Cardinal interpolation and spline functions VIII.

The Budan–Fourier Theorem for splines and applications

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Dedicated to M.G. Krein

Introduction.

The present paper is the reference [8] in the monograph [15], which was planned but not yet written when [15] appeared. The paper is divided into four parts called A, B, C, and D. We aim here at three or four different results. The unifying link between them is that they all involve the sign structure of what one might call a "Green's spline", i.e., a function which consists of two null–splines pieced together at a certain point to satisfy at that point several homogeneous conditions and just one inhomogeneous condition, much as (any section of) a (univariate) Green's function consists of two solutions to a homogeneous ordinary differential equation which are pieced together at a point in just that way. The different results are further linked by the fact that we use an extension of the Budan–Fourier theorem to splines in an essential way. In each of our applications of this theorem, the circumstances are such that the inequality furnished by the theorem becomes taut, i.e., must be an equality, and this provides an unexpected amount of precise information.

In Part A, we state and prove the Budan–Fourier theorem for splines with simple knots in the form in which we need it. We also apply it right away to the "Green's function" for odd–degree spline interpolation at arbitrarily spaced knots in a finite interval, i.e., to the Peano kernel for the error in that interpolation process.

In Part B, we develop the information about the sign structure of cardinal nullsplines required for later applications, using the Gantmacher–Krein Theory of oscillation matrices in an essential way.

Part C: The study of the remainder of cardinal spline interpolation for odd degree $n = 2m-1$, as given in [17], depended on the behavior of the remainder $K(x,t)$ of the interpolation of the function $(x-t)_+^{2m-1}$ where t is a parameter, $0 < t < 1$. The assertion (Theorem 3 of [17]) was that $\operatorname{sgn} K(x,t) = (-)^m \operatorname{sgn}\sin \pi x$ for all real x, and this was stated in [17] without proof. A proof is given in Part C, where we also discuss the remainder of *even degree* cardinal spline interpolation as well as the fundamental function of this interpolation process.

Part D: The elementary cases of the Landau–Kolmogorov problem were discussed in [16] by means of appropriate formulae of approximate differentiation with integral remainders. However, [16] was restricted to the orders $n = 2$ and $n = 3$, when only finitely many of the ordinates of the function appear in the differentiation formula. In [16], and also in [15, Lecture 9. §1], the first non–elementary case $n = 4$ was briefly mentioned. In Part D, we study the general case. Cavaretta gave in [4] an elegant proof of Kolmogorov's theorem that uses only Rolle's theorem. Our approach is much more elaborate, but provides information on the extremizing functions.

Part A. **The Budan–Fourier theorem for splines and spline interpolation on a finite interval.**

1. The Budan–Fourier theorem for splines with simple knots. We begin with the introduction of some standard notation.

For $v = (v_i)_1^n \in \mathbb{R}^n$, S^-v and S^+v denote the minimal, respectively maximal, number of sign changes in the sequence v achievable by appropriate assignment of signs to the zero entries (if any) in v. Hence, always $S^-v \le S^+v$. Further,

$$S^-v \le \liminf_{u\to v} S^-u \le \limsup_{u\to v} S^+u \le S^+v.$$

Should $S^-v = S^+v$, then it is customary to denote their common value by Sv. The identity

$$S^-(v_i)_1^n + S^+((-)^i v_i)_1^n = n-1 \tag{1}$$

will be used repeatedly.

If v is, more generally, a real valued function on some domain $G \subseteq \mathbb{R}$, then, with $*$ standing for $-$ or $+$,

$$S^* v := \sup\{S^*(v(t_i)) : (t_i)_1^n \text{ in } G, \quad n \in \mathbb{N}, \quad t_1 < \cdots < t_n\} .$$

If $E \subseteq G$, then we will write $S^*_E v$ for $S^*(v|_E)$.

Induction establishes the following useful lemma which is essentially Lemma 1.2 of Karlin and Micchelli [7].

Lemma 1. *If $f \in C^{(n)}[0, \delta]$ and $f^{(n)}(0) \neq 0$, then, for some positive ε, $f^{(j)}$ does not vanish on $(0, \varepsilon]$, $j = 0, \ldots, n$, and*

$$S^-(f(0), \ldots, f^{(n)}(0)) = \lim_{t \downarrow 0} S^+(f(t), \ldots, f^{(n)}(t)).$$

Therefore, with (1),

$$S^+(f(0), \ldots, f^{(n)}(0)) = \lim_{t \uparrow 0} S^-(f(t), \ldots, f^{(n)}(t))$$

in case $f^{(n)}(0) \neq 0$ and $f \in C^{(n)}[-\delta, 0]$.

Next, we define the multiplicity of a zero of a spline function f of degree n on $[a, b]$ with simple knots, i.e., f is composed of polynomial pieces of degree $\le n$ in such a way that $f \in C^{(n-1)}[a, b]$. Then $f \in \mathbb{L}_\infty^{(n)}[a, b]$. If $n = 0$, i.e., if f is piecewise constant, then we say that the (possibly degenerate) interval $[\sigma, \tau]$ in (a, b) is a zero of f of multiplicity $\left\{ {0 \atop 1} \right\}$ iff f vanishes on (σ, τ) and $f(\sigma^-)f(\tau^+) \left\{ {> \atop <} \right\} 0$. With this definition, *the number of zeros counting multiplicity of a piecewise constant function in (a, b) equals the number of its strong sign changes on (a, b)*. If $n > 0$, then we say that the (possibly degenerate) interval $[\sigma, \tau]$ in (a, b) is a zero of f of multiplicity r iff either $r = 0$ and $\sigma = \tau$ and $f(\sigma) \neq 0$ or else $r > 0$ and f vanishes on $[\sigma, \tau]$ and $[\sigma, \tau]$ is a zero of $f^{(1)}$ of multiplicity $r - 1$. We denote the total number of zeros, counting multiplicities, of f in (a, b) by

$$Z_f(a, b).$$

To give an example, $Z_f(0, 12) = 6$ for the linear spline f drawn in Figure 1, with a double zero at $[3, 4]$,

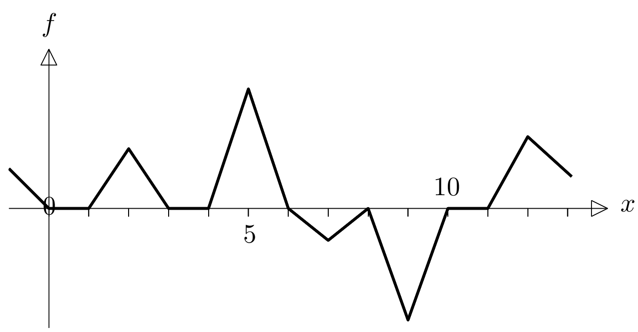

Figure 1

a simple zero at $[6, 6]$, a double zero at $[8, 8]$ and a simple zero at $[10, 11]$, and no other zeros of positive multiplicity in $(0, 12)$. In particular, the interval $[0, 1]$ is not counted as a zero in $(0, 12)$ for this f. Its first derivative has simple zeros at $[2, 2]$, $[3, 4]$, $[5, 5]$, $[7, 7]$, $[8, 8]$, $[9, 9]$ and no other zeros of positive multiplicity in $(0, 12)$, so $Z_{f^{(1)}}(0, 12) = 6$. In particular, $[10, 11]$ is a zero of 0 multiplicity for $f^{(1)}$, and, again, $[0, 1]$ is not zero in $(0, 12)$ for $f^{(1)}$. For this particular f, we would have equality in (2) below.

The number $Z_f(a, b)$ is necessarily finite if f has only finitely many knots in (a, b). Also, if $f(a)f(b) \neq 0$, then $f(a)f(b) < 0$ iff $Z_f(a, b)$ is odd. Further, $Z_f(a, b) + Z_f(b, c) \le Z_f(a, c)$ for $a < b < c$, and, as we said earlier, $Z_{f^{(n)}}(a, b) = S^-_{(a,b)} f^{(n)}$.

Theorem 1 (Budan-Fourier for Splines)†. *If f is a polynomial spline function of exact degree n on (a,b) (*i.e., of degree n with $f^{(n)}(t) \neq 0$ for some $t \in (a,b)$*) with finitely many (active) knots in (a,b), all simple, then*

$$Z_f(a,b) \le Z_{f^{(n)}}(a,b) + S^-(f(a),\dots,f^{(n-1)}(a), f^{(n)}(\sigma^+)) - S^+(f(b),\dots,f^{(n-1)}(b), f^{(n)}(\tau^-)), \tag{2}$$

with $\begin{Bmatrix}\sigma\\ \tau\end{Bmatrix} := \begin{Bmatrix}\inf\\ \sup\end{Bmatrix}\{t \in (a,b) : f^{(n)}(t) \neq 0\}$.

Proof: There is nothing to prove for $n = 0$, hence suppose that $n > 0$. Further, suppose first that $f(a)f(b)f^{(1)}(a^+)f^{(1)}(b^-) \neq 0$. We claim that then

$$Z_f(a,b) \le Z_{f^{(1)}}(a,b) + S(f(a), f^{(1)}(a^+)) - S(f(b), f^{(1)}(b^-)). \tag{3}$$

Indeed, if $Z_f(a,b) = 0$, there is nothing to prove unless also $Z_{f^{(1)}}(a,b) = 0$, but then $S(f(a), f^{(1)}(a^+)) = S(f(b), f^{(1)}(b^-))$ and (3) holds trivially. On the other hand, if $Z_f(a,b) > 0$, then we can find α and β so that $(\alpha,\beta) \supset \{t \in [a,b] : f(t) = 0\}$ while $f(\alpha)f(\beta) \neq 0$, and $f(a)/f(\alpha)$ and $f(b)/f(\beta)$ are both greater than 1. But then

$$Z_{f^{(1)}}(a,\alpha) \ge 1 - S(f(a), f^{(1)}(a^+)), \quad Z_{f^{(1)}}(\beta,b) \ge S(f(b), f^{(1)}(b^-))$$

while, by Rolle's theorem and our definition of multiplicity of zeros,

$$Z_f(a,b) - 1 \le Z_{f^{(1)}}(\alpha,\beta).$$

Hence, (3) holds in this case, too.

If now $f^{(j)}(a^+)f^{(j)}(b^-) \neq 0$ for $j = 0,\dots,n$, then

$$S(f(x),\dots,f^{(n)}(x)) = \sum_{j=1}^{n} S(f^{(j-1)}(x), f^{(j)}(x))$$

for $x = a^+$, b^-, while by (3),

$$Z_{f^{(j-1)}}(a,b) \le Z_{f^{(j)}}(a,b) + S(f^{(j-1)}(a), f^{(j)}(a^+)) - S(f^{(j-1)}(b), f^{(j)}(b^-)), \qquad j = 1,\dots,n,$$

which proves (2) for this case. From this, a limit process establishes (2) with the aid of Lemma 1 in case merely $f^{(n)}(a^+)f^{(n)}(b^-) \neq 0$.

If, finally, $f^{(n)}(a^+) = 0$, then, as f is of exact degree n on (a,b), there exists $\sigma \in (a,b)$ so that $f^{(n)}$ vanishes on (a,σ), but $f^{(n)}(\sigma^+) \neq 0$. Note that $[a,\sigma]$ is not counted as a zero of $f^{(n)}$ in (a,b) by our definition. By Lemma 1, we can find $\widehat{\sigma} > \sigma$ so that $f^{(j)}$ does not vanish on $(\sigma,\widehat{\sigma}]$ for $j = 0,\dots,n$ and so that

$$S^-(f(\sigma),\dots,f^{(n)}(\sigma^+)) = S(f(\widehat{\sigma}),\dots,f^{(n)}(\widehat{\sigma})). \tag{4}$$

Then

$$Z_f(a,b) = Z_f(a,\widehat{\sigma}) + Z_f(\widehat{\sigma},b), \quad Z_{f^{(n)}}(a,b) = Z_{f^{(n)}}(\widehat{\sigma},b). \tag{5}$$

We claim that

$$Z_f(a,\widehat{\sigma}) \le S^-(f(a),\dots,f^{(n-1)}(a), f^{(n)}(\sigma^+)) - S(f(\widehat{\sigma}),\dots,f^{(n)}(\widehat{\sigma})). \tag{6}$$

† The theorem as published was stated as if $[\sigma,\tau] = [a,b]$. In all applications of this theorem in the present paper, this is, in fact, the case. However, the added reference, [19], points out that the theorem as published is in general wrong when $[\sigma,\tau] \neq [a,b]$ and proposes the present version. The present proof reflects this change, mainly in (7).

For this, let

$$j := \begin{cases} 0 & \text{if } f \text{ vanishes identically on } [a,\sigma], \\ \max\{i \in [0,n-1] : f^{(i)}(a) \neq 0\} & \text{otherwise.} \end{cases}$$

Then

$$S^-(f^{(j)}(a),\ldots,f^{(n)}(a^+)) = 0, \quad S^-(f^{(j)}(\sigma),\ldots,f^{(n)}(\sigma^+)) = S^-(f^{(j)}(a), f^{(n)}(\sigma^+)), \tag{7}$$

the latter since $f^{(n)}$ is constant on $[a,\sigma]$. Now, (4) implies (6) in case f vanishes identically on $[a,\sigma]$ since then $[a,\sigma]$ is not counted as a zero of f in $(a,\widehat{\sigma})$, therefore $Z_f(a,\widehat{\sigma}) = 0$. Otherwise, $f^{(j)}$ is a nonzero constant on $[a,\sigma]$, therefore

$$\begin{aligned} Z_f(a,\widehat{\sigma}) &\le S^-(f(a),\ldots,f^{(j)}(a)) - S^+(f(\sigma),\ldots,f^{(j)}(\sigma)) + \text{mult. of } \sigma \text{ as a zero of } f \\ &\le S^-(f(a),\ldots,f^{(j)}(a)) - S^-(f(\sigma),\ldots,f^{(j)}(\sigma)) \\ &= S^-(f(a),\ldots,f^{(n-1)}(a), f^{(n)}(\sigma^+)) - S^-(f(\sigma),\ldots,f^{(n)}(\sigma^+)), \end{aligned}$$

the last equality by (7), and (4) now gives again (6).

These considerations also imply, by going from t to $-t$, that, for some $\widehat{\tau} \in (\widehat{\sigma},\tau)$, $f^{(j)}(\widehat{\tau}) \neq 0$ for $j = 0,\ldots,n$, and

$$\begin{aligned} &Z_f(\widehat{\sigma},b) = Z_f(\widehat{\sigma},\widehat{\tau}) + Z_f(\widehat{\tau},b), \quad Z_{f^{(n)}}(\widehat{\sigma},b) = Z_{f^{(n)}}(\widehat{\sigma},\widehat{\tau}), \\ &Z_f(\widehat{\tau},b) \le S(f(\widehat{\tau}),\ldots,f^{(n)}(\widehat{\tau}^-)) - S^+(f(b),\ldots,f^{(n-1)}(b), f^{(n)}(\tau^-)), \end{aligned}$$

and combining these facts with (5) and (7) and with (2) as already proved establishes (2) in the general case. □

It is possible to derive from this theorem appropriate statements concerning splines with multiple knots, e.g., monosplines, by an appropriate limit process. But we will not pursue this further here, as the theorem is sufficient for the purposes of this paper.

If f has its support in (a,b), then we obtain that

$$Z_f(a,b) \le S^-_{(a,b)} f^{(n)} - n.$$

Since $Z_f(a,b) \ge 0$, we recover in this way well known facts about B-splines, such as their sign structure or the fact that B-splines have minimal support. Further, we obtain $Z_f(a,b) \le S^-_{(a,b)} f^{(n+1)} - (n+1)$, which is the polynomial case of a more general result for Chebyshev splines due to H. Burchard [3].

We note that the particular choice of an f without any active knots in (a,b) and the replacement of S^+ by (the weaker) S^- leads to the classical Budan–Fourier Theorem [14]. The further specialization $a = 0$ and $b = \infty$ produces Descartes' familiar Rule of Signs.

2. A simple application to complete spline interpolation. If n is odd and greater than 1,

$$n = 2m - 1 > 1,$$

and $(x_i)_0^N$ is a sequence in $[a,b]$ with

$$a = x_0 < \cdots < x_N = b,$$

then there exists for given $f \in C^{(k-1)}[a,b]$ exactly one spline Sf of degree n with simple knots $x_1,\ldots,x_{N-1}$ in (a,b) which agrees with f in the sense that

$$Sf(x_i) = f(x_i), \quad i = 1,\ldots,N-1 \tag{1}$$

$$(Sf)^{(j)}(x_i) = f^{(j)}(x_i), \quad j = 0,\ldots,m-1 \text{ and } i = 0, N. \tag{2}$$

This spline has been called the **complete spline interpolant** (of degree $2m-1$ with knots $x_1, \ldots, x_{N-1}$) **to** f.

An imitation of the error analysis carried out for cubic spline interpolation in [1] leads directly to the statement that, for $f \in \mathbb{L}_1^{(n+1)}[a,b]$,

$$f(x) - Sf(x) = \int_a^b K(x,t) f^{(n+1)}(t)\, \mathrm{d}t/n! \tag{3}$$

with the Peano kernel K also equal to the interpolation error when applying complete spline interpolation to $(x-t)_+^n$ for fixed x as a function of t, i.e.,

$$K(x,t) = (x-t)_+^n - S(x-\cdot)_+^n(t). \tag{4}$$

Hence, if, for fixed x,

$$g(t) := K(x,t) \text{ for } t \in [a,b],$$

then g vanishes at least once at $x_1, \ldots, x_{N-1}$ and vanishes m–fold at a and at b. Because of this latter fact,

$$S^-(g(a), \ldots, g^{(n)}(a^+)) = S^-(g^{(m)}(a), \ldots, g^{(n)}(a^+)) \le n - m = m - 1 \tag{5}$$

and

$$S^+(g(b), \ldots, g^{(n)}(b^-)) \ge S^+(g(b), \ldots, g^{(m)}(b)) = m. \tag{6}$$

If $x = x_i$ for some $i \in [1, N-1]$, then $g = 0$ since then $(x-t)_+^n$ is its own spline interpolant. Otherwise, g is a spline of exact degree n (since $g^{(n)}$ has a nonzero jump at x) with simple knots at $x_1, \ldots, x_{N-1}$ and at x and nowhere else in (a,b), and vanishes (at least) at all but one of these. Now let $[\widehat{a}, \widehat{b}]$ be an interval in $[a,b]$ which is maximal with respect to the property that it contains only isolated zeros of g. Then either $\widehat{a} = a$ or else $S^-(g(\widehat{a}), \ldots, g^{(n)}(\widehat{a}^+)) = 0$, and either $\widehat{b} = b$ or else $S^+(g(\widehat{b}), \ldots, g^{(n)}(\widehat{b}^-)) = n$. Therefore, by the Budan–Fourier theorem for splines and by (5) and (6),

$$\begin{aligned} Z_g(\widehat{a}, \widehat{b}) &\le Z_{g^{(n)}}(\widehat{a}, \widehat{b}) + S^-(g(\widehat{a}), \ldots, g^{(n)}(\widehat{a}^+)) - S^+(g(\widehat{b}), \ldots, g^{(n)}(\widehat{b}^-)) \\ &\le Z_{g^{(n)}}(\widehat{a}, \widehat{b}) + S^-(g(a), \ldots, g^{(n)}(a^+)) - S^+(g(b), \ldots, g^{(n)}(b^-)) \\ &\le Z_{g^{(n)}}(\widehat{a}, \widehat{b}) + m - 1 - m \\ &\le \text{ number of knots of } g \text{ in } (\widehat{a}, \widehat{b}) - 1 \\ &\le Z_g(\widehat{a}, \widehat{b}), \end{aligned}$$

the last inequality since g fails to vanish at only one knot and since the preceding inequality already shows that $(\widehat{a}, \widehat{b})$ contains at least one knot. It follows that *all inequalities used to establish this string of inequalities must have been equalities*.

In particular, $S^-(g(\widehat{a}), \ldots, g^{(n)}(\widehat{a}^+)) = m - 1$, hence, as $m > 1$ by our assumption that $n > 1$, we must have $\widehat{a} = a$, and, similarly, $S^+(g(\widehat{b}), \ldots, g^{(n)}(\widehat{b}^-)) = m$ and so $\widehat{b} = b$. Further, these equalities produce a wealth of information about the x–section $g(t) = K(x,t)$ of the Peano kernel in case $x_i < x < x_{i+1}$†:

(i) *Since* $Z_{g^{(n)}}(a,b) = N$, $g^{(n)}$ *changes sign strongly across each of its simple knots. Since* $\text{jump}_x g^{(n)} = -(-)^n n! < 0$††, *this implies that*

$$\begin{aligned} &-(-)^{i-j} g^{(n)} > 0 \text{ on } (x_j, x_{j+1}) \text{ for } j = 0, \ldots, i-1 \\ &-(-)^{j-i} g^{(n)} < 0 \text{ on } (x_j, x_{j+1}) \text{ for } j = i+1, \ldots, N-1. \end{aligned}$$

† published version has $x_i = x < x_{i+1}$

†† *published version has* $= (-)^n$

(ii) $S^-(g(a),\dots,g^{(n)}(a^+)) = S^-(g^{(m)}(a),\dots,g^{(n)}(a^+)) = n-m$, *therefore* $g^{(j)}(a^+) \neq 0$ *for* $j = m,\dots,n$, *and, with* (i),

$$(-)^{i+n-j} g^{(j)}(a^+) > 0 \text{ for } j = m,\dots,n.$$

Since also $g^{(j)}(a) = 0$ *for* $j = 0,\dots,m-1$, *it follows that*

$$-(-)^{i+1-m} g(a+\varepsilon) > 0 \text{ for positive } \varepsilon \text{ near } 0.$$

(iii) g *has a simple zero at* $x_1,\dots,x_{N-1}$ *and vanishes nowhere else in* (a,b), *hence changes sign across each* x_j *and nowhere else. Therefore, with* (ii),

$$-(-)^{i+1-m+j} g(t) > 0 \text{ for } x_j < t < x_{j+1}, \quad j = 0,\dots,N-1.$$

(iv) $S^+(g(b),\dots,g^{(n)}(b^-)) = S^+(g(b),\dots,g^{(m)}(b)) = m$, *therefore, with* (i) *or* (iii),

$$-(-)^{i+N} g^{(j)}(b^-) > 0 \text{ for } j = m,\dots,n.$$

It follows, in particular, that, in the usual pointwise estimate

$$(8) \qquad |f(x) - Sf(x)| \le \int_a^b |K(x,t)|\,\mathrm{d}t \|f^{(n+1)}\|_\infty / n!$$

obtained from (3), we have equality iff $|f^{(n+1)}| = \|f^{(n+1)}\|_\infty$ and $f^{(n+1)}$ changes sign across each of the interpolation points $x_1,\dots,x_{N-1}$, i.e., f is a perfect spline of degree $n+1$ with simple knots at $x_1,\dots,x_{N-1}$ and nowhere else. If such a spline has a positive $(n+1)$st derivative in (a,x_1), then

$$(-)^{i+1-m}(f(x) - Sf(x)) > 0.$$

The sign structure of the *fundamental functions* for complete spline interpolation can be completely analysed in the same way. More interestingly, such an analysis of the sign structure of Peano kernel and fundamental functions can be carried out just as easily for spline interpolation with a variety of other side conditions, such a matching of value and odd derivatives at the boundary, matching of value and even derivatives at the boundary, matching of value and the m–th through $(n-1)$st derivatives at the boundary etc. The essential feature shared by these side conditions is that they are of the form

$$\lambda_i Sf = \lambda_i f, \quad i = 1,\dots,2m$$

with $(\lambda_i)_1^{2m}$ a sequence of linear functionals linearly independent over $\mathbb{P}_{2m} = \ker D^{2m}$:= polynomials of degree $< 2m$, and $(\lambda_i)_1^m$ a "good" m–sequence at a and $(\lambda_i)_{m+1}^{2m}$ a "good" m–sequence at b.

Here, we call an m–sequence $(\mu_i)_1^m$ of linear functionals a **"good" m–sequence at α**, provided $(\mu_i)_1^m$ has the following properties:

(i) $\mu_i f = \sum_{j=0}^{2m-1} a_{ij} f^{(j)}(\alpha)$ *for appropriate* a_{ij}'s, $i = 1,\dots,m$.

Further, with

$$\ker(\mu_i) := \{g \in C^{(2m-1)} \text{ near } \alpha : \mu_i g = 0, \textbf{ nocomma?} \quad \text{for } i = 1,\dots,m\},$$

(ii) $g \in \ker(\mu_i)$ *implies that* $S^+(g(\alpha),\dots,g^{(2m-1)}(\alpha)) \ge m$,

(iii) $g \in \ker(\mu_i)$ *implies that* $g^* \in \ker(\mu_i)$, *with* $g^*(\alpha + x) = g(\alpha - x)$, all x;

(iv) f, $g \in \ker(\mu_i)$ *implies that*

$$\sum_{j=0}^{2m-1} (-)^j f^{(j)}(\alpha) g^{(2m-1-j)}(\alpha) = 0.$$

We note that (ii) and (iii) together give

(ii)' $g \in \ker(\mu_i)$ *implies that* $S^-(g(\alpha),\dots,g^{(2m-1)}(\alpha)) \le m-1$ because of (1.1).

A particularly simple way of choosing a "good" m–sequence $(\mu_i)_1^m$ at α is to choose a strictly increasing subsequence $\mathbf{r} = (r_i)_1^m$ of $(0,\dots,2m-1)$ so that, for every $j = 0,\dots,2m-1$, either j or $2m-1-j$ occurs in $\mathbf{r}$ and then to take $\mu_i f = f^{(r_i)}(\alpha)$, $i = 1,\dots,m$. All of the examples mentioned earlier are of this simple form. For more complicated examples, we note that property (ii) is insured by having all m–minors of the $m \times 2m$ matrix (a_{ij}) of (i) of one sign with at least one of them nonzero.

Theorem 2. *Let* $m > 1$, $n := 2m-1$, *let* $a = x_0 < \cdots < x_N = b$, *and let* $(\lambda_i)_1^{2m}$ *be a sequence of linear functionals, linearly independent over* $\mathbb{P}_{2m} = \ker D^{2m}$ *and so that* $(\lambda_i)_1^m$ *is a "good"* m*–sequence at* a *and* $(\lambda_i)_{m+1}^{2m}$ *is a "good"* m*–sequence at* b. *Then*

(i) *For every* $f \in C^{(n)}[a,b]$ *there exists exactly one spline* Sf *of degree* n *with simple knots* $x_1, \ldots, x_{N-1}$ *in* (a,b) *which agrees with* f *in the sense that*

$$(Sf)(x_i) = f(x_i), \quad i = 1, \ldots, N-1, \tag{9}$$

$$\lambda_i Sf = \lambda_i f, \quad i = 1, \ldots, 2m. \tag{10}$$

(ii) *If* L_j *is the unique spline of degree* n *with simple knots* $x_1, \ldots, x_{N-1}$ *in* (a,b) *for which*

$$\begin{aligned} Lj(x_i) &= \delta_{ji}, \quad i = 1, \ldots, N-1, \\ \lambda_i L_j &= 0, \quad i = 1, \ldots, 2m, \end{aligned}$$

with $j \in [1, N-1]$, *(i.e., if* L_j *is a fundamental function for the interpolation process), then* L_j *has simple zeros at* x_i *for* $i \neq j$ *and vanishes nowhere else in* (a,b), *and its* n*–th derivative changes sign strongly across each knot.*

(iii) *For* $f \in \mathbb{L}_1^{(2m)}[a,b]$ *and for* $x \in (a,b)$,

$$f(x) - Sf(x) = \int_a^b K(x,t) f^{(2m)}(t)\,\mathrm{d}t/n! \tag{11}$$

where $g := K(x,\cdot)$ *is the error in interpolating* $(x-\cdot)_+^n$, *i.e.,*

$$K(x,t) = (x-t)_+^n - S_{(t)}(x-t)_+^n. \tag{12}$$

This x*–section* $g = K(x,\cdot)$ *of the kernel* K *is a spline of degree* n *with simple knots* $x_1, \ldots, x_{N-1}$ *and* x. *If* $x \notin \{x_1, \ldots, x_{N-1}\}$, *then* g *has simple zeros at* $x_1, \ldots, x_{N-1}$ *and vanishes nowhere else in* (a,b), *and its* n*–th derivative changes sign strongly across each knot in* (a,b); *otherwise* g *vanishes identically. Hence, for* $f \in \mathbb{L}_\infty^{(2m)}[a,b]$,

$$|f(x) - Sf(x)| \leq \|f^{(2m)}\|_\infty \int_a^b |K(x,t)|\,\mathrm{d}t/n! \tag{13}$$

with equality if and only if f *is a perfect spline of degree* $2m$ *with simple active knots at* $x_1, \ldots, x_{N-1}$ *in* (a,b) *and no other active knots in* (a,b), *i.e.,* $f^{(2m)}$ *is absolutely constant and changes sign strongly at* $x_1, \ldots, x_{N-1}$ *and nowhere else in* (a,b).

Proof: The proof parallels closely the earlier arguments for the special case

$$\lambda_i f = \begin{cases} f^{(i-1)}(a), & i = 1, \ldots, m, \\ f^{(i-m-1)}(b), & i = m+1, \ldots, 2m. \end{cases}$$

Property (iv) of a "good" m–sequence insures the selfadjointness of the problem

$$D^{2m} f = y, \quad \lambda_i f = 0, \quad i = 1, \ldots, 2m,$$

which then gives (11) and (12), and the sign structure of L_j and of $K(x,\cdot)$ follows from the Budan–Fourier theorem for splines and properties (ii) and (ii)' of a "good" m–sequence. We omit the details.

Remark. For the particular side conditions of matching even derivatives at a and b, (iii) of the theorem was stated by C. Hall and W. W. Meyer in [6], with the proof of the sign structure of $K(x,\cdot)$ deferred to an as yet unpublished paper (see Lemma 3 of [6]).

We add here that we learned only recently of a paper by Avraham A. Melkman entitled "The Budan–Fourier theorem for splines" which will appear eventually in the Israel Journal of Mathematics. In it, the author establishes such a theorem even for splines with multiple knots.

Part B. **The sign structure of cardinal null splines**

1. Introduction. A **cardinal spline function of order** k is a piecewise polynomial function of degree $< k$ in $C^{(k-2)}(\mathbb{R})$ with knots $\alpha + m$, for all $m \in \mathbb{Z}$ and some α. We denote their collection by

$$\$_{k,\alpha+\mathbb{Z}}.$$

A **cardinal null spline** is a cardinal spline which vanishes at all points of the form $\tau + m$ for $m \in \mathbb{Z}$ and some τ. Cardinal null splines have been analysed by Schoenberg [15] who showed them to be linear combinations of finitely many eigensplines. Here, an **eigenspline** is a nontrivial solution in $\$_{k,\alpha+\mathbb{Z}}$ of the functional equation

$$f(x+1) = zf(x),$$

shown by Schoenberg to exist for certain exceptional values of z called **eigenvalues**.

We wish to note in passing the work of Nilson [12] and earlier work by Ahlberg, Nilson and Walsh referred to therein where this eigenvalue problem is considered from another point of view.

Schoenberg used methods from the theory of difference equations for his analysis. We will proceed here somewhat differently and without reference to Schoenberg's earlier work. We start from the observation that a cardinal nullspline is completely determined by any one of its polynomial pieces, and study the linear transformation which carries such a polynomial piece into its neighboring polynomial piece. A convenient matrix representation for this linear transformation can be shown to be an *oscillation matrix*, and Gantmacher and Krein's [5] theory of such matrices then provides the detailed information about the sign structure of nullsplines needed in the later parts of this paper.

We wish to bring to the reader's attention the beautiful recent work by C. Micchelli [11] in which he, too, uses oscillation matrices in the analysis of cardinal nullsplines and eigensplines, but covers much more general splines and much more general interpolation conditions. We became aware of his work after we had completed the following sections and decided then to retain our arguments as that seemed more convenient for the reader than being told how to specialize Micchelli's more general results.

2. Cardinal splines which vanish at all knots. With the usual convention that $\binom{j}{i} = 0$ for $j < i$, we have, for a polynomial p of degree $< k$,

$$p^{(i)}(x+h)/i! = \sum_{j=0}^{k-1} \binom{j}{i} h^{j-i} p^{(j)}(x)/j!, \quad i = 0, \dots, k-1.$$

Hence, if such a polynomial vanishes at x and at $x+h$, then, for $i = 0$,

$$0 = \sum_{j=1}^{k-1} \binom{j}{0} h^j p^{(j)}(x)/j!$$

and, on subtracting $\binom{k-1}{i}/h^i$ times this equation from the i–th equation, we obtain

$$p^{(i)}(x+h)/i! = \sum_{j=1}^{k-2} (\binom{j}{i} - \binom{k-1}{i}) h^{j-1} p^{(j)}(x)/j!, \quad i = 1, \dots, k-2.$$

This we write as

$$\mathbf{p}(x+h) = -H_h^{-1} A_k H_h \mathbf{p}(x) \tag{1}$$

with $\mathbf{p}(x)$ the vector

$$\mathbf{p}(x) := (p^{(1)}(x), p^{(2)}(x)/2, \dots, p^{(k-2)}(x)/(k-2)!), \tag{2}$$

H_h the diagonal matrix

(3) $$H_h := \operatorname{diag}\lceil 1, h, \ldots, h^{k-3} \rfloor = H_{1/h}^{-1},$$

and A_k the matrix

(4) $$A_k := \left(\binom{k-1}{i} - \binom{j}{i}\right)_{i,j=1}^{k-2}.$$

If p is a polynomial of degree $< k$ vanishing at x and at $x+1$, then it vanishes also at $x+1$ and at $(x+1)+(-1)$, therefore (1) implies

$$1 = -H_{-1}^{-1} A_k H_{-1} (-H_1^{-1} A_k H_1)$$

or, with $H_1 = 1$,

(5) $$A_k^{-1} = H_{-1}^{-1} A_k H_{-1},$$

showing A_k to be invertible and similar to its own inverse. Therefore,

(6) $$\operatorname{spectrum}(A_k) = 1/\operatorname{spectrum}(A_k) = \{1/\lambda : \lambda \in \operatorname{spectrum}(A_k)\}.$$

Suppose that $\mathbf{u}$ is a (nonzero) eigenvector of A_k belonging to some nonzero eigenvalue λ. Let p be the polynomial of degree $< k$ which vanishes at 0 and at 1 and for which $\mathbf{p}(0) = \mathbf{u}$. Then $\mathbf{p}(1) = -A_k \mathbf{u} = -\lambda \mathbf{p}(0)$, hence,

$$p^{(j)}(1) = -\lambda p^{(j)}(0), \quad j = 0, \ldots, k-2.$$

The rule

$$S(x+n) := (-\lambda)^n p(x), \quad n \in \mathbb{Z}, \quad x \in [0,1)$$

therefore defines a spline of degree $< k$ with simple knots at $\mathbb{Z}$, and this spline is evidently an eigenspline belonging to the eigenvalue $-\lambda$. This explains our interest in the eigenvalue–eigenvector structure of the matrix A_k.

In [2], the matrix A_k was claimed to be totally positive. Actually, A_k *is an oscillation matrix*, i.e., A_k is totally positive and some power of A_k is strictly totally positive. To see this, observe that

$$A_k = -B'_k \begin{pmatrix} 1, \ldots, k-2 \\ 1, \ldots, k-2 \end{pmatrix}$$

with B'_k the matrix obtained from

$$B_k := \left(\binom{j}{i}\right)_{i,j=0}^{k-1}$$

by subtracting the last column of B_k from all other columns. This implies that

$$\det B_k \begin{pmatrix} 0, i_1, \ldots, i_r \\ j_1, \ldots, j_r, k-1 \end{pmatrix} = \det B'_k \begin{pmatrix} 0, i_1, \ldots, i_r \\ j_1, \ldots, j_r, k-1 \end{pmatrix}$$

and that the first row of B'_k vanishes except for a 1 in column $k-1$. Hence, further,

$$\det B'_k \begin{pmatrix} 0, i_1, \ldots, i_r \\ j_1, \ldots, j_r, k-1 \end{pmatrix} = (-)^r \det B'_k \begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix}$$
$$= \det A_k \begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix}$$

provided $0 < i_1, \ldots, i_r, j_1, \ldots, j_r < k-1$. This shows A_k to be totally positive since B_k is known to be totally positive (cf., e.g., [14]). Since A_k is also invertible, A_k is proven to be an oscillation matrix once we show that none of its entries on the first subdiagonal and the first superdiagonal vanishes (Gantmacher and Krein [5, Theorem 10 in Chap. 2, par. 7]). The numbers in question are

$$\binom{k-1}{i} - \binom{i+1}{i}, \quad i = 1, \ldots, k-3, \quad \text{and } \binom{k-1}{i}, \quad i = 2, \ldots, k-2,$$

and clearly all are positive.

The fact that A_k is an oscillation matrix allows the following conclusions (cf. [5, Theorem 6 in Chap. 2, par. 5]):

Theorem 1. (i) *The spectrum of A_k consists of $k-2$ (different) positive numbers which we will think of as ordered:*

$$0 < \lambda_{k-2} < \cdots < \lambda_1.$$

Further, by (6),

$$\lambda_i \lambda_j = 1 \; for \; i + j = k - 1. \tag{7}$$

(ii) *If $(\mathbf{u}^{(i)})_1^{k-2}$ is a corresponding (necessarily complete) sequence of (nonzero) eigenvectors for A_k and $c_p, \dots, c_q$ are numbers not all zero, then*

$$p - 1 \le S^-(\sum_{i=p}^{q} c_i \mathbf{u}^{(i)}) \le S^+(\sum_{i=p}^{q} c_i \mathbf{u}^{(i)}) \le q - 1.$$

As a particular consequence of (ii), we have

$$S^-(\mathbf{u}^{(i)}) = S^+(\mathbf{u}^{(i)}) = i - 1, \quad i = 1, \dots, k-2, \tag{8}$$

therefore

$$u_1^{(i)} \neq 0 \text{ and } u_{k-2}^{(i)} \neq 0 \tag{9}$$

(since otherwise $S^-(\mathbf{u}^{(i)}) < S^+(\mathbf{u}^{(i)})$.) Further, let p be the polynomial of degree $< k$ which vanishes at 0 and at 1 and for which

$$\mathbf{p}(0) = \mathbf{u}^{(i)}.$$

Then

$$\mathbf{p}(1) = -A_k \mathbf{u}^{(i)} = -\lambda_i \mathbf{p}(0)$$

hence

$$p^{(k-1)} = p^{(k-2)}(1) - p^{(k-2)}(0) = -(\lambda_i + 1) u_{k-2}^{(i)}$$

therefore, with (9),

$$p^{(k-2)}(0) p^{(k-1)}(0) < 0 < p^{(k-2)}(1) p^{(k-1)}(1)$$

and so, again with (9),

$$S^-(p(0), \dots, p^{(k-1)}(0)) = i = S^+(p(1), \dots, p^{(k-1)}(1)).$$

Finally, these statements about the sign structure of p persist if we perturb p slightly because of (8) as long as we keep $p(0) = p(1) = 0$.

Corollary. *If $\mathbf{u}^{(i)}$ is a nonzero eigenvector belonging to the i–th eigenvalue of A_k (ordered as in the theorem), then, for any polynomial p of degree $< k$ vanishing at 0 and at 1 and with $\mathbf{p}(0)$ close to $\mathbf{u}^{(i)}$,*

$$S^-(p(0), \dots, p^{(k-1)}(0)) = i = S^+(p(1), \dots p^{(k-1)}(1)).$$

In particular, such a p has no zeros in $(0, 1)$, by the Budan–Fourier Theorem.

If now f is a cardinal spline function of order k with knots $\mathbb{Z}$ which vanishes at its knots, then

$$\mathbf{f}(m) = -A_k \mathbf{f}(m-1) \text{ for all } m \in \mathbb{Z}$$

hence

$$\mathbf{f}(m) = (-A_k)^m \mathbf{f}(0) \text{ for all } m \in \mathbb{Z}.$$

Further, expanding $\mathbf{f}(0)$ in terms of a complete eigenvector sequence $(\mathbf{u}^{(i)})_1^{k-2}$ for A_k, we have

$$\mathbf{f}(m) = \sum_{i=1}^{k-2} (-\lambda_i)^m c_i \mathbf{u}^{(i)} = \sum_{i=1}^{k-2} c_i \mathbf{U}_i(m)$$

with U_i the unique cardinal spline of order k which vanishes at its knots $\mathbb{Z}$ and satisfies $\mathbf{U}_i(0) = \mathbf{u}^{(i)}$, hence satisfies

$$\mathbf{U}_i(m) = (-\lambda_i)^m \mathbf{U}_i(0), \quad \text{all } m \in \mathbb{Z},$$

i.e., U_i is the eigenspline belonging to the eigenvalue $-\lambda_i$. Therefore

$$f = \sum_{i=1}^{k-2} c_i U_i.$$

3. Cardinal nullsplines which vanish between knots. The analysis is slightly more complicated when the nullspline vanishes at $\tau + \mathbb{Z}$ with τ not a knot.

Let $\tau \in (0, 1)$ and let p be any spline of order k vanishing at τ and at $\tau + 1$ and with simple knot at 1. Then

$$\sum_{j=1}^{k-1} (p^{(j)}(\tau)/j!) \binom{j}{i} (1-\tau)^{j-i} = p^{(i)}(1)/i!, \quad i = 0, \ldots, k-2,$$

and

$$\sum_{j=0}^{k-2} (p^{(j)}(1)/j!) [\binom{j}{i} - \binom{k-1}{i}] \tau^{j-i} = p^{(i)}(\tau+1)/i!, \quad i = 1, \ldots, k-1.$$

Hence, with

$$B_k := (\tbinom{j}{i})_{i,j=0}^{k-1}$$

and B'_k the matrix derived from B_k by subtracting the last column from all other columns as before, we conclude that

$$-A_{k,\tau} \mathbf{p}(\tau) = \mathbf{p}(\tau + 1)$$

where

$$(1) \qquad \begin{aligned} A_{k,\tau} &: = -(H_\tau^{-1} B'_k H_\tau) \binom{1, \ldots, k-1}{0, \ldots, k-2} (H_{1-\tau}^{-1} B_k H_{1-\tau}) \binom{0, \ldots, k-2}{1, \ldots, k-1} \\ &= -\frac{1-\tau}{\tau} H_\tau^{-1} B'_k \binom{1, \ldots, k-1}{0, \ldots, k-2} H_\tau H_{1-\tau}^{-1} B_k \binom{0, \ldots, k-2}{1, \ldots, k-1} H_{1-\tau} \end{aligned}$$

and now

$$\mathbf{p}(x) := (p^{(1)}(x)/1!, \ldots, p^{(k-1)}(x)/(k-1)!).$$

By considering the spline q given by

$$q(1+x) = p(1-x), \quad \text{all } x,$$

which then vanishes at $1-\tau$ and at $1 + (1-\tau)$ and has a simple knot at 1, we find that

$$-H_{-1}^{-1} A_{k,1-\tau} H_{-1} (-A_{k,\tau}) = 1,$$

hence $A_{k,\tau}$ is invertible and

$$A_{k,\tau}^{-1} = H_{-1}^{-1} A_{k,1-\tau} H_{-1}.$$

It follows that

$$(2) \qquad \text{spectrum}(A_{k,\tau}) = 1/\text{spectrum}(A_{k,1-\tau}).$$

Both $B_k \binom{0,\ldots,k-2}{1,\ldots,k-1}$ and $-B'_k \binom{1,\ldots,k-1}{0,\ldots,k-2}$ are easily seen to be oscillation matrices with the aid of arguments used earlier to establish that $-B'_k \binom{1,\ldots,k-2}{1,\ldots,k-2}$ is an oscillation matrix. $A_{k,\tau}$ is therefore also an oscillation matrix.

Theorem 2. (i) *The spectrum of $A_{k,\tau}$ consists of $k-1$ (different) positive numbers which we will think of as ordered:*

$$0 < \lambda_{k-1}(\tau) < \cdots < \lambda_1(\tau).$$

Further, by (2),

$$\lambda_i(\tau)\lambda_j(1-\tau) = 1 \text{ for } i+j=k. \tag{3}$$

(ii) *If $(\mathbf{u}^{(i,\tau)})_{i=1}^{k-1}$ is a corresponding (necessarily complete) sequence of eigenvectors for $A_{k,\tau}$, then*

$$S^-(\mathbf{u}^{(i,\tau)}) = S^+(\mathbf{u}^{(i,\tau)}) = i-1, \quad i=1,\ldots,k-1.$$

(iii) *If p is a spline of degree $< k$ which vanishes at τ and at $\tau+1$, has a simple knot at 1, and for which $\mathbf{p}(\tau)$ is close to $\mathbf{u}^{(i,\tau)}$, then*

$$S^-(p(\tau),\ldots,p^{(k-1)}(\tau)) + 1 = i = S^+(p(\tau+1),\ldots,p^{(k-1)}(\tau+1)).$$

In particular, such a p does not vanish on $(\tau,\tau+1)$, by the Budan–Fourier theorem for splines.

Proof: In view of the preceding discussion, only assertion (iii) needs argument. For this, we note that $p(\tau)=p(\tau+1)=0$ while, by (ii), $S(p^{(1)}(\tau),\ldots,p^{(k-1)}(\tau)) = i-1$. □

Since $A_{k,\tau}$ is an analytic function of $\tau\in(0,1)$, so is each $\lambda_i(\tau)$. Further,

$$\prod_{i=1}^{k-1}\lambda_i(\tau) = (-)^{k-1}\det A_{k,\tau} = (1-\tau)/\tau$$

so that

$$\lim_{\tau\to 0}\lambda_1(\tau) = +\infty, \quad \lim_{\tau\to 1}\lambda_{k-1}(\tau) = 0.$$

Also

$$\lim_{\tau\to 1} H_{1-\tau}^{-1}B_k H_{1-\tau} = 1, \quad \lim_{\tau\to 1} H_\tau^{-1}B_k' H_\tau = B_k',$$

therefore

$$\lim_{\tau\to 1} A_{k,\tau} = A_{k-1} := B_k'\binom{1,\ldots,k-1}{0,\ldots,k-2}1\binom{0,\ldots,k-2}{1,\ldots,k-1} = \left(\begin{array}{c|c} A_k & 0 \\ \hline -1\cdots-1 & 0\end{array}\right), \text{ asyouadvised, Ididnotworryaboutit}$$

with A_k the oscillation matrix discussed earlier. Hence

$$\lim_{\tau\to 1}\lambda_i(\tau) = \lambda_i(1) := \begin{cases}\lambda_i, & i=1,\ldots,k-2\\ 0, & i=k-1\end{cases}$$

where $\lambda_{k-2} < \cdots < \lambda_1$ are the eigenvalues of A_k. Consequently, from (3) and since $\lambda_i\lambda_j = 1$ for $i+j=k-1$ by (2.7) above,

$$\lim_{\tau\to 0}\lambda_i(\tau) = 1/\lim_{\tau\to 1}\lambda_{k-i}(\tau) = 1/\lambda_{k-i} = \lambda_{i-1} = \lim_{\tau\to 1}\lambda_{i-1}(\tau)$$

for $i=2,\ldots,k-1$.

Theorem 3. *The function Λ_k defined on $(0,\infty)$ by*

$$\Lambda_k(\tau) := \begin{cases}\lambda_i(\tau-i-1), & i-1\le\tau\le i, \quad i=1,\ldots,k-1\\ 0, & k-1\le\tau\end{cases}$$

is continuous (in fact in $C^{(k-2)}(0,\infty)$) and maps $(0,k-1)$ to $(0,\infty)$. Also, Λ_k is strictly monotonely decreasing on $(0,k-1)$, and $\Lambda_k(\tau) = \Lambda_k(k-1-\tau)$.

Proof: We only have to establish the strict monotonicity of Λ_k on $(0,k-1)$. For this, it is sufficient to show:

if τ_1, $\tau_2 \in (0,1)$ are such that $\lambda_{i_1}(\tau_1) = \lambda_{i_2}(\tau_2) = \lambda$, say, then $\tau_1 = \tau_2$ and $i_1 = i_2$.

For this, let q_r be a spline of order k vanishing at τ_r and at $1+\tau_r$, with a simple knot at 1 and such that $\mathbf{q}_r(\tau_r)$ is a nontrivial eigenvector for A_{k,τ_r} corresponding to $\lambda = \lambda_{i_r}(\tau_r)$. Then

$$q_r^{(j)}(1) = -\lambda q_r^{(j)}(0), \ \ \textbf{isthe = afterthecommacorrect?} \quad j = 0,\ldots,k-2.$$

If now $q_r(0) = 0$, then $(q_r^{(1)}(0),\ldots,q_r^{(k-2)}(0)/(k-2)!)$ would be a nontrivial eigenvector of A_k, hence q_r would have to be nonzero on $(0,1)$ by the Corollary to Theorem 1, a contradiction. Hence, $q_r(0) \neq 0$, and (iii) of Theorem 2 implies that

$$q_r(\tau)/q_r(0) \gtrless 0 \ \text{ for } \ \tau \lessgtr \tau_r. \tag{4}$$

With this, the spline

$$q := q_1/q_1(0) - q_2/q_2(0)$$

is of degree $< k$, vanishes at 0 and at 1 and satisfies

$$q^{(j)}(1) = -\lambda q^{(j)}(0), \quad j = 0,\ldots,k-2,$$

while

$$q(\tau_1)q(\tau_2) = -\Big(q_1(\tau_2)/q_1(0)\Big)\Big(q_2(\tau_1)/q_2(0)\Big) \leq 0$$

by (4). Hence q vanishes in $(0,1)$, and therefore must vanish identically since otherwise $(q^{(1)}(0),\ldots,q^{(k-2)}(0)/(k-2)!)$ would be a nontrivial eigenvector for A_k, hence q couldn't vanish in $(0,1)$. It follows that $q_1/q_1(0) = q_2/q_2(0)$, therefore $\tau_1 = \tau_2$ by (4), and $i_1 = i_2$ follows from (i) of Theorem 2. □

A different proof of the theorem can be found in Schoenberg [18]. The theorem itself is due to C. Micchelli who proved it in [10] for the much more general case of cardinal $\mathcal{L}$–splines.

If we combine the earlier statement

$$\lambda_i(\tau)\lambda_j(1-\tau) = 1 \text{ for } i+j = k \tag{3}$$

with the strict decrease, just proven, of each $\lambda_i(\tau)$ as τ goes from 0 to 1, then we obtain the following corollary which will be helpful in the discussion of even degree cardinal spline interpolation in Part C.

Corollary. *Let $p = p(\tau,k)$ be the smallest integer so that $\lambda_p(\tau) \leq 1$, and let q be the largest integer so that $\lambda_q(\tau) \geq 1$ and let $\tau \in (0,1]$. Then, with $m := \lfloor k/2 \rfloor$,*

$$(p,q) = \begin{cases} (m+1,\ m) & \text{for } k \text{ odd and } \tau \in (0,1) \\ (m,\ m) & \text{for } k \text{ odd and } \tau = 1 \\ (m+1,\ m) & \text{for } k \text{ even and } \tau \in (0,\frac{1}{2}) \\ (m,\ m) & \text{for } k \text{ even and } \tau = \frac{1}{2} \\ (m,\ m-1) & \text{for } k \text{ even and } \tau \in (\frac{1}{2},1) \end{cases}$$

In particular, $\lambda_i(\tau) \neq 1$ unless $i = m$ and

$$\tau = \begin{cases} \frac{1}{2}, & k \text{ even} \\ 1, & k \text{ odd.} \end{cases}$$

It is now a simple matter to describe a cardinal spline of order k with knots $\mathbb{Z}$ which vanishes at $\tau + \mathbb{Z}$ for some $\tau \in (0,1)$. If f is such a spline, then

$$f = \sum_{i=1}^{k-1} c_i U_{i,\tau}$$

with $U_{i,\tau}$ the eigenspline belonging to the eigenvalue $-\lambda_i(\tau)$ and satisfying

$$\mathbf{U}_{i,\tau}(\tau) = \mathbf{u}^{(i,\tau)},$$

and the c_i's so chosen that $\mathbf{f}(\tau) = \sum_1^{k-1} c_i \mathbf{u}^{(i,\tau)}$.

Part C. **The sign structure of the fundamental function and the Peano kernel for cardinal spline interpolation**

1. A theorem on cardinal Green's functions. The application of the Budan–Fourier theorem for splines to cardinal spline interpolation takes the form of the following theorem.

Theorem 1. *Let K be a real valued function on $\mathbb{R}$ having the following properties:*

(i) *K is a polynomial spline of order $k > 2$ not identically zero and has simple knots at all integers n with $|n| \ge N$ for some N. K has no more than r additional knots, all of which lie in $(-N, N)$ and are simple.*

(ii) *For some $\tau \in (0, 1]$ with*

$$\tau \neq k/2 - \lfloor k/2 \rfloor + 1/2 \tag{1}$$

and for all $n \ge N$, K vanishes at $\tau + n$ and at $\tau - n - 1$. In addition, there are at least r distinct points in $(-n, N)$ at which K vanishes and no more than one of these occurs in any interval $(\alpha, \beta]$ formed by neighboring knots of K.

(iii) *K is of power growth at $\pm\infty$.*

Then, K vanishes exactly r times in $(-N, N]$ and has simple zeros at the points $\tau + n$ and $\tau - n - 1$ for $n = N, N + 1, \ldots,$ and vanishes nowhere else, and its $(k - 1)$st derivative changes sign strongly across each knot. Also, with $(\lambda_i(\tau))$ the strictly decreasing sequence of eigenvalues of

$$A := \begin{cases} A_{k,\tau}, & \tau < 1 \\ A_k \ \ , & \tau = 1 \end{cases} \tag{2}$$

as described in Part B, we have

$$\begin{aligned} 0 < \limsup_{x\to\infty} |K(x)|/|\lambda_p(\tau)|^x < \infty \\ 0 < \limsup_{x\to-\infty} |K(x)|/|\lambda_q(\tau)|^x < \infty \end{aligned} \tag{3}$$

with

$$(p, q) = \begin{cases} (m + 1,\ m) & \textit{if } k \textit{ odd} \\ (m + 1,\ m) & \textit{if } k \textit{ even and } \tau \in (0, \frac{1}{2}) \\ (m,\ m - 1) & \textit{if } k \textit{ even and } \tau \in (\frac{1}{2}, 1] \end{cases} \tag{4}$$

and

$$m := \lfloor k/2 \rfloor.$$

In particular, $K(x)$ decays exponentially as $|x| \to \infty$.

Proof: We intend to apply the Budan–Fourier theorem to the spline K and therefore begin with the observation that K is necessarily of exact degree $k - 1$ since otherwise K would be a polynomial (of degree $< k - 1$) which vanishes infinitely often, hence would have to vanish identically.

We recall from Part B the abbreviation

$$\mathbf{f}(x) := \begin{cases} (f(x), \ldots, f^{(k-1)}(x)/(k-1)!), & \tau < 1, \\ (f(x), \ldots, f^{(k-2)}(x)/(k-2)!), & \tau = 1. \end{cases}$$

Since A ($= A_{k,\tau}$ or A_k) is diagonalizable, we can write $\mathbf{K}(\tau + N)$ as

$$\mathbf{K}(\tau + N) = \mathbf{u}^{(p,\tau)} + \sum_{i>p} c_i \mathbf{u}^{(i,\tau)}$$

with $(\mathbf{u}^{(i,\tau)})$ an appropriate complete eigenvector sequence for A corresponding to the eigenvalue sequence $(\lambda_i(\tau))$. Then

$$\begin{aligned}\mathbf{K}(\tau+N+n) &= (-A)^n \mathbf{K}(\tau+N)\\ &= (-\lambda_p(\tau))^n \mathbf{u}^{(p,\tau)} + \sum_{i>p} c_i(-\lambda_i(\tau))^n \mathbf{u}^{(i,\tau)}\end{aligned}$$

for $n = 1, 2, \ldots$, hence

$$\mathbf{K}(\tau+N+n) = (-\lambda_p(\tau))^n \mathbf{u}^{(p,\tau)} + o((\lambda_p(\tau))^n) \text{ as } n \to \infty. \tag{5}$$

Since K is of power growth at ∞, it now follows that

$$\lambda_p(\tau) \le 1,$$

hence, by the Corollary to Theorem B3 we must have

$$p \ge \begin{cases} m+1, & \text{if } k \text{ is odd}\\ m+1, & \text{if } k \text{ is even and } \tau \in (0, \frac{1}{2})\\ m\ , & \text{if } k \text{ is even and } \tau \in (\frac{1}{2}, 1]\end{cases} \tag{6}$$

while, by (iii) of Theorem B2 (if $\tau < 1$) and by the Corollary to Theorem B1 (if $\tau = 1$),

$$S(K^{(1)}(\tau+n), \ldots, K^{(k-1)}(\tau+n^-)) = p-1, \tag{7}$$

therefore

$$S^+(K(\tau+n), \ldots, K^{(k-1)}(\tau+n^-)) = p \tag{8}$$

for all large $n \in \mathbb{N}$. One proves analogously that, for a possibly differently normalized eigenvector sequence for A,

$$\mathbf{K}(\tau-N-1-n) = (-\lambda_q(\tau))^{-n} \mathbf{u}^{(q,\tau)} + o((\lambda_q(\tau))^{-n}) \text{ as } n \to \infty \tag{9}$$

with

$$q \le \begin{cases} m, & \text{if } k \text{ is odd}\\ m, & \text{if } k \text{ is even and } \tau \in (0, \frac{1}{2})\\ m-1, & \text{if } k \text{ is even and } \tau \in (\frac{1}{2}, 1],\end{cases} \tag{10}$$

hence

$$S^-(K(\tau-n-\varepsilon), \ldots, K^{(k-1)}(\tau-n-\varepsilon)) = q \tag{11}$$

for large $n \in \mathbb{N}$ and small $\varepsilon > 0$. Now take n large enough so that (8) and (11) hold and abbreviate

$$a := \tau - n - 1 - \varepsilon, \quad b := \tau + n.$$

Let $[\widehat{a}, \widehat{b}]$ be an interval in $[a, b]$ which is maximal with respect to the property that it contains only isolated zeros of K. Then either $\widehat{a} = a$ or else $S^-(K(\widehat{a}), \ldots, K^{(k-1)}(\widehat{a}^+)) = 0$, and either $\widehat{b} = b$ or else $S^+(K(\widehat{b}), \ldots, K^{(k-1)}(\widehat{b}^-)) = k-1$. Therefore, we obtain from assumption (ii), from the Budan-Fourier theorem for splines, from equations (8) and (11) and from inequalities (6) and (10) that

$$\begin{aligned} Z_{K^{(k-1)}}(\widehat{a}, \widehat{b}) &\le \text{number of active knots of } K \text{ in } (\widehat{a}, \widehat{b})\\ &\le \text{number of knots of } K \text{ in } (\widehat{a}, \widehat{b})\\ &\le Z_K(\widehat{a}, \widehat{b}) + 1\\ &\le Z_{K^{(k-1)}}(\widehat{a}, \widehat{b}) + 1 + S^-(K(\widehat{a}), \ldots, K^{(k-1)}(\widehat{a}^+)) - S^+(K(\widehat{b}), \ldots, K^{(k-1)}(\widehat{b}^-))\\ &\le Z_{K^{(k-1)}}(\widehat{a}, \widehat{b}) + 1 + S^-(K(a), \ldots, K^{(k-1)}(a^+)) - S^+(K(b), \ldots, K^{(k-1)}(b^-))\\ &\le Z_{K^{(k-1)}}(\widehat{a}, \widehat{b}) + 1 + q - p\\ &\le Z_{K^{(k-1)}}(\widehat{a}, \widehat{b}),\end{aligned}$$

hence *equality must hold in all inequalities used to establish this string of inequalities*. In particular, $[\widehat{a}, \widehat{b}] = [a, b]$ since $k > 2$, and (6) and (10) must be equalities, hence (4) holds and (3) follows from (5) and (9). □

Remark. If (1) is violated, then the assumption (iii) of power growth at $\pm\infty$ is not sufficient to conclude the exponential decay of $K(x)$ as $|x| \to \infty$. For, then $\lambda_m(\tau) = 1$ by the Corollary to Theorem B3, and even boundedness of K would only imply that $q \le m \le p$, and would therefore not lead to equality in the Budan-Fourier inequality. Yet, replacing assumption (iii) by a stronger assumption such as that $K \in \L_s(\mathbb{R})$ for some $s < \infty$ would force $q \le m-1 < m+1 \le p$, and application of the Budan–Fourier theorem would lead to a contradiction unless we add additional freedom to K. We exploit this further in Part D.

Corollary. *Under the theorem's assumptions, let (t_i) be the knot sequence for K, and let (τ_i) be the increasing sequence of its zeros, numbered so that*

$$t_n \in (\tau_{n-1}, \tau_n] \text{ for all large } n,$$

as can be done by assumption. If also $t_1 \le \tau_1 < t_2$, then

$$\text{sign } K^{(1)}(\tau_1) K^{(k-1)}(\tau_1^-) = (-)^{p-1}.$$

Proof: Since all zeros of K are simple, and K changes sign strongly across each knot, we have

$$\text{sign } K^{(1)}(\tau_1) K^{(1)}(\tau_n) = (-)^{1-n} = \text{sign } K^{(k-1)}(t_1^-) K^{(k-1)}(t_n^-),$$

hence sign $K^{(1)}(\tau_1)K^{(k-1)}(\tau_1^-) = \text{sign } K^{(1)}(\tau_n)K^{(k-1)}(\tau_n^-)$ in case $t_1 \le \tau_1 < t_2$ and $t_n \le \tau_n < t_{n+1}$. On the other hand, $S(K^{(1)}(\tau_n), \ldots, K^{(k-1)}(\tau_n^-)) = p-1$ for all large n, by (7). □

2. Cardinal spline interpolation. The k–th order cardinal spline interpolant $S_k f$ to a given function f on $\mathbb{R}$ of power growth at $\pm\infty$ is, by definition, the unique spline of order k with knots $\mathbb{Z}+k/2$ which agrees with f at all integers and is of power growth at $\pm\infty$. A detailed discussion of this interpolation process can be found in Schoenberg's monograph [15].

The **fundamental function** of the process is, by definition, the unique cardinal spline L_k of power growth at $\pm\infty$ with knots $\mathbb{Z} + k/2$ which satisfies

$$L_k(n) = \delta_{0n} \text{ for all } n \in \mathbb{Z} \tag{1}$$

and so allows one to write the interpolant as

$$(S_k f)(x) = \sum_{\nu \in Z} f(\nu) L_k(x-\nu).$$

L_k is a cardinal Green's function in that

$$L_k(x) = \begin{cases} L(x), & x \le -1 \\ R(x), & x \ge 1 \end{cases}$$

with both L and R cardinal nullsplines. For even k, $K := L_k$ satisfies the hypotheses of Theorem 1 with $N=1$, $r=1$ and $\tau=1$. For odd k, $K := L_k(\cdot - \frac{1}{2})$ satisfies these hypotheses with $N=1$, $r=1$ and $\tau = \frac{1}{2}$. It is therefore a consequence of Theorem 1 that L_k *has a simple zero at every nonzero integer and vanishes nowhere else.* Therefore

$$\text{sign } L_k(x) = \text{sign } \frac{\sin \pi x}{\pi x} \text{ for all } x \in \mathbb{R}, \tag{2}$$

a fact apparently known (see, e.g., F. Richards [13]) but not proved anywhere as far as we know. Further, $L_k^{(1)}(1)$ must be negative since L_k is positive on $(-1,1)$, hence, by the Corollary to Theorem 1, $(-)^p L_k^{(k-1)}(1^-)$ must be positive, with p given by (1.4). In particular, $L_k^{(k-1)}(0^+) = L_k^{(k-1)}(1^-)$ and $p = k/2$ in case k is even, while $p = (k+1)/2$, and $L_k^{(k-1)}(0^+)$ and $L_k^{(k-1)}(1^-)$ have opposite sign in case k is odd. So,

$$\text{sign } L_k^{(k-1)}(0^+) = (-)^{\lfloor k/2 \rfloor}. \tag{3}$$

Theorem 1 also implies the known fact that

$$0 < \lim_{x\to\infty} |L_k(x)|/\gamma_k^{|x|} < \infty \tag{4}$$

with γ_k the largest eigenvalue less than one of A_k or of $A_{k,\frac{1}{2}}$ as k is even or odd.

Schoenberg [17] has obtained sharp estimates for the interpolation error $f - S_k f$ in terms of $\|f^{(k)}\|_\infty$ for even k. He uses the representation of the error

$$f(x) - (S_k f)(x) = \int_{-\infty}^{\infty} K_k(x,t) f^{(k)}(t)\,\mathrm{d}t/(k-1)! \tag{5}$$

with

$$K_k(x,t) := (x-t)_+^{k-1} - \sum_\nu (\nu - t)_+^{k-1} L_k(x-\nu) \tag{6}$$

which he shows to be valid for $f \in \mathbb{L}_{1,\mathrm{loc}}^{(k)}(\mathbb{R})$ with $f^{(k)}$ of power growth at $\pm\infty$, and for even k. He leaves to the present paper the proof of the following theorem.

Theorem 3 of [17]. *For even k greater than 2 and for $t \in (0,1)$, the function $K(x) := K_k(x,t)$ has simple zeros at all integer values of x and vanishes nowhere else.*

Proof: For fixed t, $K_k(x,t)$ is the error in interpolating $(x-t)_+^{k-1}$ in x by cardinal splines of order k, hence $K_k(\cdot,t) = 0$ in case $t \in \mathbb{Z}$ since then $(\cdot - t)_+^{k-1}$ is its own cardinal spline interpolant. Further, for $t \in (0,1)$, $K_k(\cdot,t)$ is of power growth at $\pm\infty$, vanishes at $\mathbb{Z}$ and has simple knots at $\mathbb{Z}$ and at t, and is of exact degree $k-1$ since its $(k-1)$st derivative has a nonzero jump at t. In short, $K := K_k(\cdot,t)$ satisfies the assumptions of Theorem 1 with $N = 1$, $r = 2$, and $\tau = 1$ and must, therefore, satisfy its conclusions. □

3. Even degree cardinal spline interpolation. We discuss now in more detail the interpolation error for *odd* k, i.e., for even degree cardinal spline interpolation,

$$k = 2m+1,$$

say. Let K_k be as defined in (2.6). Then, as $(\nu - t)_+^{k-1}$ is a spline of order k in t with a simple knot at ν, and is of power growth, we can write

$$(\nu - t)_+^{k-1} = \sum_\mu (\nu - \mu - \tfrac{1}{2})_+^{k-1} L_k(t - \mu - \tfrac{1}{2}),$$

with the series converging uniformly on compact sets, by (2.4). Therefore

$$\begin{aligned}\sum_\nu (\nu - t)_+^{k-1} L_k(x-\nu) &= \sum_\nu \Big(\sum_\mu (\nu - \mu - \frac{1}{2})_+^{k-1} L_k(t-\mu-\tfrac{1}{2})\Big) L_k(x-\nu) \\ &= \sum_\mu \Big(\sum_\nu (\nu - \mu - \frac{1}{2})_+^{k-1} L_k(x-\nu)\Big) L_k(t-\mu-\tfrac{1}{2}) \\ &= \sum_\mu (x - \mu - \tfrac{1}{2})_+^{k-1} L_k(t-\mu-\tfrac{1}{2})\end{aligned}$$

with the interchange permitted because of the absolute convergence of the series involved, and the last equality justified by the fact that $(x - \mu - \frac{1}{2})_+^{k-1}$ is a cardinal spline of order k in x. Since

$$(x-t)_+^{k-1} - (t-x)_+^{k-1} = (x-t)^{k-1}$$

for odd k and since cardinal spline interpolation of order k reproduces the right side of this identity, we conclude that

$$K_k(x,t) = -K_k(t-\tfrac{1}{2}, x-\tfrac{1}{2}). \tag{1}$$

Further, we conclude that, for fixed $x \in (0,1)$,

$$K_k(x, t+\tfrac{1}{2}) = (x-\tfrac{1}{2}-t)_+^{k-1} - \sum_\mu (x-\frac{1}{2}-\mu)_+^{k-1} L_k(t-\mu),$$

i.e., $K_k(x,t+\frac{1}{2})$ is the error in cardinal spline interpolation in t to $(x-\frac{1}{2}-t)_+^{k-1}$, hence is of power growth, vanishes at $\mathbb{Z}$, and has simple knots at $\mathbb{Z}-\frac{1}{2}$ and at $x-\frac{1}{2}$ and nowhere else. The function

$$K(t) := K_k(x,t)$$

therefore satisfies the hypotheses of Theorem 1 with $\tau = \frac{1}{2}$, $N = 1$, and $r = 2$, and must therefore also satisfy its conclusions. In particular, $K(t)$ has simple zeros at $\mathbb{Z}+\frac{1}{2}$ and vanishes nowhere else, i.e.,

$$\text{sign } K_k(x,t) = \varepsilon_k(x)\omega(t-\tfrac{1}{2})$$

with

$$\omega(t) := \text{sign } \sin \pi t$$

and $\varepsilon_k(x)$ equal to 1 or -1 or 0. In order to determine $\varepsilon_k(x)$, we observe that $\varepsilon_k(n) = 0$ for all $n \in \mathbb{Z}$. Further, for $x \in (0,1)$,

$$\text{jump}_x K^{(k-1)} = (k-1)!(-)^{k-1} = (k-1)!$$

since k is odd, therefore

$$\text{sign } K^{(k-1)}(x^+) = 1.$$

If now $x < \frac{1}{2}$, then the Corollary to Theorem 1 applies to K with $t_1 = x$ and $\tau_1 = \frac{1}{2}$, i.e.,

$$\text{sign } K^{(1)}(\tau_1)K^{(k-1)}(\tau_1^-) = (-)^{p-1}$$

with $p = m+1$, hence, as $K^{(k-1)}(x^+) = K^{(k-1)}(\tau_1^-)$, we have

$$\text{sign } K_k(x,x) = -\text{sign } K^{(1)}(\tau_1) = -(-)^m,$$

showing that $\varepsilon_k(x) = (-)^m$ in this case. If, on the other hand, $x \ge \frac{1}{2}$, then K satisfies the assumptions of the Corollary to Theorem 1 with $t_1 = 1$ and $\tau_1 = 3/2$, i.e., $\text{sign}K^{(1)}(3/2)K^{(k-1)}(3/2^-) = (-)^m$, while $K^{(k-1)}(x^+)$ and $K^{(k-1)}(3/2^-)$ have opposite sign, therefore

$$\text{sign } K_k(x,x) = -\text{sign } K^{(1)}(\tau_1) = (-)^m$$

showing that $\varepsilon_k(x) = (-)^m$ also in this case. Since

$$K_k(x,t) = K_k(x+1,t+1)$$

trivially, this proves that

$$\text{sign } K_k(x,t) = (-)^m\omega(x)\omega(t-1/2) \text{ for all } x \text{ and } t. \tag{2}$$

We also obtain from Theorem 1 that, for each x, there exists a constant $a = a(x)$ so that

$$|K_k(x,t)| \le a(x)|\lambda_m(\tfrac{1}{2})|^{|t|} \text{ for all } t. \tag{3}$$

Here, $\lambda_m(\frac{1}{2})$ is the largest eigenvalue less than 1 of $A_{k,\frac{1}{2}}$. It follows that, for $j = 1, \ldots, k-1$, $(\mathrm{d}/\mathrm{d}t)^j K_k(x,t)$ is (piecewise) continuous in t and decays exponentially as $|t| \to \infty$. Hence, if f has a locally integrable k–th derivative of power growth, then

$$E(x) := \int_{-\infty}^{\infty} K_k(x,t) f^{(k)}(t)\,\mathrm{d}t/(k-1)!$$

defines a function E on $\mathbb{R}$ which vanishes at $\mathbb{Z}$. Further, for $x \notin \mathbb{Z}$, we can evaluate $E(x)$ by repeated integration by parts, obtaining

$$E(x) = (-)^{k-1} \int_{-\infty}^{\infty} (d/\,\mathrm{d}t)^{k-1} K_k(x,t) f^{(1)}(t)\,\mathrm{d}t/(k-1)!.$$

But, since

$$(-)^{k-1}(\,\mathrm{d}/\mathrm{d}t)^{k-1} K_k(x,t)/(k-1)! = (x-t)_+^0 - \sum_\nu (\nu - t)_+^0 L_k(x-\nu),$$

we obtain that

$$\begin{aligned} E(x) &= \int_{-\infty}^{\infty} [(x-t)_+^0 - \sum_\nu (\nu-t)_+^0 L_k(x-\nu) f^{(1)}(t)]\,\mathrm{d}t \\ &= f(x) - \sum_\nu f(\nu) L_k(x-\nu) = f(x) - (S_k f)(x). \end{aligned}$$

Theorem 2. *Let $k = 2m+1$ be odd. If f has $k-1$ locally absolutely continuous derivatives on $\mathbb{R}$ and $f^{(k)}$ is of power growth at $\pm\infty$, then*

$$f(x) = (S_k f)(x) + \int_{-\infty}^{\infty} K_k(x,t) f^{(k)}(t)\,\mathrm{d}t/(k-1)! \tag{4}$$

with K_k given by (2.6). Further

$$|K_k(x,t)| \le a(x) \exp(-b_k|t|)$$

for some function a and some positive constant b_k, and, for $k \ge 3$,

$$\text{sign } K_k(x,t) = (-)^m \omega(x)\omega(t - \tfrac{1}{2})$$

with $\omega(r) :=$ sign $\sin r\pi$.

Specific choices for f in (4) give much information about K_k, much as in the discussion of K_{2m} in [17]. E.g., $f(x) := \sin \nu\pi x$ vanishes at $\mathbb{Z}$ and is bounded, hence $S_k f = 0$ and (4) gives

$$\sin \nu\pi x = (-)^m (\nu\pi)^{2m+1} \int_{-\infty}^{\infty} K_{2m+1}(x,t) \cos \nu\pi t\,\mathrm{d}t/(2m)!. \tag{5}$$

If we choose $f(x) := x^k/k!$, then $f^{(k)} = 1$ and $f - S_k f$ is known to be equal to the k–degree Bernoulli monospline $\overline{B}_k/k!$ (see [15, Lecture 4, §6C]), therefore

$$\int_{-\infty}^{\infty} K_{2m+1}(x,t)\,\mathrm{d}t/(2m)! = \overline{B}_{2m+1}(x)/(2m+1)!, \quad \text{all } x \in \mathbb{R}. \tag{6}$$

Finally, Hölder's inequality gives at once the following corollary.

Corollary. *If $f \in \mathbb{L}_\infty^{(k)}(\mathbb{R})$ with $k = 2m+1$, then, for any particular x,*

$$f(x) - S_k f(x) \le \|f^{(k)}\|_\infty \int_{-\infty}^{\infty} |K_k(x,t)| \, \mathrm{d}t/(k-1)!$$

with equality iff either $x \in \mathbb{Z}$ (in which case both sides vanish) or

$$f^{(k)}(t) = (-)^m \omega(x) \|f^{(k)}\|_\infty \omega(t - \tfrac{1}{2}). \tag{7}$$

One function f satisfying (7) is a shifted version of the k–th degree **Euler spline** (see [15, Lecture 4, §6B]). To recall, the k–th degree Euler spline $\mathcal{E}_k$ is a particular cardinal null spline, an eigenspline belonging to the eigenvalue -1, and normalized to satisfy

$$\mathcal{E}_k(\nu) = (-)^\nu, \quad \text{all } \nu \in \mathbb{Z}.$$

It has its knots at $\mathbb{Z} + (k+1)/2$, i.e., at $\mathbb{Z}$ since we took $k = 2m+1$, and, being an eigenspline with eigenvalue -1, must satisfy

$$\mathcal{E}_k(x+1) = -\mathcal{E}_k(x), \quad \text{all } x \in \mathbb{R},$$

therefore

$$\mathcal{E}_k(\nu + \tfrac{1}{2}) = 0, \quad \text{all } \nu \in \mathbb{Z}$$

$$\mathcal{E}_k^{(k)}(x) = (-)^m \|\mathcal{E}_k^{(k)}\|_\infty \omega(x).$$

It follows that $f(x) := \mathcal{E}_k(x - \frac{1}{2})$ has 0 for its cardinal spline interpolant and, except for the factor $\omega(x)$, satisfies (7), hence

$$\omega(x)\mathcal{E}_{2m+1}(x - \tfrac{1}{2}) = \|\mathcal{E}_{2m+1}^{(2m+1)}\|_\infty \int_{-\infty}^{\infty} |K_{2m+1}(x,t)| \, \mathrm{d}t/(2m)!. \tag{8}$$

Theorem 3. *Let $k = 2m+1$. If $f \in \mathbb{L}_\infty^{(k)}(\mathbb{R})$, then*

$$|f(x) - S_k f(x)| \le \frac{|\mathcal{E}_k(x - \frac{1}{2})|}{\|\mathcal{E}_k^{(k)}\|_\infty} \|f^{(k)}\|_\infty$$

and this inequality is sharp since it becomes equality for $f = \mathcal{E}_k(\cdot - \frac{1}{2})$. Moreover, if, for some $x \notin \mathbb{Z}$ and for some $f \in \mathbb{L}_\infty^{(k)}(\mathbb{R})$ with $\|f^{(k)}\|_\infty \le 1$,

$$|f(x) - S_k f(x)| = |\mathcal{E}_k(x - \tfrac{1}{2})| / \|\mathcal{E}_k^{(k)}\|_\infty,$$

then f must be of the form

$$f = \pm \mathcal{E}_k / \|\mathcal{E}_k^{(k)}\|_\infty + p$$

for some polynomial p of degree $< k$.

Part D. **A proof of Kolmogorov's theorem**

1. The Euler splines and statement of Kolmogorov's theorem. We already discussed the Euler splines in Section 3 of Part C, referring the reader to Schoenberg [15, Lecture 4] for background and proofs. For $k = 0, 1, 2, \ldots$, the Euler spline $\mathcal{E}_k$ is the unique spline function of degree k with simple knots at $\mathbb{Z} + (k+1)/2$ satisfying

$$\mathcal{E}_k(\nu) = (-)^\nu \text{ for all } \nu \in \mathbb{Z}, \tag{1}$$

$$\|\mathcal{E}_k\|_\infty \le 1. \tag{2}$$

Except for the name "Euler spline", these functions are very well known, their polynomial components being essentially the classical Euler polynomials. Our conditions normalize these functions in a way that is convenient for our purpose. In Schoenberg [16], the reader will find a direct recursive derivation of the $\mathcal{E}_k$.

The function $\mathcal{E}_k(x)$ is a kind if stylized version of $\cos \pi x$ to which it converges as $k \to \infty$. Its sign structure is described by the inequalities

$$(-)^{\nu + \lfloor (j+1)/2 \rfloor} \mathcal{E}_k^{(j)}(x) > 0 \text{ in } \begin{cases} \nu - \frac{1}{2} < x < \nu + \frac{1}{2} & \text{if } j \text{ is even} \\ \nu < x < \nu + 1 & \text{if } j \text{ is odd.} \end{cases} \tag{3}$$

In particular, we find that for the supremum norm we have

$$\|\mathcal{E}_k^{(j)}\|_\infty = (-)^{\lfloor (j+1)/2 \rfloor} \begin{cases} \mathcal{E}_k^{(j)}(0) & \text{if } j \text{ is even,} \\ \mathcal{E}_k^{(j)}(\frac{1}{2}) & \text{if } j \text{ is odd,} \end{cases} \quad j = 0, \ldots, k. \tag{4}$$

For convenience, we write

$$\|\mathcal{E}_k^{(j)}\|_\infty =: \gamma_{k,j} \quad (j = 0, \ldots, k). \tag{5}$$

These are rational numbers expressible in terms of the Euler numbers, and in particular $\gamma_{k,0} = 1$ by (1) and (2).

We also need the class

$$\mathbb{L}_\infty^{(k)}(\mathbb{R}) := \{ f \in C^{(k-1)}(\mathbb{R}) : f^{(k-1)} \text{ satisfies a Lipschitz cond. on } \mathbb{R} \}. \tag{6}$$

Evidently, $\mathcal{E}_k \in \mathbb{L}_\infty^{(k)}(\mathbb{R})$. By $\|f\|_\infty$ we mean the essential supremum of f on $\mathbb{R}$.

Theorem of Kolmogorov [8]. *If $f \in \mathbb{L}_\infty^{(k)}(\mathbb{R})$ is such that*

$$\|f\|_\infty \le \|\mathcal{E}_k\|_\infty \text{ and } \|f^{(k)}\| \le \|\mathcal{E}_k^{(k)}\|_\infty, \tag{7}$$

then

$$\|f^{(j)}\|_\infty \le \|\mathcal{E}_k^{(j)}\|_\infty \text{ for } j = 1, \ldots, k-1. \tag{8}$$

The right sides of (8) are the best constants for each j because the Euler spline $\mathcal{E}_k$ satisfies the conditions (7) and also the conclusions (8) with the equality signs. Note the Corollary 2 in Section 3 where it is shown that in a certain sense the Euler splines are the only functions for which we can have equality in (8), even for a single value of j.

In Section 2, we derive certain approximate differentiation formulae. These are applied in Section 3 to establish Kolmogorov's theorem. In Section 4, we establish the needed properties of the formulae of Section 2. Finally, in Section 5, we give a characterization of these differentiation formulae.

2. Some approximate differentiation formulae. In this section, we consider a cardinal Green's function K of order k which fails to be a spline function only because we require

(1) $$\text{jump}_{\alpha} K^{(k-j-1)} = (-)^{k-j}$$

for

(2) $$\alpha := \begin{cases} 0, & j \text{ even}, \\ \frac{1}{2}, & j \text{ odd}. \end{cases}$$

More explicitly, except for the jump condition (1), K is a spline function of order k with simple knots at $\mathbb{Z}$ and vanishes at $\mathbb{Z} + \tau$ where

(3) $$\tau := \begin{cases} \frac{1}{2}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

This choice of τ was explicitly excluded in Theorem C1 since it would allow *bounded* eigensplines, *viz* the Euler splines $\mathcal{E}_k$ already used in Part C and again in the previous section. This exclusion was the subject of the remark following Theorem C1. The required "additional freedom" for K mentioned there is provided here by the condition (1).

Theorem 1. *Let $k \geq 2$, and let $1 \leq j \leq k-1$. There exists a unique function K on $\mathbb{R}$ with the following three properties:*

(i) *K is a spline function of order k with simple knots at $\mathbb{Z}$ except that*

$$\text{jump}_{\alpha} K^{(k-j-1)} = (-)^{k-j},$$

with $\alpha = 0$ or $\frac{1}{2}$ as j is even or odd.

(ii) *K vanishes at $\mathbb{Z} + \tau$, with $\tau = \frac{1}{2}$ or 0 as k is even or odd, except that K is not required to vanish at τ in case $j + 1 = k$ as then K is not continuous at τ, by (i).*

(iii) *$K \in \mathbb{L}_1(\mathbb{R})$.*

Theorem 1 will be established in Section 4. Observe that the inequalities $1 \leq j \leq k-1$ imply that

$$0 \leq k - j - 1 \leq k - 2.$$

Hence K fails to be in $C^{(k-2)}(\mathbb{R})$ and this is the reason why K is not a cardinal spline.

Let us now assume that Theorem 1 were established and use the function K as a kernel as follows: If $f \in \mathbb{L}_\infty^{(k)}(\mathbb{R})$ and if we integrate by parts repeatedly the integral

$$\int_{-\infty}^{\infty} K(x) f^{(k)}(x)\,\mathrm{d}x$$

and use the jump condition (1), we obtain the following corollary.

Corollary 1. *For $f \in \mathbb{L}_\infty^{(k)}(\mathbb{R})$,*

(4) $$f^{(j)}(\alpha) = \sum_{\nu \in \mathbb{Z}} A_\nu f(\nu) + \int_{-\infty}^{\infty} K(x) f^{(k)}(x)\,\mathrm{d}x$$

with

(5) $$A_\nu := (-)^{k-1} \text{jump}_{\nu} K^{(k-1)}, \quad \textit{all } \nu \in \mathbb{Z}.$$

The derivation of this differentiation formula by integration by parts requires the following remark: Our construction of K in Sec. 4 will prove that all derivatives $K^{(\nu)}(x)$ ($\nu = 0, \ldots, k-1$) decay exponentially as $x \to \pm\infty$. On the other hand, $f \in \mathbb{L}_\infty^{(k)}(\mathbb{R})$ implies that the derivatives $f^{(\nu)}$ ($\nu = 0, \ldots, k$) can be of power growth at most at $\pm\infty$. This explains the vanishing of all "finite parts" at $\pm\infty$ and also the convergence of the series in (4). Clearly, K and A_ν depend also on k and j, but the values of k and j will be obvious from the context.

In our application of Corollary 1 to a proof of Kolmogorov's theorem, the following additional information on the A_ν and on K is vital.

Theorem 2. *We assume that $k \geq 4$.*

(i) *For certain positive constants A and B,*

$$|K^{(r)}(x)| < A\mathrm{e}^{-B|x|} \text{ for all } x \in \mathbb{R} \text{ and for } r = 0, \ldots, k-1. \tag{6}$$

(ii) *The coefficients A_ν of the formula (4) satisfy*

$$(-)^{\nu + \lfloor (j+1)/2 \rfloor} A_\nu > 0 \tag{7}$$

for all $\nu \in \mathbb{Z}$.

(iii) *The kernel K of Theorem 1 satisfies the inequality*

$$(-)^{\nu + \lfloor (k+1)/2 \rfloor + \lfloor (j+1)/2 \rfloor} K(x) > 0 \text{ for } x \in (\nu - \tau, \nu + 1 - \tau) \tag{8}$$

for all $\nu \in \mathbb{Z}$.

(iv) *The kernel K is symmetric around α. Specifically,*

$$K(\alpha + x) = (-)^{k-j} K(\alpha - x) \text{ for all } x \in \mathbb{R} \backslash \{0\}. \tag{9}$$

By (7), the coefficients A_ν alternate strictly in sign if $k \geq 4$. If $k = 2$ or 3, this is no longer the case, since then only a finite number of the A_ν are $\neq 0$ (see [16]). By (8), the kernel K vanishes only at the points $\mathbb{Z} + \tau$ (with the exception mentioned in (ii) of Theorem 1) and alternates strictly in sign as we cross from one unit interval to the next.

In the next section, we use Corollary 1 and Theorem 2 to establish Kolmogorov's theorem and to describe its extremizing functions. In Section 4, we establish Theorem 1 and Theorem 2 jointly by constructing K and then applying the Budan-Fourier theorem to its two pieces.

3. A proof of Kolmogorov's theorem.

We retain the definitions

$$\alpha := \begin{Bmatrix} 0 \\ \frac{1}{2} \end{Bmatrix} \text{ if } j \text{ is } \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}, \quad \tau := \begin{Bmatrix} \frac{1}{2} \\ 0 \end{Bmatrix} \text{ if } k \text{ is } \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$$

introduced in the preceding section. Our earlier description (1.3) – (1.4) of certain properties of the Euler spline then give that

$$(-)^{\nu + \lfloor (k+1)/2 \rfloor} \mathcal{E}_k^{(k)}(x) > 0 \quad \text{on } (\nu - \tau, \nu + 1 - \tau), \quad \text{for all } \nu \in \mathbb{Z} \tag{1}$$

and

$$(-)^{\lfloor (j+1)/2 \rfloor} \mathcal{E}_k^{(j)}(\alpha) = \|\mathcal{E}_k^{(j)}\|_\infty =: \gamma_{k,j} \quad \text{for } j = 0, \ldots, k. \tag{2}$$

We apply the differentiation formula (2.4) to the special function

$$f_0 := (-)^{\lfloor (j+1)/2 \rfloor} \mathcal{E}_k.$$

By (2),

$$f_0^{(j)}(\alpha) = (-)^{\lfloor (j+1)/2 \rfloor} \mathcal{E}_k^{(j)}(\alpha) = \gamma_{k,j} > 0. \tag{3}$$

The interpolation property (1.1) shows that $f_0(\nu) = (-)^{\nu + \lfloor (j+1)/2 \rfloor}$ and Theorem 2.(ii) shows then that

$$A_\nu f_0(\nu) = |A_\nu| > 0 \text{ for all } \nu \in \mathbb{Z}. \tag{4}$$

From (1) and (2), we find that

$$(-)^{\nu + \lfloor (k+1)/2 \rfloor + \lfloor (j+1)/2 \rfloor} f_0^{(k)}(x) = \gamma_{k,k} > 0 \text{ on } (\nu - \tau, \nu + 1 - \tau)$$

for all $\nu \in \mathbb{Z}$, and Theorem 2.(iii) then shows that

$$K(x) f_0^{(k)}(x) = |K(x)| \gamma_{k,k} > 0 \text{ for } x \in \mathbb{R} \backslash (\mathbb{Z} + \tau). \tag{5}$$

By (3), (4) and (5), the relation (2.4) becomes

$$f_0^{(j)}(\alpha) = \sum_{\nu=-\infty}^{\infty} |A_\nu| + \gamma_{k,k} \int_{-\infty}^{\infty} |K(x)| \, \mathrm{d}x. \tag{6}$$

If f is an arbitrary function in $\mathbb{L}_\infty^{(k)}(\mathbb{R})$ satisfying the assumptions (1.7), we may also assume that $f^{(j)}(\alpha) \geq 0$, for if not we consider $-f$ instead, which also satisfies all assumptions. Applying the differentiation formula (2.4) to our f, we obtain from (1.7) that

$$\begin{aligned} 0 \leq f^{(j)}(\alpha) &= \sum_{\nu=-\infty}^{\infty} A_\nu f(\nu) + \int_{-\infty}^{\infty} K(x) f^{(k)}(x) \, \mathrm{d}x \\ &\leq \sum_{\nu} |A_\nu| + \gamma_{k,k} \int_{-\infty}^{\infty} |K(x)| \, \mathrm{d}x = f_0^{(j)}(\alpha), \end{aligned} \tag{7}$$

the last equality following from (6). We have just shown that

$$|f^{(j)}(\alpha)| \leq |f_0^{(j)}(\alpha)| = \gamma_{k,j}. \tag{8}$$

If x_0 is real and if we apply our result to $f(x + x_0 - \alpha)$, then we obtain that $|f^{(j)}(x_0)| \leq \gamma_{k,j}$ and the conclusion (1.8) is established.

Let us now assume that in (7) we have $f^{(j)}(\alpha) = f_0^{(j)}(\alpha)$. This implies the equality of the two middle terms of (7), and this we may write as

$$\sum_{\nu} (|A_\nu| - A_\nu f(\nu)) + \int_{-\infty}^{\infty} \{\gamma_{k,k} - \frac{K(x)}{|K(x)|} f^{(k)}(x)\} |K(x)| \, \mathrm{d}x = 0. \tag{9}$$

By (1.7), we see that all terms of the series are nonnegative, and so is the integrand almost everywhere. From (4), we conclude that

$$A_\nu f(\nu) = A_\nu f_0(\nu) \text{ for all } \nu \in \mathbb{Z}$$

and (5) shows that

$$K(x) f^{(k)}(x) = K(x) f_0^{(k)}(x) \text{ for almost all } x.$$

Therefore

$$f(\nu) = f_0(\nu) \text{ for all } \nu, \quad f^{(k)} = f_0^{(k)} \quad \text{a.e. .} \tag{10}$$

Integrating the last relation k times, we conclude that f and f_0 may differ only by a polynomial of degree $< k$, and the first relations (10) show that this polynomial is identically zero. Hence

$$f = (-)^{\lfloor (j+1)/2 \rfloor} \mathcal{E}_k. \tag{11}$$

This completes our proof of Kolmogorov's theorem and also a proof of

Corollary 2. *Let f satisfy the assumptions (1.7) and therefore also the conclusions (1.8). If, for some value of $j \in [1, k-1]$, the equality sign holds in (1.8), and if the extremum $\|f^{(j)}\|_\infty = \gamma_{k,j}$ is attained in the sense that*

$$|f^{(j)}(x_0)| = \|f^{(j)}\|_\infty \text{ for some } x_0, \tag{12}$$

then f is necessarily of the form

$$f(x) = \pm \mathcal{E}_k(x - c). \tag{13}$$

Remarks. (i) Since we have used Theorem 2 in our proof, we have also implicitly assumed that $k \geq 4$. For the elementary cases when $k = 2$ and 3 see [16]. Corollary 2 is valid also for $k = 3$, but requires a special proof given in [16, §8]. For a discussion of the extremizing functions in the weak sense (when there is no x_0 satisfying (12)) see [16, §8].

(ii) For the case $j = 1$, Kolmogorov proved the stronger result: *If $f \in \mathbb{L}_\infty^{(k)}(\mathbb{R})$, and a, b and c are such that the function $g(x) := a\mathcal{E}_k(bx + c)$ satisfies $\|g\|_\infty = \|f\|_\infty$, $\|g^{(k)}\|_\infty = \|f^{(k)}\|_\infty$, and $g(x_0) = f(x_0)$, then $|g^{(1)}(x_0)| \geq |f^{(1)}(x_0)|$.*

(iii) For the general formulation of Kolmogorov's theorem see, e.g., [16, §6].

(vi) For the special case $k = 4$, $j = 1$, we find that $\alpha = \frac{1}{2}$ and $\text{jump}_\alpha K^{(2)} = -1$. This shows that the right hand side of (1.13) in Lecture 9 of [15] has the wrong sign as does the first line of the right hand side on (1.21) in the same section.

4. A proof of Theorems 1 and 2. The proof is divided into six parts.

(α) A proof of the unicity of K. If we have two functions, K_1 and K_2 say, satisfying the conditions of Theorem 1 for the same k and j, then, in forming their difference

$$S := K_1 - K_2,$$

the discontinuity in the $(k-j-1)$st derivative at α cancels out, and we conclude that S is a spline of order k with simple knots at $\mathbb{Z}$, and vanishes at $\mathbb{Z} + \tau$; in short, S is a **cardinal nullspline**. Hence, with

$$\mathbf{S}(x) := \begin{cases} (S(x), \ldots, S^{(k-2)}(x)/(k-2)!), & \text{if } \tau = 0, \\ (S(x), \ldots, S^{(k-1)}(x)/(k-1)!), & \text{if } \tau = \frac{1}{2}, \end{cases}$$

we infer from Part B that

$$\mathbf{S}(\nu + \tau) = \sum_{i=p}^{q} c_i(-\lambda_i(\tau))^\nu \mathbf{u}^{(i,\tau)}, \quad \text{all } \nu \in \mathbb{Z},$$

for a certain basis $(\mathbf{u}^{(i,\tau)})_i$ and a certain strictly decreasing positive sequence $(\lambda_i(\tau))_i$. On the other hand, S must be in $\mathbb{L}_1(\mathbb{R})$ since both K_1 and K_2 are, by assumption (iii). Therefore, letting $\nu \to \infty$, we see that $c_i = 0$ for all i with $\lambda_i(\tau) \geq 1$, while, letting $\nu \to -\infty$, we see that $c_i = 0$ for all i with $\lambda_i(\tau) \leq 1$. In short, $S = 0$.

(β) A proof of the symmetry relation (iv) of Theorem 2. We are to show that the unique K of Theorem 1, *if it exists*, must satisfy

$$K(\alpha + x) = (-)^{k-j} K(\alpha - x) \text{ for all positive } x. \tag{1}$$

For this, define K_1 by

$$K_1(\alpha + x) := (-)^{k-j} K(\alpha - x) \text{ for all } x \neq 0.$$

Then K_1 is in $\mathbb{L}_1(\mathbb{R})$ since K is, K_1 vanishes at $\mathbb{Z}+\tau$ since K does, and K_1 is a spline of order k with knots at $\mathbb{Z}$, K having this property, except for the discontinuity in the $(k-j-1)$st derivative at α. But

$$\begin{aligned}\text{jump}_{\alpha} K_1^{(k-j-1)} &= K_1^{(k-j-1)}(\alpha^+) - K_1^{(k-j-1)}(\alpha^-)\\ &= (-)^{k-j}(-)^{k-j-1}\{K^{(k-j-1)}(\alpha^-) - K^{(k-j-1)}(\alpha^+)\}\\ &= \text{jump}_{\alpha} K^{(k-j-1)} = (-)^{k-j},\end{aligned}$$

so that K_1 also satisfies the jump condition (2.1). But now (1) follows from the unicity of K just proven.

(γ) A proof of existence of K when k is odd. Motivated by the symmetry (1), we actually construct K by determining a spline function S in $\mathbb{L}_1[\alpha,\infty)$ of order k with simple knots at the positive integers which vanishes at $\nu+\tau$ for all nonnegative integers ν, and which satisfies

$$S^{(r)}(\alpha^+) = (-)^{k-j}\delta_{r,k-j-1}/2 \text{ for } r = \begin{cases} 0,2,\ldots,k-3 & \text{if } j \text{ even}\\ 1,3,\ldots,k-2 & \text{if } j \text{ odd}\end{cases} \tag{2}$$

Any such S will give rise to a K of the required sort by

$$K(x) := \begin{cases} S(x) & x>\alpha\\ (-)^{k-j}S(2\alpha-x), & x<\alpha\end{cases} \tag{3}$$

with the conditions (2) guaranteeing that

$$\text{jump}_{\alpha} K^{(r)} = (-)^{k-j}\delta_{r,k-j-1} \text{ for } r=0,\ldots,k-\begin{Bmatrix}2\\1\end{Bmatrix} \text{ if } \alpha = \begin{Bmatrix}0\\ \frac12\end{Bmatrix}.$$

Since k is odd, $\tau=0$, i.e., S is to vanish at its knots $1,2,\ldots$. With

$$\mathbf{S}(x) := (S(x),\ldots,S^{(k-2)}(x)/(k-2)!)$$

as in Part B, the condition that S have simple knots at the positive integers and vanish at these implies that

$$\mathbf{S}(\nu+1) = (-A_k)^{\nu}\mathbf{S}(1) \text{ for } \nu=1,2,3,\ldots, \tag{4}$$

where A_k is the matrix described in Section B2. Hence, S is determined on $[1,\infty)$ once we choose $\mathbf{S}(1)$. In particular, with $(\mathbf{u}^{(i)})_1^{k-2}$ a complete eigenvector sequence for A_k corresponding to the decreasing eigenvalue sequence $(\lambda_i)_1^{k-2}$, any $\mathbf{S}(1)$ of the form

$$\mathbf{S}(1) = \sum_{i=m+1}^{k-2} c_i\mathbf{u}^{(i)} \tag{5}$$

will give rise to an S in $\mathbb{L}_1[1,\infty)$ since $\lambda_i<1$ for $i>m:=\lfloor k/2\rfloor$ (see Theorem B1). On the other hand,

$$S^{(r)}(\alpha^+) = \sum_{i=r}^{k-2} S^{(i)}(1)(\alpha-1)^{i-r}/(i-r)! + S^{(k-1)}(1^-)(\alpha-1)^{k-1-r}/(k-1-r)! \tag{6}$$

for $r=0,\ldots,k-1$, so that, with the choice (5) for $\mathbf{S}(1)$, (2) constitutes an inhomogeneous linear system of $(k-1)/2$ equations in the

$$k-2-m+1 = (k-1)/2$$

unknowns $c_{m+1},\ldots,c_{k-2}$ and $S^{(k-1)}(1^-)$.

We are therefore assured of the existence of exactly one solution (necessarily nontrivial) in case the corresponding homogeneous equations have only the trivial solution. But that is certainly so here, since a nontrivial solution would give rise via (3) to a cardinal nullspline in $\mathbb{L}_1(\mathbb{R})$, a possibility already rejected when proving unicity.

We conclude that (2), considered via (5) and (6) as a linear system for $c_{m+1}, \ldots, c_{k-2}$ and $S^{k-1}(1^-)$, has exactly one solution, proving the existence of K for this case.

(δ) A proof of Theorem 2 when k is odd. We obtain the exponential decay as described in (i) of Theorem 2 at once from (4) and (5) above with $\mathrm{e}^{-B} = \lambda_{m+1}$. As to (ii) and (iii), we begin with the observation that roughly half the numbers $S(\alpha), \ldots, S^{(k-1)}(\alpha^+)$ vanish. Precisely, as we saw in the existence proof, (2) comprises $(k-1)/2$ equations all but one being homogeneous, hence $(k-3)/2$ of the k numbers $S(\alpha), \ldots, S^{(k-1)}(\alpha^+)$ must be zero. Therefore,

$$S^-(S(\alpha), \ldots, S^{(k-1)}(\alpha^+)) \le k - 1 - (k-3)/2 = m + 1, \tag{7}$$

with $m := \lfloor k/2 \rfloor = (k-1)/2$ as before. Let now $p+1$ be the smallest integer $\ge m+1$ for which $c_{p+1} \ne 0$ in (5). Then

$$\mathbf{S}(\nu) = (-\lambda_{p+1})^{\nu-1} c_{p+1} \mathbf{u}^{(p+1)} + o((\lambda_{p+1})^{\nu-1}) \text{ as } \nu \to \infty,$$

therefore, by the Corollary to Theorem B1,

$$S^+(S(\nu), \ldots, S^{(k-1)}(\nu^-)) = p + 1 \ge m + 1 \text{ for } \nu \text{ near } \infty. \tag{8}$$

But, on (α, ν), S is a spline of order k with simple knots only, and with at least as many zeros as knots, and all these zeros must be isolated since, by (ii) of Theorem B1 and by (5),

$$S^-(S^{(1)}(\nu), \ldots, S^{(k-2)}(\nu)) \ge p \ge m > 0 \text{ for } \nu = 1, 2, \ldots,$$

hence S cannot vanish identically on a positive interval. Also, S is not just a polynomial of degree $< k-1$ since $S \ne 0$. Therefore, from the Budan–Fourier theorem for splines, and from (7) and (8) we have for ν near ∞ that

$$\begin{aligned} Z_{S^{(k-1)}}(\alpha, \nu) &\le \text{number of active knots of } S \text{ in } (\alpha, \nu) \\ &\le \text{number of knots of } S \text{ in } (\alpha, \nu) \\ &\le Z_S(\alpha\nu) \\ &\le Z_{S^{(k-1)}}(\alpha, \nu) + S^-(S(\alpha), \ldots, S^{(k-1)}(\alpha^+)) \\ &\qquad\qquad - S^+(S(\nu), \ldots, S^{(k-1)}(\nu^-)) \\ &\le Z_{S^{(k-1)}}(\alpha, \nu) + (m+1) - (m+1) \\ &= Z_{S^{(k-1)}}(\alpha, \nu), \end{aligned} \tag{9}$$

showing that *equality must hold in all inequalities used to establish this string of inequalities.*

We harvest the fruits of this statement one at a time. Equality in (7) implies that *all entries of the sequence $S(\alpha), \ldots, S^{(k-1)}(\alpha^+)$ not explicitly set to zero by (2) must be nonzero and alternate in sign.* Since we know that

$$S^{(k-j-1)}(\alpha^+) = (-)^{k-j}/2,$$

we therefore know that

$$(-)^{k-j-r} S^{(k-j-2r)}(\alpha) > 0 \text{ for } r = 1, 2, \ldots \tag{10a}$$

and

$$(-)^{k-j+r} S^{(k-j+2r)}(\alpha^+) < 0 \text{ for } r = 0, 1, 2, \ldots . \tag{10b}$$

If now j is odd, then $\alpha = \frac{1}{2}$ and $k - j$ is even and (10a) implies that

$$(-)^{(k-j)/2} S(\tfrac{1}{2}) > 0 \text{ for } j \text{ odd.} \tag{11}$$

Further, $k - 1 = k - j + 2r$ with $r = (j-1)/2$, hence (10b) gives that

$$(-)^{(j-1)/2} S^{(k-1)}(\tfrac{1}{2}) < 0 \text{ if } j \text{ is odd.} \tag{12}$$

If, on the other hand, j is even, then $\alpha = 0$ and $k - j$ is odd, and $1 = k - j - 2r$ with $r = (k - j - 1)/2$, so (10a) implies (for $r \ge 1$) that $-(-)^{(k-j-1)/2} S^{(1)}(0) > 0$, therefore

$$(-)^{(k-j-1)/2} S(\tfrac{1}{2}) > 0 \text{ for } j \text{ even,} \tag{13}$$

since this follows directly in case $j = k - 1$. Also, $k - 2 = k - j + 2r$ with $r = (j-2)/2$, so $-(-)^{(j-2)/2} S^{(k-2)}(0) < 0$ by (10b), hence

$$(-)^{j/2} S^{(k-1)}(0^+) > 0 \text{ for } j \text{ even.} \tag{14}$$

Further, since the number of active knots of S in (α, ν) must equal the number of zeros of $S^{(k-1)}$ there, it follows that $S^{(k-1)}$ *changes sign strongly across each integer* $1, 2, 3, \ldots$. But then $K^{(k-1)}$ must change sign strongly across each $\nu \in \mathbb{Z}$: This is obvious in case $\alpha = \frac{1}{2}$; but it is also true in case $\alpha = 0$ for then j is even, hence $k - j$ is odd, and therefore all even derivatives of K are odd around $\alpha = 0$, hence $K^{(k-1)}$ is odd around $\alpha = 0$, showing that $K^{(k-1)}$ changes sign strongly also across $\alpha = 0$. It follows that

$$A_\nu A_{\nu+1} < 0 \text{ for all } \nu \in \mathbb{Z}$$

and it remains only to show (2.7) for a particular value of ν, say for $\nu = 1$, in which case (2.7) asserts that

$$(-)^{\lfloor (j+1)/2 \rfloor} \operatorname{jump}_1 K^{(k-1)} < 0.$$

But that is now a consequence of the fact that, by (12) and (14),

$$(-)^{\lfloor (j+1)/2 \rfloor} S^{(k-1)} > 0 \text{ on } (0, 1).$$

Finally, even counting multiplicities, S must have exactly as many zeros in (α, ν) as it has knots, hence S changes sign strongly at all positive integers and nowhere else in (α, ∞). K therefore *changes sign at the integers and nowhere else*: This is clear for $\alpha = \frac{1}{2}$. But it is also true for $\alpha = 0$, since then, as we just said, K must be odd around 0, hence must change sign strongly across 0. It remains to verify (2.8) for some ν, say for $\nu = 0$, in which case (2.8) asserts that

(15) $\quad (-)^{\lfloor (k+1)/2 \rfloor - \lfloor (j+1)/2 \rfloor} K > 0$ on $(0, 1)$. **thefirst⌊h⌋asonlyone), butIputintwo**

But $\lfloor (k+1)/2 \rfloor = (k+1)/2$. Further, for odd j, $\lfloor (j+1)/2 \rfloor = (j+1)/2$ while, by (11), $(-)^{(k-j)/2} K > 0$ on $(0, 1)$, proving (15) for this case. If, on the other hand, j is even, then $\lfloor (j+1)/2 \rfloor = j/2$ while, by (13), $(-)^{(k+1-j)/2} K > 0$ on $(0, 1)$, thus proving (15) for this case, too.

This proves all assertions about K made in Theorem 2, for odd k.

(ε) A proof of existence of K when k is even. In this case, it becomes convenient (and perhaps more diverting) to construct K in the form

$$K(x) := \begin{cases} (-)^{k-j} S(2\alpha - x), & x > \alpha \\ S(x), & x < \alpha \end{cases} \tag{16}$$

with S a spline of order k in $\mathbb{L}_1(-\infty,\alpha]$ with simple knots at the nonpositive integers and which vanishes at $\nu+\frac{1}{2}$ for all negative integers ν and satisfies

$$(17)\qquad S^{(r)}(\alpha^-) = -(-)^{k-j}\delta_{r,k-j-1}/2 \text{ for } r = \begin{cases} 1,3,\ldots,k-3 & \text{if } j \text{ even} \\ 0,2,\ldots,k-2 & \text{if } j \text{ odd}. \end{cases} .$$

As S is to vanish at $\nu+\tau$ for $-\nu \in \mathbb{N}$ and $\tau=\frac{1}{2}$, we recall from Sec. B3 the abbreviation

$$\mathbf{S}(x) := (S(x),\ldots,S^{(k-1)}(x)/(k-1)!),$$

in terms of which then

$$(18)\qquad \mathbf{S}(\nu-\tfrac{1}{2}) = (-A_{k,\tau})^\nu \mathbf{S}(-\tfrac{1}{2}), \quad \nu=-1,-2,\ldots,$$

where $A_{k,\tau}$ is the matrix described in Section B3. Hence, S is determined on $(-\infty,-\frac{1}{2}]$ once we have chosen $\mathbf{S}(-\frac{1}{2})$. In particular, with $(\mathbf{u}^{(i,\tau)})_1^{k-1}$ a complete eigenvector sequence for $A_{k,\tau}$ corresponding to the decreasing eigenvalue sequence $(\lambda_i(\tau))_1^{k-1}$, any $\mathbf{S}(-\frac{1}{2})$ of the form

$$(19)\qquad \mathbf{S}(-\tfrac{1}{2}) = \sum_{i=1}^{m-1} c_i \mathbf{u}^{(i,\tau)}$$

gives rise to an S in $\mathbb{L}_1(-\infty,-\frac{1}{2}]$, since $\lambda_i(\tau)>1$ for $i<m:=\lfloor k/2\rfloor$ and $\tau=\frac{1}{2}$, by Theorem B2 or by the Corollary to Theorem B3. On the other hand,

$$(20)\qquad S^{(r)}(\alpha^-) = \sum_{i=r}^{k-1} S^{(i)}(-\tfrac{1}{2})(\alpha+\frac{1}{2})^{i-r}/(i-r)! + S^{(k-1)}(0^+)\alpha^{k-1-r}/(k-1-r)!$$

for $r=0,\ldots,k$, so that, with the choice (19) for $\mathbf{S}(-\frac{1}{2})$, (17) constitutes an inhomogeneous linear system of $\left\{\begin{smallmatrix} m-1 \\ m \end{smallmatrix}\right\}$ equations for $\left\{\begin{smallmatrix} \text{even} \\ \text{odd} \end{smallmatrix}\right\}$ j in the unknowns $c_1,\ldots,c_{m-1}$, and also in $S^{(k-1)}(0^+)$ in case $\alpha\neq 0$. Hence, in terms of (19) and (20), (17) constitutes an inhomogeneous linear system in as many unknowns as equations and is therefore uniquely solvable (since a nontrivial solution to the homogeneous system would give rise to a nontrivial null spline in $\mathbb{L}_1(\mathbb{R})$, an impossibility.) This proves the existence of K when k is even.

(ζ) A proof of Theorem 2 when k is even. The argument parallels closely that given when k is odd. The exponential decay is again obvious from the construction. Further, Equations (17) set to zero $\left\{\begin{smallmatrix} m-2 \\ m-1 \end{smallmatrix}\right\}$ terms in the sequence $S(\alpha),\ldots,S_{(k-1)}(\alpha^-)$ for $\left\{\begin{smallmatrix} \text{even} \\ \text{odd} \end{smallmatrix}\right\}$ j. Hence, choosing the sign of these zeros to alternate in conjunction with the nonzero term $S^{(k-j-1)}(\alpha^-)$, we see that

$$(21)\qquad S^+(S(\alpha),\ldots,S^{(k-1)}(\alpha^-)) \ge \begin{Bmatrix} m-2 \\ m-1 \end{Bmatrix} \text{ for } j \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}.$$

Also, with $p-1$ the largest integer $\le m-1$ for which $c_{p-1}\neq 0$ in (19), we have

$$\mathbf{S}(\nu-\tfrac{1}{2}) = (-\lambda_{p-1}(\tau))^\nu \mathbf{u}^{(p-1,\tau)} + o((\lambda_{p-1}(\tau)^\nu) \text{ as } \nu\to-\infty.$$

Therefore, by Theorem B2. (iii),

$$(22)\qquad S^-(S(\nu-\tau),\ldots,S^{(k-1)}(\nu-\tau^+)) = p-2 \le m-2$$

for all integers ν near $-\infty$. Further, on $(\nu-\tau,\alpha)$, S is a spline of order k (and certainly not just a polynomial of degree $<k-1$) with simple knots at $\nu-1$, $\nu-2$,$\ldots,-1$, and also at 0 in case $\alpha=\frac{1}{2}$, i.e., when j is odd, and nowhere else, while S vanishes in $(\nu-\tau,\alpha)$ at $\nu-1-\frac{1}{2},\ldots,-\frac{1}{2}$. Since these zeros are necessarily isolated, we have

$$\text{number of knots of } S \text{ in } (\nu-\tau,\alpha) \le Z_S(\nu-\tau,\alpha) + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \text{ for } j \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}.$$

The Budan–Fourier theorem for splines, and the inequalities (21) and (22) now give, with $\beta := \nu - \tau$ and ν near $-\infty$,

$$
\begin{aligned}
Z_{S^{(k-1)}}(\beta,\alpha) &\le \text{number of active knots of } S \text{ in } (\beta,\alpha) \\
&\le \text{number of knots of } S \text{ in } (\beta,\alpha) \\
&\le Z_S(\beta,\alpha) + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \\
&\le Z_{S^{(k-1)}}(\beta,\alpha) + S^-(S(\beta),\dots,S^{(k-1)}(\beta^+)) \qquad (23)\\
&\qquad\qquad - S^+(S(\alpha),\dots S^{(k-1)}(\alpha^-)) \\
&\le Z_{S^{(k-1)}}(\beta,\alpha) + m - 2 - \begin{Bmatrix} m-2 \\ m-1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \\
&= Z_{S^{(k-1)}}(\beta,\alpha)
\end{aligned}
$$

showing that *equality must hold in all inequalities used to establish this string of inequalities.*

In particular, S changes sign strongly across $\nu + \frac{1}{2}$ for each negative integer ν, and changes sign nowhere else in $(-\infty,\alpha)$. K must therefore change sign strongly across $\nu + \frac{1}{2}$ for each $\nu \in \mathbb{Z}$ and nowhere else, which reduces the proof of (iii) to checking (2.8) for $\nu = 0$. Also, $S^{(k-1)}$, and therefore $K^{(k-1)}$, changes sign strongly across each knot, which reduces the proof of (ii) to checking (2.7) for $\nu = 0$. But since there must be equality in (21), and since $K(\alpha + x) = (-)^{k-j} K(\alpha - x)$, we have with A.1.(1) that

$$S^-(K(\alpha),\dots,K^{(k-1)}(\alpha^+)) = k/2 + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ for } j \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}.$$

This forces the $k - m + \left\{ {2 \atop 1} \right\}$ of the k terms $K(\alpha),\dots,K^{(k-1)}(\alpha^+)$ not explicitly set to zero by (17) to be in fact nonzero and to alternate in sign, and the verification of (ii) and (iii) proceeds from this and from the fact that

$$(-)^{k-j} K^{(k-j-1)}(\alpha^+) > 0$$

much as in the case k odd.

5. A characterization of the differentiation formulae of §2. If α and τ are defined by (2.2) and (2.3), $1 \le j \le k-1$, and $f \in \mathbb{L}_\infty^{(k)}(\mathbb{R})$, then we know that

$$f^{(j)}(\alpha) = \sum_{\nu\in\mathbb{Z}} A_\nu f(\nu) + \int_{-\infty}^{\infty} K(x) f^{(k)}(x)\,\mathrm{d}x. \tag{1}$$

Here $K(x)$ is the kernel of Theorem 1, having the properties

$$K(\tau + \nu) = 0 \text{ if } \nu \in \mathbb{Z}, \tag{2}$$

and

$$\text{jump}_\alpha K^{(k-j-1)} = (-)^{k-j}. \tag{3}$$

Let us first assume that

$$j \le k-2. \tag{4}$$

Since $K \in C(\mathbb{R})$, it follows that we can write (1) in the form

$$f^{(j)}(\alpha) = \sum_{\nu\in\mathbb{Z}} A_\nu f(\nu) + \int_{-\infty}^{\infty} K(x)\,\mathrm{d}f^{(k-1)}(x) \tag{5}$$

where we interpret the remainder as a Stieltjes integral. Then (5) is surely valid if we only assume that $f \in \mathbb{L}_\infty^{(k-1)}(\mathbb{R})$, where $f^{(k-1)}$ is uniformly locally of bounded variation, meaning that the total variation of $f^{(k-1)}$ on the interval $[a, a+\ell]$ is bounded for every fixed ℓ and all a. This is surely the case if

$$f \in \$_{k,\tau+\mathbb{Z}} \cap \mathbb{L}_\infty^{(k-1)}(\mathbb{R}). \tag{6}$$

In this case, $f^{(k-1)}$ is a step–function with jumps at $\tau + \mathbb{Z}$, and (2) shows that the remainder term of (5) vanishes *because of* (4).

Lemma 1. *If (6) holds then*

(7) $$f^{(j)}(\alpha) = \sum_{\nu\in\mathbb{Z}} A_\nu f(\nu),$$

where in case that

(8) $$j = k-1, \quad \textit{hence } \alpha = \tau,$$

we interpret $f^{(k-1)}(\alpha) = f^{(k-1)}(\tau)$ *to mean*

(9) $$f^{(k-1)}(\alpha) = \square f^{(k-1)}(\alpha) := (f^{(k-1)}(\alpha^+) + f^{(k-1)}(\alpha^-))/2.$$

Proof: Since the case when (4) holds has already been established before stating the lemma, we may assume (8) to hold, and are to show that

(10) $$\square f^{(k-1)}(\alpha) = \sum_{\nu\in\mathbb{Z}} A_\nu f(\nu).$$

The only difficulty is that by (3), or $\text{jump}_\alpha K = -1$, the Stieltjes integral in (5) is not defined. However, it is defined if

(11) *the point* $x = \alpha (= \tau)$ *is not an active, or actual, knot of the spline.*

In this case again (10) holds.

Let us remove the restriction (11). Observe that, whatever the parity of k may be, the Euler spline $\mathcal{E}_{k-1}(x - \frac{1}{2})$ has its knots at $\tau + \mathbb{Z}$, and $\text{jump}_\tau \mathcal{E}_{k-1}^{(k-1)} \neq 0$. It follows that for some appropriate constant c the spline

$$f_0(x) := f(x) - c\mathcal{E}_{k-1}(x - \tfrac{1}{2})$$

will satisfy the condition (11). Moreover, $\mathcal{E}_{k-1}(\nu - \frac{1}{2}) = 0$ $(\nu \in \mathbb{Z})$ and therefore $f(\nu) = f_0(\nu)$ for all ν. It follows that $f(x) = f_0(x) + c\mathcal{E}_{k-1}(x - \frac{1}{2})$ has the property that

$$\begin{aligned}\square f^{(k-1)}(\alpha) &= f_0^{(k-1)}(\alpha) + \square c\mathcal{E}_{k-1}^{(k-1)}(\alpha - \tfrac{1}{2}) \\ &= \sum_{\nu\in\mathbb{Z}} A_\nu f_0(\nu) + 0 = \sum_{\nu\in\mathbb{Z}} A_\nu f(\nu),\end{aligned}$$

which proves (10). □

We may now establish the

Theorem 3. *The differentiation formula (7) is the unique diff. formula having absolutely summable coefficients* A_ν *and which is valid for all splines* $f(x)$ *satisfying (6).*

Proof: Suppose that also the formula

(12) $$f_{(j)}(\alpha) = \sum_{\nu\in\mathbb{Z}} A'_\nu f(\nu)$$

shares all these properties with (7). Subtracting them and setting $\widetilde{A}_\nu = A_\nu - A'_\nu$, we conclude that

(13) $$\sum_\nu \widetilde{A}_\nu f(\nu) = 0$$

for all f satisfying (6). If we apply (13) to the sequence of B-splines (see, e.g. [15, p. 11])

$$f(x) = Q_k(n - x + \tau) \quad (n \in \mathbb{Z})$$

we obtain that

(14) $$\sum_{\nu\in\mathbb{Z}} \widetilde{A}_\nu Q_k(n - \nu + \tau) = 0 \quad (n \in \mathbb{Z}).$$

This shows that the cardinal spline $g := \sum_{\nu\in\mathbb{Z}} \widetilde{A}_\nu Q_k(\cdot - \nu)$ of order k vanishes at $\mathbb{Z} + \tau$ while also, by assumption on the $\widetilde{A}_\nu$, being in $\mathbb{L}_1(\mathbb{R})$. But this implies, as in the proof of unicity of K (see Section 4.(α) above) that $g = 0$, therefore $\widetilde{A}_\nu = 0$, for all ν. □

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