Note

The B-spline recurrence relations of Chakalov and of Popoviciu

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Abstract We discuss the early history of B-splines with an arbitrary knot sequence, and of their recurrence relations. These seem to have first appeared in papers of Popoviciu and Chakalov from the 1930's.

key words B-spline, recurrence, divided difference

B-splines have been the object of mathematical study well before they made their appearance in I. J. Schoenberg's fundamental paper [Sc46]. In particular, cardinal B-splines, that is, B-splines for a uniform knot sequence, introduced in [Sc46] have a long history, nicely detailed in [BSS88], and already Schoenberg [Sc46: p. 68] traces them back to Laplace (presumably to [L20: pp. 165–169]).

B-splines for an arbitrary knot sequence also appear in the literature well before their introduction in [C47] and [CS47] and study in [CS66]; a prominent example is [Fa40]. Even the recurrence relations can be found in papers published well before [dB72] and [Co72], though it does take the power of hindsight to see this clearly, as we hope to show in this note.

Popoviciu's B-spline recurrence relation

In his doctoral thesis [P33]=[P34a] and various follow-ups, Tiberiu Popoviciu* is concerned with interpolation to given data

$$(x_i, y_i), \quad \text{with} \quad x_1 < \cdots < x_m$$

by $(n+1)$**-convex**, or, more generally, $(n+1)$**-non-concave** functions (which he calls **convex**, respectively **non-concave**, **of order** n). The latter, by definition, are those functions whose $(n+1)$st divided difference

$$\Delta(\tau_0, \ldots, \tau_{n+1})f$$

(in W. Kahan's literal notation) is nonnegative for any choice of $n+2$ (distinct) points $\tau_0, \ldots, \tau_{n+1}$ in its domain. Popoviciu ascribes the introduction of this notion to Eberhard Hopf's dissertation [Ho26], claiming for himself only the credit of having considered domains more general than just an interval. To be sure, Popoviciu is also the first to use a specific terminology ('convex of order n') for what Hopf merely refers to as having all $(n+1)$st divided differences positive.

Neither Popoviciu's thesis nor his later lectures [P37] make any mention of the functions we now call B-splines, and the much later summarizing book, [P44], only mentions them in passing (on p. 21). So, the following discussion is based entirely on his paper [P34b] which he calls a completion of the work begun in his thesis.

In the main part of the paper, Popoviciu only considers divided differences at distinct points. This permits him to define the divided difference by

$$\Delta(\tau_0, \ldots, \tau_{n+1})f = \frac{U(\tau; f)}{V(\tau)},$$

with

$$V(\tau) := V(\tau_0, \ldots, \tau_{n+1}) := \det(\tau_i^j : i, j = 0, \ldots, n+1)$$

the Vandermonde determinant, and

$$U(\tau; f)$$

the determinant of the matrix obtained from the above Vandermonde matrix by replacing its last column by the vector $(f(\tau_i) : i = 0, \ldots, n+1)$.

This definition readily shows that $\Delta(\tau_0, \ldots, \tau_{n+1})f$ is the leading coefficient in the power form of the polynomial of degree $\le n+1$ that agrees with f at the τ_j, but Popoviciu makes no use of that fact.

On page 89, and for $i = 1, \ldots, m-n-1$, he introduces functions Ψ_i as follows:

$$(1) \qquad \Psi_i(x) := \begin{cases} 0 \quad \text{on} \quad (x_1 \mathinner{\ldotp\ldotp} x_i), \\ \sum_{r=0}^{k} (-1)^r \frac{V_i^{(i+r)}}{V_i} (x - x_{i+r})^n \quad \text{on} \quad (x_{i+k} \mathinner{\ldotp\ldotp} x_{i+k+1}), \quad k = 0, \ldots, n, \\ 0 \quad \text{on} \quad (x_{i+n+1} \mathinner{\ldotp\ldotp} x_m), \end{cases}$$

with

$$(a \mathinner{\ldotp\ldotp} b)$$

* `http://www.mathesis.ro/capital/en/htm/pers/personalitati/popoviciu.htm`

the open interval with end points a and b, and with

$$V_i := V(x_i, \ldots, x_{i+n+1})$$

and

$$V_i^{(k)} := V(x_i, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{i+n+1}).$$

Now, directly from the above definition of the divided difference as a ratio of determinants, we deduce that

$$\sum_{r=0}^{n+1} (-1)^r \frac{V_i^{(i+r)}}{V_i} f(x_{i+r}) = (-1)^{n+1} \Delta\!\!\!\Delta(x_i, \ldots, x_{i+n+1}) f.$$

So, fortunate in having the truncated power notation at our disposal, and realizing that $(x-\cdot)_+^n = (x-\cdot)^n + (-1)^{n+1}(\cdot - x)_+^n$, we recognize (as Popoviciu eventually may have, e.g., when writing [P44: p. 21]) that

$$\Psi_i(x) = \Delta\!\!\!\Delta(x_i, \ldots, x_{i+n+1})(\cdot - x)_+^n, \tag{2}$$

hence

$$(n+1)\Psi_i = M_{i,n+1},$$

the ith Curry-Schoenberg B-spline of order $n+1$ for the knot sequence $(x_1, \ldots, x_m)$, normalized to integrate to 1, that is, $\Psi_i = M(\cdot|x_i, \ldots, x_{i+n+1})/(n+1)$. Working directly with his definition of the Ψ_i, and using nothing more than the well-known explicit expression

$$V(x_i, \ldots, x_j) = \prod_{i \le \mu < \nu \le j} (x_\nu - x_\mu) \tag{3}$$

for the Vandermonde determinant, Popoviciu proves (see page 93) the relation

$$(x_{n+j+1} - x_j)\Psi_j(x) = (x - x_j)\Psi'_j(x) + (x_{n+j+1} - x)\Psi'_{j+1}(x), \tag{4}$$

where, as he says, the Ψ'_j are defined just like the Ψ_j, except that n is replaced by $n-1$. With that, we recognize in (4) the (for us) standard B-spline recurrence relations. His proof of (4) (see page 91) rests on nothing more than the ready consequence ("qu'on vérifie facilement") of (3) that

$$\begin{gathered} V'^{(r)}(x_{n+2} - x_2)(x_{n+2} - x_3) \cdots (x_{n+2} - x_{n+1})(x - x_1) - \\ -V''^{(r)}(x_2 - x_1)(x_3 - x_1) \cdots (x_{n+1} - x_1)(x_{n+2} - x) = V_1^{(r)}(x - x_1), \end{gathered} \tag{5}$$

with

$$V'^{(r)} := V(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_{n+1})$$

and

$$V''^{(r)} := V(x_2, \ldots, x_{r-1}, x_{r+1}, \ldots, x_{n+2}).$$

To be sure, the last subscript in (5) should be r rather than 1 but, with that correction, (5) can be verified and readily yields his equation (18):

$$\begin{aligned} \sum_{r=1}^{k} (-1)^{r-1} V_1^{(r)} (x - x_r)^n &= (x_{n+2} - x_2)(x_{n+2} - x_3) \cdots \\ &(x_{n+2} - x_{n+1})(x - x_1) \sum_{r=1}^{k} (-1)^{r-1} V'^{(r)} (x - x_r)^{n-1} + \\ +(x_2 - x_1)(x_3 - x_1) \cdots (x_{n+1} - x_1)&(x_{n+2} - x) \sum_{r=2}^{k} (-1)^{r-2} V''^{(r)} (x - x_r)^{n-1}, \end{aligned}$$

which is (4) written out for the interval $(x_k \,.\,.\, x_{k+1})$ and for $i = 1$ in full detail.

Popoviciu uses the recurrence relation (4) to prove two things: (i) the strict positivity of the Ψ_j on their support (see pages 90-91), and (ii) the fact that, for $x_{n+1} < \xi < x_{n+2}$,

$$\sum_{i=1}^{n+1} (x_{n+i+1} - x_i)\Psi_i(\xi)(x - x_{i+1}) \cdots (x - x_{i+n}) = (x - \xi)^n, \tag{6}$$

(see page 93), that is, what we now call Marsden's identity because of [Ma70].

The positivity of the Ψ_j is of importance to Popoviciu since he has the following

Theorem ([P34b: p. 94]). *Assume that the data $((x_i, y_i) : i = 1, \ldots, m)$ are non-concave of order n. Then these are the restriction to the x_j of a function f non-concave of order n if and only if*

$$\sum_j \lambda_j \Psi_j \geq 0 \quad \textit{on} \quad (x_1 \mathinner{\ldotp\ldotp} x_m)$$

implies that

$$\sum_j \lambda_j \Delta(x_j, \ldots, x_{j+n+1}) f \geq 0.$$

These conditions for extendability of such data to a non-concave function of order n on the closed interval $[x_1 \mathinner{\ldotp\ldotp} x_m]$ he calls the **convexity constraints of order** n. The theorem is now known to follow from general duality considerations.

He is interested in the strict positivity of the Ψ_j on their support since this permits him to uniformly approximate non-concave functions of order n by functions that are (strictly) convex of order n.

The Marsden identity (6), Popoviciu uses to conclude that any f defined on $(x_1, x_2, \ldots, x_m)$ and satisfying $\Delta(x_j, \ldots, x_{j+n+1}) f = \Psi_j(\xi)$ for all j and some ξ has an extension of the form $P + (\cdot - \xi)_+^n$, with P the polynomial of degree $\leq n$ that matches $f - (\cdot - \xi)_+^n$ at $x_1, \ldots, x_{n+1}$, and this extension is evidently non-concave of order n. (To be sure, Popoviciu actually considers various cases since he doesn't realize that (see (2)) $\Delta(x_j, \ldots, x_{j+n+1})(\cdot - \xi)_+^n = \Psi_j(\xi)$ for all j and that therefore f and $(\cdot - \xi)_+^n$ differ on $(x_1, \ldots, x_m)$ only by a polynomial of degree $\leq n$.) With this result in hand, he proves the sufficiency part of his Theorem by observing that the convexity constraints of order n demand that, whenever a hyperplane in $\mathbb{R}^{m-n-1}$ through the origin leaves the entire curve $[x_1 \mathinner{\ldotp\ldotp} x_m] \to \mathbb{R}^{m-n-1} : \xi \mapsto \Psi(\xi) := (\Psi_j(\xi) : j = 1, \ldots, m-n-1)$ to one side, then also the vector $(\Delta(x_j, \ldots, x_{j+n+1}) f : j = 1, \ldots, m-n-1)$ must lie on that same side. This puts the latter vector into the closed cone spanned by that curve, hence writeable as a nonnegative combination of suitable points $\Psi(\xi_k)$, $k = 1, \ldots, m-n-1$, thus showing that such f is extendable to a function, non-concave of order n, of the form

$$P + \sum_{k=1}^{m-n-1} \mu_k (\cdot - \xi_k)_+^n$$

for some polynomial P of degree $\leq n$, some nonnegative μ_k, and some $\xi_k \in (x_1 \mathinner{\ldotp\ldotp} x_m)$.

Any such function (but with arbitrary μ_k), Popoviciu calls an **elementary function of degree** n **with** $m - n - 1$ **vertices**. Of course, we now call such a function a spline of order $n+1$ with simple (interior) knots. Popoviciu takes it as obvious (see top of page 90) that any elementary function of degree n with vertices at the m points $x_1, \ldots, x_m$ and vanishing outside the interval $[x_1 \mathinner{\ldotp\ldotp} x_m]$ can be written as $\sum_j \lambda_j \Psi_j$.

In a footnote on page 91, Popoviciu claims that the recurrence relation also provides immediately a proof of the fact that the Ψ_j are 'very positive' ("très positif") in the sense that, for each $k = j, \ldots, j+n$, the Bernstein form $\sum_{i=0}^n A_i (\cdot - x_k)^i (x_{k+1} - \cdot)^{n-i}$ for the polynomial that agrees with Ψ_j on the interval $(x_k \mathinner{\ldotp\ldotp} x_{k+1})$ has all its coefficients positive. To be sure, this is not quite true for $k = j$ since then $A_i = 0$ for $i < n$, nor for $k = j+n$ since then $A_i = 0$ for $i > 0$.

In a lengthy last chapter, Popoviciu considers in complete detail the problem when $x_1 = \cdots = x_{n+1} = a < b = x_{n+2} = \cdots = x_{2n+2}$, using the fact that, in this case,

$$\Psi_j = \binom{n}{j-1} (b - \cdot)^{n-j+1} (\cdot - a)^{j-1} / (b-a)^{n+1}.$$

Finally, noteworthy from the spline point of view is the observation that, in [P34a: p. 7], Popoviciu records (in less suggestive notation) the following almost immediate consequence

$$(7) \qquad (t_n - t_0) \Delta(t_0, \ldots, t_n) = (t_n - \xi) \Delta(\xi, t_1, \ldots, t_n) + (\xi - t_0) \Delta(t_0, \ldots, t_{n-1}, \xi), \quad \text{all } \xi,$$

of the divided difference recurrence and deduces from it by induction that, for any increasing refinement σ of the increasing sequence τ,

$$\Delta(\tau_0, \ldots, \tau_n) = \sum_j \alpha_j(\tau, \sigma) \Delta(\sigma_j, \ldots, \sigma_{j+n}),$$

with the $\alpha_j(\tau, \sigma)$ nonnegative and summing to 1. To be sure, for $n = 1$, this observation can already be found in Cauchy's work [Ca40]. Since $x \mapsto n \Delta(t_0, \ldots, t_n)(\cdot - x)_+^{n-1}$ is the B-spline with knots $t_0, \ldots, t_n$ that integrates to 1, we recognize, in hindsight, in (7) Boehm's [B80] now standard formula for knot insertion.

Chakalov's B-spline recurrence relation

In [C38a] (see [C36] for an earlier announcement of these results and [C38b] for an extended summary in German), Liubomir Chakalov* discusses divided differences in full generality, that is, permitting any kind of coincidence among the points. He writes the general divided difference in terms of a strictly increasing sequence $(a_0, \ldots, a_m)$ with a corresponding sequence ν of natural numbers, the multiplicities, and so considers

$$N[f] := \sum_{r=0}^{m} \sum_{0 \le \lambda < \nu_r} A_{r\lambda} D^\lambda f(a_r), \tag{8}$$

with D^λ indicating λ-fold differentiation and with the $A_{r\lambda}$ set so that, with $()^j$ denoting the function $x \mapsto x^j$,

$$N[()^j] = \delta_{jn}, \quad j = 0, 1, \ldots, n := (\sum_r \nu_r) - 1. \tag{9}$$

Hence, assuming there exist such $A_{r\lambda}$ (their existence is proved below),

$$N[f] = \Delta(\tau_0, \ldots, \tau_n) f =: \Delta(\tau) f,$$

with

$$\tau := (\ldots, \underbrace{a_r, \ldots, a_r}_{\nu_r \text{ terms}}, \ldots)$$

the nondecreasing sequence that contains a_r exactly ν_r times, all r.

Chakalov is interested in 'minimal sets'. This notion, introduced by him in [C34], concerns the possible values of ξ in the well-known formula

$$\Delta(\tau) f = D^n f(\xi)/n! \tag{10}$$

as f varies over a given class F. For this, he develops the integral representation

$$\Delta(\tau) f = \int u(x) D^n f(x)\, \mathrm{d}x.$$

Of course, in modern notation,

$$u(x) = M(x|\tau)/n! = \Delta(\tau)(\cdot - x)_+^{n-1}/(n-1)! =: M_{0,n}(x)/n!,$$

that is, u is the B-spline with knots τ that integrates to $1/n!$. Chakalov proves that u is piecewise polynomial of degree $n-1$ with breaks only at the τ_j, and is zero outside the interval $(\tau_0 \,..\, \tau_n)$. He also proves that u is positive on that interval, and does this by induction on n. For this, he proves (with the aid of the calculus of residues; see below) that

$$D(u/(a_m - \cdot)^{n-1}) = u_1/(a_m - \cdot)^n, \tag{11}$$

with $u_1 := M_{0,n-1}/(n-1)!$.

In more current terms and with the differentiation on the product carried out and recalling that $a_m = \tau_n$, this reads:

$$(n-1)M_{0,n} + (\tau_n - \cdot) D M_{0,n} = n M_{0,n-1}.$$

Using the fact (whether or not known to Chakalov) that

$$D M_{0,n} = n \frac{M_{0,n-1} - M_{1,n-1}}{\tau_n - \tau_0},$$

* `http://wwww.math.technion.ac.il/hat/chakalov.html`

this implies that

$$\begin{aligned}(n-1)M_{0,n} &= n\left(M_{0,n-1} - \frac{\tau_n - \cdot}{\tau_n - \tau_0}(M_{0,n-1} - M_{1,n-1})\right)\\ &= n\left(\frac{\cdot - \tau_0}{\tau_n - \tau_0}M_{0,n-1} + \frac{\tau_n - \cdot}{\tau_n - \tau_0}M_{1,n-1}\right),\end{aligned}$$

the well-known B-spline recurrence relation (4) in slightly different garb.

Chakalov obtains his recurrence relation (11) in the following manner. It is well-known that

$$\Delta(\tau)(z - \cdot)^{-1} = 1/P(z) := 1/\prod_j (z - \tau_j) =: \sum_{r=0}^{m} \sum_{0 \le \lambda < \nu_r} \frac{\lambda! A_{r\lambda}}{(z - a_r)^{\lambda+1}},$$

with the double sum the partial fraction expansion of $1/P$. On the other hand, the double sum is also exactly what one obtains when applying (8) to the function $f = 1/(z - \cdot)$. Hence, the coefficients in (8) are uniquely determined by the conditions (9), that is, by the requirement that N be the divided difference on τ, and can be computed via the partial fraction expansion of $1/P$. This is Chakalov's Theorem 1.

Chakalov makes the point that, for any polynomial f, its remainder on division by P is the unique polynomial of degree $\le n$ that matches f at τ. From this, he derives the standard formula for the divided differences of an arbitrary power.

On page 361, he applies his explicit expression for $\Delta(\tau)$ to the function

$$x \mapsto \int_{a_0}^{x} [\![x - t]\!]^{n-1} \varphi(t)\,\mathrm{d}t,$$

with φ an arbitrary continuous function on the interval $[a_0 \,..\, a_m]$ and

$$[\![y]\!]^s := y^s/s!$$

a convenient abbreviation for the normalized power function that would have saved him some writing. This gives him

$$u(x) = \sum_{r=0}^{m} \sum_{0 \le \lambda < \nu_r} A_{r\lambda} [\![(a_r - x)_+]\!]^{n-\lambda-1}.$$

From this, he obtains the piecewise polynomial character of u and, because of his explicit formula (8), perhaps even the smoothness across breaks (though, offhand, he does not comment on that).

His major contribution occurs on page 363 where he states the formula

$$u(x) = \frac{1}{2\pi \mathrm{i}} \int_{C_s} [\![z - x]\!]^{n-1} \frac{\mathrm{d}z}{P(z)}, \tag{12}$$

which holds for $a_{s-1} < x < a_s$ in case C_s is any curve that excludes $a_1, \ldots, a_{s-1}$ and includes $a_s, \ldots, a_m$. (12) holds, as he points out, since $\sum_{0 \le \lambda < \nu_r} A_{r\lambda} [\![a_r - x]\!]^{n-\lambda-1}$ is the residual at $z = a_r$ of the function $z \mapsto [\![z - x]\!]^{n-1}/P(z)$. (12) was rediscovered and put to good use much later by Meinardus; see [Me74].

From (12), Chakalov then derives his recurrence relation (11) by differentiation.

To be sure, the expression of a divided difference, at an arbitrary sequence of points including possible coincidences, as a contour integral occurs already in [Fr71] (though it takes hindsight to notice this since there is no mention of polynomial interpolation, let alone divided differences), and before Hermite's introduction, in [He78], of what we now call Hermite interpolation and whose analysis there is firmly based on the use of the calculus of residues. Further facts concerning the history of this approach to divided differences can be found in [N24: pp. 199f]. A fair exposition and discussion, in English, of Chakalov's results can be found in [BHS93: pp. 6–8, 18, 39–41, 43–44], and in [B96: pp. 22–26].

Finally, noteworthy from the spline point of view is the fact that, in the last part of [C38a], Chakalov derives what we now would call the Peano kernel of the error in quadrature, in particular Gauss quadrature, showing it to be what we now would call a monospline, and showing its positivity.

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