Approximation Order without Quasi-Interpolants

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Abstract. In the study of approximation order, particularly in a multivariable setting, quasi-interpolants have played a major role. This report points out some limitations of quasi-interpolants and describes some recent results on approximation order obtained without the benefit of the quasi-interpolant idea.

§1. Approximation Order

In most general terms, "approximation order" is defined as follows.

Definition 1.1. *The indexed collection* (S_h) *(with* $h \to 0$*) of linear subspaces of some normed linear space* X *has (exact) approximation order* k*, in symbols:*

$$\mathbf{ao}((S_h)_h) \; = \; k \; ,$$

provided

(i) *for all "smooth"* f, $\operatorname{dist}(f, S_h) = O(h^k)$ *(lower bound)*
(ii) *for some "smooth"* f, $\operatorname{dist}(f, S_h) \neq o(h^k)$ *(upper bound)*

This definition raises many questions.

• **norm?** In this report, I will usually consider $X = L_p(G)$, with G some 'suitable' subset of $\mathbb{R}^d$, e.g., either a bounded convex body, or else all of $\mathbb{R}^d$. In fact, the major results reported are for $G = \mathbb{R}^d$ and $p = 2$ or $p = \infty$. With X such a function space,

$$X_{\mathbf{c}}$$

denotes the subspace of *compactly supported* $f \in X$.

• **"smooth"?** With X as chosen, a typical choice for "smooth" is that f be in the Sobolev space $W_p^k(G)$ (written $W^{k,p}(G)$ in [1]. If G is bounded and

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there is no specification of the expected constant in $O(h^k)$, then it is usually sufficient to define "smooth" to mean "polynomial". In that case, it is usually a polynomial of homogeneous degree k that furnishes the upper bound.

• **how does $O(h^k)$ depend on f?** The definition of approximation order permits, offhand, the possibility that the constant in the $O(h^k)$ term of 1.1(i) depends in some entirely unspecified way on f. It is more satisfactory, though, if this dependence can be made explicit, for example in the terms that specify "smoothness". Thus a desirable strengthening of 1.1(i) is that

$$\sup_{h,f} \frac{\operatorname{dist}(f, S_h)}{h^k \|f\|_{(k)}} < \infty ,$$

with the finiteness of $\|f\|_{(k)}$ defining that f is "smooth". Theorems 7.1, 6.3 and 6.4 below give such results.

• **S_h?** In this report, I will deal only with the following choices:

◦ Each S_h is a space of piecewise polynomial (=: pp) or, more generally, piecewise exponential (=: pe) or piecewise analytic (=: pa) functions, and h is the "meshsize" of the underlying partition Δ (consisting, typically, of convex bodies, such as simplices and the like).

◦ (S_h) is a *scale*, i.e.,

$$S_h = \sigma_h S := \{f(\cdot/h) : f \in S\} ,$$

with S some fixed space of functions. In this case, I will use the abbreviation

$$\mathbf{ao}(S) \;:=\; \mathbf{ao}((\sigma_h S)_h) .$$

Such an indexed collection (S_h) is called *stationary*, in order to distinguish it from the next example.

◦ More generally, we might have $S_h = \sigma_h S^h$, a case referred to as *non-stationary* (in case the S^h do change with h). Note that, in either case, the space S_h is given as the h-dilate of some space. This is done since, in certain arguments, it is more efficient to deal with the *scale-ups* $\sigma_{1/h} S_h$ than with the spaces S_h themselves. In the stationary case, this amounts to considering the approximation of

$$f_h := \sigma_{1/h} f = f(\cdot h)$$

from the *fixed* space S.

◦ Of particular interest in this report (and in much current work in approximation theory, in part because of the current interest in wavelets) is the case when each S^h is shift-invariant, i.e., closed under *shifts* := integer translations.

§2. Shift-invariance

A collection S of functions on $\mathbb{R}^d$ is called *shift-invariant* if

$$g \in S \quad \Longrightarrow \quad g(\cdot + \alpha) \in S \ \text{ for all } \ \alpha \in \mathbb{Z}^d$$

(where $\mathbb{Z}^d$ is the set of d-vectors whose entries are integers).

For example, the space

$$\Pi^{\rho}_{<k,\Delta}$$

of all pp C^{ρ}-functions of total degree $< k$ on some partition Δ is shift-invariant in case the partition is shift-invariant in the sense that

$$\Delta + \alpha = \Delta \ \text{ for all } \ \alpha \in \mathbb{Z}^d \ .$$

Examples of interest include the three- and four-direction mesh popular in the bivariate box spline literature.

The simplest (nontrivial) example of a shift-invariant space is the space

$$\mathcal{S}_0(\varphi) := \left\{ \sum_{\alpha \in \mathbb{Z}^d} \varphi(\cdot - \alpha)\, c(\alpha) : c \in \ell_0(\mathbb{Z}^d) \right\}$$

of all finite linear combinations of the shifts of one function, φ. This is the *shift-invariant space generated by* φ since it is the smallest shift-invariant space containing φ. If $\mathcal{S}_0(\varphi)$ is contained in our normed linear space X of interest, then we follow [6] and write

$$\mathcal{S}(\varphi) \ := \ \mathcal{S}_0(\varphi)^-$$

for the closure of $\mathcal{S}_0(\varphi)$ in X and call it the *principal shift-invariant*, or *PSI*, space generated by φ.

More generally, if Φ is a finite collection of functions on $\mathbb{R}^d$, then we set

$$\mathcal{S}_0(\Phi) \ := \ \sum_{\varphi \in \Phi} \mathcal{S}_0(\varphi)$$

and call

$$\mathcal{S}(\Phi) \ := \ \mathcal{S}_0(\Phi)^-$$

a *finitely generated shift-invariant*, or *FSI*, space, and call Φ its set of generators. The structure of PSI and FSI spaces in $L_2(\mathbb{R}^d)$ is detailed in [6] and [7], with particular emphasis on the construction of generating sets for a given FSI space having good properties (such as 'stability' or 'linear independence').

It is natural to consider approximations from $\mathcal{S}(\varphi)$ in the form

$$\varphi{*}c := \sum_{\alpha \in \mathbb{Z}^d} \varphi(\cdot - \alpha)\, c(\alpha)$$

for a suitable coefficient sequence c. However, offhand, such a sum makes sense only for finitely supported c, and one of the technical difficulties in ascertaining the approximation order of $\mathcal{S}(\varphi)$ derives from the fact that, in general, $\mathcal{S}(\varphi)$ may contain elements which cannot be represented in the form $\varphi{*}c$ for some sequence c, with the series $\varphi{*}c$ converging in norm.

§3. Quasi-interpolants

In the spline and finite-element literature, lower bounds for $\mathbf{ao}((S_h)_h)$ are usually obtained with the aid of a corresponding sequence $(Q_h)_h$ of linear maps, with $\operatorname{ran} Q_h \subseteq S_h$, which is a 'good quasi-interpolant sequence of order k' in the sense of the following definition.

Definition 3.1. *$(Q_h)_h$ is a good quasi-interpolant sequence of order k if it satisfies the following two conditions:*
(i) *uniformly local: For some h-independent finite ball B and all $x \in G$, $|(Q_h f)(x)| \le \text{const} \, \|f_{|x+hB}\|$;*
(ii) *polynomial reproduction: $Q_h f = f$ for all $f \in \Pi_{<k}$.*

Here,

$$\Pi_{<k}$$

denotes the collection of all polynomials in d arguments of total degree $< k$.

The term 'quasi-interpolant' is used in the finite element literature (see, e.g., [26] to stress the fact that $Q_h f$ does not necessarily match function values at all the nodes of the finite elements used, but 'merely' reproduces certain polynomials. [4] contains a recent survey of the use of quasi-interpolants in spline theory.

To recall, the standard use made of such a good quasi-interpolant sequence is to observe that, for arbitrary f and arbitrary $g \in \Pi_{<k}$,

$$|f(x) - Q_h f(x)| = |(1 - Q_h)(f - g)(x)| \le \text{const} \, \|(f - g)_{|x+hB}\| \ ,$$

which provides a bound on $\|f - Q_h f\|$ in terms of how well f can be approximated from $\Pi_{<k}$ on a set of the form $x + hB$, giving the error bound $\text{const}_B \, h^k \|f\|_{(k)}$ in which $\|f\|_{(k)}$ measures the 'size' of the k-th derivatives of f and which provides the desired $O(h^k)$. If our space X is L_p for some $p < \infty$, then this argument has to be fleshed out a bit (see, e.g., [20]).

Since this argument is so simple and effective, there have been various generalizations. For example, since the argument relies on how well f can be approximated locally from $\Pi_{<k}$, it has been observed (e.g., in [15], [11], [22]) that it is sufficient to have Q_h reproduce a translation-invariant space H (e.g., a space of exponentials) which is 'locally close' to $\Pi_{<k}$ (in the sense defined at this section's end).

As another example, if $S_h = \sigma_h \mathcal{S}(\varphi)$, then it is natural to construct $Q_h f$ in the form

$$\sigma_h Q f_h$$

(recall that $f_h := \sigma_{1/h} f$) with

$$Qf := \sum_{\alpha \in \mathbb{Z}^d} \varphi(\cdot - \alpha) \lambda f(\cdot + \alpha)$$

for some suitable linear functional λ. Since, for any linear functional λ (defined at least on $\Pi_{<k}$) and any $f \in \Pi_{<k}$, $\alpha \mapsto \lambda f(\cdot + \alpha)$ is polynomial of degree $< k$

in α, this approach requires that $\varphi * c$ be at least well-defined for sequences c with some growth at infinity. In the original context of a compactly supported φ (e.g., as in [27]), this is no problem. However, with the recent interest in radial basis functions (see, e.g., [24]) and wavelets, also noncompactly supported φ have to be considered and, for these, the quasi-interpolant approach (as used, e.g., in [23], [16], [20], and [2]) requires that φ satisfy the condition $\varphi(x) = O(|x|^{-d-k-\varepsilon})$ at ∞ for some positive ε (and forces one to make do with Q which is only 'essentially local'). In particular, the higher the desired approximation order, the faster must φ decay at infinity.

There are other costs associated with the quasi-interpolant approach. For example, it works, offhand, only with integer values of k. Also, it requires that

$$\cap_h S_h \neq \{0\} .$$

The artificiality of this last restriction is nicely illustrated by the following simple example, from [15]:

Example 3.2. (Dyn, Ron). Let $d = 1$, $p = \infty$, and let S_h be the span of the $h\mathbb{Z}$-translates of the piecewise linear function

$$\varphi_h : x \mapsto \begin{cases} x+1 , & 0 \le x < h ; \\ 0 , & \text{otherwise} . \end{cases}$$

Thus S_h consists of certain piecewise linear functions, with breakpoint sequence $h\mathbb{Z}$, but the only polynomial (hence the only analytic function) it contains is the zero polynomial. In particular, it is not possible to construct a quasi-interpolant of positive order for it. Nevertheless, the approximation

$$Q_h f := \sum_{j \in h\mathbb{Z}} \varphi_h(\cdot - j) f(j)$$

has the error

$$f - Q_h f = f - \sum_{j \in h\mathbb{Z}} \chi_h(\cdot - j) f(j) \quad + \quad \sum_{j \in h\mathbb{Z}} (\chi_h - \varphi_h)(\cdot - j) f(j) ,$$

with χ_h the characteristic function of the interval $[0\,..\,h)$. Since $\|\chi_h - \varphi_h\|_\infty = h$,

$$\|f - Q_h f\|_\infty \le \omega_f(h) + \|f\|_\infty h ,$$

where ω_f is the modulus of continuity of f. It follows that $Q_h f$ converges to f uniformly in case f is uniformly continuous and bounded. ■

This example could still be treated by an appropriate generalization of the notion of quasi-interpolant. Specifically, one could consider a good quasi-interpolant sequence (Q_h) of positive *local* order k, meaning that (Q_h) is uniformly local and that

$$Q_h f = f + O(\|f_{|B}\| \, |h|^k)$$

on hB for any $f \in \Pi_{<k}$. A sufficient condition for this is that $Q_h = 1$ on a D-invariant space H of entire functions which is *locally close to* $\Pi_{<k}$ in the sense that its 'limit at the origin' (cf. [10]), $H_\downarrow$, contains $\Pi_{<k}$. Here,

$$H_\downarrow := \operatorname{span}\{f_\downarrow : f \in H\} , \tag{3.3}$$

where $f_\downarrow$ is the *initial*, i.e., the first nonzero, term in the expansion $f = f_0 + f_1 + f_2 + \cdots$ of f into homogeneous polynomials f_j of degree j, all j. Thus, for any $f \in \Pi_{<k}$, there exists $g \in H$ with $f = g + O(|h|^k)$ on hB, hence, on hB, $Q_h f = Q_h g + O(|h|^k) = g + O(|h|^k) = f + O(|h|^k) = f + O(\|f_{|B}\| |h|^k)$ (the last equality by the fact that $\Pi_{<k}$ is finite-dimensional).

Still, the point of the example should be clear.

Finally, the quasi-interpolant approach is of no help with upper bounds.

§4. Upper Bounds

Upper bounds for $\mathbf{ao}((S_h)_h)$ have to be fashioned separately for each case (much as the details of a quasi-interpolant sequence have to be so fashioned). The general principle employed is duality, which provides the following well-known observation.

If Y is a linear subspace of the normed linear space X, and $\lambda \in X^*$ with $\lambda \perp Y$ (i.e., λ is a continuous linear functional on X which vanishes on all of Y), then, for any $x \in X$ and any $y \in Y$, $\lambda x = \lambda(x - y) \le \|\lambda\| \|x - y\|$, hence $|\lambda x| \le \|\lambda\| \operatorname{dist}(x, Y)$. In other words,

$$\lambda \perp Y \quad \Longrightarrow \quad \operatorname{dist}(x, Y) \ge \frac{|\lambda x|}{\|\lambda\|} .$$

As a simple application, consider $\mathbf{ao}(S)$ for

$$X = L_\infty(G), \quad S = \Pi^\rho_{<k,\Delta} .$$

Assume without loss of generality that G is the d-dimensional cube,

$$G = C := [-1 \,.\,.\, 1]^d ,$$

let δ be any element in the partition Δ, and let g be any nontrivial homogeneous polynomial of degree k. If e is the error in the best $L_2(\delta)$-approximation to g from $\Pi_{<k}$, then the mapping

$$\lambda : L_\infty \to \mathbb{R} : f \mapsto \int_\delta ef$$

(i) is a bounded linear functional;
(ii) is orthogonal to S, since all λ sees of $f \in S$ is its restriction to δ, and on δ each $f \in S$ is just a polynomial of degree $< k$;
(iii) satisfies $\lambda g = \int_\delta ee > 0$.

Now consider $\lambda_h f := \int_\delta e f(h\cdot)$. Then

(i) λ_h is a bounded linear functional, with h-independent norm

$$\|\lambda_h\| = \int_\delta |e| \;=\; \lambda\,\mathrm{signum}(e)\ ,$$

where $\mathrm{signum}(e) : x \mapsto \mathrm{signum}(e(x))$.

(ii) $\lambda_h \perp S_h := \sigma_h S$, since $g \in S_h$ is of the form $f(\cdot/h)$ for some $f \in S$.

(iii) Using the homogeneity of g, one computes that $\lambda_h g = \int_\delta e g(h\cdot) = h^k \int_\delta eg$ $= h^k \lambda g$ with $\lambda g > 0$.

So, altogether,

$$\mathrm{dist}(g, S_h) \ge h^k(\lambda g/\lambda\,\mathrm{signum}(e))\ ,$$

showing that $\mathbf{ao}(\Pi^\rho_{<k,\Delta}) \le k$.

If we try the same argument for $p < \infty$, we hit a little snag. Take, in fact, p at the other extreme, $p = 1$. There is no difficulty with (ii) or (iii), but the conclusion is weakened because (i) now reads

(i)′ $\|\lambda_h\| = \sup_{f\in L_1} |\int_\delta e f(h\cdot)|/\|f\|_1 \le \|e_{|\delta}\|_\infty \sup_{f\in L_1(\delta)} \int_\delta |f(h\cdot)|/\|f\|_1\ ,$

and the best we can say about that last supremum is that it is at most h^{-d} since $\int_\delta f(h\cdot) = \int_{h\delta} f/h^d$. Hence, altogether, $\|\lambda_h\| \le \mathrm{const}/h^d$.

Thus, now our bound reads

$$\mathrm{dist}_1(g, S_h) \ge h^k\,\mathrm{const}/(\mathrm{const}/h^d) \ne o(h^{k+d})$$

which is surely correct, but not very helpful.

What we are witnessing here is the fact that the error in a max-norm approximation is indeed localized, i.e., it occurs at a point, while, for $p < \infty$, the error 'at a point' is less relevant; the error is more global; one needs to consider the error over a good part of G. Further, in the argument below, I need some kind of uniformity of the partition Δ, of the following (very weak) sort (in which $|A|$ denotes the d-dimensional volume of the set A, and C continues to denote the cube $[-1\,..\,1]^d$):

Assumption 4.1. *There exists an open set b and a locally finite set $I \subset \mathbb{R}^d$ (meaning that I meets any bounded set only in finitely many points) so that*

(α) *$b + I$ is the disjoint union of $b + i$, $i \in I$, with each $b + i$ lying in some $\delta \in \Delta$ (the possibility of several lying in the same δ is not excluded);*

(β) *for some* $\mathrm{const} > 0$ *and all n, $|(b + I) \cap nC| \ge \mathrm{const}\,|nC|$.*

For example, any uniform partition of $\mathbb{R}$ satisfies this condition. As another example, if $d = 2$ and Δ is the three-direction mesh, then Δ consists of triangles of two kinds, and taking b to be the interior of one of these triangles and $I = \mathbb{Z}^2$ guarantees (α), while (β) holds with $\mathrm{const} = 1/2$. On the other hand, Shayne Waldron (a student at Madison) has constructed a neat example to show that the Assumption 4.1 is, in general, necessary for the conclusion that $\mathbf{ao}(\Pi^\rho_{<k,\Delta}) \le k$. The example uses $\rho = -1$ and arbitrary k,

$d = 1$, $G = [-1\,..\,1]$, $p = 1$, and Δ obtained from $\mathbb{Z}$ by subdividing $[j\,..\,j+1]$ into $2^{|j|}$ equal pieces, $j \in \mathbb{Z}$.

With Assumption 4.1 holding, define λ as before, but with b replacing the element δ of Δ. Further, assume without loss that $C \subseteq G$, and define

$$\lambda_h f := \int_b e \sum_{i \in I_h} f(h \cdot +i) \ ,$$

where

$$I_h := \{i \in I : b + i \subseteq C/h\} \ .$$

This gives:

(i)$_1$

$$\|\lambda_h\| \le \sup_{f \in L_1} \frac{\sum_{i \in I_h} \int_{b+i} |e||f(h\cdot)|}{\sum_{i \in I_h} \int_{h(b+i)} |f|} = \|e_{|b}\|_\infty / h^d \ ,$$

using the fact that the sum $b + I_h$ is disjoint.

Hence, we didn't worsen our situation here. We also didn't sacrifice (ii) because, by assumption, each $b + i$ lies in the interior of some $\delta \in \Delta$, and therefore $\int_b ef(h \cdot +i) = 0$ for every $f \in S_h$. But we materially improved the situation as regards (iii), for we now obtain

(iii)$_1$

$$\lambda_h g = \int_b e \sum_{i \in I_h} g(h \cdot +i) = h^k \int_b e \sum_{i \in I_h} g = h^k \text{ const } \#I_h$$

with

$$\#I_h = |b + I_h|/|b| \ge \text{const}\,|C/h|/|b| = \text{const}\,/h^d \ .$$

With this, our conclusion is back to what we want:

$$\text{dist}_1(g, S_h) \ge (h^k \text{ const}\,/h^d)/(\text{const}\,/h^d) \ne o(h^k) \ .$$

Note that this lower bound on the distance only sees S as a space of pp's of degree $< k$, hence is valid even when we take the biggest such space, i.e., the space

$$\Pi_{<k,\Delta}$$

of *all* pp functions of degree $< k$ on the partition Δ. For this space, it is not hard to show that the approximation order is at least k, since approximations can be constructed entirely locally. Thus,

$$\mathbf{ao}(\Pi_{<k,\Delta}) = k \ .$$

For this reason, this is called the *optimal* approximation order for a pp space of degree $< k$.

Such a local construction of approximations is still possible for $\Pi^0_{<k,\Delta}$ (at least in the uniform norm; it would be interesting to run down this argument for the 1-norm), hence, at least in the uniform norm,

$$\mathbf{ao}(\Pi^{\rho}_{<k,\Delta}) = k \qquad \text{for} \quad \rho \le 0 \ .$$

However, for $\rho > 0$, the story is largely unknown, with first results in [5] and [19].

I became sensitized to the issue that the derivation of upper bounds for the approximation order from pa spaces requires much more care for $p < \infty$ than for $p = \infty$ by the paper [22] in which $\mathbf{ao}((S_h)_h)$ is carefully studied for the case that each S_h is a piecewise exponential space. Here is their result concerning upper bounds (in which the term 'exponential' is meant to describe any function which is a linear combination, with polynomial coefficients, of functions of the form $x \mapsto \exp(\theta{\cdot}x)$).

Theorem 4.2. (Lei, Jia). *Let $(S_h)_h$ be an indexed collection of piecewise exponential spaces on $\mathbb{R}^d$ with the property that, for some open subset Ω of $(0\,..\,1)^d$ and every h and every $\alpha \in \mathbb{Z}^d$, $S_{h|(\Omega+\alpha)h} \subseteq H_{|(\Omega+\alpha)h}$ for some fixed D-invariant finite-dimensional space H of exponentials for which $\Pi_k \not\subseteq H_{\downarrow}$ (as defined in (3.3)). Then, for any p in the range $1 \le p \le \infty$, $\mathbf{ao}((S_h)_h) \le k$.*

Here is my version of their proof (in which $\|f\|_p(B)$ denotes the $L_p(B)$-norm of $f_{|B}$ while $\|a\|$ is any norm of the n-vector a, and B_h is the Euclidean ball with radius h centered at the origin). The special case of pp S_h treated earlier is simpler since, in that case, H is also scale-invariant.

Let

$$V := [v_1, v_2, \ldots, v_n] : \mathbb{R}^n \to H_{\downarrow} : a \mapsto \sum_j v_j a(j)$$

be any homogeneous basis for $H_{\downarrow}$.

I claim that any $F = [f_1, f_2, \ldots, f_n] : \mathbb{R}^n \to H$ with $f_{j\downarrow} = v_j$, all j, is a basis for H. For the proof (which also proves the inequality (4.3) of use later), observe that $\|Va\|_p(B_h) = \|Va^h\|_p(B_1) \ge \|a^h\|/\|V^{-1}\|$, where

$$a^h := (h^{d/p+\deg v_j} a(j))_{j=1}^n \ , \qquad \|V^{-1}\| := \sup_c \|c\|/\|Vc\|_p(B_1) \ ,$$

and $\|V^{-1}\|$ is certainly finite. On the other hand, $(v_j - f_j)(x) = O(|x|^{\deg v_j + 1})$ since $v_j = f_{j\downarrow}$, hence

$$\|(F-V)a\|_p(B_h) \le h \operatorname{const}_F \|a^h\| \ .$$

Therefore, altogether,

$$\begin{aligned} \|Fa\|_p(B_h) \ &\ge \|Va\|_p(B_h) - \|(F-V)a\|_p(B_h) \\ &\ge (1/\|V^{-1}\| - h \operatorname{const}_F)\|a^h\| \ =: \operatorname{const}_{h,F} \|a^h\| \ , \end{aligned} \tag{4.3}$$

which shows that F is one-to-one (since the last expression is positive for all sufficiently small h). Since $\dim H_\downarrow = \dim H$ by [10], this finishes the proof.

Now let q be a homogeneous polynomial not in the range of V. Then $[q, V]$ is one-to-one, and is made up of the initial terms of the columns of $[q, F]$. This permits substitution of $[q, F]$ for F in (4.3) (with $\mathrm{const}_{[q,F]} = \mathrm{const}_F$), and so gives the conclusion that, for all $a \in \mathbb{R}^n$,

$$\|q - Fa\|_p(B_h) = \|[q, F](1, -a)\|_p(B_h) \geq \mathrm{const}_{h,F}\, \|(h^{d/p+\deg q}, a^h)\| \\ \geq \mathrm{const}_{h,F}\, h^{d/p+\deg q}\ ,$$

hence

$$\mathrm{dist}_p(q, H)(B_h) = \min_a \|q - Fa\|_p(B_h) \geq \mathrm{const}_{h,F}\, h^{d/p+\deg q}\ , \tag{4.4}$$

with $\lim_{h\to 0} \mathrm{const}_{h,F} = 1/\|V^{-1}\| > 0$. Since we can choose $\deg q = k$ by assumption, this proves the upper bound when $p = \infty$. (I note in passing that this argument could also have been phrased explicitly in terms of annihilating linear functionals.)

As to the L_p-argument, start with the observation that it is sufficient to prove an upper bound for the L_1-approximation order on any bounded G since this implies the same upper bound for any $p > 1$ (including $p = \infty$) and for any G, bounded or not.

Thus, to establish the desired upper bound, it is sufficient to prove that

$$\mathrm{dist}_1(q, S_h)(B_\rho) \geq \mathrm{const}\, h^k$$

for some smooth q, some positive const, and any particular positive ρ.

For this, we now choose q to be any homogeneous polynomial of *minimal* degree not in $H_\downarrow$. Then, for any z, $q(\cdot + z) = q + V a_z$, with $\|a_z\| \leq \mathrm{const}\, \|z\|$, and $q(\cdot + z) - Fa = q - F(a - a_z) + (V - F)a_z$, therefore

$$\mathrm{dist}_1(q(\cdot + z), H)(B_h) \geq \mathrm{dist}_1(q, H)(B_h) - h\, \mathrm{const}_F\, \|a_z^h\|.$$

This implies with (4.4) that there exist positive constants const, h_0, R (depending on F and q) so that

$$\mathrm{dist}_1(q(\cdot + z), H)(B_h) \geq \mathrm{const}\, h^{d+\deg q} \tag{4.5}$$

for all $h < h_0$, $\|z\| < R$.

By the translation-invariance of H (which follows from the assumed D-invariance),

$$\mathrm{dist}(q, H)(\Omega h + z) = \mathrm{dist}(q(\cdot + z), H)(\Omega h)$$

while, by assumption, $S_h \subseteq H$ on each $(\Omega + \alpha)h$ with $\alpha \in \mathbb{Z}^d$. Thus, from (4.5) and using the fact that Ω contains some ball of positive radius, we find that

$$\mathrm{dist}_1(q, S_h)(B_\rho) \geq \sum_{\alpha\in N} \mathrm{dist}_1(q(\cdot + \alpha h), H)(\Omega h) \geq \mathrm{const}\, h^{\deg q}\, h^d\, \#N\ ,$$

with

$$N := \{\alpha \in \mathbb{Z}^d : (\Omega + \alpha)h \subseteq B_\rho, \|\alpha h\| < R\}$$

and with $h < h_0$, where const > 0 and $R > 0$ are independent of h. Since $\#N = O(h^{-d})$ for all small h, and $\deg q \le k$, we are done. ■

Further illustrations of the use of duality in the derivation of upper bounds on $\mathbf{ao}(S)$ (albeit only for bivariate pp S) can be found in [9] and its references.

§5. The Strang-Fix Condition

The literature on $\mathbf{ao}(\mathcal{S}(\varphi))$ for a compactly supported φ has been dominated by the Strang-Fix condition. It concerns the behavior of the Fourier transform

$$\widehat{\varphi} : \xi \mapsto \int_{\mathbb{R}^d} \varphi\, e_{-\xi}$$

of φ at the points of $2\pi\mathbb{Z}^d$. Here and below,

$$e_\theta : \mathbb{R}^d \to \mathbb{C} : x \mapsto \exp(i\theta\cdot x)$$

denotes the exponential function (with purely imaginary frequency $i\theta$). In one of its many versions, the Strang-Fix condition reads as follows.

Definition 5.1. *We say that φ satisfies* SF_k *in case*

(i) $\widehat{\varphi}(0) = 1$;

(ii) *For all multi-indices α satisfying $|\alpha| < k$ we have $D^\alpha \widehat{\varphi} = 0$ on $2\pi\mathbb{Z}^d\backslash 0$.*

Its importance derives from the following theorem, in which we use the convenient notation

$$\varphi *' f := \sum_{j\in\mathbb{Z}^d} \varphi(\cdot - j) f(j)$$

for the *semidiscrete convolution* of the two functions φ and f even if it requires further discussion of just what exactly is meant by it when the sum is not (locally) finite. Also, recall that $L_1(\mathbb{R}^d)_{\mathbf{c}}$ denotes the compactly supported functions in $L_1(\mathbb{R}^d)$.

Theorem 5.2. (Schoenberg (d=1), Fix and Strang). *For $\varphi \in L_1(\mathbb{R}^d)_{\mathbf{c}}$, the following are equivalent:*

(a) *$\varphi *'$ is degree-preserving on $\Pi_{<k}$: for all p in $\Pi_{<k}$, $\varphi *' p \in p + \Pi_{<\deg p}$;*

(b) *φ satisfies* SF_k.

The proof is via the Poisson summation formula. Starting with [27], the theorem is used to construct a good quasi-interpolant sequence (Q_h) of order k with $\operatorname{ran} Q_h \subseteq \sigma_h \mathcal{S}(\varphi)$. More than that, it forms part of an argument that seems to show that $\mathbf{ao}(\mathcal{S}(\varphi)) \ge k$ if and only if $\varphi/\widehat{\varphi}(0)$ satisfies SF_k. The precise statement of this equivalence for $X = L_2(\mathbb{R}^d)$ (see [27]) involves, unfortunately, a restricted notion of approximation order called 'controlled' approximation. (According to [25], this restriction can be dropped for $X = L_\infty(\mathbb{R}^d)$ provided $\widehat{\varphi}(0) \ne 0$.)

On a related issue, [27] reports the following

Conjecture 5.3. (Babuška). *The approximation order in $L_2(\mathbb{R}^d)$ of the FSI space $\mathcal{S}(\Phi)$ with $\Phi \subset L_2(\mathbb{R}^d)_{\mathbf{c}}$ is already attained by some PSI space $\mathcal{S}(\varphi)$ with $\varphi \in \mathcal{S}_0(\Phi)$.*

The actual version of this conjecture reported in [27] involves controlled approximation and was eventually shown to be invalid by Jia in [18]. The following correct version, involving yet another restricted notion of approximation order called 'local' approximation, can be found in [8], with some details actually attended to only in [20].

Theorem 5.4. (de Boor, Jia). *Let Φ be a finite subset of $L_p(\mathbb{R}^d)_{\mathbf{c}}$, and let $X = L_p(\mathbb{R}^d)$. Then the following are equivalent:*

(a) *$(\sigma_h \mathcal{S}(\Phi))$ has 'local' approximation order k;*
(b) *some $\varphi \in \mathcal{S}_0(\Phi)$ satisfies SF_k.*

This theorem verifies the version of the Babuška conjecture used in [14]. Further, [21] shows that (b) is equivalent to the statement

(b)′ *Some sequence (φ_n) in $\mathcal{S}_0(\Phi)$ satisfies SF_k "in the limit".*

Finally, [19] contains the following extension of work begun in [5]:

Theorem 5.5. (Jia). *Let S be a* univariate, *shift-invariant, locally finite-dimensional set of functions, closed under convergence on compact sets. Then the following are equivalent:*

(a) $\mathbf{ao}(S \cap L_p(\mathbb{R})) \ge k$;
(b) *Some $\varphi \in S_{\mathbf{c}}$ satisfies SF_k.*

§6. Approximation Order in L_∞

In [13], Chui, Jetter and Ward introduce the *commutator* for $\varphi \in C(\mathbb{R}^d)_{\mathbf{c}}$ as the linear map

$$C(\mathbb{R}^d) \to C(\mathbb{R}^d) : f \mapsto \varphi *' f - f *' \varphi$$

and use it for the construction of a good quasi-interpolant sequence (Q_h) of order k with $\operatorname{ran} Q_h \subseteq \sigma_h \mathcal{S}(\varphi)$. For this, they prove the following.

Proposition 6.1. (Chui, Jetter, Ward). *If φ belongs to $C(\mathbb{R}^d)_{\mathbf{c}}$ and satisfies SF_k, then*

$$\text{for all } \ f \in \Pi_{<k} \ , \quad \varphi *' f = f *' \varphi \ .$$

Subsequently, it was observed in [3] that actually

$$\text{for all } \ f \in \mathcal{S}(\varphi) \ , \quad \varphi *' f = f *' \varphi \ , \tag{6.2}$$

and this observation was exploited by A. Ron in [25] in the following simple and surprising way. He observes that, as a consequence of (6.2),

$$\text{for all } \ f \in \mathcal{S}(\varphi) \ , \quad \varphi *' e_\theta - e_\theta *' \varphi = \varphi *' (e_\theta - f) - (e_\theta - f) *' \varphi \ ,$$

(recall that $e_\theta : x \mapsto \exp(i\theta{\cdot}x)$), and this leads to the conclusion that

$$\|\varphi{*}'e_\theta - e_\theta{*}'\varphi\|_\infty \le 2\|\varphi{*}'\|_\infty \,\mathrm{dist}_\infty(e_\theta, \mathcal{S}(\varphi))$$

(with $\|\varphi{*}'\|_\infty = \|\sum_{\alpha\in\mathbb{Z}^d}|\varphi(\cdot-\alpha)|\|_\infty$). Since (as pointed out by A. Ron)

$$\frac{\varphi{*}'e_\theta - e_\theta{*}'\varphi}{e_\theta} \;\sim\; c + \sum_{\alpha\in\mathbb{Z}^d\backslash 0} \widehat{\varphi}(\theta+2\pi\alpha)\, e_\alpha \ ,$$

this throws new light on the connection between $\mathbf{ao}(\mathcal{S}(\varphi))$ in L_∞ and the behavior of $\widehat{\varphi}$ 'at' $2\pi\mathbb{Z}^d$.

[12] exploits this idea in the more general context of a $\varphi \in X := L_\infty(\mathbb{R}^d)$ with the only requirement that $\varphi{*}'$ be a bounded map from ℓ_∞ to X. Further, while $\mathcal{S}(\varphi)$ is still taken to be the 'closure' of $\mathcal{S}_0(\varphi)$, it is not taken as the norm-closure but, in effect, as the largest shift-invariant space containing $\mathcal{S}_0(\varphi)$ and satisfying (6.2).

Here is the main result of [12] concerning *upper bounds*.

Theorem 6.3. ([12]). *Let (φ_h) be an indexed collection of elements of $X := L_\infty(\mathbb{R}^d)$. Assume that $\varphi_h{*}' : \ell_\infty \to X$ is defined and bounded independently of h, and that $\theta \in \mathbb{R}^d$. If $\mathrm{dist}(e_\theta, \sigma_h\mathcal{S}(\varphi_h)) = O(h^k)$, then*

$$\sum_{\alpha\in\mathbb{Z}^d\backslash 0} |\widehat{\varphi}_h(h\theta+2\pi\alpha)|^2 \le \mathrm{const}_\theta\, h^{2k} \ .$$

In particular, then

$$|\widehat{\varphi}_h(h\theta+2\pi\alpha)| \le \mathrm{const}_\theta\, h^k \ \ \textit{for all nonzero} \ \ \alpha \ \ \textit{in} \ \ \mathbb{Z}^d \ .$$

The following points should be stressed:

- ◦ There is some latitude here for the definition of "smooth" since it need only include complex exponentials.
- ◦ Only mild decay of φ_h is needed (enough to make $\varphi{*}' : \ell_\infty \to L_\infty$ well-defined).
- ◦ Nothing is said here about $\widehat{\varphi}_h(0)$ (which is particularly important if $\widehat{\varphi}_h(0)$ is zero).
- ◦ It is easy to recover the rest of SF_k in the stationary case, i.e., in case $\varphi_h = \varphi$, for all h.
- ◦ Even if "smooth" is taken to mean "compactly supported, but infinitely smooth", the same condition is obtained, provided φ_h has a certain decay at ∞.

The results of [12] concerning *lower bounds* on $\mathbf{ao}(\mathcal{S}(\varphi))$ make use of the following definition of "smooth": $f \in X$ is "smooth" if its Fourier transform is a Radon measure for which

$$\|f\|_{(k)} := \|(1+|\cdot|^k)\widehat{f}\|_1 < \infty \ ,$$

with the suffix '1' intended to indicate that the total variation of the measure in question is meant.

Here is a sample result.

Theorem 6.4. ([12]). *Assume that $\varphi_h *' : \ell_\infty \to L_\infty$ is bounded for every h. Then, for any positive η,*

$$\operatorname{dist}(f, \sigma_h \mathcal{S}(\varphi_h)) \ \le\ h^k\,(2\pi)^{-d}\,\|f\|_{(k)}\,A \ +\ o(h^k)$$

with

$$A := \sup_h \sum_{\alpha\in\mathbb{Z}^d\backslash 0} \Big\| \frac{1}{(h^k + |\cdot|^k)} \frac{\widehat{\varphi}_h(\cdot + 2\pi\alpha)}{\widehat{\varphi}_h} \Big\|_{L_\infty(B_\eta)} .$$

Since this theorem gives $\mathbf{ao}((\sigma_h\mathcal{S}(\varphi_h))_h) \ge k$ only if $A < \infty$, this focuses attention on the behavior near zero of each of the functions

$$\widehat{\varphi}_h(\cdot + 2\pi\alpha)/\widehat{\varphi}_h\ , \qquad \alpha \in \mathbb{Z}^d\backslash 0\ . \tag{6.5}$$

Specifically, in the stationary case, if this ratio is a smooth function in a neighborhood of 0, then the finiteness of A would require the ratio to have a zero of order k at 0, and conversely, provided $\widehat{\varphi}$ has some decay. From this vantage point, the Strang-Fix condition SF_k is seen to be neither necessary nor sufficient for $\mathbf{ao}(\mathcal{S}(\varphi)) \ge k$, but to come close to being necessary and sufficient for appropriately restricted φ.

Note that the finiteness of A requires the infinite sum in its definition to be finite, and such finiteness can, in general, only be deduced when $\widehat{\varphi}_h$, in addition to being "small" near $2\pi\mathbb{Z}^d\backslash 0$, decays appropriately (and this requires some smoothness of φ_h).

The fact that the finiteness of A involves only the *ratios* (6.5) makes the conclusion of the theorem *independent of localization*, i.e., independent of which difference operators were applied to the original generator for $\mathcal{S}(\varphi_h)$ in order to obtain the appropriately decaying φ_h.

The proof in [12] of results like this theorem makes use of an approximation from $\mathcal{S}(\varphi)$ of the form

$$f \approx Rf := (2\pi)^{-d} \int_{\mathbb{R}^d} \varepsilon_\theta \widehat{f}(\theta)\, d\theta \quad \in \mathcal{S}(\varphi)$$

in which the approximation

$$e_\theta \approx \varepsilon_\theta := \varphi *' e_\theta / \widetilde{\varphi}(\theta) \in \mathcal{S}(\varphi)$$

is suggested by

$$e_\theta *' \varphi = e_\theta \sum_j \exp(-ij)\varphi(j) \quad =: e_\theta\ \widetilde{\varphi}(\theta)\ .$$

§7. Approximation Order in L_2

For an arbitrary $\varphi \in X := L_2(\mathbb{R}^d)$, the approximation order of $\mathcal{S}(\varphi)$ can be characterized completely, in terms of $\widehat{\varphi}$. This is due to the fact (proved in [6] but also derivable from more general results in [17]) that, if P_S is the orthogonal projector onto $\mathcal{S}(\varphi)$, then

$$\widehat{P_S f} = \frac{[\widehat{f}, \widehat{\varphi}]}{[\widehat{\varphi}, \widehat{\varphi}]} \widehat{\varphi} \ ,$$

where

$$[\widehat{f}, \widehat{\varphi}] : \mathbb{T}^d \to \mathbb{C} : x \mapsto \sum_{\alpha \in 2\pi\mathbb{Z}^d} \widehat{f}(x+\alpha) \overline{\widehat{g}}(x+\alpha)$$

is the very convenient "bracket product" of $\widehat{f}, \widehat{\varphi} \in X$, and $\mathbb{T}^d$ is the d-dimensional torus, i.e.,

$$\mathbb{T}^d \ := \ [-\pi \mathinner{.\,.} \pi]^d$$

with the appropriate identification of boundary points.

The definition of f being "smooth" employed in [6] is that

$$\|f\|_{W_2^k(\mathbb{R}^d)} := \|(1 + |\cdot|)^k \widehat{f}\|_2 < \infty \ .$$

The characterization uses the following abbreviation

$$\Lambda_\varphi := 1 - \frac{|\widehat{\varphi}|^2}{[\widehat{\varphi}, \widehat{\varphi}]} = \frac{\sum_{\alpha \in \mathbb{Z}^d \backslash 0} |\widehat{\varphi}(\cdot + 2\pi\alpha)|^2}{\sum_{\alpha \in \mathbb{Z}^d} |\widehat{\varphi}(\cdot + 2\pi\alpha)|^2} \ .$$

Theorem 7.1. ([6]). *For any $(\varphi_h)_h$ in $X = L_2(\mathbb{R}^d)$,*

$$\mathbf{ao}((\sigma_h \mathcal{S}(\varphi_h))_h) \geq k \quad \iff \quad \sup{}_h \Big\| \frac{\Lambda_{\varphi_h}}{(h + |\cdot|)^{2k}} \Big\|_{L_\infty(\mathbb{T}^d)} < \infty \ .$$

This result focuses attention on the behavior of Λ_φ near 0, hence, if $\widehat{\varphi}$ is bounded away from zero near 0, it focuses, once again, attention on the ratios (6.5). Here is a typical

Corollary 7.2. ([6]). *If $\varphi \in L_2(\mathbb{R}^d)$, and $1/\widehat{\varphi}$ is essentially bounded near 0, and $\widehat{\varphi} \in W_2^\rho(U)$ for some $\rho > k + d/2$ and some nbhd U of $2\pi\mathbb{Z}^d \backslash 0$, and if φ satisfies SF_k, then $\mathbf{ao}(\mathcal{S}(\varphi)) \geq k$.*

For a general closed shift-invariant subspace of $L_2(\mathbb{R}^d)$, there is the following result.

Theorem 7.3. ([6]). *Let S be a closed shift-invariant subspace of $L_2(\mathbb{R}^d)$, and let $f, g \in L_2(\mathbb{R}^d)$. Then*

$$\mathrm{dist}(f, S) \ \leq \ \mathrm{dist}(f, \mathcal{S}(P_S g)) \ \leq \ \mathrm{dist}(f, S) + 2\, \mathrm{dist}(f, \mathcal{S}(g)) \ .$$

This theorem shows that the approximation power of a general shift-invariant subspace of L_2 is already attained by some PSI subspace of it, provided we can, for given k, supply an element $g \in L_2(\mathbb{R}^d)$ for which $\mathbf{ao}(\mathcal{S}(g)) > k$. But that is easy to do:

Lemma 7.4. *There are simple functions g (e.g., the inverse Fourier transform of the characteristic function of some small neighborhood of the origin) for which, for any k,*

$$\operatorname{dist}(f, \sigma_h \mathcal{S}(g)) \;=\; o(h^k \|f\|_{W_2^k(\mathbb{R}^d)}) \ .$$

§8. The Babuška Conjecture Revisited

Theorem 7.1 is used in [7] to provide a proof of the Babuška Conjecture 5.3, as follows.

Let $S = \mathcal{S}(\Phi)$, where Φ is a finite subset of $L_2(\mathbb{R}^d)_{\mathbf{c}}$.

(i) Since each $\varphi \in \Phi$ is compactly supported, hence $\widehat{\varphi}$ is analytic, it can be assumed, after going to a subset of Φ if need be, that, for almost every $x \in \mathbb{T}^d$, the set of $\ell_2(\mathbb{Z}^d)$-vectors

$$\widehat{\varphi}_{\|x} := (\widehat{\varphi}(x + 2\pi\alpha))_{\alpha\in\mathbb{Z}^d} \ , \qquad \varphi \in \Phi \ ,$$

is linearly independent, hence is a basis for $\widehat{S}_{\|x}$.

(ii) For any $g \in L_2(\mathbb{R}^d)$,

$$\widehat{P_S g} = \sum_{\varphi\in\Phi} \frac{\det G_\varphi(g)}{\det G(\Phi)} \widehat{\varphi}$$

where

$$G(\Phi) \;:=\; \big([\widehat{\varphi}, \widehat{\psi}]\big)_{\varphi,\psi\in\Phi}$$

and $G_\varphi(g)$ is obtained from this by replacing the row $[\widehat{\varphi}, \ \cdot\ ]$ by the row $[\widehat{g}, \ \cdot\ ]$.

(iii) Since

$$[\widehat{f}, \widehat{g}] \;=\; \sum_{j\in\mathbb{Z}^d} \langle f, g(\ \cdot\ + j)\rangle \, e_j \ ,$$

each entry of $G(\Phi)$ is a trigonometric polynomial, hence so is $\det G(\Phi)$, and $\det G(\Phi) \neq 0$ a.e. (by (i)).

(iv) If $g \in L_2(\mathbb{R}^d)_{\mathbf{c}}$, then $\mathcal{S}(P_S g) = \mathcal{S}(g_\star)$ (it is shown in [6] that $\mathcal{S}(\psi') = \mathcal{S}(\psi)$ in case $\psi' \in \mathcal{S}(\psi)$ and $\operatorname{supp}\widehat{\psi'} \supseteq \operatorname{supp}\widehat{\psi}$), where

$$\widehat{g_\star} := \det G(\Phi)\ \widehat{P_S g} \;=\; \sum_{\varphi\in\Phi} \det G_\varphi(g)\ \widehat{\varphi} \ ,$$

by (ii), hence $g_\star \in \mathcal{S}_0(\Phi)$, by (iii).

(v) By Theorem 7.3 and Lemma 7.4, we can choose g so that

$$\operatorname{dist}(f, \mathcal{S}(g_\star)) \;\sim\; \operatorname{dist}(f, \mathcal{S}(\Phi)) \ ,$$

hence *Babuška was right.*

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