MINIMAL DEGREE UNIVARIATE PIECEWISE POLYNOMIALS WITH PRESCRIBED SOBOLEV REGULARITY

Amal Al-Rashdan & Michael J. Johnson*

Department of Mathematics
Kuwait University
P.O. Box: 5969 Safat 13060 Kuwait
yohnson1963@hotmail.com*

Abstract. For $k \in \{1, 2, 3, \dots\}$, we construct an even compactly supported piecewise polynomial ψ_k whose Fourier transform satisfies $A_k(1+\omega^2)^{-k} \le \widehat{\psi}_k(\omega) \le B_k(1+\omega^2)^{-k}$, $\omega \in \mathbb{R}$, for some constants $B_k \ge A_k > 0$. The degree of ψ_k is shown to be minimal, and is strictly less than that of Wendland's function $\phi_{1,k-1}$ when $k > 2$. This shows that, for $k > 2$, Wendland's piecewise polynomial $\phi_{1,k-1}$ is not of minimal degree if one places no restrictions on the number of pieces.

1. Introduction

A function $\Phi \in L_1(\mathbb{R}^d)$ is said to have *Sobolev regularity* $k > 0$ if its Fourier transform $\widehat{\Phi}(\omega) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \Phi(x) e^{-\imath x \cdot \omega}\, dx$ satisfies

$$A(1 + \|\omega\|^2)^{-k} \le \widehat{\Phi}(\omega) \le B(1 + \|\omega\|^2)^{-k}, \quad \omega \in \mathbb{R}^d,$$

for some constants $B \ge A > 0$. Such functions are useful in radial basis function methods since the generated native space will equal the Sobolev space $W_2^k(\mathbb{R}^d)$. The reader is referred to Schaback [3] for a description of the current state of the art in the construction of compactly supported functions Φ having prescribed Sobolev regularity. Wendland (see [4] and [5]) has constructed radial functions $\Phi_{d,\ell}(x) = \phi_{d,\ell}(\|x\|)$, where $\phi_{d,\ell}$ is a piecewise polynomial of the form $\phi_{d,\ell}(t) = \begin{cases} p(|t|), & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$, p being a polynomial. For $d \in \{1, 2, 3, \dots\}$ and $\ell \in \{0, 1, 2, \dots\}$, with the case $d = 1$, $\ell = 0$ excluded, $\Phi_{d,\ell}$ has Sobolev regularity $k = \ell + (d+1)/2$ and the degree of the piecewise polynomial $\phi_{d,\ell}$, namely $\lfloor d/2 \rfloor + 3\ell + 1$, is minimal with respect to this property. A natural question to ask is whether the degree of $\phi_{d,\ell}$ would still be minimal if we considered functions of the form $\Phi(x) = \phi(\|x\|)$ where ϕ is a piecewise polynomial having bounded support. In this note, we answer this question in the univariate case $d = 1$. Specifically, we construct a compactly supported even piecewise

Key words and phrases. positive definite function, compactly supported, B-spline.

polynomial ψ_k, with Sobolev regularity k (see Theorem 2.8), and we show that the degree of ψ_k, namely $2k$, is minimal (see Theorem 2.10). In comparison with Wendland's function $\Phi_{1,k-1}$ (which has Sobolev regularity k when $k>1$), we see that $\deg\psi_k = \deg\phi_{1,k-1}$, if $k=2$, while $\deg\psi_k = 2k < 3k-2 = \deg\phi_{1,k-1}$ when $k>2$.

2. Results

Wendland's piecewise polynomial $\phi_{d,\ell}$ can be identified as a constant multiple of the B-spline having $\ell+1$ knots at the nodes -1 and 1 and $\lfloor d/2\rfloor+\ell+1$ knots at 0. This can be verified simply by observing that $\phi_{d,\ell}$ and the above-mentioned B-spline have the same degree, $\lfloor d/2\rfloor+3\ell+1$, and satisfy the same number of continuity conditions across each of the nodes $-1,0,1$, namely $\lfloor d/2\rfloor+2\ell+1$ at $-1,1$ and $2\ell+1$ at 0. It is well understood in the theory of B-splines that multiple knots are to be avoided if one wishes to keep the degree low, and with this in mind, we define ψ_k as follows. For $k=1,2,3,\ldots$, let ψ_k be the B-spline having knots $-k,\ldots,-2,-1,0;0,1,2,\ldots,k$ (note that 0 is the only double knot). For easy reference, we display $\psi_k(t)$ (normalized) for $t\in[0,k]$ and $k=1,2,3$:

$$\psi_1(t)=(1-t)^2, \qquad \psi_2(t)=\begin{cases} 8-24t^2+24t^3-7t^4, & t\in[0,1] \\ (2-t)^4, & t\in(1,2] \end{cases}$$

$$\psi_3(t)=\begin{cases} 198-270t^2+270t^4-180t^5+37t^6, & t\in[0,1] \\ 153+270t-945t^2+900t^3-405t^4+90t^5-8t^6, & t\in(1,2] \\ (3-t)^6, & t\in(2,3] \end{cases}$$

We begin by mentioning several salient facts about the B-spline ψ_k which can be found in [1, pp. 108–131]. The piecewise polynomial ψ_k is supported on $[-k,k]$, positive on $(-k,k)$, even and of degree $2k$. Furthermore, it is $2k-1$ times continuously differentiable on $\mathbb{R}\backslash\{0\}$ and $2k-2$ times continuously differentiable on all of $\mathbb{R}$. It follows from this that the $2k-1$ order derivative, $D^{2k-1}\psi_k$, is a piecewise linear function which is supported on $[-k,k]$ and is continuous except at the origin where it has a jump discontinuity. Consequently, the $2k$ order derivative has the form

$$D^{2k}\psi_k = \sqrt{2\pi}a_0\delta_0 + \sum_{j=1}^{k}\sqrt{2\pi}a_j(\chi_{[j-1,j)}+\chi_{[-j,1-j)}),$$

for some constants $a_0,a_1,a_2,\ldots,a_k$ and where δ_0 is the Dirac δ-distribution defined by $\delta_0(f)=f(0)$. We can thus express the Fourier transform of $D^{2k}\psi_k$ as

$$\left(D^{2k}\psi_k\right)\widehat{\ }(\omega) = a_0 + 2\sum_{j=1}^{k}a_j\frac{\sin(j\omega)-\sin((j-1)\omega)}{\omega} = a_0+\sum_{j=1}^{k}2(a_j-a_{j+1})\frac{\sin(j\omega)}{\omega},$$

with $a_{k+1}:=0$, whence it follows that

$$\widehat{\psi}_k(\omega) = (\imath\omega)^{-2k}\left(D^{2k}\psi_k\right)\widehat{\ }(\omega) = \frac{(-1)^k}{\omega^{2k+1}}\left(a_0\omega+\sum_{j=1}^{k}2(a_j-a_{j+1})\sin(j\omega)\right). \tag{2.1}$$

Lemma 2.2. *Let $\beta \in \mathbb{R}$. Then there exist unique scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$ such that*

$$\left|\beta + \sum_{j=1}^{k} c_j \cos(j\omega)\right| = O(|\omega|^{2k}) \text{ as } \omega \to 0. \tag{2.3}$$

Proof. Define $g(w) = \beta + \sum_{i=1}^{k} c_i \cos(i\omega)$. Since $g \in C^\infty(\mathbb{R})$ is even, (2.3) holds if and only if $D^{2\ell}g(0) = 0$ for $\ell = 0, 1, 2, \ldots, k-1$. These conditions form the system of linear equations $[c_1, c_2, \ldots, c_k]A = [-\beta, 0, 0, \ldots, 0]$, where A is the $k \times k$ matrix given by $A(i,j) = (-1)^{j-1}i^{2j-2}$. Writing $A(i,j) = (-i^2)^{j-1}$, we recognize A as a nonsingular Vandermonde matrix, and therefore, (2.3) holds if and only if $[c_1, c_2, \ldots, c_k] = [-\beta, 0, 0, \ldots, 0]A^{-1}$. □

Theorem 2.4. *Let $\beta, c_1, c_2, \ldots, c_k \in \mathbb{R}$ be such that (2.3) holds. Then*

$$\beta + \sum_{j=1}^{k} c_j \cos(j\omega) = \beta\alpha_k(1 - \cos\omega)^k, \quad \omega \in \mathbb{R}, \tag{2.5}$$

where $\alpha_k > 0$ is defined by $\frac{1}{\alpha_k} = \frac{1}{\pi}\int_0^\pi (1-\cos\omega)^k\, d\omega$.

Proof. Since $\cos^j \omega \in span\{1, \cos\omega, \cos 2\omega, \ldots, \cos k\omega\}$ for $j = 0, 1, \ldots, k$, it follows that there exist $b_j \in \mathbb{R}$ such that $(1-\cos\omega)^k = b_0 + \sum_{j=1}^{k} b_j \cos(j\omega)$. Note that

$$0 < \frac{1}{\alpha_k} = \frac{1}{\pi}\int_0^\pi (1-\cos\omega)^k\, d\omega = \frac{1}{\pi}\int_0^\pi b_0\, d\omega + \sum_{j=1}^{k} b_j \frac{1}{\pi}\int_0^\pi \cos(j\omega)\, d\omega = b_0,$$

and hence $\beta\alpha_k(1-\cos\omega)^k = \beta + \sum_{j=1}^{k} \beta\alpha_k b_j \cos(j\omega)$. Since $\left|\beta\alpha_k(1-\cos\omega)^k\right| = O(|\omega|^{2k})$ as $\omega \to 0$, it follows from the lemma that $c_j = \beta\alpha_k b_j$ for $j = 1, 2, \ldots, k$, and therefore (2.5) holds. □

Corollary 2.6. *Let a_0 be as in (2.1). Then $(-1)^k a_0 > 0$ and*

$$\widehat{\psi}_k(\omega) = \frac{(-1)^k a_0 \alpha_k}{\omega^{2k+1}} \int_0^\omega (1-\cos t)^k\, dt, \quad \omega \neq 0. \tag{2.7}$$

Proof. It follows from (2.1) that $\widehat{\psi}_k(\omega) = \frac{(-1)^k}{\omega^{2k+1}} f(\omega)$, where $f(\omega) := a_0\omega + \sum_{j=1}^{k} 2(a_j - a_{j+1})\sin(j\omega)$. Since ψ_k is supported on $[-k, k]$ and positive on $(-k, k)$, it follows that $\widehat{\psi}_k$ is continuous (in fact entire) and $\widehat{\psi}_k(0) > 0$. Consequently, $|f(\omega)| = O(|\omega|^{2k+1})$ as $\omega \to 0$. Since f is infinitely differentiable, it follows that $|f'(\omega)| = \left|a_0 + \sum_{j=1}^{k} 2j(a_j - a_{j+1})\cos(j\omega)\right| = O(|\omega|^{2k})$ as $\omega \to 0$, and so by Theorem 2.4, $f'(\omega) = a_0\alpha_k(1-\cos\omega)^k$. Since $f(0) = 0$, we can write $f(\omega) = \int_0^\omega f'(t)\, dt = a_0\alpha_k \int_0^\omega (1-\cos t)^k\, dt$, and hence obtain (2.7). That $(-1)^k a_0 > 0$ is now evident since $0 < \widehat{\psi}_k(0) = \lim_{\omega\to 0^+} \widehat{\psi}_k(w)$. □

Remark. At this point, it is also easy to show that

$$\widehat{\psi}_k(\omega) = \frac{(-1)^k a_0}{\omega^{2k+1}}(\omega + \sum_{j=1}^{k} b_j \sin(j\omega)), \quad \omega \neq 0,$$

where the scalars $\{b_j\}$ are determined by the fact that $\widehat{\psi}_k$ is continuous at 0.

Theorem 2.8. *For $k \in \{1, 2, 3, \ldots\}$, ψ_k has Sobolev regularity k; that is, there exist constants $B_k \geq A_k > 0$ such that*

$$A_k(1 + |\omega|^2)^{-k} \leq \widehat{\psi}_k(\omega) \leq B_k(1 + |\omega|^2)^{-k}, \quad \omega \in \mathbb{R}. \tag{2.9}$$

Proof. As in the proof of Corollary 2.6, let us write $\widehat{\psi}_k(\omega) = \frac{(-1)^k}{\omega^{2k+1}} f(\omega)$, where $f(\omega) := a_0\omega + \sum_{j=1}^{k} 2(a_j - a_{j+1}) \sin(j\omega)$. Since $\lim_{\omega\to\infty} \frac{f(\omega)}{\omega} = a_0$, it follows that $\lim_{w\to\infty} w^{2k}\psi_k(w) = (-1)^k a_0$. Since $(-1)^k a_0 > 0$ (by Corollary 2.6), it follows that there exists $N > 0$ such that (2.9) holds for $\omega \geq N$. That $\widehat{\psi}_k(\omega) > 0$ for all $\omega > 0$ follows easily from Corollary 2.6, and since $\widehat{\psi}_k$ is continuous and $\widehat{\psi}_k(0) > 0$, we see that (2.9) holds for $0 \leq \omega \leq N$. We finally conclude that (2.9) holds for all $\omega \in \mathbb{R}$ since $\widehat{\psi}_k$ is an even function. □

We now show that the degree of ψ_k is minimal.

Theorem 2.10. *If ψ is an even, compactly supported, piecewise polynomial satisfying (2.9), then the degree of ψ is at least $2k$.*

Proof. Let ψ be an even, compactly supported piecewise polynomial satisfying (2.9) and let the ℓ-th derivative of ψ be the first discontinuous derivative of ψ (if ψ is itself discontinuous then $\ell = 0$). Then $D^{\ell+1}\psi$ can be written as

$$D^{\ell+1}\psi = g + \sum_{j=1}^{n} \sqrt{2\pi} c_j \delta_{x_j}, \tag{2.11}$$

where $g \in L_1(\mathbb{R})$ and c_j is the height (possibly 0) of the jump discontinuity at x_j. We can then express the Fourier transform of ψ as

$$\widehat{\psi}(\omega) = (\imath\omega)^{-\ell-1} \left(D^{\ell+1}\psi\right)\widehat{\ }(\omega) = (\imath\omega)^{-\ell-1}(\widehat{g}(\omega) + \Theta(\omega)),$$

where $\Theta(\omega) = \sum_{j=1}^{n} c_j e^{-\imath x_j \omega}$. Since Θ is bounded and $|\widehat{g}(\omega)|$ has limit 0 as $|\omega| \to \infty$ (by the Riemann-Lebesgue lemma), it follows that $\left|\widehat{\psi}(\omega)\right| = O(|\omega|^{-\ell-1})$ as $|\omega| \to \infty$. With the left side of (2.9) in view, we conclude that $\ell + 1 \leq 2k$. Since Θ is a non-trivial almost periodic function (see [2, pp.9–14]), it follows that $|\Theta(\omega)| \neq o(1)$ as $|\omega| \to \infty$, and with the right side of (2.9) in view, we see that $\ell + 1 \geq 2k$. Therefore, $\ell + 1 = 2k$ and we conclude that ψ_k is $2k - 2$ times continuously differentiable. Since ψ_k is compactly supported (ie. ψ_k is not a polynomial), it follows that ψ_k has degree at least $2k - 1$ (see [1, pp. 96-120]). In order to show that the degree of ψ_k is at least $2k$, let us assume to the contrary that the degree equals $2k - 1$. In this case the $\ell = 2k - 1$ derivative of ψ_k is piecewise constant and hence $g = 0$ and $\widehat{\psi}(\omega) = (-1)^k\omega^{-2k}\Theta(\omega)$. Since $\widehat{\psi}$ is continuous at 0, it follows that $\Theta(0) = 0$. Since Θ is an almost periodic function, there exist values $\omega_1 < \omega_2 < \cdots$ such that $\lim_{n\to\infty} \omega_n = \infty$ and $\lim_{n\to\infty} \Theta(\omega_n) = 0$; but this contradicts the left side of (2.9). Therefore, ψ has degree at least $2k$. □

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