**On determining the foot of the continental slope**

Carl de Boor, Nov. 95 (amended Feb. 96)

The official definition of the foot of the continental slope seems to have been made with *univariate* imagery in mind and without input from mathematicians.

Specifically, the image seems to be one of a *profile*, dropping from the flat of the continental shelf steeply down the continental slope to meet, eventually, the rather flat rise toward it. The point where they meet would be the foot, and it is, in this view, marked by a rather fast change of the gradient, from steep to flat. In other words, the curvature at that point (or, more precisely, small region) is maximally positive.

The resulting identification of these points with points of maximum positive normal curvature in the *surface* which describes the sea bottom ignores the original imagery of a profile. While the maximum normal curvature will be large at a well-defined foot, it can also be large at other points.

Rather, to capture the original imagery, one might want to do the following. Let $z(x, y)$ be the depth (or height) of the sea bed at the point $(x, y)$.

(i) Determine the direction in which the sea bed goes down at that point $(x, y)$. This is $-Dz(x, y)$, with $Dz$ the **gradient**, i.e.,

$$Dz := (D_x z, D_y z).$$

It is the direction perpendicular to the *level line* or *contour line* of the sea bed through the point $(x, y)$.

(ii) Determine the normalized second derivative of $z$ in the direction of steepest descent at $(x, y)$. This is given by the number

$$N := v^t\, D^2 z\, v = v_x^2 D_{xx} z + 2 v_x v_y D_{xy} z + v_y^2 D_{yy} z, \tag{1}$$

with the second partials, $D_{xx}z$, $D_{xy}z$, $D_{yy}z$, of the function $z$ all to be evaluated at $(x, y)$, and with

$$v = (v_x, v_y) := Dz/\|Dz\|$$

the normalized gradient of $z$ at $(x, y)$ (with $\|v\|^2 := v_x^2 + v_y^2$).

The calculation of $N$ is certainly easier than the calculation of the accepted *surface of maximum curvature* (see below).

In addition, note that, for any $v$, the right-hand side of (1) gives the second directional derivative of $z$ in the direction $v$. Further, with $v = Dz/\|Dz\|$, we can rewrite (1) as

$$N = \frac{(Dz)^t H Dz}{(Dz)^t Dz},$$

with

$$H := \begin{bmatrix} D_{xx}z & D_{xy}z \\ D_{yx}z & D_{yy}z \end{bmatrix} = D^2 z$$

the **Hessian** of $z$. In other words, $N$ is the **Rayleigh Quotient**

$$R_H(v) := \frac{v^t H v}{v^t v}$$

for $H$, evaluated at $v = Dz$.

This gives an opportunity to compare use of this **gradient-directed second derivative** with that of the maximum curvature.

First, if by 'curvature' we mean nothing more than the second derivative in any particular normalized direction, then, by the argument just given, the maximum curvature would be the maximum of the Rayleigh quotient of the Hessian, i.e., the maximum eigenvalue of $H$. In particular, if the gradient-directed second derivative is 'large', then so must be this maximum 'curvature'. On the other hand, this maximum 'curvature' may well be 'large' in places where the gradient-directed second derivative, i.e., the 'curvature' in the gradient direction, is not large.

However, the **surface of maximum curvature** or **SMC** proposed in [VWH] as a means for determining the foot of the continental slope is based on the actual curvature of the surface

$$S : (x, y) \mapsto (x, y, z(x, y)),$$

hence may be even further removed from the original intent of the definition of the foot of the continental slope. Specifically, the SMC is obtained as

$$(x, y) \mapsto (x, y, \max\{0, \max \kappa(x, y)\}),$$

with $\max \kappa(x, y)$ the maximum normal curvature of $S$ at the point $(x, y, z(x, y))$. Elementary differential geometry applied to the particular surface $S$ shows that the normal curvature of $S$ in the direction $v$ equals the value at $v$ of the Rayleigh quotient

$$K(v) := R_{\widehat{H},G}(v) := \frac{v^t \widehat{H} v}{v^t G v} = \frac{v^t H v}{v^t G v} / \sqrt{(D_x z)^2 + (D_y z)^2 + 1},$$

with $\widehat{H}$ the matrix of the second fundamental form for $S$, i.e.,

$$\widehat{H} = H / \|(-Dz, 1)\|,$$

and $G$ the matrix of the first fundamental form for $S$, i.e.,

$$G = (DS)^t DS = \begin{bmatrix} 1 + (D_x z)^2 & D_x z D_y z \\ D_y z D_x z & 1 + (D_y z)^2 \end{bmatrix}.$$

For the specific choice $v = Dz$ of the gradient direction, one computes

$$K(Dz) = N / \|(-Dz, 1)\|^3,$$

thus affording a simple comparison between the gradient-directed second derivative, $N$, and the SMC, $\max_v K(v)$. In particular, since $\|(-Dz, 1)\|$ may vary widely, there may be no connection between the maxima of $N$ and those of the SMC, making the use of the SMC even more doubtful (for the purpose of determining the foot of the continental slope). In any case, since either way uses second-derivative information, only a carefully smoothed version of the original data has any hope of leading to a correct identification of the foot of the continental slope.

Since both $\widehat{H}$ and $G$ are real symmetric, the maximum normal curvature at a point is the larger of the two principal curvatures at that point, i.e., the larger of the two eigenvalues $\kappa_1$, $\kappa_2$ of the eigenvalue problem

$$\widehat{H} - \kappa G.$$

Equivalently, $\max \kappa$ is the larger of the two solutions of the quadratic equation

$$\det(\widehat{H} - \kappa G) = 0,$$

which, on expanding the determinant and collecting terms according to powers of $\kappa$, gives exactly the equation (2) of [VWH] (as it should).

Finally, simple real examples show that the original definition of the foot of the continental slope does not always cover reality since the passage from steep decline to flattish continental rise or sea bottom can also be quite gradual, with no particular area of sharp change of gradient. For such a situation, an alternative definition seems needed.

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[VWH] Petr Vaniček, David E. Wells, and Tianhang Hou, "Determination of the Foot of the Continental Slope", Department of Geodesy and Geomatics Engineering, University of New Brunswick, Fredericton NB, E3B 5A3, March 15, 1994.