcal{F}(X')$ be a facet hyperplane of X' and let η be normal to H. Then

$$\#(X' \setminus H) = \#(X \setminus H) + \#(B_0 \setminus H) \geq \#(X \setminus H) + 1 = \#(X \setminus \operatorname{span} Y) + 1,$$

where $Y := X \cap H$. Applying Corollary 4.9, we see that $p_\eta^{\#(X' \setminus H)} \in \mathcal{I}_+(X)$. Thus all generators of $\mathcal{I}(X')$ are in $\mathcal{I}_+(X)$ and therefore $\mathcal{I}(X') \subset \mathcal{I}_+(X)$. Consequently,

$$\ker \mathcal{I}_+(X) \subset \ker \mathcal{I}(X') = \mathcal{P}(X').$$

Step II. We order X' in a way that B_0 is placed after X, we let $\{Q_B := p_{X'(B)} : B \in \mathbb{B}(X')\}$ be the homogeneous basis for $\mathcal{P}(X')$ (per the given order). We define

$$\mathbb{B}' := \mathbb{B}(X') \setminus \mathbb{B}_+(X), \quad F := \operatorname{span}\{Q_B \, : \, B \in \mathbb{B}'\}.$$

We will now prove that $F \cap \ker \mathcal{I}_+(X) = \{0\}$, hence that the quotient map $\mathcal{P}(X') \to \mathcal{P}(X')/F$ is an injection on $\ker \mathcal{I}_+(X)$, and

$$\dim \ker \mathcal{I}_+(X) \leq \dim \mathcal{P}(X') - \dim F = \#\mathbb{B}_+(X) = \#\mathbb{I}(X).$$

Let

$$f := \sum_{B \in \mathbb{B}'} a(B) Q_B \in F.$$

Assume $f \in \ker \mathcal{I}_+(X)$. We claim that $a(B) = 0, \quad \forall B \in \mathbb{B}'$. To this end, we grade the bases in $\mathbb{B}'$ according to the location in B_0 of their maximal element, with respect to our fixed order. Note that the maximal element must be in B_0, since otherwise $B \in \mathbb{B}(X)$. Assume that there exists $B_1 \in \mathbb{B}'$ such that $a(B_1) \neq 0$. Assume further, without loss of generality, that $a(B) = 0$ for every basis $B \in \mathbb{B}'$ with higher grade. We then choose $H := \operatorname{span}(B_1 \backslash \max\{B_1\})$, let η be normal to H, and consider the differential operator D_η^k, with

$$k := \#\{x \in X' \backslash H : x \notin \operatorname{span}\{b \in B_1 : \ b \preceq x\}\}.$$

Recall that the basis B_1 is *not* obtained by a greedy completion of an independent subset of X and that B_0 is in general position with respect to X. This implies

$$k \geq \#(X \backslash Y) + 1, \qquad \text{where} \quad Y := X \cap H.$$

By Proposition 4.8, $p_\eta^k \in \mathcal{I}_+(X)$, therefore D_η^k annihilates $\ker \mathcal{I}_+(X)$ and, in particular, f.

Finally, consider the set $\mathbb{B}'' \subset \mathbb{B}'$ of bases B such that (i) their grade does not exceed the grade of B_1, and (ii) $D_\eta^k Q_B \neq 0$. These are the bases with the property $\#(X'(B) \backslash H) \geq k$ or, equivalently,

$$\#\{x \in X' \backslash H : x \notin \operatorname{span}\{b \in B : b \preceq x\}\} \geq \#\{x \in X' \backslash H : x \notin \operatorname{span}\{b \in B_1 : b \preceq x\}\}.$$

Since no element of such a basis B is located further than the maximum element $\max\{B_1\}$ of B_1, each such basis B must consist of a basis for H augmented by the vector $\max\{B_1\}$ itself. Hence, the set

$$\{D_\eta^k Q_B : B \in \mathbb{B}''\}$$

consists of (nonzero multiples of) elements of the homogeneous basis of $\mathcal{P}(X' \cap H)$. This implies that $a(B) = 0$ for each $B \in \mathbb{B}''$, which leads to a contradiction, since $B_1 \in \mathbb{B}''$. □

We now state formally the main theorem of this section.

Theorem 4.10

(1) $\dim \mathcal{P}_+(X) = \dim \mathcal{D}_+(X) = \#\mathbb{I}(X)$.

(2) The map $p \mapsto \langle p, \cdot \rangle$ *is a bijection between* $\mathcal{D}_+(X)$ *and* $\mathcal{P}_+(X)'$.

(3) $\mathcal{D}_+(X) = \Pi(V_+(X)) = \ker \mathcal{J}_+(X)$.

(4) The set $V_+(X, \lambda)$ *is correct for the space* $\mathcal{D}_+(X)$*, as well as for the space* $\mathcal{P}_+(X)$.[7]

(5) $\mathcal{P}_+(X) = \ker \mathcal{I}_+(X)$.

(6) $\mathcal{P}_+(X) \oplus \mathcal{J}_+(X) = \Pi$.

[7] Note that the set $V_+(X, \lambda)$ depends on X, on λ and on the augmented order basis B_0. The space $\mathcal{D}_+(X)$ depends on X and B_0, but not on λ. Finally, $\mathcal{P}_+(X)$ depends only on X.

Proof. The proof is analogous to that of Theorem 3.8. We put together inequalities obtained in Corollaries 4.2, 4.4, 4.6 and in Theorem 4.7 to get

$$\#\mathbb{B}_+(X) \le \dim \mathcal{D}_+(X) \le \dim \mathcal{P}_+(X) \le \dim \ker \mathcal{I}_+(X) \le \#\mathbb{B}_+(X).$$

This shows that equalities must hold throughout. We then invoke Theorems 4.1, 4.3 and 4.5, along with Results 2.3 and 2.11 and Corollary 2.6, to prove the remaining claims of this theorem. $\square$

Theorem 4.11 *Let $\mathcal{Z}_+(X)$ be the integer points in the closed zonotope $Z(X)$. Then*

$$\Pi(Z_+(X)) = \mathcal{P}_+(X) = \ker \mathcal{I}_+(X),$$

provided that X is unimodular.

Proof. We first recall that, according to Result 2.2,

$$\#\mathcal{Z}_+(X) = \#\mathbb{I}(X)$$

in case X is unimodular. That implies, by invoking Theorem 4.10, that

$$\dim \Pi(\mathcal{Z}_+(X)) = \#\mathbb{B}_+(X) = \dim \mathcal{P}_+(X).$$

Hence our claim follows from the fact that

$$\Pi(\mathcal{Z}_+(X)) \subset \ker \mathcal{I}_+(X).$$

The proof of this latter inclusion follows closely that of Theorem 3.9, hence is merely outlined: we need to show that, given any generator $q := p_{\eta_H}^{m(H)+1}$, $H \in \mathcal{F}(X)$, of $\mathcal{I}_+(X)$, there exists $p \in \Pi$ that vanishes on $\mathcal{Z}_+(X)$ and satisfies $p_\uparrow = q$. The existence of such p follows from the fact that, whatever facet hyperplane H we choose, the set $\mathcal{Z}_+(X)$ lies in the union

$$\cup_{j=0}^{m(H)} (a_j + H),$$

with $a_j := \sum_{k=1}^{j} y_k$, and where $\{y_j\}_{j=1}^{m(H)} := X \backslash H$. It is the unimodularity that guarantees that the above hyperplanes do not depend on the order, and that the entire set $\mathcal{Z}_+(X)$ lies in their union. (The description above assumes that $X \backslash H$ all lie on the same side of H; the modifications that are needed for the general case are notational.) $\square$

4.2 Homogeneous basis and Hilbert series for $\mathcal{P}_+(X)$

As before, we order X, and define, for each $I \in \mathbb{I}(X)$,

$$X(I) := \{x \in X \,:\, x \notin \operatorname{span}\{b \in I \,:\, b \le x\}\}.$$

Our goal now is to show that the set

$$\{Q_I := p_{X(I)} \,:\, I \in \mathbb{I}(X)\}$$

is a basis for $\mathcal{P}_+(X)$.

Theorem 4.12 *The set* $\{Q_I := p_{X(I)} : I \in \mathbb{I}(X)\}$ *is a basis for* $\mathcal{P}_+(X)$.

Proof. Since the cardinality of the given set of polynomials is $\#I(X) = \#B_+(X) = \dim \mathcal{P}_+(X)$, and since obviously each one of these polynomials lies in $\mathcal{P}_+(X)$, we only need to show that the set $\{Q_I : I \in \mathbb{I}(X)\}$ is linearly independent. For the proof of this part, we order the augmented set $X' := X \cup B_0$ such that X retains its internal order, and B_0 is placed after X. Recall that each $I \in \mathbb{I}(X)$ has a well-defined extension to a basis $\mathrm{ex}(I) \in \mathbb{B}(X')$:

$$\mathbb{I}(X) \to \mathbb{B}_+(X) \subset \mathbb{B}(X') : I \mapsto \mathrm{ex}(I).$$

We therefore examine the known homogeneous basis for $\mathcal{P}(X')$. The polynomials in that latter basis are $p_{X'(B)}$, $B \in \mathbb{B}(X')$, with

$$X'(B) := \{x \in X' : x \notin \mathrm{span}\{b \in B : b \preceq x\}\}.$$

Now, given $I \in \mathbb{I}(X)$, since $\mathrm{ex}(I) \in \mathbb{B}(X')$ is a *greedy* extension of I, it easily follows that

$$X(I) = X'(\mathrm{ex}(I)),$$

hence that

$$Q_I := p_{X(I)} = p_{X'(\mathrm{ex}(I))}.$$

Thus, the set $\{Q_I\}_{I \in \mathbb{I}(X)}$ is a subset of the basis $\{p_{X'(B)} : B \in \mathbb{B}(X')\}$ for $\mathcal{P}(X')$, and the requisite linear independence thus follows. $\square$

Note that the basis we just constructed is a homogeneous extension of the homogeneous basis that was constructed for $\mathcal{P}(X)$ in Section 3.2. Moreover, the valuation val that was defined there on $\mathbb{B}(X)$ has just been extended also in the most natural way to $\mathbb{I}(X)$:

$$\mathrm{val}(I) := \#X(I), \quad I \in \mathbb{I}(X).$$

This motivates us to associate X with *an external Hilbert series:*

$$h_+ := h_{X,+} : \mathbb{N} \to \mathbb{N} \quad : \quad j \mapsto \#\{I \in \mathbb{I}(X) : \ \mathrm{val}(I) = j\}.$$

$h_+(j)$ equals thus to the dimension of $\mathcal{P}_+(X) \cap \Pi^0_j$ and, by duality with $\mathcal{D}_+(X)$, also to the dimension of $\mathcal{D}_+(X) \cap \Pi^0_j$. The external Hilbert function is very special, in the sense that its last non-zero entry always equals one. This fact is not easy to observe by examining either of $\mathcal{I}_+(X)$, $\mathcal{J}_+(X)$ or $\mathcal{D}_+(X)$. However, it trivially follows from the structure of $\mathcal{P}_+(X)$: the unique polynomial of maximal degree of the form p_Y, $Y \subset X$, is p_X. One can use the above construction of a basis for $\mathcal{P}_+(X)$ to conclude that $h_{X,+}(\#X - 1)$ is the number of equivalence classes of X under the equivalence ($x \sim y$ iff $\{x, y\}$ is a dependent set).

Example 4.13 *Let*

$$X = [x_1, x_2, x_3] := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

This X corresponds to a complete graph of three vertices, and is unimodular, as is every graphical X. The zonotope $Z(X)$ has 7 vertices in its closure. A basis for $\mathcal{P}_+(X)$ is given by

$$\{p_Y : Y \in 2^X \backslash \{\{x_3\}\}\}.$$

The external Hilbert series is $h_{X,+} = (1,2,3,1)$. With $x_4 := x_3\perp = (1,1)'$, the ideal $\mathcal{I}_+(X)$ is generated by the three polynomials

$$p_{x_i}^3, \quad i = 1,2,4.$$

It is clear that $h_{X,+}$ is indeed the correct Hilbert series for this ideal.

In contrast, the space $\mathcal{D}_+(X)$ and its ideal $\mathcal{J}_+(X)$ are not unique, and depend on the choice of the augmented basis B_0. If we choose $B_0 = (y,z)$ with y,z in general position with respect to X, then the generators of $\mathcal{J}_+(X)$ become

$$p_{X\cup z}, \quad \textit{and} \quad p_{(X\backslash x)\cup y},\ x \in X.$$

$\square$

Theorem 4.10 yields the following characterization $\mathcal{P}_+(X)$:

Theorem 4.14

$$\mathcal{P}_+(X) = \bigcap_{B\in\mathbb{B}(X)} \mathcal{P}(X\cup B).$$

Proof. The fact that $\mathcal{P}_+(X) \subset \mathcal{P}(X\cup B)$, for any fixed basis B for $\mathbb{R}^n$ follows trivially from the definitions of $\mathcal{P}(X)$ and $\mathcal{P}_+(X)$ (and was used multiple times). We prove that every polynomial p in the intersection lies in $\ker \mathcal{I}_+(X)$ $(= \mathcal{P}_+(X))$. Let p be such a polynomial, and let $H \in \mathcal{F}(X)$. We need to show that, with $\eta_H \perp H$, $D_{\eta_H}^{m(H)+1} p = 0$, with $m(H) := \#(X\backslash H)$. To this end, we choose a basis $B \in \mathbb{B}(X)$ such that $\operatorname{span}(B\cap H) = H$ (such a basis exists, since H is a facet hyperplane.) Then, with $X' := X \cup B$, $p \in \mathcal{P}(X') = \ker \mathcal{I}(X')$, hence is annihilated by $D_{\eta_H}^{\#(X'\backslash H)}$. Since only one vector of B lies outside H, we get that $\#(X'\backslash H) = m(H)+1$, and the result follows. $\square$

5 Internal algebra

5.1 Main results

We first recall the definition of internal bases. We impose an (arbitrary but fixed) ordering $\prec$ on X. Let $B \in \mathbb{B}(X)$. If, for each $b \in B$,

$$b \neq \max\{X\backslash H\}, \quad H := \operatorname{span}\{B\backslash b\} \in \mathcal{F}(X),$$

then B is called an *internal basis*. We denote the set of all internal bases by $\mathbb{B}_-(X)$.

For each $B \in \mathbb{B}(X)$, we define the *dual valuation* as follows:

$$\operatorname{val}^*(B) := \#\{b \in B \,:\, b \neq \max\{X\backslash \operatorname{span}(B\backslash b)\}\}.$$

Then,

$$\mathbb{B}_-(X) = \{B \in \mathbb{B}(X) \,:\, \operatorname{val}^*(B) = n\}.$$

We remind the reader about the fact mentioned in Section 2.2:

$$\#\mathbb{B}_-(X) = \sum_{I\in\mathbb{I}(X)} (-1)^{n-\#I}.$$

Remarks. (i) In matroid theory, b is said to be *internally active in* B in the situation encountered above, i.e., if, for $b \in B$ and $H := \mathrm{span}\{B\backslash b\}$, we have that $b = \max\{X\backslash H\}$. The number $n - \mathrm{val}^*(B)$ is known as the *internal activity* of B. (ii) The valuation val^* is then (matroid-)dual to the valuation val in the sense that it coincides with the valuation val on the dual matroid of X. We make no use of this duality since we do not develop zonotopal spaces on the dual matroid within this paper.

With a given order on X, we define the set of *barely long subsets of* X:

$$L_-(X) := \{Y \subset X \;:\; Y \cap B \neq \emptyset, \quad B \in \mathbb{B}_-(X)\}.$$

The corresponding ideal is defined as

$$\mathcal{J}_-(X) := \mathrm{Ideal}\{p_Y \;:\; Y \in L_-(X)\},$$

and the notation for its kernel is

$$\mathcal{D}_-(X) := \ker \mathcal{J}_-(X).$$

It is clear that $\mathcal{J}_-(X) \supset \mathcal{J}(X)$ hence that $\mathcal{D}_-(X) \subset \mathcal{D}(X)$. As is the case with external theory, one easily finds that $\mathcal{D}_-(X)$ depends on the ordering of X. (To recall, $\mathcal{D}(X)$ does not depend on any ordering.)

Given a generic X-hyperplane arrangement $\mathcal{H}(X, \lambda)$, we pick those vertices of it that are associated with the internal bases $B \in \mathbb{B}_-(X)$ and call the resulting set $V_-(X, \lambda)$. Then Theorem 2.7 and its Corollary 2.8 apply to the space $\mathcal{D}_-(X)$ and its associated vertex set $V_-(X, \lambda)$ to yield the following two results.

Theorem 5.1 $\Pi(V_-(X, \lambda)) \subseteq \mathcal{D}_-(X)$. □

Corollary 5.2 $\dim \mathcal{D}_-(X) \geq \#\mathbb{B}_-(X)$. □

We now define a polynomial space and its ideal that will serve as duals to the space $\mathcal{D}_-(X)$ and its ideal.

$$\begin{aligned} \mathcal{P}_-(X) &:= \cap_{x \in X} \mathcal{P}(X\backslash x), \\ \mathcal{I}_-(X) &:= \mathrm{Ideal}\{p_{\eta_H}^{m(H)-1} \;:\; \eta_H \perp H, \quad H \in \mathcal{F}(X)\}. \end{aligned} \tag{14}$$

Note that $\mathcal{P}_-(X)$ and $\#\mathbb{B}_-(X)$ are independent of the order $\prec$. Needless to say, the set $\mathbb{B}_-(X)$ itself depends on that order.

Theorem 5.3

$$\mathcal{P}_-(X) = \ker \mathcal{I}_-(X).$$

Moreover, for every $B \in \mathbb{B}(X)$,

$$\mathcal{P}_-(X) = \cap_{x \in B} \mathcal{P}(X\backslash x).$$

Proof. We prove this result by examining the corresponding ideals: since $\mathcal{P}(X\backslash x) = \ker \mathcal{I}(X\backslash x)$, Theorem 3.8, the stated result is proved once we show that (i) $\mathcal{I}(X\backslash x) \subset \mathcal{I}_-(X)$, for every $x \in X$, and (ii) Given any $B \in \mathbb{B}(X)$, $\mathcal{I}_-(X) \subset \mathrm{Ideal}\{\cup_{b \in B} \mathcal{I}(X\backslash b)\}$.

For the proof of (i), fix $x \in X$, and denote $X' := X\backslash x$. A generator Q in the ideal $\mathcal{I}(X')$ is of the form $Q := p_{\eta_H}^{\#(X'\backslash H)}$, with $H \in \mathcal{F}(X')$. Then H is also a facet hyperplane of X, and obviously $\#(X'\backslash H) \geq \#(X\backslash H) - 1$, and hence $Q \in \mathcal{I}_-(X)$, and (i) follows.

For (ii), we fix $B \in \mathbb{B}(X)$, and pick a generator of $\mathcal{I}_-(X)$: $Q = p_{\eta_H}^{m(H)-1}$, $H \in \mathcal{F}(X)$, (14). We then choose $x \in B \backslash H$, and denote $X' := X \backslash x$. Since $x \notin H$, it is clear that $H \in \mathcal{F}(X')$, and then $\#(X' \backslash H) = \#(X \backslash H) - 1 = m(H) - 1$. Thus the polynomial Q lies in $\mathcal{I}(X')$, and (ii) follows. □

Going back to the order we impose on X (which is required for the definition of $\mathcal{J}_-(X)$), we recall the homogeneous construction of a basis $(Q_B)_{B \in \mathbb{B}(X)}$ for $\mathcal{P}(X)$ (see Theorem 3.12), per that order. Since $\mathbb{B}(X)$ is decomposed into internal and non-internal bases, it makes sense to decompose $\mathcal{P}(X)$ accordingly, viz.,

$$\mathcal{P}_{in}(X) := \operatorname{span}\{Q_B : B \in \mathbb{B}_-(X)\}, \quad \mathcal{P}_{ex}(X) := \operatorname{span}\{Q_B : \ B \in \mathbb{B}(X) \backslash \mathbb{B}_-(X)\}.$$

Then

$$\mathcal{P}(X) = \mathcal{P}_{in}(X) \oplus \mathcal{P}_{ex}(X),$$

with the "internal summand" $\mathcal{P}_{in}(X)$ having the "right dimension", i.e., $\#\mathbb{B}_-(X)$. However, in general that space differs from $\mathcal{P}_-(X)$ (whose dimension will be shown to equal $\#\mathbb{B}_-(X)$, too). On the other hand, the complementary inclusion is true:

Lemma 5.4 $\mathcal{P}_{ex}(X) \subset \mathcal{J}_-(X)$, *hence*

$$\operatorname{codim} J_-(X) \leq \#\mathbb{B}_-(X).$$

Proof. Fix $B \in \mathbb{B}(X) \backslash \mathbb{B}_-(X)$. Then B contains an internally active b: with $H := \operatorname{span}(B \backslash b) \in \mathcal{F}(X)$, b is the last vector in $X \backslash H$. Examining the definition (9) of the set $X(B)$, it is then clear that $X \backslash (X(B) \cup B) \subset H$. (Indeed, if $x \in X \backslash (H \cup B)$, then $x \prec b$, hence it is impossible that $x \in \operatorname{span}\{b' \in B : b' \prec x\}$, since the latter span lies in H.) Next, it easily follows that b belongs to, and is internally active in every basis $B' \subset X \backslash X(B)$. Thus $X \backslash X(B)$ does not contain an internal basis, hence $X(B) \in L_-(X)$.

Recall now that $\Pi = \mathcal{P}(X) \oplus \mathcal{J}(X)$ according to Theorem 3.8, and since $\mathcal{J}(X) \subset \mathcal{J}_-(X)$, we conclude that $\mathcal{P}_{ex}(X) + \mathcal{J}(X) \subset \mathcal{J}_-(X)$, hence

$$\Pi = \mathcal{P}_{in}(X) \oplus \mathcal{P}_{ex}(X) \oplus \mathcal{J}(X) = \mathcal{P}_{in} + \mathcal{J}_-(X).$$

Consequently,

$$\operatorname{codim} \mathcal{J}_-(X) \leq \dim \mathcal{P}_{in}(X) = \#\mathbb{B}_-(X).$$

□

Theorem 5.5

$$\dim \mathcal{D}_-(X) = \#\mathbb{B}_-(X),$$

and $\Pi(V_-(X, \lambda)) = \mathcal{D}_-(X)$.

Proof. Since $\mathcal{D}_-(X) = \ker \mathcal{J}_-(X)$, by definition, we have that $\dim \mathcal{D}_-(X) = \operatorname{codim} \mathcal{J}_-(X)$. However, by Corollary 5.2, $\dim \mathcal{D}_-(X) \geq \#\mathbb{B}_-(X)$, while by Lemma 5.4, $\operatorname{codim} \mathcal{J}_-(X) \leq \#\mathbb{B}_-(X)$. Thus:

$$\dim \mathcal{D}_-(X) = \#\mathbb{B}_-(X).$$

But then, $\Pi(V_-(X, \lambda))$ is a subspace of $\mathcal{D}_-(X)$ (Theorem 5.1) of dimension $\#\mathbb{B}_-(X)$, so we have that $\Pi(V_-(X, \lambda)) = \mathcal{D}_-(X)$. □

Corollary 5.6 $\mathcal{J}_-(X) = \mathcal{J}(X) \oplus \mathcal{P}_{ex}(X)$.

Theorem 5.7 $\mathcal{P}_-(X) + \mathcal{J}_-(X) = \Pi$. *In particular,* $\dim \mathcal{P}_-(X) \geq \#\mathbb{B}_-(X)$.

Proof. Since we already know that $\operatorname{codim} \mathcal{J}_-(X) = \#\mathbb{B}_-(X)$, the second claim in the theorem follows from the first. Let us thus prove the first.

From Lemma 5.4, we know that $\mathcal{P}_{in}(X) + \mathcal{J}_-(X) = \Pi$. Thus, it is enough to show that $\mathcal{P}_{in} \subset \mathcal{P}_-(X) + \mathcal{J}_-(X)$. We achieve this latter relation by showing that every polynomial Q_B, $B \in \mathbb{B}_-(X)$, lies in $\mathcal{P}_-(X) + \mathcal{J}_-(X)$, and use the following general approach. Fixing $B \in \mathbb{B}_-(X)$, we know that $Q_B = p_{X(B)}$, for suitable $X(B) \subset X$. We decompose $X(B)$ in a certain way $X(B) = Z \cup W$. Thus

$$Q_B = p_Z p_W.$$

We then replace each $w \in W$ by a vector w' (not necessarily in X), to obtain a new polynomial

$$\widetilde{Q}_B := p_Z p_{W'},$$

and prove that (i) $\widetilde{Q}_B \in \mathcal{P}_-(X)$, and (ii) $Q_B - \widetilde{Q}_B \in \mathcal{J}_-(X)$.

So, let $Q_B = p_{X(B)}$ be given. If $Q_B \in \ker \mathcal{I}_-(X) = \mathcal{P}_-(X)$, there is nothing to prove. Otherwise, let $\mathbf{H} \subset \mathcal{F}(X)$ be the collection of *all* facet hyperplanes for which $D_{\eta_H}^{m(H)-1} Q_B \neq 0$. The set $\mathbf{H}$ is not empty, since otherwise $Q_B \in \ker \mathcal{I}_-(X)$. Given $H \in \mathbf{H}$, we conclude that $\#(X(B)\backslash H) \geq m(H) - 1$, hence that, with $Y := X\backslash X(B)$, $\#(Y\backslash H) \leq 1$. Since $B \subset Y$, the set $Y\backslash H$ must be a singleton $x_H \in B$. We denote

$$X_{\mathbf{H}} := \{x_H : \ H \in \mathbf{H}\}.$$

Define

$$W := \{\max\{X\backslash H\} : \ H \in \mathbf{H}\}.$$

We index the vectors in W according to their order in X: $W = \{w_1 \prec w_2 \prec \ldots \prec w_k\}$. For each $1 \leq i \leq k$, we define

$$X_i := \{x_H : \ H \in \mathbf{H}, \ \max\{X\backslash H\} = w_i\}, \quad \mathbf{H}_i := \{H \in \mathbf{H} : \ x_H \in X_i\}.$$

Thus, $X_{\mathbf{H}} = \bigcup_{i=1}^k X_i$.

Setting all these notations, we first observe that $W \cap X_{\mathbf{H}} = \emptyset$, i.e., w_i does not lie in X_i. Indeed, the set $X_{\mathbf{H}}$ is a subset of every $B' \in \mathbb{B}(Y)$, with $\operatorname{span}(B'\backslash x_H) = H$ for each $x_H \in X_{\mathbf{H}}$. If some x_H is $\max\{X\backslash H\}$, it will be internally active in every $B' \in \mathbb{B}(Y)$, which would imply that $\mathbb{B}(Y)$ does not contain internal bases, which is impossible since $B \in \mathbb{B}(Y)$. Thus, $W \subset X(B)$, and we define $Z := X(B)\backslash W$, to obtain

$$Q_B = p_Z p_W.$$

Define further:

$$S_i := \cap\{H : H \in \cup_{j=1}^i \mathbf{H}_j\}, \quad S_0 := \mathbb{R}^n.$$

Then, for $i = 1, \ldots, k$, $S_{i-1} = S_i \oplus \operatorname{span} X_i$, and $w_i \in S_{i-1}\backslash S_i$. Thus, for $i = 1, \ldots k$, the vector w_i admits a unique representation of the form

$$w_i = w_i' + \sum_{x \in X_i} a_x x, \quad w_i' \in S_i, \ a_x \in \mathbb{R}\backslash\{0\}. \tag{15}$$

Define

$$W' = \{w'_1, \ldots, w'_k\}, \text{ and } \widetilde{Q}_B := p_Z p_{W'}.$$

We prove first that

$$\widetilde{Q}_B - Q_B = p_Z(p_{W'} - p_W)$$

lies in $\mathcal{J}_-(X)$. To this end, we multiply out the product

$$p_{W'} = \prod_{i=1}^{k} p_{w'_i} = \prod_{i=1}^{k} (p_{w_i} - \sum_{x \in X_i} a_x p_x). \tag{16}$$

Every summand in the above expansion is of the form p_Ξ, with Ξ a suitable mix of W-vectors and $X_{\mathbf{H}}$-vectors. The summand p_W in the above expansion in canceled when we subtract Q_B. Any other Ξ is obtained from W by replacing at least once a w_i vector by some vector in X_i, which we denote by x_i. Let $w_{i_1} \prec w_{i_2} \prec \ldots \prec w_{i_j}$ be all the w-vectors in $W \backslash \Xi$, and let H_1 be the facet hyperplane that corresponds to x_{i_1} ($H_1 := \text{span}(B \backslash x_{i_1})$.) Then, we have that $w_{i_1} \in X \backslash (Z \cup \Xi) =: Y'$, and we claim that $Y' \backslash w_{i_1} \subset H_1$. To this end, we write $Y' \backslash H_1 = ((Y' \cap Y) \backslash H_1) \cup (Y' \backslash Y) \backslash H_1$. Now, $Y \backslash H_1 = x_{i_1}$, and since $x_{i_1} \notin Y'$ (as it was replaced by w_{i_1}), the term $(Y' \cap Y) \backslash H_1$ is empty. The second term consists of $(w_{i_m})_{m=1}^{j} \backslash H_1$. However, $w_{i_m} \in S_{i_m - 1} \subset S_{i_1} \subset H_1$, for every $m \geq 2$. Thus, w_{i_1} is the only vector in $Y' \backslash H_1$. Being also the last vector in $X \backslash H_1$, we conclude that w_{i_1} is internally active in every $B \in \mathbb{B}(Y')$, hence that $p_{Z \cup W} \in \mathcal{J}_-(X)$. This being true for every summand in $\widetilde{Q}_B - Q_B$, we conclude that this latter polynomial lies in $\mathcal{J}_-(X)$.

We now prove that $\widetilde{Q}_B = p_{Z \cup W'} \in \ker \mathcal{I}_-(X)$. To this end, we need to show that, for every $H \in \mathcal{F}(X)$, $\#((Z \cup W') \backslash H) < m(H) - 1$. We divide the discussion here to three cases. As before, $Y := X \backslash X(B)$.

Assume first that $H \in \mathbf{H}_i$ for some $1 \leq i \leq k$. Then, for $X(B) = Z \cup W$ we had that $\#((Z \cup W) \backslash H) = m(H) - 1$. Now, x_H is the only vector in $Y \backslash H$, and $x_H \in X_i$. Thus, the subset $X_j \subset Y$, must lie in H for every $j \neq i$, which means that we conclude that, $w_j \in H$ iff $w'_j \in H$ (since $w_j - w'_j \in \text{span} X_j \subset H$). Finally, while $w_i \notin H$, $w'_i \in S_i \subset H$, hence, altogether, $\#(W' \backslash H) < \#(W \backslash H)$, and we reach the final conclusion that

$$\#((Z \cup W') \backslash H) < \#((Z \cup W) \backslash H) = m(H) - 1.$$

Secondly, assume that $S' := S_k \cap H \neq S_k$. Then, necessarily, $U := (Y \cap S) \backslash S'$ contains at least two vectors (otherwise, all the vectors of Y but one lie in the rank deficient set $(Y \cap S') \cup (Y \backslash S_k)$). Now, with

$$m_1 := \#\{w \in W : w \in H \wedge w' \notin H\}, \text{ and } m_2 := \#(\cup_{i=1}^{k} (X_i \backslash H)),$$

we know that $\#((Z \cup W') \backslash H) = m(H) - \#U + m_1 - m_2$. However, we must have that $m_1 \leq m_2$: if $w'_i \notin H$, while $w_i \in H$, then, since $w_i - w'_i \in \text{span} X_i$, we have that $\#(X_i \backslash H) > 0$.

Finally, we assume $H \in \mathcal{F}(X) \backslash \mathbf{H}$, and $S_k \subset H$. Let $j \geq 1$ be the minimal index i for which $S_i \subset H$. We modify the definition of m_1 and m_2 from the second case by replacing W by $W \backslash w_j$ in the definition of m_1, and removing X_j from $\cup_{i=1}^{k} X_i$ in the definition of m_2. We still have that $m_1 \leq m_2$, by the same argument as above. However, the set U that we used in the previous case is not available for us. Instead, we examine the relation

$$w_j - w'_j \in \text{span} X_j.$$

We know *a priori* that $S_j \oplus \mathrm{span} X_j = S_{j-1}$. Since $S_j \subset H$, while $S_{j-1} \not\subset H$, we must have that $X_j \backslash H \neq \emptyset$. But, $w'_j \in H$, hence, with $U := (w_j \cup X_j) \backslash H$, $\# U \geq 2$, hence the argument used for the previous case works here, with U, m_1 and m_2 modified as explained. $\square$

We now establish our second non-trivial theorem in this section.

Theorem 5.8
$$\mathcal{P}_-(X) \cap \mathcal{P}_{ex}(X) = \{0\}.$$
In particular, $\dim \mathcal{P}_-(X) \leq \# \mathbb{B}_-(X)$.

Proof. The second claim follows from the first: since $\mathcal{P}_-(X) \subset \mathcal{P}(X) = \mathcal{P}_{in}(X) \oplus \mathcal{P}_{ex}(X)$, the first claim implies that

$$\dim \mathcal{P}_-(X) \leq \dim \mathcal{P}(X)/\mathcal{P}_{ex}(X) = \dim \mathcal{P}_{in}(X) = \# \mathbb{B}_-(X).$$

In order to prove the first claim, we denote

$$\mathbb{B}' := \mathbb{B}(X) \backslash \mathbb{B}_-(X).$$

Then, with Q_B the polynomial in the homogeneous basis for $\mathcal{P}(X)$ that corresponds to $B \in \mathbb{B}(X)$, we pick a generic function $f \in \mathcal{P}_{ex}(X)$:

$$f = \sum_{B \in \mathbb{B}'} a(B) Q_B,$$

and assume that $f \in \ker \mathcal{I}_-(X)$. The proof that $f = 0$ will be done as follows. In addition to the existing order $\prec$ on X, we will impose a full order $\prec\!\!\!\prec$ on the bases in $\mathbb{B}'$. Assuming $f \neq 0$, we will then select $B' \in \mathbb{B}'$ which is minimal (in the full order $\prec\!\!\!\prec$) with respect to the condition $a(B) \neq 0$. Thus
$$f - a(B') Q_{B'} = \sum_{B \in \mathbb{B}',\ B \succ\!\!\!\succ B'} a(B) Q_B. \tag{17}$$
We will then select a facet hyperplane $H \in \mathcal{F}(X)$, and, with $\eta \in \mathbb{R}^n$ a normal to that hyperplane, apply to both side of (17) the differential operator $D_\eta^{m(H)-1}$. Since $f \in \ker \mathcal{I}_-(X)$, by assumption, $D_\eta^{m(H)-1} f = 0$. Therefore,

$$-D_\eta^{m(H)-1} a(B') Q_{B'} = \sum_{B \in \mathbb{B}',\ B \succ\!\!\!\succ B'} a(B) D_\eta^{m(H)-1} Q_B.$$

The key of the proof will be to show, with the proper selection of the full order, and with proper selection of the hyperplane H, that the polynomial $D_\eta^{m(H)-1} Q_{B'}$ is independent of the polynomials $D_\eta^{m(H)-1} Q_B$, $B \in \mathbb{B}'$, $B \succ\!\!\!\succ B'$. This will imply that $a(B') = 0$, hence will provide the sought-for contradiction.

Here are the details. We start with the introduction of the order on $\mathbb{B}'$. To this end, given $B \in \mathbb{B}'$, and $b \in B$, we recall that b to is said to be *internally active in* B if b is the maximal vector in $X \backslash \mathrm{span}\{B \backslash b\}$. We denote by $\alpha(B)$ the number of internally active vectors in B. Note that $B \in \mathbb{B}'$ if and only if $\alpha(B) > 0$. We choose the order on $\mathbb{B}'$ to respect the number of internally active vectors, i.e.,
$$B \prec\!\!\!\prec \widetilde{B} \Longrightarrow \alpha(B) \leq \alpha(\widetilde{B}).$$

Next, let $B \in \mathbb{B}'$ and $x \in B$. Set $H := \mathrm{span}(B\backslash x)$. We say that x is an *H-shield* of B if $X\backslash\{X(B) \cup x\}$ is not full rank (hence lies in H), but x is not the maximal vector in $X\backslash H$.

Now, in order to show that $a(B') = 0$, we select any internally active $b' \in B'$ (there must be at least one, since $B \in \mathbb{B}'$), define $H' := \mathrm{span}(B'\backslash b')$, and take η to be normal to H'. With $G := D_\eta^{m(H')-1}$, we know that $Gf = 0$ (since $f \in \ker \mathcal{I}_-(X)$, by assumption). Moreover, since $m(H') = \#(X\backslash H')$, and since, for any $B \in \mathbb{B}(X)$, $X(B)$ is disjoint of B (while B contains at least one vector from $X\backslash H'$), we have $\#(X(B)\backslash H') < m(H)$. If $\#(X(B)\backslash H') < m(H) - 1$, then $GQ_B = 0$. Otherwise, up to a non-zero multiplicative constant, $GQ_B = p_{X(B)\cap H'}$. Note that, with $X' := X \cap H'$, we get $X(B) \cap H' = X'(B \cap H')$, i.e., the set $B \cap H'$ spans H' (otherwise, $GQ_B = 0$), hence lies in $\mathbb{B}(X \cap H')$; with X' retaining its X-order $\prec$, the construction of a homogeneous basis for $\mathcal{P}(X \cap H')$ associates the basis $B \cap H'$ with the polynomial $p_{X(B)\cap H'}$, i.e., with the polynomial GQ_B (up to the aforementioned constant). The selection of H' clearly implies that $GQ_{B'} \neq 0$, hence, in particular, that $GQ_{B'} = c' p_{X(B')\cap H'}$ for some $c' \neq 0$. Set

$$\mathbb{B}'' := \{B \in \mathbb{B}' : B \succ B',\ \#(X(B)\backslash H') = m(H') - 1\}.$$

We get

$$-c' a(B') p_{X(B')\cap H'} = \sum_{B\in\mathbb{B}''} c(B) a(B) p_{X(B)\cap H'}. \tag{18}$$

By our argument above, all the polynomial summands on both sides of (18) belong to the homogeneous basis of $\mathcal{P}(X')$. However, *a priori* we cannot conclude that all the coefficients in (18) equal 0, since we have not excluded the possibility that polynomials from the aforementioned homogeneous basis make multiple appearances in (18). Since we only need to prove that $a(B') = 0$, we need only to show that $p_{X(B')\cap H'}$ is not one of the summands in the right hand side of (18). This is equivalent to proving that, for each $B \in \mathbb{B}''$, $B' \cap H' \neq B \cap H'$.

Let us assume that $B \in \mathbb{B}''$ and $B' \cap H' = B \cap H' =: A$. Obviously, $b := B\backslash A \neq b'$ (otherwise, $B = B'$), hence b is an H'-shield of B. We will show that the existence of H'-shield in B implies that $\alpha(B) < \alpha(B')$, which will contradict the assumption that $B' \prec B$.

It remains to prove the crucial thing: that $\alpha(B) < \alpha(B')$. The argument for that is as follows: We recall that $B\backslash b = B'\backslash b' =: A$, and A is a basis for H'. We already know that b is not internally active in B, while b' is internally active in B': b' is the last vector in $X\backslash H'$, and $b \prec b'$. We will show that if $x \in A$ is internally active in B then it is also internally active in B'. Then, all the internally active vectors in B are internally active in B', while B' contains an additional internally active element, viz., b'.

So, let $x \in A$ be internally active in B. Set $S := A\backslash x$. Note that $\mathrm{rank}\, S = n - 2$. If x is not internally active in B', then there exists $y \succ x$ such that $y \notin \mathrm{span}\{S \cup b'\}$. Assume y to be maximal element outside $\mathrm{span}\{S \cup b'\}$. We get the contradiction to the existence of such y by showing that it is impossible to have $y \succ b'$, and it is also impossible to have $y \prec b'$.

If $y \succ b'$, then, since b' is maximal outside $\mathrm{span}\{B\backslash b'\} = \mathrm{span}\, A = \mathrm{span}\{S \cup x\}$, we have that $y \in \mathrm{span}\{S \cup x\}$. Also, since $y \succ x$, and x is maximal outside $\mathrm{span}\{B\backslash x\} = \mathrm{span}\{S \cup b\}$, we have $y \in \mathrm{span}\{S \cup b\}$. But $S \cup b \cup x = B$, and B is independent, hence $y \in \mathrm{span}\, S$, which is impossible since we assume y to be outside $\mathrm{span}\{S \cup b'\}$.

Otherwise, $y \prec b'$. The maximality of y then implies that $x \prec y \prec b'$. The maximality of x outside $\mathrm{span}\{S \cup b\}$ implies that $b' \in \mathrm{span}\{S \cup b\}$. Since $b' \notin S$, we obtain that $\mathrm{span}\{S \cup b\} = \mathrm{span}\{S \cup b'\}$, which is impossible since y lies in exactly one of these two spaces. $\square$

We now state formally the main theorem of this section.

Theorem 5.9

(1) $\dim \mathcal{P}_-(X) = \dim \mathcal{D}_-(X) = \#\mathbb{B}_-(X)$.

(2) *The map* $p \mapsto \langle p, \cdot \rangle$ *is a bijection between* $\mathcal{D}_-(X)$ *and* $\mathcal{P}_-(X)'$.

(3) $\mathcal{D}_-(X) = \Pi(V_-(X,\lambda)) = \ker \mathcal{J}_-(X)$.

(4) *The vertex set* $V_-(X,\lambda)$ *is correct for* $\mathcal{D}_-(X)$ *as well as for* $\mathcal{P}_-(X)$.

(5) $\mathcal{P}_-(X) = \ker \mathcal{I}_-(X)$.

(6) $\mathcal{P}_-(X) \oplus \mathcal{J}_-(X) = \Pi$.

Proof. This proof is not directly parallel but still quite similar to that of Theorem 3.8. We put together inequalities and equalities obtained in Corollary 5.2 and Theorems 5.3, 5.5 and 5.8 to get

$$\#\mathbb{B}_-(X) = \dim \mathcal{D}_-(X) \leq \dim \mathcal{P}_-(X) = \dim \ker \mathcal{I}_-(X) \leq \#\mathbb{B}_-(X).$$

This shows that equalities must hold throughout. We then invoke Theorems 5.3, 5.5 and 5.7, along with Result 2.3, Corollary 2.6 and Result 2.11, to obtain the remaining claims of this theorem. □

Theorem 5.10 *Let* $\mathcal{Z}_-(X)$ *be the integer points in the interior of the zonotope* $Z(X)$*. Then*

$$\Pi(Z_-(X)) = \mathcal{P}_-(X) = \ker \mathcal{I}_-(X),$$

provided that X *is unimodular.*

Proof. The proof is analogous to the proofs of Theorems 3.9 and 4.11 before. We first recall the count

$$\#\mathcal{Z}_-(X) = \#\mathbb{B}_-(X),$$

which is true for a unimodular X. That implies, by invoking Theorem 5.9, that

$$\dim \Pi(\mathcal{Z}_-(X)) = \dim \mathcal{P}_-(X).$$

Hence our claim follows from the fact that

$$\Pi(\mathcal{Z}_-(X)) \subset \ker \mathcal{I}_-(X).$$

The proof of this latter inclusion requires us to show that, given any generator $q := p_{\eta_H}^{m(H)-1}$, $H \in \mathcal{F}(X)$, of $\mathcal{I}_-(X)$, there exists $p \in \Pi$ that vanishes on $\mathcal{Z}_-(X)$ and satisfies $p_\uparrow = q$. The existence of such p follows from the fact that, whatever facet hyperplane H we choose, the set $\mathcal{Z}_-(X)$ lies in the union

$$\cup_{j=1}^{m(H)-1}(a_j + H),$$

with $a_j := \sum_{k=1}^{j} x_k$, and where $\{x_j\}_{j=1}^{m(H)} = X \backslash H$; the hyperplanes in the above union do not depend on the order we impose on $X\backslash H$. As before, we can assume without loss of generality that $X\backslash H$ all lie on the same side of H. □

5.2 Homogeneous basis and Hilbert series for $\mathcal{P}_-(X)$

The *internal Hilbert series* $h_{X,-}$ records the homogeneous dimensions of $\mathcal{P}_-(X)$:

$$h_-(j) := h_{X,-}(j) := \dim(\mathcal{P}_-(X) \cap \Pi_j^0) = \dim(\mathcal{D}_-(X) \cap \Pi_j^0), \quad j \in \mathbb{N}.$$

While it is not true in general that the polynomials $Q_B := p_{X(B)}$, $B \in \mathbb{B}_-(X)$, form a basis for $\mathcal{P}_-(X)$, they *can* be used for computing $h_{X,-}$:

$$h_{X,-}(j) = \#\{B \in \mathbb{B}_-(X) : \mathrm{val}(B) = \#X(B) = \deg Q_B = j\}.$$

In other words, the homogeneous dimensions of the (order-dependent) space $\mathcal{P}_{in}(X)$ coincide with those of $\mathcal{P}_-(X)$:

$$\dim(\mathcal{P}_{in}(X) \cap \Pi_j^0) = \dim(\mathcal{P}_-(X) \cap \Pi_j^0), \quad \forall j.$$

The simplest way to observe this fact, is to follow the proof of Theorem 5.7: Every Q_B there was proved to be writable as

$$Q_B = \widetilde{Q}_B + f_B$$

with $f_B \in \mathcal{J}_-(X)$ and $\widetilde{Q}_B \in \mathcal{P}_-(X)$. The fact that $\widetilde{Q}_B$, $B \in \mathbb{B}_-(X)$, are independent follows directly from the independence of Q_B, $B \in \mathbb{B}_-(X)$, and the fact that the sum $\mathcal{P}_{in}(X) + \mathcal{J}_-(X)$ is direct. Since we know by now that $\dim \mathcal{P}_-(X) = \#\mathbb{B}_-(X)$, we conclude that

Corollary 5.11 *The polynomials $\widetilde{Q}_B$, $B \in \mathbb{B}_-(X)$, from the proof of Theorem 5.7 form a basis for $\mathcal{P}_-(X)$.*

Now, each $\widetilde{Q}_B$ is obtained by replacing some of the factors p_w, $w \in X$ of Q_B, by polynomials $p_{w'}$, $w' \in \mathbb{R}^n \backslash 0$. Thus, trivially, $\deg Q_B = \deg \widetilde{Q}_B$, hence we may indeed compute $h_{X,-}$ via the polynomials $(Q_B)_{B \in \mathbb{B}_-(X)}$.

The fact that the spaces $\mathcal{P}_{in}(X)$ and $\mathcal{P}_-(X)$ are different is somewhat less trivial. For example, in two dimensions they are actually the same. In three dimensions, however, they may not be the same, as the following example shows:

Example 5.12 *Let*

$$X = [x_1, \ldots, x_5] := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then

$$\mathbb{B}_-(X) = \{[x_1, x_2, x_3], [x_1, x_3, x_4], [x_1, x_2, x_4]\} =: (B_1, B_2, B_3).$$

Our theory asserts, then, that $\dim \mathcal{P}_-(X) = 3$. Indeed, one verifies directly that

$$\mathcal{P}_-(X) = \mathrm{span}\{1, p_{x_2}, p_{x_4}\}.$$

The polynomials $Q_{B_1} = 1$, $Q_{B_2} = p_{x_2}$ and $Q_{B_3} = p_{x_3}$ span the space $\mathcal{P}_{in}(X) = \mathrm{span}\{1, p_{x_2}, p_{x_3}\}$. The two spaces, $\mathcal{P}_-(X)$ and $\mathcal{P}_{in}(X)$ are different, but they produce the same Hilbert series:

$$h_{X,-} = (1, 2, 0, 0, \ldots).$$

□

6 Concluding remarks

A key component of the theory of zonotopal algebra is the explicit use of the polynomials p_Y, $Y \subset X$, in the construction of the $\mathcal{J}$-ideals, as well as in the construction of the $\mathcal{P}$-spaces. In the context of the $\mathcal{J}$-ideal, the only deviation is the use of the external basis B_0 for defining $\mathcal{J}_+(X)$: a generator of $\mathcal{J}_+(X)$ is of the form $p_Y p_b$, with $Y \subset X$, and $b \in B_0$. For example, if $\mathrm{rank}(X \backslash Y) = n-1$, then b is the first vector in $B_0 \backslash \mathrm{span}(X \backslash Y)$.

In the context of the $\mathcal{P}$-spaces, the deviation from a direct use of polynomials p_Y occurs in the case of $\mathcal{P}_-(X)$. In an earlier formulation of the internal theory, we defined the internal $\mathcal{P}$-space as

$$\widetilde{\mathcal{P}}_-(X) := \mathrm{span}\{p_Y : Y \in S_-(X)\},$$

with the subset of *very short X-sets* defined as

$$S_-(X) := \{Y \subset X : \mathrm{rank}(X \backslash (Y \cup x)) = n, \ \forall x \in X \backslash Y\}.$$

While this variant *is* spanned by polynomials of the form p_Y, $Y \subset X$, and while it is straightforward to check that this space is a subspace of $\mathcal{P}_-(X)$, we did not prove that the two variants coincide. We conjecture, however, that the two spaces do coincide:

Conjecture 6.1 *For every X, $\mathcal{P}_-(X) = \mathrm{span}\{p_Y : Y \in S_-(X)\}$.*

Note that proving the above conjecture is tantamount to showing that the polynomials $p_Y \in \mathcal{P}_-(X)$, $Y \subset X$, span $\mathcal{P}_-(X)$: a polynomial p_Y, $Y \subset X$ lies in $\mathcal{P}_-(X)$ iff Y is very short. In any event, the proof of Theorem 5.7 reveals the following information about $\mathcal{P}_-(X)$:

Corollary 6.2 *The space $\mathcal{P}_-(X)$ is spanned by polynomials of the form p_Y, $Y \subset \mathbb{R}^n$. Moreover, there exists a basis for $\mathcal{P}_-(X)$ such that each polynomial in that basis is of the form*

$$p_Y p_Z,$$

with $Y \subset X$, $Z \subset \mathbb{R}^n$, and $\#Z \leq n-2$.

A second remark concerns the $\mathcal{I}$-ideals and $\mathcal{D}$-spaces. While the $\mathcal{I}$-ideals admit a simple set of generators, we do not know of any simple algorithm for constructing an *explicit* basis of a $\mathcal{D}$-space. Another remark, of a different flavor, concerns a special property of the central $\mathcal{D}(X)$ space: $\mathcal{D}(X)$ is the smallest translation-invariant subspace of Π that contains $\mathcal{D}(X) \cap \Pi^0_{\#X-n}$. This property does not extend to other $\mathcal{D}$-spaces. For example, with $N := \#X$, $\dim(\mathcal{D}_+(X) \cap \Pi^0_N) = 1$, while $\dim(\mathcal{D}_+(X) \cap \Pi^0_{N-1}) > n$, unless X is a tensor product, i.e., consists of n different vectors, each appearing with arbitrary multiplicity.

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