Frames and Stable Bases for Shift-Invariant Subspaces of $L_2(\mathbb{R}^d)$

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ABSTRACT

Let X be a countable fundamental set in a Hilbert space H, and let T be the operator

$$T : \ell_2(X) \to H : c \mapsto \sum_{x\in X} c(x)x.$$

Whenever T is well-defined and bounded, X is said to be a Bessel sequence. If, in addition, $\operatorname{ran} T$ is closed, then X is a frame. Finally, a frame whose corresponding T is injective is a stable basis (also known as a Riesz basis).

This paper considers the above three properties for subspaces H of $L_2(\mathbb{R}^d)$, and for sets X of the form

$$X = \{\phi(\cdot - \alpha) : \ \phi \in \Phi, \ \alpha \in \mathbb{Z}^d\},$$

with Φ either a singleton, a finite set, or, more generally, a countable set. The analysis is performed on the Fourier domain, where the two operators TT^* and T^*T are decomposed into a collection of simpler "fiber" operators. The main theme of the entire analysis is the characterization of each of the above three properties in terms of the analogous property of these simpler operators.

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1. Introduction

1.1. General

We study in this paper certain types of "bases" for shift-invariant subspaces of $L_2(\mathbb{R}^d)$. Our primary objective is to connect among three important families of "basis" sets: shift-invariant sets, Weyl-Heisenberg sets, and affine (wavelet) sets. The present paper is the first in a series of three, and is concerned with the basic theory of shift-invariant bases for shift-invariant spaces. The two other papers, [RS1] and [RS2], will focus on the applications of the theory developed here to Weyl-Heisenberg and affine sets.

Given $X \subset L_2(\mathbb{R}^d)$, we say that X is **a shift-invariant** (SI, for short) set if it is invariant under all possible **shifts**, i.e., invariant under all integer translations. A **shift-invariant subspace** S of $L_2(\mathbb{R}^d)$ is a closed subspace which is also a shift-invariant set. Such spaces play an important role in the areas of Multivariate Splines, Wavelets, Radial Function Approximation and Sampling Theory.

The following terminology is commonly used in the context of shift-invariant spaces. First, for a given $\Phi \subset L_2(\mathbb{R}^d)$, the space **generated** by Φ, denoted by

$$S(\Phi),$$

is the smallest (closed) shift-invariant space that contains Φ. The set of shifts of Φ

$$E_\Phi := \{E^\alpha \phi : \ \phi \in \Phi, \ \alpha \in \mathbb{Z}^d\}, \tag{1.1.1}$$

with

$$E^\alpha f \mapsto f(\cdot - \alpha), \tag{1.1.2}$$

is then clearly fundamental in $S(\Phi)$, and is a natural candidate for the previously discussed X. The space S is a **principal shift-invariant** (PSI) space in case $S = S(\Phi)$ for a *singleton* Φ, and, more generally, is a **finitely generated shift-invariant** (FSI) space if Φ above is finite. Many articles are devoted, wholly or in part, to the study of *Riesz (=unconditional=stable) bases* for PSI and FSI spaces (cf. e.g. [JM], [BDR1]). In particular, a complete characterization of such bases is given in [BDR1], which, further, introduces and analyses the more general notion of *quasi-stable bases.* These results form the starting point of the present paper.

We provide here a complete characterization of frames and tight frames in FSI spaces, and draw interesting connections between these notions and the notions of quasi-stability and quasi-orthogonality of [BDR1]. We further give a comprehensive analysis of *infinitely* generated SI spaces, and employ in that course two complementary approaches termed here as "Gramian Analysis" and "dual Gramian Analysis".

1.2. Notations

The Fourier transform of a tempered distribution f is denoted here by $\widehat{f}$, and is defined, for $f \in L_1(\mathbb{R}^d)$, by

$$\widehat{f}(w) := \int_{\mathbb{R}^d} f(t) e_{-w}(t)\, dt,$$

where

$$e_w : t \mapsto e^{iw\cdot t}.$$

The inverse Fourier transform of f is denoted by $f^\vee$.

We frequently discuss in this paper functions that are defined on $\mathbb{T}^d$, the d-dimensional torus. Those functions can be viewed as 2π-periodic functions, via the standard transformation $\mathbb{R}^d \ni w \mapsto e^{iw} := (e^{iw_1}, ..., e^{iw_d}) \in \mathbb{T}^d$. Though we may refer to such functions as being defined on $\mathbb{T}^d$, we always treat their argument as *real*. Thus, "multiplying a function defined on $\mathbb{T}^d$ by a function defined on $\mathbb{R}^d$" simply means "multiplying a 2π-periodic function by ...". Following this slight abuse of terminology, we write "$\Omega \subset \mathbb{T}^d$" and mean "$\Omega \subset [-\pi\, d\pi]^d$". The 2π-periodic extension, $\Omega + 2\pi\mathbb{Z}^d$, of Ω is denoted by

$$\Omega^\circ.$$

The inner product (norm) of any Hilbert space H discussed in this paper is denoted by $\langle \cdot, \cdot \rangle_H$ ($\| \cdot \|_H$, respectively). The default inner product and norm are these of $L_2(\mathbb{R}^d)$. We may also suppress the subscripts in $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$ if they are clear from the context.

Given a set X, the notation

$$\ell_2(X)$$

stands (as usual) for the space of square-summable sequences on X, with the standard inner product. Also, if $Y \subset X$, we embed $\ell_2(Y)$ canonically in $\ell_2(X)$ (i.e., by defining each $c \in \ell_2(Y)$ to be zero on $X \backslash Y$). The space

$$\ell_0(X)$$

is the space of all finitely supported sequences in $\ell_2(X)$, and is considered as a subspace of the latter (i.e., equipped with the same norm).

Vectors in $\mathbb{R}^d$ are considered as either row vectors or column vectors, and the exact meaning should be clear from the context.

For a countable $\Phi \subset L_2(\mathbb{R}^d)$, we define the Hilbert space of $L_2(\mathbb{T}^d)$-valued Φ-vectors as follows

$$L_2^\Phi := \{(\tau_\phi)_{\phi\in\Phi} : \tau_\phi \in L_2(\mathbb{T}^d); \quad \sum_{\phi\in\Phi} \|\tau_\phi\|^2_{L_2(\mathbb{T}^d)} < \infty\}.$$

The inner product here is

$$\langle \tau, \tau' \rangle_{L_2^\Phi} := \sum_{\phi\in\Phi} \langle \tau_\phi, \tau'_\phi \rangle_{L_2(\mathbb{T}^d)}.$$

If $\tau \in L_2^\Phi$, then $\tau(w) \in \ell_2(\Phi)$, for almost all $w \in \mathbb{T}^d$.

The space L_2^Φ enters the discussion in this paper as the image under the Fourier transform of the sequence space $\ell_2(\mathbb{Z}^d \times \Phi)$. Indeed, given $c \in \ell_2(\mathbb{Z}^d \times \Phi)$, we denote by c_ϕ, $\phi \in \Phi$, the restriction of c to $\mathbb{Z}^d \times (\phi)$. The Fourier series $\widehat{c_\phi}$ of c_ϕ is defined as

$$\widehat{c_\phi} := \sum_{\alpha \in \mathbb{Z}^d} c_\phi(\alpha) e_{-\alpha}.$$

Accordingly, the Fourier transform of $c \in \ell_2(\mathbb{Z}^d \times \Phi)$ is defined as the element

$$\widehat{c} := (\widehat{c_\phi})_{\phi \in \Phi} \in L_2^\Phi.$$

Note that this Fourier transformation is an isometry between $\ell_2(\mathbb{Z}^d \times \Phi)$ and L_2^Φ.

The following **bracket product** plays an important role in the analysis of shift-invariant spaces: given f and g in $L_2(\mathbb{R}^d)$, the bracket product is defined as

$$[f, g] := \sum_{\alpha \in 2\pi\mathbb{Z}^d} f(\cdot + \alpha)\overline{g(\cdot + \alpha)}. \tag{1.2.1}$$

Then, $[f, g]$ is a well-defined element of $L_1(\mathbb{T}^d)$, and satisfies

$$\|[f, f]\|_{L_1(\mathbb{T}^d)} = \|f\|^2_{L_2(\mathbb{R}^d)}. \tag{1.2.2}$$

Also, a standard periodization argument yields that

$$(\langle f, g(\cdot - \alpha)\rangle = 0,\ \forall \alpha \in \mathbb{Z}^d) \quad \Longleftrightarrow \quad ([f, g] = 0, \quad a.e.). \tag{1.2.3}$$

Finally, we find it convenient to define g/f as follows:

$$g/f : x \mapsto \begin{cases} g(x)/f(x), & x \in \operatorname{supp} f \cap \operatorname{supp} g, \\ 0, & \text{otherwise}. \end{cases}$$

1.3. Preliminaries

In this section we briefly recall some elementary facts concerning fundamental sets in Hilbert spaces. While most of the material here can be found in [C], [D1,2], [DS], [HW] and in several other references, it makes the paper more self-contained, and allows us to introduce the basic terminology in its natural setup. Only occasional proofs are given here.

Let H be a separable Hilbert space and X a countable subset of H. We attempt to introduce the operator

$$T := T_X : \ell_2(X) \to H : c \mapsto \sum_{x \in X} c(x)x. \tag{1.3.1}$$

T is certainly well-defined on the finitely supported elements of $\ell_2(X)$. X is said to be a **Bessel sequence/set** if T is bounded on the subspace of finitely supported sequences. In such a case, it is continuously extended to a bounded operator on $\ell_2(X)$.

Associated with T_X is the map $T^* := T_X^* : H$ defined by

$$T^* : h \mapsto \{\langle h, x\rangle_H\}_{x \in X}.$$

Proposition 1.3.2. *T^* is a bounded map from H into $\ell_2(X)$ if and only if X is a Bessel set. In such a case T^* is the adjoint of T and $\|T\| = \|T^*\|$.*

Now, let T be *any* bounded operator from a Hilbert space H' into a Hilbert space H. Then the set

$$C_T := H' \ominus \ker T. \tag{1.3.3}$$

(i.e., the orthogonal complement of $\ker T$ in H') is well-defined, T is injective on C_T, $\operatorname{ran} T = \operatorname{ran}(T_{|C_T})$, and $\operatorname{ran} T^*$ is dense in C_T. In this paper, we use the notation $T_|{}^{-1}$ to denote the inverse map from $\operatorname{ran} T$ to C_T and, similarly, denote by $T^*{}_|{}^{-1}$ the inverse map from $\operatorname{ran} T^*$ to $H \ominus \ker T^*$. These maps are usually referred to as **partial** (or **pseudo) inverses**. With these conventions, we have the following result.

Proposition 1.3.4. *Let X be a Bessel set, and $T := T_X$, $T^* := T^*_X$ as before. Then the following conditions are equivalent:*
(a) $\operatorname{ran} T$ *is closed.*
(b) T is bounded below on C_T.
(c) T^* is onto C_T.
(d) T^* is bounded below on $H \ominus \ker T^*$.
When one (hence all) of these conditions holds, we have $\|T^*{}_|{}^{-1}\| = \|T_|{}^{-1}\|$.

Definition 1.3.5. *Let H be a Hilbert space and X a fundamental Bessel set in H. We say that X is a* **frame** *for H if one (hence all) of the conditions of Proposition 1.3.4 holds. A frame X is called* **tight** *if $\|T\|\|T_|{}^{-1}\| = 1$. We call a frame for $H := L_2(\mathbb{R}^d)$* **a fundamental frame**.

Thus, X is a frame if and only if there exist constants C_1, C_2 such that the inequalities

$$C_1\|h\|^2 \le \sum_{x\in X} |\langle h, x\rangle_H|^2 \le C_2\|h\|^2$$

hold (for all $h \in H$). The sharpest possible constants are $C_2 = \|T\|^2 = \|T^*\|^2$ and $C_1 = 1/\|T_|{}^{-1}\|^2 = 1/\|T^*{}_|{}^{-1}\|^2$ and are usually referred to as the **frame bounds**. A frame is tight if and only if its frame bounds coincide.

A notion closely related to frames is that of a *stable basis* for H (also known as a Riesz or unconditional basis) defined as follows:

Definition 1.3.6. *A* **stable basis** *X for H is a frame for H whose corresponding T_X is injective. Equivalently, it is a frame whose corresponding T^*_X is onto $\ell_2(X)$.*

Given a frame X for H, the map

$$TT^* : H \to H : h \mapsto \sum_{x\in X} \langle h, x\rangle_H\, x$$

is called the **frame operator**. TT^* is continuously invertible, and we use

$$\mathrm{R} := \mathrm{R}_X$$

for its inverse. Since the map R maps X 1-1 onto $\mathrm{R}X$, we may identify canonically the spaces $\ell_2(X)$ and $\ell_2(\mathrm{R}X)$, as we do hereafter, without further notice.

Since R is self-adjoint, $T_X^*\mathrm{R} = T_{\mathrm{R}X}^*$, and hence (i): $T_{\mathrm{R}X}^*$ is a right inverse of T_X, and (ii): $\mathrm{R}X$ is a frame (the latter since $T_{\mathrm{R}X}^*$ is composed of two continuously invertible maps). The frame $\mathrm{R}X$ is known as **the dual frame** of X, and some basic facts concerning dual frames are collected in the following proposition. *

Proposition 1.3.7. *Let* $\mathrm{R}X$ *be the dual frame of the frame* X*. Then:*

(a) X *is the dual frame of* $\mathrm{R}X$ *(i.e., duality is reflexive).*

(b) $T_X T_{\mathrm{R}X}^* = T_{\mathrm{R}X} T_X^* = I_H$*, with* I_H *the identity map on* H*.*

(c) $\ker T_X = \ker T_{\mathrm{R}X}$ *and* $C_{T_X} = C_{T_{\mathrm{R}X}}$*.*

(d) The dual frame $\mathrm{R}X$ *is the only Bessel set* $\mathrm{R}'X$ *in* H *that satisfies* $T_X T_{\mathrm{R}'X}^* = I_H$ *and* $\ker T_X = \ker T_{\mathrm{R}'X}$*.*

Proof: Since $\mathrm{R}T_X = T_{\mathrm{R}X}$, we have $T_{\mathrm{R}X}T_{\mathrm{R}X}^* = \mathrm{R}T_X T_X^*\mathrm{R} = \mathrm{R}$, hence the dual of the frame $\mathrm{R}X$ is $\mathrm{R}^{-1}\mathrm{R}X = X$, which shows (a).

For (b), we already know that $T_X T_{\mathrm{R}X}^* = I_H$. Taking adjoints (or, alternatively, interchanging the roles of X and $\mathrm{R}X$, which is possible thanks to (a)), we get that $T_{\mathrm{R}X}T_X^* = I_H$.

The relation $T_{\mathrm{R}X} = \mathrm{R}T_X$ shows also that $\ker T_X = \ker T_{\mathrm{R}X}$, and hence $C_{T_X} = C_{T_{\mathrm{R}X}}$, which proves (c).

Finally, assume $\mathrm{R}' : X \to H$ satisfies the conditions in (d). Define (on X) a map $K := \mathrm{R} - \mathrm{R}'$. Then KX is Bessel, and $T_X T_{KX}^* = T_X(T_{\mathrm{R}X}^* - T_{\mathrm{R}'X}^*) = 0$, showing that $\ker T_X \supset C_{T_{KX}}$. Further, since $\ker T_{\mathrm{R}'X} = \ker T_{\mathrm{R}X} = \ker T_X$ (by assumption), we have $\ker T_{KX} \supset \ker T_X$. Thus, $\ker T_{KX}$ contains its orthogonal complement $C_{T_{KX}}$. This implies that $T_{KX} = 0$, hence, $KX = 0$. □

The above proposition allows us to represent the orthogonal projector onto H with the aid of a frame and its dual:

Proposition 1.3.8. *Let* S *be a closed subspace of a Hilbert space* H*. Suppose that* X *is a frame of* S *with a dual frame* $\mathrm{R}X$*. Then* $T_X T_{\mathrm{R}X}^*$ *is the orthogonal projector* $\mathcal{P}_S : H \to S$*, i.e.,*

$$\mathcal{P}_S h = \sum_{x \in X} \langle h, \mathrm{R}x\rangle x.$$

Proof: The definition of $T_X T_{\mathrm{R}X}^*$ directly implies that its range lies in S, and hence, by (b) of Proposition 1.3.7, it is, indeed, a projector. It is also orthogonal, since $T_{\mathrm{R}X}^*$, hence $T_X T_{\mathrm{R}X}^*$, obviously vanish on the orthogonal complement of S in H. □

* The symbol $\widetilde{x}$ which is commonly used in the literature to denote the dual frame is used in this paper for a totally different purpose. In any case, the use of $\widetilde{x}$ to denote the dual of x is an abuse of mathematical notations, since it suppresses the dependence of $\mathrm{R}_X x$ on $X \backslash x$. The notation $\widetilde{x}$ for the dual has many other drawbacks. To see one of them, try to rewrite the discussion here on dual frames using it instead of R.

Part (d) of Proposition 1.3.7 provides a criterion for checking whether a certain Bessel set $\mathrm{R}X$ is the dual frame of X, or not. However, that criterion might be hard to implement, since it requires the identification of $\ker T_X$ and $\ker T_{\mathrm{R}X}$. The following corollary provides us with partial remedy to that difficulty.

Corollary 1.3.9. *Let H be a Hilbert space, H' a closed subspace of H, X a frame for H', and R a map from X to H'. Assume that $\mathrm{R}X$ is a Bessel set which is fundamental in H'. Then the following conditions are equivalent:*
(a) $\mathrm{R}X$ *is the dual frame of* X.
(b) $T_{\mathrm{R}X}^* T_X$, $T_X^* T_{\mathrm{R}X}$, $T_X T_{\mathrm{R}X}^*$, *and* $T_{\mathrm{R}X} T_X^*$ *are orthogonal projectors.*
(c) $T_{\mathrm{R}X}^* T_X$, *and* $T_{\mathrm{R}X} T_X^*$ *are orthogonal projectors.*

Proof: The equivalence of (b) and (c) follows from the fact that every orthogonal projector is, in particular, self-adjoint, and hence, assuming (c), we get that $T_{\mathrm{R}X}^* T_X = T_X^* T_{\mathrm{R}X}$, and $T_X T_{\mathrm{R}X}^* = T_{\mathrm{R}X} T_X^*$ verifying thereby (b).

Assume (a). The fact that $T_X T_{\mathrm{R}X}^*$ is then an orthogonal projector is the statement of Proposition 1.3.8. This implies that $T_X^* T_{\mathrm{R}X}$ is a projector. Since $\mathrm{R}X$ is a frame, $T_{\mathrm{R}X}$ maps $\ell_2(X)$ onto H', and since X is a frame, T_X^* maps H' onto C_{T_X}. Hence, $T_X^* T_{\mathrm{R}X}$ must be the indentity on C_{T_X}. The orthogonal complement of C_{T_X} is $\ker T_X = \ker T_{\mathrm{R}X}$ (the equality by (c) of Proposition 1.3.7), and $T_X^* T_{\mathrm{R}X}$ certainly vanish on $\ker T_{\mathrm{R}X}$. Hence it is orthogonal.

Now, assume (b). By statement (d) of Proposition 1.3.7, in order to prove that $\mathrm{R}X$ is the dual frame of X, we only need show that $C_{T_X} = C_{T_{\mathrm{R}X}}$. For that, we first observe that, since both X and $\mathrm{R}X$ are fundamental in H', $T_X^* T_{\mathrm{R}X}$ maps $C_{T_{\mathrm{R}X}}$ 1-1 densely into C_{T_X}. Since that operator certainly vanishes on $\ker T_{\mathrm{R}X}$ and is assumed to be orthogonal, we must have $C_{T_X} = C_{T_{\mathrm{R}X}}$. □

For a shift-invariant set $X = E_\Phi$ (with E_Φ as in (1.1.1)), we use the abbreviated notations

$$\mathcal{T}_\Phi := T_{E_\Phi}, \quad \mathcal{T}_\Phi^* := T_{E_\Phi}^*.$$

For this case, the search for the dual frame is simpler due to the following proposition.

Proposition 1.3.10. *The dual $\mathrm{R}(E_\Phi)$ of a shift-invariant frame E_Φ is the shift-invariant frame $E_{\mathrm{R}\Phi}$ generated by $\mathrm{R}\Phi$. In particular, the dual of a principal (respectively, finite) shift-invariant frame is also a principal (finite) shift-invariant frame.*

Proof: We need to show that R commutes with shifts $E^\alpha : f \mapsto f(\cdot - \alpha)$, $\alpha \in \mathbb{Z}^d$. For that, it suffices to show that the map

$$\mathcal{T}_\Phi \mathcal{T}_\Phi^* : f \mapsto \sum_{x \in E_\Phi} \langle f, x \rangle x$$

commutes with shifts E^α (and use the fact that R is the inverse of that map). Indeed, for $\alpha \in \mathbb{Z}^d$,

$$(\mathcal{T}_\Phi \mathcal{T}_\Phi^*)(E^\alpha f) = \sum_{x \in E_\Phi} \langle E^\alpha f, x \rangle x = \sum_{x \in E_\Phi} \langle f, E^{-\alpha} x \rangle x = \sum_{x \in E_\Phi} \langle f, x \rangle E^\alpha x = E^\alpha \mathcal{T}_\Phi \mathcal{T}_\Phi^* f,$$

with the fact that $E^\alpha E_\Phi = E_\Phi$ being used in the penultimate equality. □

1.4. The Gramian matrices

The central notions in this paper are the *pre-Gramian matrix, the Gramian matrix, and the dual Gramian matrix.* In principle, the objective is to decompose the involved operators $\mathcal{T}_\Phi$ and $\mathcal{T}_\Phi^*$ into a collection of simpler operators ("fibers"), indexed by $w \in \mathbb{T}^d$. Each one of the "fiber" operators acts from a sequence space to (the same or another) sequence space and its matrix representation can be explicitly described in terms of the Fourier transforms of the generators Φ. The main theme of the entire analysis is as follows: *every property of the set* E_Φ (such as being a Bessel set, a frame, a stable basis etc.) *is equivalent to the "fiber" operators satisfying an analogous property in a uniform way (here "uniformity" refers to the norms of the underlying operators).*

The **pre-Gramian operator** J_Φ is simply the Fourier transform analog of the operator $\mathcal{T}_\Phi$. If $c \in \ell_2(E_\Phi)$ is finitely supported, we see that

$$(1.4.1) \qquad (\mathcal{T}_\Phi c)\widehat{\ } = \sum_{\phi\in\Phi} \widehat{c_\phi}\widehat{\phi}.$$

Hence, we may introduce an operator J_Φ, which is defined, at least, on the space

$$(1.4.2) \qquad L_0^\Phi := \{\widehat{c} : \ c : \ E_\Phi \mapsto \mathbb{C} \text{ is finitely supported}\},$$

by the rule

$$(1.4.3) \qquad J_\Phi : \tau \mapsto \sum_{\phi\in\Phi} \tau_\phi \widehat{\phi}.$$

Since the Fourier transform is an isometry, the boundedness, invertibility, and other properties of $\mathcal{T}_\Phi$ can be equally studied via J_Φ.

The definition of J_Φ extends naturally to spaces larger than L_0^Φ; for instance, if Φ is finite, the rule in (1.4.3) can be extended to the entire L_2^Φ (In such a case, $J_\Phi\tau$ need not be a $L_2(\mathbb{R}^d)$-function, but is always defined a.e.).

More relevant to our purposes, the pre-Gramian can be "evaluated" on $\mathbb{T}^d$ in the following way: we define **the value** $J_\Phi(w)$ **of** J_Φ **at** $w \in \mathbb{T}^d$ as the $(2\pi\mathbb{Z}^d \times \Phi)$-matrix

$$J_\Phi(w) := (\widehat{\phi}(w+\alpha))_{\alpha,\phi}.$$

Since each $\widehat{\phi}$ is well-defined only up to a null-set, so is the function $w \mapsto J_\Phi(w)$. In a natural way, the matrix $J_\Phi(w)$ can be viewed as a densely defined operator on $\ell_2(\Phi)$. In any case, (1.4.1) together with (1.4.3) show that, for $c \in \ell_0(E_\Phi)$,

$$(1.4.4) \qquad ((\mathcal{T}_\Phi c)\widehat{\ }(w+\alpha))_{\alpha\in 2\pi\mathbb{Z}^d} = J_\Phi(w)\widehat{c}(w).$$

In summary, we have decomposed $\mathcal{T}_\Phi$, on the Fourier domain, into a collection of operators $\{J_\Phi(w) : \ w \in \mathbb{T}^d\}$, defined for almost every w, each of which acts on a dense subspace of $\ell_2(\Phi)$ and represents the action of J_Φ on the coset $w + 2\pi\mathbb{Z}^d$. Because of the explicit matrix representation of each $J_\Phi(w)$, questions like its boundedness, invertibility etc., are by far more accessible than their $\mathcal{T}_\Phi$ counterparts. Thus, our goal is to study $\mathcal{T}_\Phi$ via the behaviour of the "fibers" $J_\Phi(w)$, $w \in \mathbb{T}^d$.

The **spectrum** of the space $S(\Phi)$ generated by Φ is defined (up to a null-set) as

$$\sigma\Phi := \sigma S(\Phi) := \{w \in \mathbb{T}^d : \ J_\Phi(w) \neq 0\}.$$

An equivalent definition of the spectrum is:

$$\sigma\Phi := \{w \in \mathbb{T}^d : \ [\widehat{\phi}, \widehat{\phi}](w) \neq 0, \text{ for some } \phi \in \Phi\}. \tag{1.4.5}$$

For a FSI space, it was proved in [BDR1] that the spectrum of S only relies on the space and is independent of any particular selection of the generators of the space. That proof can be carried on to infinitely generated SI spaces.

Next, we want to decompose the operator $\mathcal{T}_\Phi^*$. Since the Fourier transform is an isometry, the (formal, say) relation $J_\Phi = \widehat{\mathcal{T}_\Phi}$, leads to the relation

$$\widehat{\mathcal{T}_\Phi^*} = J_\Phi^*.$$

In §2 (cf. (2.1.1)) we show that, given $\phi \in \Phi$ and $f \in L_2(\mathbb{R}^d)$, the sequence $\mathcal{T}_\phi^* f$, though need not be in $\ell_2(E_\phi)$, is always in the Wiener algebra of E_ϕ, more precisely, consists of the Fourier coefficients of the $L_1(\mathbb{T}^d)$-function $[\widehat{f}, \widehat{\phi}]$. This leads to the conclusion that J_Φ^*, the Fourier transform analogue of $\mathcal{T}_\Phi^*$, has the form

$$J_\Phi^* : f \mapsto ([f, \widehat{\phi}])_{\phi\in\Phi}, \tag{1.4.6}$$

and allows us to introduce "point evaluation" with respect to J_Φ^*: we define $J_\Phi^*(w)$ to be the following operator acting on $\ell_2(2\pi\mathbb{Z}^d)$:

$$J_\Phi^*(w) : c \mapsto (\sum_{\alpha\in 2\pi\mathbb{Z}^d} c(\alpha)\overline{\widehat{\phi}(w+\alpha)})_{\phi\in\Phi}. \tag{1.4.7}$$

(To compare (1.4.6) and (1.4.7), choose $c(\alpha) := f(w+\alpha)$ in the latter.)

As expected, the analysis above reveals that the matrix representation of the operator $J_\Phi^*(w)$ is the adjoint of the matrix representation of the operator $J_\Phi(w)$. I.e., we had verified that "evaluation" commutes with taking adjoints. After making that observation, and with only very few necessary exceptions, we will identify J_Φ with its matrix representation $(J_\Phi(w))_{w\in\mathbb{T}^d}$.

The following lemma collects two useful facts that were just observed.

Lemma 1.4.8. *Let $\Phi \subset L_2(\mathbb{R}^d)$ be a countable set. Then for any $c \in \ell_0(E_\Phi)$ and $f \in L_2(\mathbb{R}^d)$, and for a.e. $w \in \mathbb{T}^d$,*

$$\widehat{\mathcal{T}_\Phi c}(w+\cdot)_{|2\pi\mathbb{Z}^d} = J_\Phi(w)\widehat{c}(w), \tag{1.4.9}$$

and

$$\widehat{\mathcal{T}_\Phi^* f}(w) = J_\Phi^*(w)(\widehat{f}_{|_{w+2\pi\mathbb{Z}^d}}). \tag{1.4.10}$$

Two self-adjoint operators can be constructed from J_Φ. The first is the **Gramian** $G := G_\Phi$, which is defined by

$$G := J_\Phi^* J_\Phi.$$

Previous considerations imply that G_Φ is the Fourier transform representer of $\mathcal{T}_\Phi^*\mathcal{T}_\Phi$. This fact allows us to draw the following immediate conclusions.

Proposition 1.4.11. *For the densely defined linear operators $\mathcal{T}_\Phi$ and G:*

(i) $\mathcal{T}_\Phi$ *is bounded if and only if* G*, considered as an endomorphism of* L_2^Φ*, is well-defined and bounded. Also,* $\|G\| = \|\mathcal{T}_\Phi\|^2$.

(ii) Assume $\mathcal{T}_\Phi$ *(hence,* G*) is bounded. Then,* $\mathcal{T}_\Phi$ *is partially invertible if and only if* G *is partially invertible. Also,* $\|G_|{}^{-1}\| = \|\mathcal{T}_{\Phi|}{}^{-1}\|^2$.

(iii) Assume $\mathcal{T}_\Phi$ *is bounded. Then,* $\mathcal{T}_\Phi$ *is invertible if and only if* G *is invertible. Also,* $\|G^{-1}\| = \|\mathcal{T}_\Phi{}^{-1}\|^2$.

We define the value $G(w)$ of G at $w \in \mathbb{T}^d$ as

$$G(w) := J_\Phi^*(w) J_\Phi(w) = ([\widehat{\phi}, \widehat{\phi'}](w))_{\phi', \phi \in \Phi}. \tag{1.4.12}$$

In general, for a.e. $w \in \mathbb{T}^d$, the Gramian $G(w)$ is a densely defined self-adjoint operator on $\ell_2(\Phi)$ (hopefully into itself). In order to make any good use of $G(w)$, one needs to make sure that, at least on L_0^Φ, evaluation commutes with the application of G, i.e., that

$$(G\tau)(w) = G(w)\tau(w), \quad \text{for } \tau \in L_0^\Phi, \text{ and for a.e. } w \in \mathbb{T}^d.$$

This is actually obtained by summation-by-parts, whose straightforward justification is omitted here. Hence:

Lemma 1.4.13. *For every* $c \in \ell_0(E_\Phi)$*, and for a.e.* $w \in \mathbb{T}^d$*,*

$$((\mathcal{T}_\Phi^* \mathcal{T}_\Phi c)\widehat{\ }(w))_{\phi \in \Phi} = G(w)\widehat{c}(w).$$

The notation

$$\Lambda(w) := \|G(w)\|$$

stands for the operator norm of $G(w)$, and is assumed to be ∞ whenever $G(w)$ is not well-defined or is unbounded. In case $G(w)$ is also boundedly invertible, we denote its bounded inverse by $G(w)^{-1}$, and set

$$\lambda(w) := \|G(w)^{-1}\|^{-1}.$$

Also, we set

$$\lambda^+(w) := \|G(w)_|{}^{-1}\|^{-1}.$$

In case Φ is finite, $\Lambda(w)$ and $\lambda(w)$ are clearly the largest and smallest eigenvalues of the finite-order matrix $G(w)$. A closer look may reveal that $\lambda^+(w)$ is, in such a case, the smallest *non-zero* eigenvalue of $G(w)$.

Typical results concerning the Gramian analysis can be found in Theorem 2.2.7 (PSI spaces), Theorem 2.2.14 (PSI spaces, several generators), Theorem 2.3.6 (FSI spaces), and Theorems 3.2.3 and 3.4.1 (infinitely generated SI spaces).

The Gramian approach is efficient for the study of those properties of E_Φ which are "visible" via the operator $\mathcal{T}_\Phi$, primarily orthogonality and stability properties. In contrast, other properties such as E_Φ being a fundamental frame or a fundamental tight frame are better analysed with the aid of the adjoint $\mathcal{T}_\Phi^*$. For the analysis of this adjoint operator, we introduce another self-adjoint operator which we call the **dual Gramian** $\widetilde{G}$. It is obtained by multiplying the pre-Gramians, but in reverse order, namely,

$$\widetilde{G} := \widetilde{G}_\Phi := J_\Phi J_\Phi^*. \tag{1.4.14}$$

Problems of well-definedness are more subtle here than in the Gramian case. Fully detailed discussions of that point are given in §3.3, and we mention here only two facts: first, if E_Φ is a Bessel set, then $\widetilde{G}$ is a well-defined self-adjoint bounded endomorphism of $L_2(\mathbb{R}^d)$. Second, if E_Φ is not a Bessel set, the definition (1.4.14) may not make sense, and it is safer to view $\widetilde{G}$ as a quadratic form, i.e., to define it by

$$\widetilde{G} : f \mapsto \|J_\Phi^* f\|^2_{L_2^\Phi} = \sum_{\phi\in\Phi} \|J_\phi^* f\|^2_{L_2(\mathbb{T}^d)} = \|\sum_{\phi\in\Phi} |[f,\widehat{\phi}]|^2\|_{L_1(\mathbb{T}^d)}.$$

The evaluation $\widetilde{G}(w)$ of the dual Gramian is the $(2\pi\mathbb{Z}^d \times 2\pi\mathbb{Z}^d)$-matrix whose (α,α')-entry has the form

$$\sum_{\phi\in\Phi} \widehat{\phi}(w+\alpha)\overline{\widehat{\phi}(w+\alpha')}. \tag{1.4.15}$$

The argument w may be restricted to $\mathbb{T}^d$. For a general E_Φ, the entries of $\widetilde{G}(w)$ may not be well-defined (in the sense that the sum in their definition needs not converge absolutely). Nevertheless, we will show (in §3.3) that, whenever E_Φ is a Bessel set, the sum in (1.4.15) converges absolutely for every $\alpha,\alpha' \in 2\pi\mathbb{Z}^d$ and for a.e. w. Thus, for a Bessel set E_Φ, $\widetilde{G}(w)$ is well-defined a.e., and can viewed as a densely defined operator from $\ell_2(2\pi\mathbb{Z}^d)$ (hopefully into itself). Moreover, we will show then that the basic relation

$$(\widetilde{G}f)(w) = \widetilde{G}(w) f_{|w}$$

(with $f_{|w}$ the restriction of f to $w + 2\pi\mathbb{Z}^d$) holds a.e. A similar relation is drawn in §3.3 even in the non-Bessel case, under the assumption that the entries of $\widetilde{G}(w)$ are well-defined, and with the interpretation of $\widetilde{G}$ and $\widetilde{G}(w)$ as quadratic forms.

Analogously to the Gramian case, we define here the following functions

$$\widetilde{\Lambda}(w) := \|\widetilde{G}(w)\|,$$

$$\widetilde{\lambda}(w) := \|\widetilde{G}(w)^{-1}\|^{-1},$$

$$\widetilde{\lambda}^+(w) := \|\widetilde{G}(w)_|{}^{-1}\|^{-1},$$

and attempt to study properties of E_Φ in terms of the behaviour of these functions. Our main results in this regard are Theorem 3.3.5, and Theorem 3.4.1.

The Gramian/dual Gramian analyses are also efficient for studying the connection between a frame and its dual: given two sets $\Phi,\Psi \subset L_2(\mathbb{R}^d)$, and some bijection $\mathrm{R} : \Phi \to \Psi$, this is done via the study of the matrices $J_\Phi(w)J^*_{\mathrm{R}\Phi}(w)$, and $J^*_\Phi(w)J_{\mathrm{R}\Phi}(w)$, as discussed in §4.

1.5. An example

We provide here an example, which is taken from [RS1], (and is a specific type of what we call there "self-adjoint Weyl-Heisenberg sets") that illustrates the potential power of the results to be developed in this paper.

Let $\phi \in L_2(\mathbb{R}^d)$. Let

$$\Phi := (e_\alpha \phi)_{\alpha \in 2\pi\mathbb{Z}^d}.$$

Indexing Φ by $2\pi\mathbb{Z}^d$, the pre-Gramian $J_\Phi(w)$ is found to be

$$J_\Phi(w) = (\widehat{\phi}(w + \alpha + \beta))_{\alpha,\beta \in 2\pi\mathbb{Z}^d}.$$

Therefore, $J_\Phi^*(w) = \overline{J_\Phi(w)}$, and hence

$$G_\Phi(w) = \overline{\widetilde{G}_\Phi(w)}.$$

Now, Theorem 3.2.3 characterizes the stability property of E_Φ in terms of the Gramian fibers $G_\Phi(w)$, $w \in \mathbb{T}^d$. On the other hand, the same criterion when applied to $\widetilde{G}_\Phi(w)$, $w \in \mathbb{T}^d$, is shown to be equivalent to E_Φ being the fundamental frame (Theorem 3.3.5). This recovers the following well-known fact (cf. e.g. [D1,2]):

Corollary 1.5.1. *With Φ as above, E_Φ is a stable basis if and only if it is a fundamental frame.*

1.6. An application: estimating the frame bounds

The main results of this paper are concerned with the connections between the spectrum of the operators G and $\widetilde{G}$ and the spectra of the operators $G(w)$ and $\widetilde{G}(w)$, $w \in \mathbb{T}^d$. As we mentioned before, information about the fiber operators $G(w)$ and $\widetilde{G}(w)$ is more readily available as compared to similar information concerning G and $\widetilde{G}$. Still, computing exactly, e.g., the norm of $G(w)$ (considered as a linear map from $\ell_2(\Phi)$ into itself) might appear as a hard task. However, *estimating* this norm (either from below or from above) in terms of the Fourier transforms of the functions in the generating set Φ is quite easy. This subsection is devoted to the discussion of such estimates.

To this end, we let I be a countable (or finite) index set, and let M be a complex-valued non-negative Hermitian matrix with rows and columns indexed by I, and considered as an operator from $\ell_2(I)$ into itself. We use the following estimates of $\|M\|$:

$$\sup_{i\in I}\Big(\sum_{j\in I} |M(i,j)|^2\Big)^{\frac{1}{2}} \le \|M\| \le \sup_{i\in I} \sum_{j\in I} |M(i,j)|. \tag{1.6.1}$$

Combining these estimates with Theorem 3.2.3, we obtain our first estimate for $\|\mathcal{T}_\Phi\|$:

Corollary 1.6.2. *Let Φ be a countable (or finite) subset of $L_2(\mathbb{R}^d)$.*
(a) If the function

$$B_1 : \mathbb{T}^d \times \Phi \to \mathbb{R} : (w, \phi) \mapsto \sum_{\phi' \in \Phi} \Big|\sum_{\alpha \in 2\pi\mathbb{Z}^d} \widehat{\phi}(w+\alpha)\overline{\widehat{\phi'}(w+\alpha)}\Big|$$

is essentially bounded, then E_Φ is a Bessel set, and $\|\mathcal{T}_\Phi\|^2 \le \|B_1\|_{L_\infty(\mathbb{T}^d\times\Phi)}$.

(b) If E_Φ is a Bessel set, then the function

$$B_2 : \mathbb{T}^d \times \Phi \to \mathbb{R} : (w,\phi) \mapsto \Big(\sum_{\phi'\in\Phi} \Big|\sum_{\alpha\in 2\pi\mathbb{Z}^d} \widehat{\phi}(w+\alpha)\overline{\widehat{\phi'}(w+\alpha)}\Big|^2\Big)^{\frac12}$$

is essentially bounded, and $\|\mathcal{T}_\Phi\|^2 \ge \|B_2\|_{L_\infty(\mathbb{T}^d\times\Phi)}$.

On the other hand, combining (1.6.1) with Theorem 3.3.5, we obtain different estimates:

Corollary 1.6.3. *Let Φ be a countable (or finite) subset of $L_2(\mathbb{R}^d)$.*

(a) If the function

$$\widetilde{B}_1 : \mathbb{R}^d \to \mathbb{R} : w \mapsto \sum_{\alpha\in 2\pi\mathbb{Z}^d} \Big|\sum_{\phi\in\Phi} \widehat{\phi}(w)\overline{\widehat{\phi}(w+\alpha)}\Big|$$

is essentially bounded, then E_Φ is a Bessel set, and $\|\mathcal{T}_\Phi\|^2 \le \|\widetilde{B}_1\|_{L_\infty(\mathbb{R}^d)}$.

(b) If E_Φ is a Bessel set, then the function

$$\widetilde{B}_2 : \mathbb{R}^d \to \mathbb{R} : w \mapsto \Big(\sum_{\alpha\in 2\pi\mathbb{Z}^d} \Big|\sum_{\phi\in\Phi} \widehat{\phi}(w)\overline{\widehat{\phi}(w+\alpha)}\Big|^2\Big)^{\frac12}$$

is bounded and $\|\mathcal{T}_\Phi\|^2 \ge \|\widetilde{B}_2\|_{L_\infty(\mathbb{R}^d)}$.

For the estimation of the other frame bound, we need a bound on $\|M^{-1}\|$. In what follows we employ the estimate

$$\|M^{-1}\| \le \sup_{i\in I}\Big(|M(i,i)| - \sum_{j\in I\backslash i} |M(i,j)|\Big)^{-1}, \tag{1.6.4}$$

which is valid for any Hermitian diagonally dominant M. An application of this estimate to Theorem 3.2.3 yields the following:

Corollary 1.6.5. *Let $\Phi \subset L_2(\mathbb{R}^d)$ be countable (or finite), and assume that E_Φ is a Bessel set. Then E_Φ is a stable basis if the function*

$$b_1 : \mathbb{T}^d \times \Phi :\to \mathbb{R} : (w,\phi) \mapsto \Big(\sum_{\alpha\in 2\pi\mathbb{Z}^d} |\widehat{\phi}(w+\alpha)|^2 - \sum_{\phi'\in\Phi\backslash\phi} \Big|\sum_{\alpha\in 2\pi\mathbb{Z}^d} \widehat{\phi}(w+\alpha)\overline{\widehat{\phi'}(w+\alpha)}\Big|\Big)^{-1}$$

is positive and essentially bounded. Furthermore, in this case

$$\|\mathcal{T}_\Phi{}^{-1}\|^2 \le \|b_1\|_{L_\infty(\mathbb{T}^d\times\Phi)},$$

Finally, an application of (1.6.4) to Theorem 3.3.5 yields the following:

Corollary 1.6.6. *Let $\Phi \subset L_2(\mathbb{R}^d)$ be countable (or finite), and assume that E_Φ is a Bessel set. Then E_Φ is a fundamental frame if the function*

$$\widetilde{b}_1 : \mathbb{R}^d :\to \mathbb{R} : w \mapsto \Big(\sum_{\phi\in\Phi} |\widehat{\phi}(w)|^2 - \sum_{\alpha\in 2\pi\mathbb{Z}^d\backslash 0} |\sum_{\phi\in\Phi} \widehat{\phi}(w)\overline{\widehat{\phi}(w+\alpha)}|\Big)^{-1}$$

is positive and essentially bounded. Furthermore,

$$\|\mathcal{T}_\Phi^{*-1}\|^2 \le \|\widetilde{b}_1\|_{L_\infty(\mathbb{R}^d)}.$$

The simplest example that follows from the above results (and can also be checked directly) is the following.

Example 1.6.7. Suppose that, for every $\phi \in \Phi$, for every $\alpha \in 2\pi\mathbb{Z}^d$, and for almost every $w \in \mathbb{R}^d$, $\widehat{\phi}(w)\widehat{\phi}(w+\alpha) = 0$ (e.g., each $\widehat{\phi}$ is supported in some cube $t_\phi + [0..2\pi)^d$, $t_\phi \in \mathbb{R}^d$). Then, the (square root of the) function $\widetilde{B}_1$ can be replaced by the function

$$g : \mathbb{R}^d \to \mathbb{R} : w \mapsto (\sum_{\phi\in\Phi} |\widehat{\phi}(w)|^2)^{\frac{1}{2}}.$$

Similarly, the function $\widetilde{b}_1$ can be replaced by $1/g$. Consequently, we obtain E_Φ is a fundamental frame if the two functions g and $1/g$ are essentially bounded. In fact, the results of this paper will show that the converse of this last statement is valid as well.

2. Finitely generated SI spaces

2.1. General

While general SI spaces are best analysed with simultaneous use of the Gramian and dual Gramian matrices, this is not the case for FSI spaces. The reason is easy to inspect: for a finitely generated SI space, the dual Gramian matrix is infinite, while the Gramian matrix is *finite*. This explains to a large extent the prevalence of Gramian analysis in the study of FSI spaces. Moreover, in the principal case, the Gramian matrix is reduced to a single function, providing thereby a further significant simplification in the course of study of such spaces. Therefore, we will first present (in the next subsection) a detailed analysis of bases for PSI spaces, and only then discuss the FSI counterpart of that theory. The present subsection is devoted to some simple initial observations and estimates.

In the PSI case, the generating set Φ is a singleton (ϕ), and the operator $\mathcal{T}_\phi^* := \mathcal{T}_{(\phi)}^*$ then takes the particularly simple form

$$\mathcal{T}_\phi^* : f \mapsto \{\langle f, E^\alpha\phi\rangle\}_{\alpha\in\mathbb{Z}^d}.$$

From Parseval's identity, and the 2π-periodicity of the exponentials e_α, $\alpha \in \mathbb{Z}^d$, we obtain that

$$\langle f, E^\alpha\phi\rangle = (2\pi)^{-d}\langle \widehat{f}\overline{\widehat{\phi}}, e_\alpha\rangle = (2\pi)^{-d}\langle [\widehat{f}, \widehat{\phi}], e_\alpha\rangle_{L_2(\mathbb{T}^d)}. \tag{2.1.1}$$

Therefore, $\mathcal{T}_\phi^* f$ is the set of Fourier coefficients of the $L_1(\mathbb{T}^d)$-function $[\widehat{f}, \widehat{\phi}]$, that is

$$\widehat{\mathcal{T}_\phi^* f} = [\widehat{f}, \widehat{\phi}]. \tag{2.1.2}$$

In particular,

Proposition 2.1.3. *Given $\phi, f \in L_2(\mathbb{R}^d)$,*

$$\|\mathcal{T}_\phi^* f\|_{\ell_2(\mathbb{Z}^d)} = (2\pi)^{-d/2} \|[\widehat{f}, \widehat{\phi}]\|_{L_2(\mathbb{T}^d)}.$$

Some coarse estimates can be derived directly from the above. By Schwartz inequality,

$$|[\widehat{f}, \widehat{\phi}]|^2 \leq [\widehat{f}, \widehat{f}][\widehat{\phi}, \widehat{\phi}].$$

Thus, for $\Phi \subset L_2(\mathbb{R}^d)$,

$$\|\mathcal{T}_\Phi^* f\|^2_{\ell_2(E_\Phi)} \leq (2\pi)^{-d} \|[\widehat{f}, \widehat{f}] \sum_{\phi \in \Phi} [\widehat{\phi}, \widehat{\phi}]\|_{L_1(\mathbb{T}^d)}.$$

Since $\|f\|^2 = (2\pi)^{-d} \|[\widehat{f}, \widehat{f}]\|_{L_1(\mathbb{T}^d)}$, we conclude that

$$\|\mathcal{T}_\Phi^* f\|_{\ell_2(E_\Phi)} \leq \|f\| \, \|\sum_{\phi \in \Phi} [\widehat{\phi}, \widehat{\phi}]\|^{\frac{1}{2}}_{L_\infty(\mathbb{T}^d)}.$$

Denoting

$$\widetilde{\Phi} := (\sum_{\phi \in \Phi} [\widehat{\phi}, \widehat{\phi}])^{\frac{1}{2}},$$

we have proved the following result.

Proposition 2.1.4. *Given $\Phi \subset L_2(\mathbb{R}^d)$, E_Φ is a Bessel set in case $\widetilde{\Phi} \in L_\infty(\mathbb{T}^d)$, and we then have*

$$\|\mathcal{T}_\Phi^*\| \leq \|\widetilde{\Phi}\|_{L_\infty(\mathbb{T}^d)}.$$

We will show later that equality holds in the above in case Φ is taken from some PSI subspace of $L_2(\mathbb{R}^d)$. Further, we will show that for a *finite* Φ the boundedness of $\widetilde{\Phi}$ is not only sufficient for E_Φ to be a Bessel sequence, but also *necessary*. However, the bound provided by $\|\widetilde{\Phi}\|_{L_\infty(\mathbb{T}^d)}$ is, in general, not sharp.

2.2. Frames in PSI spaces

Throughout this subsection, S is a PSI subspace of $L_2(\mathbb{R}^d)$ generated by some (fixed) function.

Motivated by the search for an explicit representation for the orthogonal projection onto shift-invariant spaces, [BDR1] introduces and studies the notions of *quasi-stable* and *quasi-orthogonal* bases for FSI spaces. For PSI spaces, in the terminology used in the present paper, its definitions are as follows:

Definition 2.2.5. ([BDR1]) *Let $\phi \in L_2(\mathbb{R}^d)$, and let $\mathcal{T}_\phi$ be the operator*

$$\mathcal{T}_\phi : \ell_2(\mathbb{Z}^d) \to S(\phi) : c \mapsto \sum_{\alpha \in \mathbb{Z}^d} E^\alpha \phi \, c(\alpha).$$

Then ϕ is called a **quasi-stable generator** *if $\mathcal{T}_\phi$ is a well-defined bounded map, and provides an isomorphism between $C_{\mathcal{T}_\phi} := (\ker \mathcal{T}_\phi)^\perp$ and $S(\phi)$. If, further, that isomorphism is an isometry, ϕ is termed a* **quasi-orthogonal** *generator.*

In view of (b) of Proposition 1.3.4, and Definition 1.3.5 of frames and tight frames we obtain the following Corollary.

Corollary 2.2.6. *Let $\phi \in L_2(\mathbb{R}^d)$. Then E_ϕ is a frame if and only if ϕ is a quasi-stable generator of $S(\phi)$. Further, this frame is tight if and only if ϕ is a scalar multiple of a quasi-orthogonal generator of $S(\phi)$.*

Thus, implicitly, [BDR1] contains an extensive discussion of frames in PSI spaces. Furthermore, as we had learnt from the referee of this paper, frames for PSI spaces were (explicitly) studied by Benedetto and Li [BL]. Indeed, Theorem 7.7 of [BW] (which is attributed there to [BL]) is essentially equivalent to Theorem 2.2.7.

We recall the definition of the spectrum σS given in (1.4.5), and recall the notation

$$\widetilde{\phi} = [\widehat{\phi}, \widehat{\phi}]^{\frac{1}{2}} = \Big(\sum_{\beta \in 2\pi \mathbb{Z}^d} |\widehat{\phi}(\cdot + \beta)|^2\Big)^{\frac{1}{2}}.$$

Theorem 2.2.7. ([BDR1], [BL]) *Let $\phi \in L_2(\mathbb{R}^d)$ be given, and let S be the PSI space generated by ϕ.*

(a) *The shifts E_ϕ of ϕ form a Bessel sequence in S if and only if $\widetilde{\phi}$ is essentially bounded.*

(b) *The shifts E_ϕ of ϕ form a frame for S if and only if $\widetilde{\phi}$ and $1/\widetilde{\phi}$ are essentially bounded on σS. Furthermore,*

$$\|\mathcal{T}_\phi\| = \|\widetilde{\phi}\|_{L_\infty(\mathbb{T}^d)} = \|\widetilde{\phi}\|_{L_\infty(\sigma S)},$$

and

$$\|\mathcal{T}_{\phi|}{}^{-1}\| = \|1/\widetilde{\phi}\|_{L_\infty(\sigma S)}.$$

Therefore, for a frame E_ϕ, the inequalities

$$\|f\|/\|1/\widetilde{\phi}\|_{L_\infty(\sigma S)} \le \Big(\sum_{\alpha \in \mathbb{Z}^d} |\langle f, E^\alpha \phi\rangle|^2\Big)^{\frac{1}{2}} \le \|\widetilde{\phi}\|_{L_\infty(\sigma S)} \|f\|, \quad f \in S,$$

are valid and sharp.

(c) *E_ϕ is a tight frame if and only if $\widetilde{\phi} = \mathrm{const}$ (a.e.) on its support.*

(d) *With $\psi := (\widehat{\phi}/\widetilde{\phi})^\vee$, the set E_ψ is a tight frame for $S(\phi)$ (and hence every PSI space is generated by some PSI tight frame).*

(e) *The frame (tight frame) E_ϕ is a stable (orthogonal) basis for S if and only if $\sigma S = \mathbb{T}^d$.*

Proof: By Corollary 2.2.6, the shifts of ϕ form a frame (tight frame) if and only if ϕ is a quasi-stable (quasi-orthogonal) generator of $S(\phi)$. Therefore, the theorem follows from the corresponding results in section 2 of [BDR1]. $\square$

We observe that the above (d) and (e) imply that S contains an orthonormal basis E_ϕ if and only if $\sigma S = \mathbb{T}^d$. That case was termed **regular** in [BDR1]. Thus (e) above shows that the notions of a stable basis and a frame coincide for a principal shift-invariant E_ϕ, provided that $S(\phi)$ is regular. It is worth mentioning that, in case ϕ is compactly supported, $S(\phi)$ is always regular.

The spaces $\ker \mathcal{T}_\phi$ and $C_{\mathcal{T}_\phi}$ were described explicitly in [BDR1] as follows:

$$\ker \mathcal{T}_\phi := \{c \in \ell_2(\mathbb{Z}^d) :\ \mathrm{supp}\,\widehat{c} \subset (\mathbb{T}^d \backslash \sigma S)\},$$

and hence

$$C_{\mathcal{T}_\phi} := \{c \in \ell_2(\mathbb{Z}^d) :\ \mathrm{supp}\,\widehat{c} \subset \sigma S\}. \tag{2.2.8}$$

Next, we need the following characterization of the Fourier transforms of the elements of $S(\phi)$:

Result 2.2.9. ([BDR2]) *Let $\phi, f \in L_2(\mathbb{R}^d)$. Then $f \in S(\phi)$ if and only if $\widehat{f} = \tau\widehat{\phi}$ for some 2π-periodic function τ.*

Corollary 2.2.10. *Let $S := S(\phi)$ be a PSI space, and assume that E_ϕ forms a frame for S. Then, given $c \in \ell_2(\mathbb{Z}^d)$, there exists $f \in S$ such that*

$$c(\alpha) = \langle f, E^\alpha \phi \rangle, \quad \alpha \in \mathbb{Z}^d$$

if and only if $\widehat{c}$ is supported in the spectrum of S. The unique solution f has the form

$$f = \sum_{\alpha \in \mathbb{Z}^d} E^\alpha \phi \, c_f(\alpha), \tag{2.2.11}$$

with the sequence $c_f \in C_{\mathcal{T}_\phi}$ being the solution of the discrete convolution equation

$$[\widehat{\phi}, \widehat{\phi}]^\vee * c_f = c.$$

Proof: By the definition of $\mathcal{T}_\phi^*$, a solution f exists if and only if c lies in the range of $\mathcal{T}_\phi^*$, i.e., if and only if $c \in C_{\mathcal{T}_\phi}$. Therefore, in view of (2.2.8), we only need to prove the statements concerning the nature of the solution f. Since E_ϕ is a frame for S, then, given any $f \in S$, there exists a unique $c_f \in C_{\mathcal{T}_\phi}$ that satisfies (2.2.11). Taking Fourier transforms, we obtain that $\widehat{f} = \widehat{c_f}\widehat{\phi}$. Invoking (2.1.2), we see that

$$\widehat{c} = \widehat{\mathcal{T}_\phi^* f} = [\widehat{f}, \widehat{\phi}] = \widehat{c_f}[\widehat{\phi}, \widehat{\phi}], \tag{2.2.12}$$

where, in the last equality, the periodicity of $\widehat{c_f}$ was used. The desired result then follows by inversion. □

Given a frame E_ϕ, Proposition 1.3.10 asserts that there exists a function $\mathrm{R}\phi \in S(\phi)$, such that $E_{\mathrm{R}\phi}$ is the dual frame of E_ϕ. Further, we can compute $\mathrm{R}\phi$ as follows: first, we seek $c_\phi \in C_{\mathcal{T}_\phi}$ such that $\mathcal{T}_\phi c_\phi = \phi$. Applying Fourier transform, then multiplying by $\overline{\widehat{\phi}}$, and periodizing over $2\pi\mathbb{Z}^d$, we obtain the equation $\widehat{c_\phi}[\widehat{\phi}, \widehat{\phi}] = [\widehat{\phi}, \widehat{\phi}]$. Since c_ϕ is in $C_{\mathcal{T}_\phi}$, it is supported on $\mathrm{supp}[\widehat{\phi}, \widehat{\phi}] = \sigma S$, and so $\widehat{c_\phi}$ is the characteristic function χ of σS. Let c be the solution of $[\widehat{\phi}, \widehat{\phi}]^\vee * ? = c_\phi$, and $\widehat{\mathrm{R}\phi} := \widehat{c}\widehat{\phi}$. Then $E_{\mathrm{R}\phi}$ is the dual basis of E_ϕ by the fact $\mathcal{T}_{\mathrm{R}\phi}^* \phi = c_\phi$ and by Corollary 2.2.10. Hence $\widehat{c}$ is defined by

$$\widehat{c} = \frac{\widehat{c_\phi}}{[\widehat{\phi}, \widehat{\phi}]} = \frac{1}{[\widehat{\phi}, \widehat{\phi}]},$$

and $\mathrm{R}\phi$ is given by

$$\widehat{\mathrm{R}\phi} = \widehat{\phi}/[\widehat{\phi}, \widehat{\phi}]. \tag{2.2.13}$$

This representation of $\mathrm{R}\phi$ is detailed in [BDR1] (using a different approach) and is well-known in the special *regular* case mentioned above (in which a frame becomes a stable basis).

The *redundancy* offered by frames does not really exist for principal shift-invariant ones. Yet, given a PSI space, one may use *several* functions from S to generate a shift-invariant frame for S. The details of that case are given in the next theorem.

Theorem 2.2.14. *Let S be a PSI space, and $\Phi \subset S$ be a countable (or finite) set. Then*
(a) E_Φ is a Bessel set if and only if the function

$$\widetilde{\Phi} := \Big(\sum_{\phi\in\Phi} [\widehat{\phi}, \widehat{\phi}]\Big)^{\frac{1}{2}} \tag{2.2.15}$$

is essentially bounded. Furthermore, $\|\mathcal{T}_\Phi\| = \|\widetilde{\Phi}\|_{L_\infty(\mathbb{T}^d)}$.
(b) E_Φ is a frame for S if and only if $\widetilde{\Phi}$ and $1/\widetilde{\Phi}$ are essentially bounded on the spectrum σS. In such a case, $\|\mathcal{T}_{\Phi|}{}^{-1}\| = \|1/\widetilde{\Phi}\|_{L_\infty(\sigma S)}$.
(c) E_Φ is a tight frame if and only if $\widetilde{\Phi}$ is constant a.e. on its support.

Proof: By Proposition 2.1.3, given $f \in L_2(\mathbb{R}^d)$,

$$\|\mathcal{T}_\Phi^* f\|^2_{\ell_2(E_\Phi)} = (2\pi)^{-d} \| \sum_{\phi\in\Phi} |[\widehat{f}, \widehat{\phi}]|^2 \|_{L_1(\mathbb{T}^d)}.$$

Let ψ be a generator of S. For $f \in S$ and $\phi \in \Phi$, Result 2.2.9 implies the existence of 2π-periodic τ_ϕ, τ_f such that

$$\widehat{f} = \tau_f \widehat{\psi}, \qquad \widehat{\phi} = \tau_\phi \widehat{\psi},\ \phi \in \Phi.$$

Therefore,

$$|[\widehat{f}, \widehat{\phi}]|^2 = |\tau_f|^2 |\tau_\phi|^2 |[\widehat{\psi}, \widehat{\psi}]|^2 = |[\widehat{f}, \widehat{f}][\widehat{\phi}, \widehat{\phi}]|.$$

Consequently,

$$\|\mathcal{T}_\Phi^* f\|^2_{\ell_2(E_\Phi)} = (2\pi)^{-d} \|[\widehat{f}, \widehat{f}]\widetilde{\Phi}^2\|_{L_1(\mathbb{T}^d)}.$$

Since $\|f\|^2_{L_2(\mathbb{R}^d)} = (2\pi)^{-d}\|[\widehat{f}, \widehat{f}]\|_{L_1(\mathbb{T}^d)}$, and since $[\widehat{f}, \widehat{f}]$ is necessarily supported on σS, the proof of the theorem relies on the comparison of

$$\|[\widehat{f}, \widehat{f}]\|^{\frac{1}{2}}_{L_1(\sigma S)}$$

and

$$\|[\widehat{f}, \widehat{f}]\widetilde{\Phi}^2\|^{\frac{1}{2}}_{L_1(\sigma S)}.$$

Further, we note that Result 2.2.9 also implies that for any closed $\Omega \subset \sigma S$, there exists $f \in S$ for which $[\widehat{f}, \widehat{f}]$ is the characteristic function of Ω. The proof can be then completed by a routine argument (cf. e.g., the proof of Theorem 2.16 in [BDR1]). □

The final theorem of this subsection provides the details concerning the dual frame of the above E_Φ and a complete description of $\ker \mathcal{T}_\Phi$ and $C_{\mathcal{T}_\Phi}$:

Theorem 2.2.16. *Let Φ be a countable subset of a PSI space S, E_Φ its corresponding shift-invariant set. If E_Φ is a frame then:*
(a) Let ψ be any generator of S (i.e., $S = S(\psi)$), and $c = (c_\phi)_{\phi\in\Phi} \in \ell_2(E_\Phi)$ (with c_ϕ the restriction of c to E_ϕ). Then $c \in C_{\mathcal{T}_\Phi}$ if and only if

$$(\widehat{c_\phi})_\phi = \tau([\widehat{\psi}, \widehat{\phi}])_\phi,$$

for some 2π-periodic function τ, that is supported on σS.

(b) The map R *from the frame E_Φ to its dual is given by*

$$\mathrm{R} : f \mapsto (\widehat{f}/\widetilde{\Phi}^2)^\vee.$$

(c) The orthogonal projector $\mathcal{P} : L_2(\mathbb{R}^d) \to S$ can be written in the form

$$\mathcal{P}f = \sum_{\phi\in\Phi,\ \alpha\in\mathbb{Z}^d} \langle f, E^\alpha((\widehat{\phi}/\widetilde{\Phi}^2)^\vee)\rangle\, E^\alpha\phi.$$

Proof: Claim (c) is immediate from (b) and Proposition 1.3.8. To prove (b), we need to show that the map R inverts $\mathcal{T}_\Phi\mathcal{T}_\Phi^*$, and this will follow as soon as we show that $(\mathcal{T}_\Phi\mathcal{T}_\Phi^* f)\widehat{\ } = \widetilde{\Phi}^2\widehat{f}$ on S. For that, note first that Result 2.2.9 implies that, for every $f, g \in S$,

$$[\widehat{f},\widehat{g}]\,\widehat{g} = [\widehat{g},\widehat{g}]\widehat{f}. \tag{2.2.17}$$

Now, given $f \in S$, we first recall that, by (2.1.2), for every $\phi\in\Phi$,

$$(\mathcal{T}_\phi\mathcal{T}_\phi^* f)\widehat{\ } = [\widehat{f},\widehat{\phi}]\widehat{\phi}.$$

This, together with (2.2.17) and the fact that $\mathcal{T}_\Phi\mathcal{T}_\Phi^* = \sum_{\phi\in\Phi}\mathcal{T}_\phi\mathcal{T}_\phi^*$, implies that

$$(\mathcal{T}_\Phi\mathcal{T}_\Phi^* f)\widehat{\ } = \sum_{\phi\in\Phi}[\widehat{f},\widehat{\phi}]\widehat{\phi} = \sum_{\phi\in\Phi}[\widehat{\phi},\widehat{\phi}]\widehat{f} = \widetilde{\Phi}^2\widehat{f}.$$

This proves (b) and thereby (c).

To prove (a), we compute $C_{\mathcal{T}_\Phi}$ using the identity

$$C_{\mathcal{T}_\Phi} = \operatorname{ran}\mathcal{T}_\Phi^*.$$

For $f \in S$, there exists, by Result 2.2.9, a function τ_f supported on σS, such that $\widehat{f} = \tau_f\widehat{\psi}$. By (2.1.2),

$$\widehat{\mathcal{T}_\phi^* f} = [\widehat{f},\widehat{\phi}] = \tau_f[\widehat{\psi},\widehat{\phi}].$$

Since $C_{\mathcal{T}_\Phi}$ is the range of $\mathcal{T}_\Phi^*$, this shows that the Fourier transform of each $c = (c_\phi)_{\phi\in\Phi} \in C_{\mathcal{T}_\Phi}$ is of the form $\widehat{c_\phi} = \tau[\widehat{\psi},\widehat{\phi}]$, $\forall\phi\in\Phi$, for some 2π-periodic τ supported on σS, i.e., $C_{\mathcal{T}_\Phi}$ contains only sequences of the required form.

Conversely, assume that $c = (c_\phi)$ satisfies $\widehat{c_\phi} = \tau([\widehat{\psi},\widehat{\phi}])$. We consider the nature of $\mathcal{T}_\Phi c = \sum_{\phi\in\Phi}\mathcal{T}_\phi c_\phi$. Applying Fourier transform, and invoking (2.2.17) once again, we obtain that

$$\widehat{\mathcal{T}_\Phi c} = \sum_{\phi\in\Phi}\widehat{c_\phi}\widehat{\phi} = \sum_{\phi\in\Phi}\tau[\widehat{\psi},\widehat{\phi}]\widehat{\phi} = \sum_{\phi\in\Phi}\tau[\widehat{\phi},\widehat{\phi}]\widehat{\psi} = \tau\widetilde{\Phi}^2\widehat{\psi}.$$

Since $\mathcal{T}_\Phi$ is bounded, $\tau\widetilde{\Phi}^2\widehat{\psi} \in L_2(\mathbb{R}^d)$. On the other hand, since E_Φ is a frame, then, by Theorem 2.2.14, $\widetilde{\Phi}$ is bounded below on $(\sigma S)^\circ \supset \operatorname{supp}\widehat{\psi}$, and therefore $\tau\widehat{\psi} \in L_2(\mathbb{R}^d)$. Thus, $f := (\tau\widehat{\psi})^\vee$ is in $L_2(\mathbb{R}^d)$, and hence, by Result 2.2.9, is also in S. Since the proof of the previous implication shows that $\mathcal{T}_\Phi^* f = c$, we obtain that $c \in \operatorname{ran}\mathcal{T}_\Phi^*$, as needed. $\square$

From (a) of Theorem 2.2.16, it easily follows that

$$\ker \mathcal{T}_\Phi = \{(c_\phi)_\phi \in \ell_2(E_\Phi) : \sum_{\phi\in\Phi} \widehat{c}_\phi[\widehat{\psi}, \widehat{\phi}] = 0\},$$

with ψ some (any) generator of S.

2.3. Frames in FSI spaces

In order to lift the results of the previous section from PSI spaces to FSI spaces, we need first the following FSI analog of Result 2.2.9 (cf. Theorem 1.7 in [BDR1]):

Result 2.3.1. *Let Φ be a finite subset of $L_2(\mathbb{R}^d)$. A function $f \in L_2(\mathbb{R}^d)$ is in $S := S(\Phi)$ if and only if there exists $\tau := (\tau_\phi)_{\phi\in\Phi}$, with each τ_ϕ a 2π-periodic function, such that*

$$\widehat{f} = \sum_{\phi\in\Phi} \tau_\phi \widehat{\phi}. \tag{2.3.2}$$

Several different approaches are available for the analysis of frames in FSI spaces. We have chosen here the one which incorporates efficiently the results on PSI frames that were established in the previous subsection. We do that by studying first the straightforward case when the finite generating set Φ of S induces an *orthogonal* decomposition of S into the sum $\oplus_{\phi\in\Phi} S(\phi)$ of PSI spaces. We then reduce the general setup to that simple case.

Recall that, by (1.2.3), the space $S(\phi)$ is orthogonal to the space $S(\psi)$ if and only if $[\widehat{\phi}, \widehat{\psi}] = 0$, a.e. Thus, the sum $\sum_{\phi\in\Phi} S(\phi)$ is orthogonal if and only if the Gramian matrix G is diagonal.

Proposition 2.3.3. *If the Gramian matrix G is diagonal, then:*

(a) E_Φ is a Bessel set if and only if, for each $\phi \in \Phi$, $\widetilde{\phi}$ is bounded on $\sigma\phi = \sigma(S(\phi))$. Furthermore,

$$\|\mathcal{T}_\Phi\| = \max_\phi \|\mathcal{T}_\phi\| = \max_\phi \|\widetilde{\phi}\|_{L_\infty(\mathbb{T}^d)}.$$

(b) E_Φ is a frame for $S(\Phi)$ if and only if, for each $\phi \in \Phi$, $\widetilde{\phi}$ and $1/\widetilde{\phi}$ are bounded on $\sigma\phi$. The frame is tight if and only if, for every ϕ, $\widetilde{\phi} =$ const on $\sigma\phi$ (with const independent of ϕ). Furthermore,

$$\|\mathcal{T}_{\Phi|}{}^{-1}\| = \max_\phi \|\mathcal{T}_{\phi|}{}^{-1}\| = \max_\phi \|1/\widetilde{\phi}\|_{L_\infty(\sigma\phi)}.$$

Proof: The orthogonal sum decomposition $\oplus_\phi S(\phi)$ of $S(\Phi)$ implies that $\mathcal{T}_\Phi^*$ agrees with $\mathcal{T}_\phi^*$ on $S(\phi)$ (recall that we naturally embed the target space $\ell_2(E_\phi)$ of the latter into the target space $\ell_2(E_\Phi)$ of the former). Since $\ell_2(E_\Phi)$ is (always) the orthogonal sum $\oplus_\phi \ell_2(E_\phi)$, we conclude that, indeed,

$$\|\mathcal{T}_\Phi\| = \|\mathcal{T}_\Phi^*\| = \max_{\phi\in\Phi} \|\mathcal{T}_\phi^*\| = \max_{\phi\in\Phi} \|\mathcal{T}_\phi\|,$$

and

$$\|\mathcal{T}_{\Phi|}{}^{-1}\| = \|\mathcal{T}_{\Phi|}^*{}^{-1}\| = \max_{\phi\in\Phi} \|\mathcal{T}_{\phi|}^*{}^{-1}\| = \max_{\phi\in\Phi} \|\mathcal{T}_{\phi|}{}^{-1}\|.$$

The result then follows by an application of parts (a-c) of Theorem 2.2.7. □

In accordance with the definitions of §1.4, we define here

$$\Lambda(w)$$

to be the largest eigenvalue of $G(w)$,

$$\lambda(w)$$

to be the smallest eigenvalue of $G(w)$, and

$$\lambda^+(w)$$

to be the smallest *non-zero* eigenvalue of $G(w)$. Then, both $\Lambda(w)$ and $\lambda^+(w)$ are non-negative and well-defined on σS. Further, Proposition 2.3.3 can be stated as follows:

If G is diagonal, then E_Φ is a Bessel set if and only if $\|\Lambda\|_{L_\infty(\sigma S)} < \infty$. E_Φ is a frame for $S(\Phi)$ if and only if

$$\Lambda \quad \text{and} \quad 1/\lambda^+ \quad \text{are (essentially) bounded on the spectrum of } \; S, \tag{2.3.4}$$

and, moreover, the frame bounds of E_Φ are $\|\Lambda\|_{L_\infty(\sigma S)}$ and $\|1/\lambda^+\|_{L_\infty(\sigma S)}$.

As Theorem 2.3.6 below asserts, the above characterizations are valid for general FSI spaces.

The proof of Theorem 2.3.6 is based on the following (technical) lemma:

Lemma 2.3.5. *Given a finite order Hermitian matrix G, whose entries are measurable functions defined on some domain Ω, there exists a matrix $U := U_{\Phi\times\Phi}$ whose entries are measurable functions defined on Ω, such that U^*GU is a diagonal matrix, and $U(w)$ is unitary for every $w \in \Omega$.*

Prior to proving the lemma, we state our theorem and show how it follows from that lemma. Part (d) of the theorem is due to [BDR1] (and was previously proved, under certain decay conditions on Φ, in [JM]). For the special case of *quasi-regular* FSI spaces (a notion that will be defined in the next subsection), Theorem 2.3.6 in its entirety was already proved in [BDR1] (cf. Corollary 3.30 there. In a quasi-regular FSI space S, $\lambda^+ = \lambda$ on σS, and hence the [BDR1]-analysis, which is based only on the functions λ and Λ, can still go through).

Theorem 2.3.6. *Let $\Phi \subset L_2(\mathbb{R}^d)$ be finite with corresponding Gramian matrix G, and corresponding eigenvalue functions Λ, λ, and λ^+. Then*

(a) E_Φ is a Bessel set if and only if Λ is essentially bounded. Furthermore,

$$\|\mathcal{T}_\Phi\|^2 = \|\Lambda\|_{L_\infty(\sigma S(\Phi))}.$$

(b) A Bessel set E_Φ is also a frame if and only if $1/\lambda^+$ is bounded on the spectrum of $S(\Phi)$. In such a case,

$$\|{\mathcal{T}_\Phi}_|{}^{-1}\|^2 = \|1/\lambda^+\|_{L_\infty(\sigma S(\Phi))}.$$

(c) E_Φ is a tight frame if and only if $\Lambda = \lambda^+ = \text{const}$ on $\sigma S(\Phi)$.

(d) *The Bessel set E_Φ is a stable basis for $S(\Phi)$ if and only if $1/\lambda$ is essentially bounded.*

Proof: Let $U := (u_{\phi,\phi'})_{\phi,\phi'\in\Phi}$ be the unitary matrix from Lemma 2.3.5 (with respect to $G := G_\Phi$). Define

$$\Psi := \{\psi_\phi : \ \widehat{\psi_\phi} := (U^T\widehat{\Phi})_\phi := \sum_{\phi'\in\Phi} u_{\phi',\phi}\widehat{\phi'}, \quad \phi\in\Phi\}.$$

Since $U(w)$ is unitary for every $w\in\mathbb{T}^d$, it follows that U, considered as an endomorphism of L_2^Φ, is also unitary. From that it easily follows that $\Psi\subset L_2(\mathbb{R}^d)$ (in fact, $\sum_{\psi\in\Psi}\|\widehat{\psi}\|^2 = \sum_{\phi\in\Phi}\|\widehat{\phi}\|^2$). Thus, $\Psi\subset S(\Phi)$ by Result 2.3.1. Similarly, since $\widehat{\Phi} = \overline{U}\widehat{\Psi}$, $\Phi\subset S(\Psi)$, and, consequently, $S(\Psi) = S(\Phi)$. Further, $G_\Psi = U^* G_\Phi U$, hence G_Φ and G_Ψ have the same eigenvalue functions.

To prove (a), we let J_Φ and J_Ψ be defined as in (1.4.3). Then $J_\Psi = J_\Phi U$. Since U is unitary, J_Ψ is bounded if and only if J_Φ is, and the two maps have the same norm. Therefore, E_Ψ is a Bessel set of $S(\Psi) = S(\Phi)$ if and only if E_Φ is so. Consequently, (a) follows from Proposition 2.3.3 and the fact that, for each w, $\{\widetilde{\psi}(w)\}_{\psi\in\Psi}$ are the eigenvalues of the diagonal matrix $G_\Psi(w)$.

The proofs of (b), (c) and (d) are similar. □

Now, we turn to proof of the Lemma.

Proof of Lemma 2.3.5. Since, for each $w\in\Omega$, the Hermitian matrix $G(w)$ can certainly be unitarily diagonalized, the actual goal of the proof is to achieve the required measurability.

Let $\Lambda_j(w)$, $w\in\Omega$, $j = 1,...,n := \#\Phi$ denote the jth smallest eigenvalue of $G(w)$. Our first goal is to show that Λ_j is a measurable function. For that we need the following claim.

Claim 2.3.7. *Let $\{a_m\}_{m=0}^{n-1}$ be a set of convergent sequences $a_m : \mathbb{N}\to\mathbb{R}$. Let $a_m(0)$ denote the limit of $(a_m(k))_{k=1}^\infty$. For each non-negative integer k, let q_k be the univariate polynomial*

$$q_k(t) := t^n + \sum_{m=0}^{n-1} a_m(k)t^m.$$

Assume that each q_k has only real roots, and let $\Lambda_{k,j}$ denotes the jth smallest root of q_k. Then $\Lambda_{k,j} \xrightarrow[k\to\infty]{} \Lambda_{0,j}$, for each $j = 1,...,n$.

Proof of Claim 2.3.7. For each $k\ge 0$, let Λ_k be the vector $(\Lambda_{k,j})_{j=1}^n$. It is clear that $(\Lambda_k)_{k\in\mathbb{N}}$ is bounded (in $\mathbb{R}^n$), hence it suffices to show that Λ_0 is the only limit point of $(\Lambda_k)_k$. In this regard, we note that a limit point l_j of the sequence $(\Lambda_{k,j})_k$, is a zero of q_0, since $\sum_{i=0}^n a_i t^i$ is a continuous function of $a_0,\ldots,a_n,t$.

To prove that the sequence $(\Lambda_k)_{k\in\mathbb{N}}$ has only one limit point, we let $l := (l_j)_{j=1}^n$ be a limit point of $(\Lambda_k)_k$. Then, it is clear that $(l_j)_{j=1}^n$ is non-decreasing, and, as observed above, all the n entries of l are roots of q_0. Since q_0 has only n roots, l will be proved to equal Λ_0 as soon as we show the following: "if θ occurs m times in l, then its multiplicity as a root of q_0 is at least m".

Assume, therefore, that, $l_{s+1} = l_{s+2} = ... = l_{s+m} = \theta$, for some s and m. Let $(k_i)_{i=1}^\infty$ be a set of increasing integers for which $(\Lambda_{k_i})_i$ converges to l. By Rolle's theorem, for each fixed $r = 0,...,m-1$, the rth order derivative $q_{k_i}^{(r)}$ of the polynomial q_{k_i} would have a zero z_{k_i} in the convex hull of $\{\Lambda_{k_i,s+j}\}_{1\le j\le m}$. Since, as $i\to\infty$, that convex hull shrinks to θ (since each $(\Lambda_{k_i,s+j})_i$ converges to $l_{s+j} = \theta$), z_{k_i} converges to θ. Thus, θ is a limit point of roots of $(q_{k_i}^{(r)})_i$, $r = 0,...,m-1$, hence θ is a root of q_0 of multiplicity $\ge m$, as claimed.

After establishing the claim, we can prove the measurability of the eigenfunctions Λ_j as follows. We approximate the matrix G by Hermitian matrices G_k whose entries are simple measurable functions that converge (say, pointwise) to the entries of G. Let $q_0(w,\cdot)$ be the characteristic polynomial of $G(w)$, and $q_k(w,\cdot)$ the characteristic polynomial of $G_k(w)$, $k=1,2,...$. Since the coefficients of $q_k(w,\cdot)$ are simple measurable functions, so is the jth smallest eigenvalue function $\Lambda_{k,j}(w)$ of $G_k(w)$. On the other hand, the coefficients of $q_k(w,\cdot)$ converge to the corresponding coefficients of $q_0(w,\cdot)$. Since $G(w)$ and $G_k(w)$, $k\in\mathbb{N}$ are Hermitian, their characteristic polynomials have only real roots. By the previous claim, this implies that, for every $j=1,...,n$, and for every w, the eigenvalue functions $(\Lambda_{k,j}(w))_k$ converge to $\Lambda_j(w)$. Thus, each Λ_j is the pointwise limit of measurable functions, hence is measurable.

Finally, we construct the columns of U inductively. Assume by induction that we already found $V=\{v_1,...,v_{j-1}\}$ vectors whose entries are measurable functions, such that $Gv_i=\Lambda_i v_i$, for each $i=1,...,j-1$, and such that $\{v_1(w),...,v_{j-1}(w)\}$ is an orthonormal set for every $w\in\Omega$.

For each w, let $k(w)$ be the largest integer that satisfies $\Lambda_j(w)=\Lambda_{j-k(w)}(w)$. For $k=0,...,n-1$, set $K_k:=\{w\in\Omega : k(w)=k\}$. Then $(K_k)_k$ forms a measurable partition of Ω. On each set K_k, we augment the matrix $\Lambda_j I-G$ by adding the row vectors $v_{j-k},...,v_{j-1}$ and obtain in this way a matrix R with measurable entries, that satisfies $\operatorname{rank} R(w)<n$, for every $w\in K_k$. Precisely, $\operatorname{rank} R(w)=n-m(w)+k$, where $m(w)>k$ is the multiplicity of $\Lambda_j(w)$. Applying the proof of Lemma 2.4 of [JS], we obtain a measurable vector v_j such that $Rv_j=0$ on K_k, and for every $w\in K_k$, $v_j(w)$ (considered as vector in $\mathbb{R}^n$) has norm 1. Since $R(w)v_j(w)=0$, $w\in K_k$, $v_j(w)$ is an eigenvector of $G(w)$, and is orthogonal to $\{v_{j-k}(w),...,v_{j-1}(w)\}$. It is also orthogonal to $v_i(w)$, $i<j-k$ as well, since $v_i(w)$ is an eigenvector that corresponds to the eigenvalue $\Lambda_i(w)$ which is different from the eigenvalue $\Lambda_j(w)$ of $v_j(w)$. Hence, v_j is (pointwise) orthogonal to each of its predecessors. This completes the inductive step, thereby the proof of the lemma. $\square$

Incidentally, the proof of Theorem 2.3.6 shows that every FSI space can be written as a finite orthogonal sum of PSI spaces. This fact was established before in [BDR1] (cf. Theorem 3.5 there). It leads to the following interesting corollary.

Corollary 2.3.8. *Given any FSI space S, there exists a finite subset $\Psi\subset S$ whose corresponding shift-invariant set E_Ψ is a tight frame for S.*

Proof: We write S as a finite orthogonal sum of PSI spaces $\{S(\eta)\}_{\eta\in\mathrm{H}}$. By (d) of Theorem 2.2.7, each $S(\eta)$ contains a function ψ_η whose shifts E_{ψ_η} form a tight frame for $S(\eta)$, say, with frame bound 1. The totality $\{\psi_\eta\}_{\eta\in\mathrm{H}}$ is the required Ψ. $\square$

In general, there are many ways to write S as an orthogonal sum, and, therefore, S contains many tight frames. Though the norms of the individual generators $\psi\in\Psi$ depend in general on the specific Ψ chosen, the sum $\sum_{\psi\in\Psi}\|\psi\|^2$ depends only on the space S, that is: it is the same for all tight frames E_Ψ whose frame bound is 1, and whose corresponding $S(\psi)$, $\psi\in\Psi$ form an orthogonal decomposition of S.

2.4. Frames in quasi-regular FSI spaces

We had proved in the last subsection that every FSI space contains a shift-invariant tight frame. However, not every FSI space contains a shift-invariant stable basis. A partial solution to that difficulty was offered in [BDR1] via the more general notion of *quasi-stable* generating sets. That notion was defined in (3.16) of [BDR1], and is closely related to the notion of frames. In fact, Definition 1.3.5 here allows us to rephrase Definition 3.16 of [BDR1] as follows:

Definition 2.4.1. *Let Φ be a finite generating set for the FSI space S. We say that (the shifts E_Φ of) Φ is (are)* **a quasi-stable generating set**, *if (i): E_Φ is a frame for S; (ii):*

$$C_{\mathcal{T}_\Phi} = \{c = (c_\phi)_{\phi\in\Phi} \in \ell_2(E_\Phi) :\ \operatorname{supp} \widehat{c_\phi} \subset \sigma S,\ \forall \phi \in \Phi\}.$$

Note that quasi-stability coincides with stability whenever $\sigma S = \mathbb{T}^d$, i.e., whenever S is regular (indeed, if S is regular and Φ is quasi-stable, then $C_{\mathcal{T}_\Phi} = \ell_2(E_\Phi)$, and hence $\ker \mathcal{T}_\Phi = \{0\}$). Even with this weakening of the stability notion, [BDR1] shows that not every FSI space has a quasi-stable basis (we have proved, in Corollary 2.3.8, that every FSI space has a shift-invariant frame, and even a tight one, therefore, the existence of a quasi-stable basis really relies on the structure of $C_{\mathcal{T}_\Phi}$). Spaces that do have quasi-stable bases are termed in [BDR1] as **quasi-regular**. We discuss here several properties of frames in quasi-regular FSI spaces, which may not be valid in more general FSI spaces. One of these is an explicit representation for the orthogonal projector onto S: [BDR1] obtains such formulas for quasi-regular spaces by a Cramer-rule-like expression (see (1.9) there). On the other hand, we know from Proposition 1.3.8 that the orthogonal projector can also be represented by using a frame for S and its dual frame, and this will lead us to an alternative representation of this projector.

Before we state our first result, we recall the definition of a **quasi-basis** from [BDR1]: The finite Φ is a quasi-basis for the FSI space S if $\det G_\Phi$ is non-zero a.e. on σS. We mention, [BDR1], that the existence of a quasi-basis for S is equivalent to the quasi-regularity of S, and that every quasi-stable basis is also a quasi-basis but not vice versa. The cardinality of the quasi-basis is the **length** len$S$ of S and is shown in [BDR1] to depend only on S.

Proposition 2.4.2. *Let Φ be a finite quasi-basis for the (quasi-regular) FSI space S. Assume that E_Φ is a Bessel set. Then,*

$$C_{\mathcal{T}_\Phi} = \{c = (c_\phi)_{\phi\in\Phi} \in \ell_2(E_\Phi) :\ \operatorname{supp} \widehat{c_\phi} \subset \sigma S\}. \tag{2.4.3}$$

Proof: Denoting the right hand side of equation (2.4.3) by C_Φ, we will show that (i): $C_{\mathcal{T}_\Phi} \subset C_\Phi$, and (ii): $\ker \mathcal{T}_\Phi \cap C_\Phi = \{0\}$. Since $C_{\mathcal{T}_\Phi}$ is the orthogonal complement of $\ker \mathcal{T}_\Phi$, (2.4.3) would then follow from (i) and (ii) combined.

The required (ii) was proved in [BDR1]: Corollary 3.11 there asserts that, since Φ is a quasi-basis, the map

$$\ell_2(E_\Phi) \ni c \mapsto \widehat{\mathcal{T}_\Phi c} = \sum_{\phi\in\Phi} \widehat{c_\phi}\widehat{\phi}$$

is 1-1 on C_Φ.

As for (i), given $f \in S$, $\operatorname{supp} \widehat{f}$ lies in the 2π-periodic extension $(\sigma S)^\circ$ of σS. Thus, if, for some $c = (c_\phi)_{\phi\in\Phi} \in \ell_2(E_\Phi)$, each $\operatorname{supp} \widehat{c_\phi}$ is disjoint of σS, we have

$$\widehat{\mathcal{T}_\Phi c} = \sum_{\phi\in\Phi} \widehat{c_\phi}\widehat{\phi} = 0.$$

This means that the space

$$K_\Phi := \{c \in \ell_2(E_\Phi) : \ \operatorname{supp} \widehat{c_\phi} \cap \sigma S \text{ is a null-set}, \ \forall \phi \in \Phi\}$$

lies in $\ker \mathcal{T}_\Phi$. Since C_Φ is clearly the orthogonal complement of K_Φ, we obtain (i) by applying orthogonal complements to the inclusion $K_\Phi \subset \ker \mathcal{T}_\Phi$. □

Theorem 2.4.4. *Let Φ be a finite generating set for the quasi-regular FSI space S. Then Φ is a quasi-stable generating set if and only if it is a quasi-basis and its corresponding shifts E_Φ form a frame for S.*

Proof: If E_Φ is quasi-stable, then, by definition, it is a frame, and it is also a quasi-basis by virtue of Proposition 3.18 of [BDR1].

Conversely, if Φ is a quasi-basis and E_Φ is a frame, then, for the quasi-stability of Φ, it remains to show that $C_{\mathcal{T}_\Phi}$ has the required structure. This follows from Proposition 2.4.2 and the assumption that Φ is a quasi-basis. □

We mention that, given a quasi-regular FSI space S, there exist shift-invariant frames E_Φ for S which are *not* quasi-stable (hence do not form a quasi-basis). For example, the length of a PSI space is 1, and hence any quasi-basis for it is formed by the shifts of single function ϕ. At the same time, frames for PSI spaces that consist of the shifts of several functions exist, and, in fact, were discussed in detail in §2.1.

The proof of the second implication in the above theorem could also be done through eigenvalue functions. The argument is as follows. Since E_Φ is a frame, Theorem 2.3.6 implies that the eigenvalue function $\Lambda(w)$ ($\lambda^+(w)$) is essentially bounded above (away from zero) on σS. However, since $G_\Phi(w)$ is invertible a.e. on σS (since Φ is a quasi-basis), it follows that $\lambda(w) = \lambda^+(w)$, a.e. on σS, where $\lambda(w)$ is the smallest eigenvalue function. Thus $\Lambda(w)$ is essentially bounded above and $\lambda(w)$ is bounded below on σS. By Corollary 3.30 of [BDR1], Φ is a quasi-stable generating set.

In the rest of the subsection, we consider frame-dual frame representations of the orthogonal projector onto a quasi-regular FSI space S. The idea is to use the fact that, given a general frame X for H and a dual frame $\mathrm{R}X$, the map $T_{\mathrm{R}X}T_X^*$ is always the identity on H. Before we develop that direction further, we point out a relevant result. If X is a stable basis, then the condition $T_{\mathrm{R}X}T_X^* = I_H$ is not only necessary but also *sufficient* for $\mathrm{R}X$ to be the dual of X. The result below shows that, in the shift-invariant setup, that sufficiency assertion extends to quasi-stable sets:

Corollary 2.4.5. *Let E_Φ be a quasi-stable basis for the FSI space S, and let R be some map from Φ into $S(\Phi)$. If $E_{\mathrm{R}\Phi}$ is a Bessel set, then $E_{\mathrm{R}\Phi}$ is the dual frame of E_Φ if (and only if) $\mathcal{T}_{\mathrm{R}\Phi}\mathcal{T}_\Phi^*$ is the identity on S, that is, if*

$$f = \sum_{\phi\in\Phi,\alpha\in\mathbb{Z}^d} \langle f, E^\alpha\phi\rangle\, E^\alpha \mathrm{R}\phi, \quad \forall f \in S. \tag{2.4.6}$$

Proof: After extending R from Φ to E_Φ by the rule $\mathrm{R}E^\alpha\phi := E^\alpha \mathrm{R}\phi$, we appeal to Proposition 1.3.7. That proposition validates the "only if" implication, and reduces the proof of the "if" implication to proving that $C_{\mathcal{T}_{\mathrm{R}\Phi}} = C_{\mathcal{T}_\Phi}$. Furthermore, Proposition 2.4.2 asserts that $C_{\mathcal{T}_\Psi}$ is the same for all *quasi-bases* Ψ of S.

Since Φ is already known to be a quasi-basis (by virtue of its quasi-stability, cf. Theorem 2.4.4), it suffices to show that $\mathrm{R}\Phi$ is also a quasi-basis. The proof of this statement goes as follows. Since $\mathrm{R}\Phi \subset S$, we have $S(\mathrm{R}\Phi) \subset S$. This, together with (2.4.6), shows that $E_{\mathrm{R}\Phi}$ is fundamental in S, and hence $S(\mathrm{R}\Phi) = S(\Phi)$. Since Φ is a quasi-basis for S, its cardinality is the length, $\mathrm{len}S$, of S. Therefore, $\#(\mathrm{R}\Phi) \le \#\Phi = \mathrm{len}S$. However, as asserted by Theorem 3.12 of [BDR1], every generating set of a quasi-regular FSI space S that contains no more than $\mathrm{len}S$ elements must be a quasi-basis. □

Theorem 2.4.7. *Assume that the shifts E_Φ of the finite Φ form a quasi-stable basis for the FSI space S. Then the Fourier transforms of the generators $\mathrm{R}\Phi$ of the dual quasi-stable basis are given, on σS, by*

$$\widehat{\mathrm{R}\Phi} = \overline{G_\Phi^{-1}}\widehat{\Phi}.$$

with G_Φ^{-1} the (pointwise) inverse of G_Φ.

Proof: Since R should invert $\mathcal{T}_\Phi\mathcal{T}_\Phi^*$, we compute first $\mathcal{T}_\Phi\mathcal{T}_\Phi^*\Phi$. Here, we use (2.1.2) (and the fact that $\mathcal{T}_\Phi\mathcal{T}_\Phi^* = \sum_{\phi\in\Phi}\mathcal{T}_\phi\mathcal{T}_\phi^*$) to conclude that

$$(\mathcal{T}_\Phi\mathcal{T}_\Phi^*\Phi)\widehat{\ } = (\sum_{\phi'\in\Phi}[\widehat{\phi},\widehat{\phi'}]\widehat{\phi'})_{\phi\in\Phi} = \overline{G_\Phi}\widehat{\Phi}.$$

Since G_Φ is invertible on σS (and is zero elsewhere), the claim follows. □

By Proposition 1.3.8, $\mathcal{T}_{\mathrm{R}\Phi}\mathcal{T}_\Phi^*$ is the orthogonal projector $\mathcal{P}_S$ of $L_2(\mathbb{R}^d)$ on S. The last result thus allows us to write

$$\widehat{\mathcal{P}_S f} = \sum_{\phi,\phi'\in\Phi}[\widehat{f},\widehat{\phi}]g_{\phi,\phi'}\widehat{\phi'},$$

with $(g_{\phi,\phi'})_{\phi,\phi'\in\Phi} = \overline{G_\Phi^{-1}}$. Instead, we could have solved the equation $G_\Phi\widehat{\mathrm{R}\Phi} = \widehat{\Phi}$ by applying Cramer's rule. That attempt would have resulted in the form for $\mathcal{P}_S$ that was discussed in [BDR1].

3. Infinitely generated SI spaces

3.1. General

The study of FSI subspaces of $L_2(\mathbb{R}^d)$ is pertinent to Approximation Theory, where one attempts to approximate from small, simple spaces of approximants. In other areas (such as wavelets) the main goal is to find an attractive basis for the *entire* $L_2(\mathbb{R}^d)$ or to a "big" subspace of it. We therefore analyse in this section shift-invariant subspaces of $L_2(\mathbb{R}^d)$ generated by a countable set of generators.

Our results on FSI spaces were stated in terms of the matrix spectrum of each of the "fiber" matrices $G(w)$, $w \in \mathbb{T}^d$. We pause here momentarily in order to have a closer look at the potential practical value of the obtained characterizations. Assuming we hold in hand the Gramian matrix, the characterization of *stability* and of the *Bessel property* are of a more favorable nature than those of *frames* and *tight frames*: in many cases, the estimation of the largest eigenvalue $\Lambda(w)$ and the smallest eigenvalue $\lambda(w)$ of $G(w)$ can be done directly in terms of the entries of $G(w)$ (as we did in §1.6). However, estimating the smallest non-zero eigenvalue $\lambda^+(w)$, would, almost certainly, require the application of a costly iterative process. Consequently, the kind of characterization of FSI frames that was obtained in Theorem 2.3.6 seems to be practically less useful than its stability counterpart. This can also be viewed as follows: the invertibility of a certain operator is a more accessible property than its partial invertibility.

A partial solution to the above problem is obtained with the addition of the complementary *dual Gramian analysis* that will be developed. Indeed, as was already explained in the introduction, the Gramian analysis is engaged with the decomposition of the operator $\mathcal{T}_\Phi^*\mathcal{T}_\Phi$, while in the dual case the operator $\mathcal{T}_\Phi\mathcal{T}_\Phi^*$ is the object. In two respects, there is a significant difference between these two operators: the stability of a Bessel set E_Φ is equivalent to the invertibility of $\mathcal{T}_\Phi^*\mathcal{T}_\Phi$, but is not so nicely reflected by $\mathcal{T}_\Phi\mathcal{T}_\Phi^*$ (this latter operator should be partially invertible and onto $\ell_2(E_\Phi)$, two hard-to-verify properties). On the other hand, a fundamental frame for $L_2(\mathbb{R}^d)$ is characterized nicely through $\mathcal{T}_\Phi\mathcal{T}_\Phi^*$ (should be invertible), and is hard to be analysed via $\mathcal{T}_\Phi^*\mathcal{T}_\Phi$. *In summary, Gramian analysis is best suited for the study of stable bases, while dual Gramian analysis is particularly good for fundamental frames for* $L_2(\mathbb{R}^d)$, hence, indeed, the two approaches complement each other.

In view of the above, one may wonder why we have not employed the dual Gramian analysis for the study of frames in FSI spaces. The answer for that is as follows: since an FSI space is always a *proper* subspace of $L_2(\mathbb{R}^d)$, a frame for it is never fundamental in $L_2(\mathbb{R}^d)$. For the analysis of frames which are not fundamental, both Gramian analysis and dual Gramian analysis require the (hard-to-verify) partial invertibility of their associated operator, hence the switch from the finite-order Gramian G to infinite-order dual Gramian $\widetilde{G}$ provides no gain.

Throughout the section, we use the notation ΣA for the spectrum of the operator A; namely, given a bounded linear endomorphism A of a Hilbert space H, we denote

$$\Sigma A := \{\lambda \in \mathbb{C} : \text{ the inverse of } \lambda I - A \text{ is undefined or unbounded}\}.$$

To make a clear distinction between this notion and the spectrum $\sigma S(\Phi)$ of $S(\Phi)$, we will always refer to the former as the **the operator spectrum**.

3.2. Gramian Analysis: SI spaces as the limit of FSI spaces

Two different approaches for the study of SI spaces are employed here. The first, that we discuss in the present subsection, attempts to extend the results from §2 on FSI spaces to general SI spaces, by viewing the latter as a certain limit of the former. That approach leads to the desired characterizations of the Bessel property and of the stability property, but is short of characterizing frames. Therefore, we will develop, (in §3.4) an alternative method, where we inspect directly the operator spectrum of each of the "fibers" $G(w)$. This latter direction is more powerful, alas, much more involved, whence our decision to present both approaches.

The "going-to-the-limit" argument is almost self-suggestive, and is based on an elementary observation. Let X be a countable subset of the Hilbert space H. Given any subset $Y \subset X$, let H_Y be the closure in H of the finite span of Y (that is, Y is fundamental in H_Y). As before, the operator T_Y is defined on $\ell_0(Y)$, and, if bounded, is extended to the entire $\ell_2(Y)$ by continuity. Further, $\ell_2(Y)$ is isometrically embedded in $\ell_2(X)$ in the usual way.

For a set $X \subset H$, a chain

$$.... \subset X_{n-1} \subset X_n \subset X_{n+1} \subset ....$$

that satisfies $\cup_n X_n = X$ is called a **filtration** of X.

Theorem 3.2.1. *Let X be a countable fundamental set of the Hilbert space H. Suppose that $\{X_n\}_n$ is a filtration of X, i.e., $X_n \subset X_{n+1}$ for all $n \in \mathbb{N}$, and $\cup_n X_n = X$. Denote $T := T_X$, $T_n := T_{X_n}$, $H_n := H_{X_n}$. Then:*

(a) X is a Bessel set if and only if the following condition holds "each X_n is a Bessel set, and $\sup_n \|T_n\| < \infty$". In such a case, $\|T\| = \sup_n \|T_n\| = \lim_{n\to\infty} \|T_n\|$.

(b) Assume X is a Bessel set. Then, X is a stable basis for H if and only if the following condition holds "each X_n is a stable basis for H_n, and $\sup_n \|{T_n}^{-1}\| < \infty$". In such a case, $\|T^{-1}\| = \sup_n \|{T_n}^{-1}\| = \lim_{n\to\infty} \|{T_n}^{-1}\|$.

(c) Assume X is a Bessel set. Then, X is a frame for H if the following condition holds "for infinitely many n, X_n is a frame for H_n, and $\liminf_n \|{T_{n|}}^{-1}\| < \infty$". In such a case, $\|{T_|}^{-1}\| \le \liminf_n \|{T_{n|}}^{-1}\|$.

Proof: The boundedness and invertibility of T (T_n) is determined by its action on the finitely supported sequences $\ell_0(X)$ ($\ell_0(X_n)$) in $\ell_2(X)$ ($\ell_2(X_n)$). Assertions (a) and (b) thus follow from the fact that, since $\{X_n\}_n$ is a filter of X, $\ell_0(X)$ is the union of $(\ell_0(X_n))_n$.

(c): Without loss, we may assume that each X_n is a frame for H_n, and that $(\|{T^*_{n|}}^{-1}\| = \|{T_{n|}}^{-1}\|)_n$ converges (otherwise, we take a subsequence). Set $A := \lim \|{T^*_{n|}}^{-1}\|^{-1}$. Since T is bounded, $A < \infty$. More importantly, by our assumptions here $A > 0$. Now, let $f \in H$. Given $\varepsilon > 0$, we can find, for all sufficiently large k, an element $f_k \in H_k$ so that $\|f - f_k\| \le \frac{\varepsilon}{\max\{\|T\|, \|{T^*_{k|}}^{-1}\|^{-1}\}}$. Then, $\|T^* f\| \ge \|T^* f_k\| - \varepsilon \ge \|T^*_k f_k\| - \varepsilon$. Also, $\|T^*_k f_k\| \ge \|{T^*_{k|}}^{-1}\|^{-1}\|f_k\| \ge \|{T^*_{k|}}^{-1}\|^{-1}\|f\| - \varepsilon$. In summary, for every $f \in H$ and for all sufficiently large k,

$$\|T^* f\| \ge \|{T^*_{k|}}^{-1}\|^{-1}\|f\| - 2\varepsilon.$$

By taking $k \to \infty$, we obtain that $\|T^* f\| \ge A\|f\| - 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the desired result follows. □

Let S be a shift-invariant space generated by the countable set Φ. Let $(\Phi_n)_n$ be a filtration of Φ by finite sets. Then, $(E_n := E_{\Phi_n})_n$ is a filtration of E_Φ that employs FSI sets. Let Λ_n, λ_n and λ_n^+ be the eigenvalue functions of E_n (cf. the paragraph after Proposition 2.3.3). Combining Theorem 2.3.6 and Theorem 3.2.1, we obtain the following result.

Corollary 3.2.2. *With $\Phi \subset L_2(\mathbb{R}^d)$ a countable set, with $(\Phi_n)_n$ a filtration of Φ that is made of finite sets, and with Λ_n, λ_n and λ_n^+ as above, we have*

(a) E_Φ is a Bessel set if and only if the function set $\{\Lambda_n\}_n$ is bounded in $L_\infty(\mathbb{T}^d)$. Furthermore, $\|\mathcal{T}_\Phi\|^2 = \sup_n \|\Lambda_n\|_{L_\infty(\mathbb{T}^d)}$.

(b) Assume E_Φ is a Bessel set. Then it is also a stable basis for S if and only if the function set $\{1/\lambda_n\}_n$ is bounded in $L_\infty(\mathbb{T}^d)$. Furthermore, $\|T^{-1}\| = \sup_n \|1/\lambda_n\|_{L_\infty(\mathbb{T}^d)}$. (Here, $1/0 := \infty$.)

(c) Assume E_Φ is a Bessel set. Then it is also a frame if the following holds: "for each n, the function $1/\lambda_n^+$ is bounded on the spectrum σ_n of the FSI space $S(\Phi_n)$, and $\liminf_n \|1/\lambda_n^+\|_{L_\infty(\sigma_n)} < \infty$."

The analysis of E_Φ for a finite Φ was done by a spectral-like decomposition of $\mathcal{T}_\Phi$ into the simpler fiber operators. For a countable Φ, we can still derive from (a) and (b) of the last corollary similar decomposition results.

We recall the functions $\Lambda(w)$, $\lambda(w)$ and $\lambda^+(w)$ that were defined in the introduction. Note that for a finite Φ these definitions coincide with the definitions of $\Lambda(w)$, $\lambda(w)$ and $\lambda^+(w)$ as eigenvalue functions. Given now a filtration $(\Phi_n)_n$ of Φ, Corollary 3.2.2 implies that $\|\mathcal{T}_\Phi\|^2 = \lim_{n\to\infty} \|\Lambda_n\|_{L_\infty(\mathbb{T}^d)}$. Moreover, it is straightforward to show that, monotonically, $\Lambda_n(w) \to \Lambda(w)$, and $\lambda_n(w) \to \lambda(w)$ a.e. on $\mathbb{T}^d$. This implies that Λ and λ are measurable, and, further, since the convergence $\Lambda_n \to \Lambda$ and $\lambda_n \to \lambda$ is monotone,

$$\|\Lambda\|_{L_\infty(\mathbb{T}^d)} = \lim_{n\to\infty} \|\Lambda_n\|_{L_\infty(\mathbb{T}^d)},$$

and

$$\|1/\lambda\|_{L_\infty(\mathbb{T}^d)} = \lim_{n\to\infty} \|1/\lambda_n\|_{L_\infty(\mathbb{T}^d)}.$$

Thus we obtain the following extension of (a) and (b) of Theorem 2.3.6:

Theorem 3.2.3. *Let Φ be a countable subset of $L_2(\mathbb{R}^d)$ with Gramian matrix G. Let $\Lambda(w) := \|G(w)\|$ and $\lambda(w) := \|G(w)^{-1}\|^{-1}$. Then:*

(a) E_Φ is a Bessel set if and only if Λ is essentially bounded. Moreover, we have $\|\mathcal{T}_\Phi\|^2 = \|\Lambda\|_{L_\infty(\mathbb{T}^d)}$.

(b) Suppose E_Φ is a Bessel set. Then E_Φ is a stable basis if and only if $1/\lambda$ is essentially bounded. Moreover, we have $\|\mathcal{T}_\Phi{}^{-1}\|^2 = \|1/\lambda\|_{L_\infty(\mathbb{T}^d)}$.

Theorem 3.2.3 provides characterizations of the Bessel property and the stability property that, though were derived with the aid of the FSI results, are stated explicitly in terms of the fiber operators $G(w)$, $w \in \mathbb{T}^d$. Such a characterization is valid for frames, but, cannot be derived with the aid of the filtration argument. Therefore, we develop in §3.4 a direct approach that decompose G without the use of a filter. Since the proofs there are lengthy and technical, we postpone that development until after the dual Gramian analysis is presented.

3.3. Dual Gramian analysis

The starting point of the Gramian analysis is the fact that both G and its fibers $(G(w))_w$ can be viewed as densely defined operator on L_2^{Φ} and $\ell_2(\Phi)$, respectively. An analogous statement about the dual Gramian is less obvious, and we need surmount here new obstacles.

The first (though, minor) difficulty that one should note is the well-definedness of the entries of the dual Gramian: while the Gramian entries $[\widehat{\phi}, \widehat{\psi}]$, $\phi, \psi \in \Phi$ are in $L_1(\mathbb{T}^d)$ hence well-defined a.e. regardless of the choice of the set Φ, the same cannot be said about the entries

$$\sum_{\phi\in\Phi} \widehat{\phi}(\cdot+\alpha)\overline{\widehat{\phi}(\cdot+\beta)}, \quad \alpha, \beta \in 2\pi\mathbb{Z}^d$$

of the dual Gramian $\widetilde{G}$. We start our discussion by settling that question.

Assume that E_Φ is a Bessel set. Then, since $\sum_{\phi\in\Phi} \|\mathcal{T}_\phi^* f\|^2 = \|\mathcal{T}_\Phi^* f\|^2 < \infty$, and since the Fourier transform is an isometry on $L_2(\mathbb{R}^d)$, we conclude from (2.1.2) that

$$\sum_{\phi\in\Phi} \|[\widehat{f}, \widehat{\phi}]\|^2_{L_2(\mathbb{T}^d)} < \infty, \quad \forall f \in L_2(\mathbb{R}^d).$$

Choosing now f as the inverse Fourier transform of the characteristic function of the cube $\alpha + [-\pi..\pi]^d$, $\alpha \in 2\pi\mathbb{Z}^d$, we compute that $[\widehat{f}, \widehat{\phi}] = \overline{\widehat{\phi}(\cdot - \alpha)}_{|C}$, and therefore,

$$\|[\widehat{f}, \widehat{\phi}]\|^2_{L_2(\mathbb{T}^d)} = \|\widehat{\phi}^2\|_{L_1(\alpha+C)}.$$

Thus, we have proved that the sum

$$\sum_{\phi\in\Phi} |\widehat{\phi}(\cdot+\alpha)|^2$$

is $L_1(\mathbb{T}^d)$-convergent, hence is also convergent pointwise a.e. Since that sum is the (α, α)-entry of the dual Gramian, we conclude the following:

Proposition 3.3.1. *Let Φ be a countable subset of $L_2(\mathbb{R}^d)$, and assume that E_Φ is a Bessel set. Then, for each $\alpha, \beta \in 2\pi\mathbb{Z}^d$, the (α, β)-entry*

$$\sum_{\phi\in\Phi} \widehat{\phi}(\cdot+\alpha)\overline{\widehat{\phi}(\cdot+\beta)}$$

of the dual Gramian matrix converges absolutely a.e. to an element of $L_1(\mathbb{T}^d)$.

Proof: For $\alpha = \beta$, the assertion was proved in the paragraph preceding the proposition. The extension to a general pair (α, β) follows from Schwartz' inequality. □

Since the Bessel property of the set E_Φ is the weakest property of that set of interest to us here, we may assume hereafter that, for all $\alpha, \beta \in 2\pi\mathbb{Z}^d$, the sum that defines the (α,β)-entry

$$\widetilde{G}_{\alpha,\beta}$$

of the dual Gramian converges absolutely a.e.

Another, more substantial, difficulty occurs upon attempting to prove that dual Gramian operator can be evaluated, i.e., that, under "reasonable assumptions"

$$(\widetilde{G}f)(w) = \widetilde{G}(w) f_{|w}, \quad \text{for a.e. } w \in \mathbb{T}^d.$$

Here, as before $f_{|w} := f_{|w+2\pi\mathbb{Z}^d}$. Recall that the dual Gramian operator $\widetilde{G}$ is defined as $\widetilde{G} := J_\Phi J_\Phi^*$, i.e.,

$$\widetilde{G} : f \mapsto \sum_{\phi\in\Phi} [f, \widehat{\phi}]\widehat{\phi}.$$

If E_Φ is a Bessel set, the above sum must converge in $L_2(\mathbb{R}^d)$, for every $f \in L_2(\mathbb{R}^d)$. However, interpreting the above sum in the non-Bessel case is a non-obvious task. On the other hand, the connection between $\widetilde{G}$ and its evaluation $\widetilde{G}(w)$ is important even when E_Φ is not Bessel, since, otherwise, we will not be able to use the fibers $\{\widetilde{G}(w)\}_{w\in\mathbb{T}^d}$ for the characterization of the Bessel property. For this reason, we view, to this end, $\widetilde{G}$ as a *quadratic form* rather than as an operator, i.e., make use of the connection

$$\langle \widetilde{G}\widehat{f}, \widehat{f}\rangle = \sum_{\phi\in\Phi} \|(T_\phi^* f)\widehat{\ }\|^2 = \|\sum_{\phi\in\Phi} |[\widehat{f}, \widehat{\phi}]|^2\|_{L_1(\mathbb{T}^d)}.$$

Assuming $\widehat{f}$ is compactly supported, we may use the a.e. finiteness of $\sum_{\phi\in\Phi} |\widehat{\phi}(\cdot+\alpha)\widehat{\phi}(\cdot+\beta)|$ to sum by parts as follows:

$$\begin{aligned}\sum_{\phi\in\Phi} |[\widehat{f}, \widehat{\phi}](w)|^2 &= \sum_{\phi\in\Phi} \sum_{\alpha,\beta\in 2\pi\mathbb{Z}^d} \widehat{f}(w+\alpha)\overline{\widehat{f}(w+\beta)}\widehat{\phi}(w+\beta)\overline{\widehat{\phi}(w+\alpha)} \\ &= \sum_{\alpha,\beta} \widehat{f}(w+\alpha)\overline{\widehat{f}(w+\beta)} \sum_{\phi\in\Phi} \widehat{\phi}(w+\beta)\overline{\widehat{\phi}(w+\alpha)} \\ &= (\widehat{f}_{|w})^* \widetilde{G}(w) \widehat{f}_{|w}.\end{aligned}$$

Therefore, we conclude that

Lemma 3.3.2. *Let Φ be a countable subset of $L_2(\mathbb{R}^d)$.*
(a) If, for some $\alpha, \beta \in 2\pi\mathbb{Z}^d$, the sum $\sum_{\phi\in\Phi} |\widehat{\phi}(\cdot+\alpha)\widehat{\phi}(\cdot+\beta)|$ is infinite on a set of positive measure, then E_Φ is not a Bessel set.
(b) If the above sum is finite a.e. for every $\alpha, \beta \in 2\pi\mathbb{Z}^d$, then, for every band-limited f,

$$\|T_\Phi^* f\|^2 = (2\pi)^{-d} \int_{\mathbb{T}^d} (\widehat{f}_{|w})^* \widetilde{G}(w) \widehat{f}_{|w}\, dw.$$

The dual Gramian analysis can now be developed along lines parallel to the development of the Gramian analysis. For that, we set, for $\alpha \in 2\pi\mathbb{Z}^d$, S_α to be the subspace of $L_2(\mathbb{R}^d)$ consisting of those functions whose Fourier transform is supported (up to a null-set) in $\alpha + [-\pi..\pi]^d$. S_α is a translation-invariant space. In fact, it is also a PSI space, and is generated by ${\chi_\alpha}^\vee$, with χ_α the support function of $\alpha + [-\pi..\pi]^d$ (cf. Result 2.2.9). We consider the restriction $\mathcal{T}^*_{\Phi,\alpha}$, $\alpha \in \mathbb{Z}^d$ of $\mathcal{T}^*_\Phi$ to the space S_α, and observe that, for $w \in \mathbb{T}^d$ and $f \in S_\alpha$, the quadratic form $\widehat{f}_{|w}{}^* \widetilde{G}(w) \widehat{f}_{|w}$, $w \in \mathbb{T}^d$, is reduced to $\overline{\widehat{f}(w+\alpha)} \widetilde{G}_{\alpha,\alpha}(w) \widehat{f}(w+\alpha) = \widetilde{G}_{\alpha,\alpha}(w) |\widehat{f}(w+\alpha)|^2$, and therefore

$$\|\mathcal{T}^*_\Phi f\|^2 = (2\pi)^{-d} \|\widetilde{G}_{\alpha,\alpha} |\widehat{f}(\cdot + \alpha)|^2\|_{L_1(\mathbb{T}^d)}.$$

Since also $\|f\|_{L_2(\mathbb{R}^d)} = (2\pi)^{-d} \||\widehat{f}(\cdot+\alpha)|^2\|_{L_1(\mathbb{T}^d)}$ (since $f \in S_\alpha$), the norm bounds on the restricted operator $\mathcal{T}^*_{\Phi,\alpha}$ and its inverse are the same as those of the map

$$L_1(\mathbb{T}^d) \ni \tau \mapsto \widetilde{G}_{\alpha,\alpha} \tau.$$

Thus, in complete analogy with Theorem 2.2.7 (cf. the argument used in the proof of Theorem 2.2.14) we have the following.

Proposition 3.3.3. *Let $\Phi \subset L_2(\mathbb{R}^d)$ be countable (or finite), and assume that the sum $\sum_{\phi\in\Phi} |\widehat{\phi}|^2$ converges a.e. Then, for every $\alpha \in 2\pi\mathbb{Z}^d$:*

(a) The restricted operator $\mathcal{T}^*_{\Phi,\alpha}$ is bounded if and only if the function $\widetilde{G}_{\alpha,\alpha}$ is essentially bounded. Furthermore,

$$\|\mathcal{T}^*_{\Phi,\alpha}\|^2 = \|\widetilde{G}_{\alpha,\alpha}\|_{L_\infty(\mathbb{T}^d)}.$$

(b) Assume $\mathcal{T}^*_{\Phi,\alpha}$ is bounded. Then it is also invertible if and only if the function $1/\widetilde{G}_{\alpha,\alpha}$ is essentially bounded. Further,

$$\|{\mathcal{T}^*_{\Phi,\alpha}}^{-1}\|^2 = \|1/\widetilde{G}_{\alpha,\alpha}\|_{L_\infty(\mathbb{T}^d)}.$$

(c) Assume $\mathcal{T}^*_{\Phi,\alpha}$ is bounded. Then it is also partially invertible if and only if $1/\widetilde{G}_{\alpha,\alpha}$ is essentially bounded on its support $\widetilde{\sigma}_\alpha \subset \mathbb{T}^d$. Further,

$$\|{\mathcal{T}^*_{\Phi,\alpha|}}^{-1}\|^2 = \|1/\widetilde{G}_{\alpha,\alpha}\|_{L_\infty(\widetilde{\sigma}_\alpha)}.$$

The dual Gramian analogue of the FSI results (i.e., Theorem 2.3.6) is obtained by restricting $\mathcal{T}^*_\Phi$ to a larger space of band-limited functions. Here, we take $\mathcal{Z}$ to be any finite subset of $2\pi\mathbb{Z}^d$, and define $\Omega_\mathcal{Z} := \mathcal{Z} + [-\pi..\pi]^d$. We then consider the restriction $\mathcal{T}^*_{\Phi,\mathcal{Z}}$ of $\mathcal{T}^*_\Phi$ to the space

$$S_\mathcal{Z} := \{f \in L_2(\mathbb{R}^d) : \ \mathrm{supp}\, \widehat{f} \subset \Omega_\mathcal{Z}\}.$$

Given g defined on $\Omega_\mathcal{Z}$, and $w \in \mathbb{T}^d$, we denote by

$$g_\mathcal{Z}(w)$$

the vector $(g(w+z) : \ z \in \mathcal{Z})$. Also,

$$\widetilde{G}_\mathcal{Z}$$

stands for the finite-order matrix obtained from the dual Gramian $\widetilde{G}_\Phi$ by deleting all rows and columns not in $\mathcal{Z}$. From Lemma 3.3.2,

$$\|\mathcal{T}^*_\Phi f\|^2 = (2\pi)^{-d} \|\widehat{f}^*_\mathcal{Z} \widetilde{G}_\mathcal{Z} \widehat{f}_\mathcal{Z}\|_{L_1(\mathbb{T}^d)}, \quad \forall f \in S_\mathcal{Z}.$$

Then, following the arguments in §2.3 (that is, establishing the analogous result of Proposition 2.3.3 and invoking then Lemma 2.3.5), we obtain the following analogue of Theorem 2.3.6:

Proposition 3.3.4. *Let $\Phi \subset L_2(\mathbb{R}^d)$ be countable and assume that $\sum_{\phi\in\Phi} |\widehat{\phi}|^2$ is finite a.e. Let $\mathcal{Z}$ be a finite subset of $2\pi\mathbb{Z}^d$, and let $\mathcal{T}^*_{\Phi,\mathcal{Z}}$ be the restriction of $\mathcal{T}^*_\Phi$ to $S_\mathcal{Z}$. Let $\widetilde{\Lambda}_\mathcal{Z}$, $\widetilde{\lambda}_\mathcal{Z}$ and $\widetilde{\lambda}^+_\mathcal{Z}$ be the eigenvalue functions defined as Λ, λ and λ^+ of §2.3, but with respect to the dual Gramian $\widetilde{G}_\mathcal{Z}$. Then:*

(a) $\mathcal{T}^*_{\Phi,\mathcal{Z}}$ is bounded if and only if $\widetilde{\Lambda}_\mathcal{Z}$ is essentially bounded on $\mathbb{T}^d$. Furthermore, $\|\mathcal{T}^*_{\Phi,\mathcal{Z}}\|^2 = \|\widetilde{\Lambda}_\mathcal{Z}\|_{L_\infty(\mathbb{T}^d)}$.

(b) Assume $\mathcal{T}^*_{\Phi,\mathcal{Z}}$ is bounded. Then it is also invertible if and only if $1/\widetilde{\lambda}_\mathcal{Z}$ is essentially bounded on $\mathbb{T}^d$. Furthermore, $\|{\mathcal{T}^*_{\Phi,\mathcal{Z}}}^{-1}\|^2 = \|1/\widetilde{\lambda}_\mathcal{Z}\|_{L_\infty(\mathbb{T}^d)}$.

(c) Assume $\mathcal{T}^*_{\Phi,\mathcal{Z}}$ is bounded. Then it is also partially invertible if and only if $\widetilde{\lambda}^+_\mathcal{Z}$ is essentially bounded on $\widetilde{\sigma}_\mathcal{Z} := \{w \in \mathbb{T}^d : \ \widetilde{G}_\mathcal{Z}(w) \neq 0\}$. Furthermore, $\|{\mathcal{T}^*_{\Phi,\mathcal{Z}|}}^{-1}\|^2 = \|1/\widetilde{\lambda}^+_\mathcal{Z}\|_{L_\infty(\widetilde{\sigma}_\mathcal{Z})}$.

To extend Proposition 3.3.4 from spaces of the form $S_\mathcal{Z}$ to the entire $L_2(\mathbb{R}^d)$, we use some filtration

$$\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \mathcal{Z}_2 \subset ...$$

of $2\pi\mathbb{Z}^d$. It induces a corresponding filtration of $\mathbb{R}^d$:

$$\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset ...,$$

where $\Omega_j := \mathcal{Z}_j + [-\pi..\pi]^d$. In this way we obtain the increasing space sequence

$$S_{\mathcal{Z}_0} \subset S_{\mathcal{Z}_1} \subset S_{\mathcal{Z}_2} \subset ...$$

whose union S is dense in $L_2(\mathbb{R}^d)$. Denoting by $\mathcal{T}^*_n$ the restriction of $\mathcal{T}^*_\Phi$ to $S_{\mathcal{Z}_n}$, we conclude that the boundedness and invertibility of $\mathcal{T}^*_\Phi$ are completely determined by its restriction to S (which is the space of all band-limited functions). Therefore, we have the following analog of Theorem 3.2.3:

Theorem 3.3.5. *Let Φ be a countable subset of $L_2(\mathbb{R}^d)$. Then:*

(a) If the sum $\sum_{\phi\in\Phi} |\widehat{\phi}|^2$ diverges on some positive measure set, E_Φ is not a Bessel set.

(b) Assume that $\sum_{\phi\in\Phi} |\widehat{\phi}|^2$ is finite a.e., and let $\widetilde{G}$ be the dual Gramian of E_Φ. Further, let $\widetilde{\Lambda}$ and $\widetilde{\lambda}$ be defined by

$$\widetilde{\Lambda}(w) := \|\widetilde{G}(w)\|, \quad \widetilde{\lambda}(w) := 1/\|\widetilde{G}(w)^{-1}\|, \quad w \in \mathbb{T}^d.$$

Then:

(b1) E_Φ is Bessel set if and only if $\widetilde{\Lambda}$ is essentially bounded. Furthermore,

$$\|\mathcal{T}_\Phi\|^2 = \|\widetilde{\Lambda}\|_{L_\infty(\mathbb{T}^d)}.$$

(b2) Assume that E_Φ is a Bessel set. Then E_Φ is a fundamental frame if and only if the following condition holds: "for a.e. w, $\widetilde{G}(w)$ is boundedly invertible, and the hence-well-defined function $1/\widetilde{\lambda}$ is essentially bounded". Furthermore, $\|{\mathcal{T}^*_\Phi}^{-1}\|^2 = \|1/\widetilde{\lambda}\|_{L_\infty(\mathbb{T}^d)}$.

Theorem 3.3.5 leads to an interesting conclusion concerning tight frames. Tight frames E_Φ are characterized by the equality $\|\mathcal{T}_\Phi\|\|{\mathcal{T}_{\Phi|}}^{-1}\| = 1$. The theorem shows that the latter condition is equivalent to the equality

$$\widetilde{\Lambda}(w) = \widetilde{\lambda}(w) = \text{const}, \quad \text{for a.e. } w \in \mathbb{T}^d.$$

The equality $\widetilde{\Lambda}(w) = \widetilde{\lambda}(w)$ says that the operator spectrum of $\widetilde{G}(w)$ consists of a single point, which can happen if and only if $\widetilde{G}(w)$ is a scalar operator. This leads to the following:

Corollary 3.3.6. *Let Φ be a countable subset of $L_2(\mathbb{R}^d)$. Then E_Φ is a fundamental tight frame for $L_2(\mathbb{R}^d)$ if and only if there exists a constant* const *such that, for every $\alpha, \alpha' \in 2\pi\mathbb{Z}^d$, and for almost every $w \in \mathbb{T}^d$,*

$$\sum_{\phi\in\Phi} \widehat{\phi}(w+\alpha)\overline{\widehat{\phi}(w+\alpha')} = \mathrm{const}\,\delta_{\alpha,\alpha'}. \tag{3.3.7}$$

Proof: If the sum in (3.3.7) does not converge absolutely for some α, α' and on a set of positive measure. then, by Theorem 3.3.5, E_Φ is not a Bessel set. Otherwise, the condition in (3.3.7) implies, Theorem 3.3.5, that E_Φ is a Bessel set. Also, that condition implies that E_Φ is fundamental: if not, there exists $f \in L_2(\mathbb{R}^d)$ so that $\mathcal{T}_\Phi^* f = 0$, hence $\widetilde{G}\widehat{f} = 0$, implying thus that $\widetilde{G}(w)\widehat{f}_{|w} = 0$, a.e., in contradiction to the assumed structure of $\widetilde{G}(w)$ in (3.3.7).

Therefore, when proving the required equivalence, we may assume, without loss, that E_Φ is a fundamental Bessel set. The claim then follows from the arguments preceding the present corollary. □

If X is a tight frame, then, up to a scalar multiple, it forms its own dual. The above result is thus a special case of a general relation between a shift-invariant fundamental frame and its dual (cf. Corollary 4.2).

3.4. Analysis of frames which are not fundamental in $L_2(\mathbb{R}^d)$

Theorems 3.2.3 and 3.3.5 provide us with the desired characterizations of the Bessel property (twice), the stability property, and the property of being a fundamental frame for $L_2(\mathbb{R}^d)$. It fails to provide similar characterizations for frames of a shift-invariant proper subspace of $L_2(\mathbb{R}^d)$ (unless that frame happens to be a stable basis). The present subsection is aimed at settling this remaining problem. After a brief introduction, we state the main theorem that will be proved here. The proof details then follow.

Let B be a bounded operator from a Hilbert space H into a Hilbert space H', and let $A := \mathrm{B}^*\mathrm{B}$. Let ΣA be the operator spectrum of A. We define

$$\lambda^+(A) := \inf\{\mu : \ \mu \in \Sigma A\backslash 0\}.$$

The operator A is partially invertible if and only if $\lambda^+(A) > 0$, and the norm of the partial inverse is $1/\lambda^+(A)$ (the "only if" implication is quite clear. The argument for the "if" statement can be found in the proof of the implication (b) $\Longrightarrow$ (a) of Theorem 3.4.1).

Given a Bessel set E_Φ with Gramian G and dual Gramian $\widetilde{G}$, our two objectives are to connect (a): between the function

$$\lambda^+(w) := \lambda^+(G(w)), \quad w \in \mathbb{T}^d,$$

and the number $\lambda^+(G)$; (b): between the function

$$\widetilde{\lambda}^+(w) := \lambda^+(\widetilde{G}(w)), \quad w \in \mathbb{T}^d,$$

and the number $\lambda^+(\widetilde{G})$. Since $\lambda^+(G) = \lambda^+(\widetilde{G}) = \|\mathcal{T}_{\Phi|}{}^{-1}\|^{-1}$ (with $\infty^{-1} := 0$), we will obtain in this way two characterizations of frames. In fact, we will prove the following:

Theorem 3.4.1. *Let Φ be a countable subset of $L_2(\mathbb{R}^d)$, and assume that E_Φ is a Bessel set. Let $\sigma\Phi := \operatorname{supp} G = \operatorname{supp} \widetilde{G} \subset \mathbb{T}^d$. Then the following conditions are equivalent:*
(a) E_Φ is a frame, and the norm of the partial inverse of $\mathcal{T}_\Phi$ is $K < \infty$.
(b) The function λ^+ is bounded away from zero on $\sigma\Phi$, and $\|1/\lambda^+\|_{L_\infty(\sigma\Phi)} = K^2$.
(c) The function $\widetilde{\lambda}^+$ is bounded away from zero on $\sigma\Phi$, and $\|1/\widetilde{\lambda}^+\|_{L_\infty(\sigma\Phi)} = K^2$.

The equivalence of (b) and (c) is quite straightforward. (Since E_Φ is Bessel, then, by Theorems 3.2.3 and 3.3.5, both $G(w)$ and $\widetilde{G}(w)$ are bounded for a.e. w. Since $G(w)$ is the product $J_\Phi^*(w)J_\Phi(w)$, and $\widetilde{G}(w)$ is the product of the same matrices in reversed order, $\Sigma(G(w))$ and $\Sigma(\widetilde{G}(w))$ can differ only by the single point $\{0\}$. Thus, λ^+ and $\widetilde{\lambda}^+$ are equal pointwise.) We will prove here the equivalence of (a) and (b). The proof of the implication (b)$\Longrightarrow$(a) is based on the following lemma.

Lemma 3.4.2. *Let E_Φ be a Bessel set, and let $\tau \in L_2^\Phi$, $G := G_\Phi$. Then,*
(a) $\tau \in \ker G$ if and only if $\tau(w) \in \ker G(w)$ for almost every w.
(b) $\tau \in C_G := (\ker G)^\perp$ if and only if $\tau(w) \in C_w := (\ker G(w))^\perp$, for a.e. w.

Proof: The first assertion is obvious, since $(G\tau)(w)$ is $G(w)\tau(w)$. As for (b), assume first that $\tau(w) \in C_w$ for a.e. w. Then, for an arbitrary $\tau' \in \ker G$,

$$\langle \tau, \tau' \rangle_{L_2^\Phi} = \int_{\mathbb{T}^d} \langle \tau(w), \tau'(w) \rangle_{\ell_2(\Phi)} dw = 0,$$

since, by (a), $\tau'(w) \in \ker G(w) = C_w{}^\perp$, a.e. Therefore, $\tau \in (\ker G)^\perp = C_G$.

Conversely, assume that $\tau \in C_G$. If $\tau \in \operatorname{ran} G$, then $\tau = G\tau_0$, for some τ_0, hence, for a.e. w (precisely, whenever $G(w)$ is bounded, and $\tau_0(w) \in \ell_2(\Phi)$), $\tau(w) = G(w)\tau_0(w) \in \operatorname{ran} G(w) \subset C_w$. If $\tau \notin \operatorname{ran} G$, it can still be approximated in L_2^Φ by a sequence $(\tau_n)_n \subset \operatorname{ran} G$ (since $\operatorname{ran} G$ is dense in C_G). By switching to a subsequence, if necessary, we may assume that, for almost every w, $(\tau_n(w))_n$ converges in $\ell_2(\Phi)$ to $\tau(w)$. Combining this with the argument in the beginning of the paragraph, we conclude that, for almost every w, $(\tau_n(w))_n$ is in C_w and converges in the $\ell_2(\Phi)$-norm to $\tau(w)$. Since C_w is certainly closed, we obtained that $\tau(w) \in C_w$, a.e. □

Proof of the implication (b)$\Longrightarrow$(a) in Theorem 3.4.1.
We will prove that, assuming (b), E_Φ is a frame, and $\|\mathcal{T}_{\Phi|}{}^{-1}\| \le \|1/\lambda^+\|_{L_\infty(\sigma\Phi)}$.

Assume that $1/\lambda^+$ is essentially bounded on $\sigma\Phi$, and let $\tau \in C_G \backslash 0$. By Lemma 3.4.2, $\tau(w) \in C_w$, a.e. on $\mathbb{T}^d$. We claim that, a.e.w, if $G(w) \neq 0$, it is partially invertible, i.e., bounded below on C_w. Indeed, the restriction $G(w)_|$ of $G(w)$ to C_w is (always) injective. Furthermore, since $\lambda^+(w) > 0$, the operator spectrum of $G(w)$ is disjoint from the non-empty interval $(0, \lambda^+(w))$. Therefore, the operator spectrum of $G(w)_|$ is also disjoint from $(0, \lambda^+(w))$. Since $G(w)_|$ is non-negative and injective, 0 cannot be an isolated point of its spectrum, hence it must be invertible. The argument also shows that $\|G(w)_|{}^{-1}\| = 1/\lambda^+(w)$.

This means that, for a.e. w, if $\tau(w) \neq 0$, then

$$\|G(w)\tau(w)\|_{\ell_2(\Phi)} \ge \frac{\|\tau(w)\|_{\ell_2(\Phi)}}{\|G(w)_|{}^{-1}\|} \ge \frac{\|\tau(w)\|_{\ell_2(\Phi)}}{\|1/\lambda^+\|_{L_\infty(\sigma\Phi)}}. \tag{3.4.3}$$

For $\tau \in L_2^\Phi$,

$$\|\tau\|_{L_2^\Phi}^2 = \int_{\mathbb{T}^d} \|\tau(w)\|_{\ell_2(\Phi)}^2,$$

hence also

$$\|G\tau\|_{L_2^\Phi}^2 = \int_{\mathbb{T}^d} \|G(w)\tau(w)\|_{\ell_2(\Phi)}^2,$$

hence (3.4.3) implies that

$$\|G\tau\|_{L_2^\Phi} \geq \frac{\|\tau\|_{L_2^\Phi}}{\|1/\lambda^+\|_{L_\infty(\sigma\Phi)}}.$$

Therefore, G is partially invertible, and hence, Proposition 1.4.11, E_Φ is a frame. Also, $\|\mathcal{T}_{\Phi|}{}^{-1}\|^2 = \|G_|{}^{-1}\| \leq \|1/\lambda^+\|_{L_\infty(\sigma\Phi)}$, with the inequality by the proof above, and the equality by Proposition 1.4.11. □

Proof of the implication (a)⟹(b) in Theorem 3.4.1.
Since we will need, in the next section, a closely related result, we will prove herein the following more general statement:

Theorem 3.4.4. *Let G be a non-negative self-adjoint bounded endomorphism of L_2^Φ. Let $(G(w))_w$ be a collection of non-negative self-adjoint bounded endomorphisms of $\ell_2(\Phi)$, that satisfy, for every $\tau \in L_2^\Phi$, and for a.e. $w \in \mathbb{T}^d$, $(G\tau)(w) = G(w)\tau(w)$. Let $\Lambda(w) := \|G(w)\|$, and assume that $\Lambda \in L_\infty(\mathbb{T}^d)$. Let $\lambda^+(w) := \inf\{\mu \in \Sigma(G(w))\backslash 0\}$. Let Ω be the set $\Omega := \{w \in \mathbb{T}^d : \ G(w) \neq 0\}$. If G is partially invertible, then $1/\lambda^+$ is essentially bounded on Ω, and*

$$\|1/\lambda^+\|_{L_\infty(\Omega)} \leq \|G_|{}^{-1}\|.$$

The fact that Theorem 3.4.4 is a generalization of the required implication (a) $\Longrightarrow$ (b) is clear. To this end, we prove Theorem 3.4.4.

In the proof, we use the following lemma, whose proof is postponed until after the proof of Theorem 3.4.4 is done.

Lemma 3.4.5. *Under the conditions of Theorem 3.4.4, there exists a countable dense subset D of $\ell_2(\Phi)$, and a null-set $Z \subset \Omega$, such that, for every $c \in D$, for every $w' \in \Omega\backslash Z$, and every $\varepsilon > 0$, the set*

$$K_{c,w',\varepsilon} := \{w \in \Omega : \ \|(G(w) - G(w'))c\|_{\ell_2(\Phi)} < \varepsilon\|c\|_{\ell_2(\Phi)}\}$$

has a positive measure.

Proof of Theorem 3.4.4. Let D and Z be the sets specified in the above lemma. Recall also the notations $C_G := (\ker G)^\perp$, $C_w := (\ker G(w))^\perp$.

Choose any $w' \in \Omega\backslash Z$, and let $\mu > 0$ be any point in the operator spectrum $\Sigma(G(w'))$. We will construct an element $\tau \in C_G$, for which

$$\|G\tau\|_{L_2^\Phi} \leq (1+\delta)\mu\|\tau\|_{L_2^\Phi}, \tag{3.4.6}$$

with δ positive and arbitrarily close to 0. This would yield that $\|G_{|}{}^{-1}\| \geq 1/\mu$, implying thus that $\lambda^+(w') > 0$, and that

$$\|G_{|}{}^{-1}\| \geq 1/\lambda^+(w').$$

Since Z is a null-set, we will then conclude that

$$\|G_{|}{}^{-1}\| \geq \|1/\lambda^+\|_{L_\infty(\Omega)},$$

which is the desired result.

The actual construction of τ in (3.4.6) is as follows: we will find $0 \neq \tau \in L_2^\Phi$, supported in $A \times \Phi$, where $A \subset \Omega$ is some set of positive measure, such that (i): $\tau(w) \in C_w$, for every $w \in \mathbb{T}^d$, and (ii): $\|G(w)\tau(w)\|_{\ell_2(\Phi)} \leq (1+\delta)\mu\|\tau(w)\|_{\ell_2(\Phi)}$. Condition (i) would imply (as in Lemma 3.4.2) that $\tau \in C_G$, while condition (ii) is needed for the conclusion that $\|G\tau\|_{L_2^\Phi} \leq (1+\delta)\mu\|\tau\|_{L_2^\Phi}$ (cf. the two displays after (3.4.3)).

In general, for the sake of (i) above, it might be hard to know whether a particular sequence lies in C_w. The most efficient way is, probably, to select elements in $\operatorname{ran} G(w)$ (and use the fact that $\operatorname{ran} G(w)$ is dense in C_w, by virtue of the self-adjointness of $G(w)$). Indeed, our element τ will be defined as

$$\tau(w) := \begin{cases} G(w)c, & w \in A, \\ 0, & \text{otherwise}, \end{cases}$$

with c some fixed sequence in $\ell_2(\Phi)$.

Here are the details: since $\mu \in \Sigma(G(w'))$, $G(w') - \mu I$ has no bounded inverse, and so we can find an element $c \in \ell_2(\Phi)$, such that $\|c\|_{\ell_2(\Phi)} = 1/\mu$, and

$$\|G(w')c - \mu c\|_{\ell_2(\Phi)} \leq \varepsilon, \tag{3.4.7}$$

with $\varepsilon > 0$ arbitrarily small. It follows then that

$$\|G(w')c\|_{\ell_2(\Phi)} \leq 1 + \varepsilon. \tag{3.4.8}$$

Since $G(w')$ is bounded and D is dense in $\ell_2(\Phi)$, we may assume that $\operatorname{span} c \cap D \neq \emptyset$. Therefore, by Lemma 3.4.5, there exists a subset A of Ω with positive measure, such that

$$\|(G(w) - G(w'))c\|_{\ell_2(\Phi)} \leq \varepsilon/\mu, \quad \forall w \in A.$$

We define $\tau \in L_2^\Phi$ by

$$\tau(w) := \begin{cases} G(w)c, & w \in A, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, condition (i) (i.e., that $\tau(w) \in C_w$, all w) is satisfied. Also, the uniform boundedness of the operators $\{G(w)\}_{w \in \mathbb{T}^d}$ easily implies that $\tau \in L_2^\Phi$. Thus, to complete the proof, it remains to show that, for almost all $w \in \mathbb{T}^d$,

$$\|G(w)\tau(w)\|_{\ell_2(\Phi)} \leq (1+\delta)\mu\|\tau(w)\|_{\ell_2(\Phi)}.$$

This last claim is trivial for $w \in \mathbb{T}^d \backslash A$, so we may assume that $w \in A$. We first choose $w = w'$. For that specific choice, we get

$$\|G(w')\tau(w')\|_{\ell_2(\Phi)} = \|G(w')G(w')c\|_{\ell_2(\Phi)} \le \|\mu G(w')c\|_{\ell_2(\Phi)} + \|G(w')(G(w')c - \mu c)\|_{\ell_2(\Phi)}. \tag{3.4.9}$$

Denoting

$$C := \|\Lambda\|_{L_\infty(\mathbb{T}^d)} < \infty,$$

we obtain from (3.4.9), (3.4.8), and (3.4.7) that

$$\|G(w')\tau(w')\|_{\ell_2(\Phi)} \le \mu(1+\varepsilon) + C\varepsilon.$$

On the other hand, by (3.4.7),

$$1 - \|\tau(w')\|_{\ell_2(\Phi)} = \mu\|c\|_{\ell_2(\Phi)} - \|G(w')c\|_{\ell_2(\Phi)} \le \varepsilon. \tag{3.4.10}$$

Altogether, we obtained for that case the inequality

$$\|G(w')\tau(w')\|_{\ell_2(\Phi)} \le \frac{\mu(1+\varepsilon) + C\varepsilon}{1-\varepsilon}\|\tau(w')\|_{\ell_2(\Phi)}.$$

By choosing ε sufficiently small (and adjusting A if necessary to that ε), we obtain that

$$\|G(w')\tau(w')\|_{\ell_2(\Phi)} \le (1+\delta)\mu\|\tau(w')\|_{\ell_2(\Phi)}. \tag{3.4.11}$$

To extend that to a general $w \in A$, we show that both $\tau(w) - \tau(w')$, and $G(w')\tau(w') - G(w)\tau(w)$ can be made arbitrarily small (in norm), and then invoke (3.4.11). First, since $\operatorname{span} c \cap D \ne \emptyset$,

$$\|\tau(w) - \tau(w')\|_{\ell_2(\Phi)} = \|(G(w) - G(w'))c\|_{\ell_2(\Phi)} \le \varepsilon/\mu. \tag{3.4.12}$$

Therefore, $\|\tau(w)\|_{\ell_2(\Phi)} \ge \|\tau(w')\|_{\ell_2(\Phi)} - \varepsilon/\mu \ge 1 - \varepsilon - \varepsilon/\mu$, the second inequality by (3.4.10). This verifies that $\tau(w) - \tau(w')$ is, indeed, small, and also means that, on A, $\tau(w)$ is being kept away from zero, a consequence that will be required shortly. Second, to estimate $G(w)\tau(w) - G(w')\tau(w')$, we write

$$G(w)^2 - G(w')^2 = G(w)(G(w) - G(w')) + (G(w) - G(w'))G(w'). \tag{3.4.13}$$

Now, since $\|(G(w) - G(w'))c\|_{\ell_2(\Phi)} \le \varepsilon/\mu$, we have that

$$\|G(w)(G(w) - G(w'))c\|_{\ell_2(\Phi)} \le C\varepsilon/\mu.$$

Also, due to (3.4.7) and the fact that $\|(G(w) - G(w'))c\|_{\ell_2(\Phi)} \le \varepsilon/\mu$,

$$\|(G(w)-G(w'))G(w')c\|_{\ell_2(\Phi)} \le \|\mu(G(w)-G(w'))c\|_{\ell_2(\Phi)} + \|(G(w)-G(w'))(\mu c - G(w')c)\|_{\ell_2(\Phi)} \le \varepsilon + 2C\varepsilon.$$

So, we conclude from (3.4.13) that

$$\|G(w)\tau(w) - G(w')\tau(w')\|_{\ell_2(\Phi)} = \|G(w)^2c - G(w')^2c\|_{\ell_2(\Phi)} \le (C/\mu + 2C + 1)\varepsilon.$$

Therefore, by (3.4.11) and (3.4.12),

$$\begin{aligned}\|G(w)\tau(w)\|_{\ell_2(\Phi)} \leq& \|G(w')\tau(w')\|_{\ell_2(\Phi)} + (C/\mu + 2C + 1)\varepsilon \\ \leq& (1+\delta)\mu\|\tau(w')\|_{\ell_2(\Phi)} + (C/\mu + 2C + 1)\varepsilon \\ \leq& (1+\delta)\mu(\|\tau(w)\|_{\ell_2(\Phi)} + \varepsilon/\mu) + (C/\mu + 2C + 1)\varepsilon \\ =& (1+\delta)\mu\|\tau(w)\|_{\ell_2(\Phi)} + (C/\mu + 2C + 2 + \delta)\varepsilon.\end{aligned}$$

Since we have already proved that $\|\tau(w)\|_{\ell_2(\Phi)}$ is kept away from zero, we can modify ε (hence A) to guarantee that, say,

$$\|G(w)\tau(w)\|_{\ell_2(\Phi)} \leq (1+2\delta)\mu\|\tau(w)\|_{\ell_2(\Phi)},$$

and the desired result then follows. □

Finally we prove Lemma 3.4.5. For that we first recall the definition of *measurable maps*:

Definition 3.4.14. *Let* M *be a measure space, and* B *a topological space. A map* $f : \mathrm{M} \to \mathrm{B}$ *is* **measurable** *provided that* $f^{-1}(\Omega)$ *is a measurable set in* M *for every open set* Ω *in* B.

Clearly, if $f : \mathrm{M} \to \mathrm{B}$ is measurable, then $f^{-1}(U)$ is measurable for every Borel set $U \subset \mathrm{B}$.

Proposition 3.4.15. *Let* M *be a positive measure space and* B *be a separable normed space. If the map* $f : \mathrm{M} \to \mathrm{B}$ *is measurable, then, there exists a null-set* $Z \subset \mathrm{M}$, *such that, for every* $w' \in \mathrm{M}\backslash Z$ *and for arbitrary* $\varepsilon > 0$, *there is a positive-measure set* $A := A_{w',\varepsilon} \subset \mathrm{M}$, *such that for arbitrary* $w \in A$,

$$\|f(w) - f(w')\|_{\mathrm{B}} < \varepsilon.$$

Proof: All norms in the proof below are B-norms.

Let X be a countable dense subset of B, and let " $<$ " be some well-ordering of X. Given $n \in \mathbb{N}$, let

$$O_{x,n} := \{u \in \mathrm{B} : \|u - x\| < 1/n\}.$$

Then $(O_{x,n})_{x\in X}$ is an open covering of B, and, defining

$$U_{x,n} := O_{x,n}\backslash(\cup_{y<x}O_{y,n}), \quad x \in X,$$

we obtain a partition of B into Borel sets. That partition induces a partition

$$(A_{x.n} := f^{-1}(U_{x,n}))_{x\in X}$$

of M into measurable sets. We then define a map $s_n : \mathrm{M} \to \mathrm{B}$ (as a matter of fact, $\operatorname{ran} s_n \subset X$) as follows:

$$s_n(w) = \sum_{x\in X} x\,\chi_{A_{x,n}}(w).$$

Then, s_n converges to f uniformly. Indeed, we have that $\|f(w) - s_n(w)\| < 1/n$ for all $n \in \mathbb{N}$, $w \in \mathrm{M}$.

Let Z be a null-set that contains all those $A_{x,n}$ ($x \in X$, $n \in \mathbb{N}$) whose measure is zero. Let $w' \in \mathrm{M}\backslash Z$. For arbitrary ε, pick n with $2/n < \varepsilon$. Since $w' \notin Z$, w' is in some positive-measure $A_{x,n}$. For $w \in A_{x,n}$,

$$\|f(w') - f(w)\| \leq \|f(w') - s_n(w')\| + \|s_n(w') - s_n(w)\| + \|s_n(w) - f(w)\| < \varepsilon.$$

□

Proof of Lemma 3.4.5. Let D be a dense countable subset of $\ell_2(\Phi)$. Given $c \in D$, let B_c be the space of all (bounded) linear operators from span$\{c\}$ into $\ell_2(\Phi)$.

Since we know that $G(w)$, $w \in \Omega$, is a bounded linear endomorphism of $\ell_2(\Phi)$, then, certainly, $G(w)_{|\,\mathrm{span}\{c\}}$ is bounded for every $w \in \Omega$. This defines a.e. the map

$$f : \Omega \to B_c : w \mapsto G(w)_{|\,\mathrm{span}\{c\}}.$$

We need to prove that this map is measurable.

Given $L \in B_c$ and $w \in \Omega$, one observes that

$$\|G(w) - L\|_{B_c} = \frac{\|G(w)c - Lc\|_{\ell_2(\Phi)}}{\|c\|_{\ell_2(\Phi)}}.$$

Further, since G is bounded, $Gc \in L_2^\Phi$, and in particular, its entries are measurable functions (for the sake of applying G to c, c should be interpreted as the element $\tau \in L_2^\Phi$ with constant entries $\tau_\phi = c_\phi$). Also, since $\|G(w)c - Lc\|_{\ell_2(\Phi)}$ is finite, the series that defines $\|G(w)c - Lc\|_{\ell_2(\Phi)}$ converges (unconditionally). Combining that with the previous observation, viz., that the entries of $Gc - Lc$ are measurable, we conclude that the map $w \mapsto \|G(w)c - Lc\|_{\ell_2(\Phi)}$ is measurable, hence so is our f.

An application of Proposition 3.4.15 with respect to the map f, yields the existence of a null-set $Z_c \subset \Omega$, such that for every $\varepsilon > 0$ and every $w' \in \Omega \backslash Z_c$, the set

$$\{w : \ \|G(w) - G(w')\|_{B_c}\} < \varepsilon$$

has a positive measure. Defining $Z := \cup_{c \in D} Z_c$, we obtain that (a): Z is a null-set, (b) the claim of the lemma holds for this Z. □

4. Dual frames

Let Φ be a countable (or finite) subset of $L_2(\mathbb{R}^d)$, and assume that E_Φ is a frame. Let $\mathrm{R} : \Phi \to L_2(\mathbb{R}^d)$ be some map, and assume that $E_{\mathrm{R}\Phi}$ is a Bessel set. Let J_Φ and $J_{\mathrm{R}\Phi}$ be the pre-Gramian of Φ and $\mathrm{R}\Phi$ respectively. Our objective in this brief section is to study the property "$E_{\mathrm{R}\Phi}$ is the dual frame of E_Φ" via the fiber matrices $J_\Phi(w)$ and $J_{\mathrm{R}\Phi}(w)$.

Our initial tool is Corollary 1.3.9. Part (b) of that Corollary says that, if $E_{\mathrm{R}\Phi}$ is the dual of E_Φ, then $\mathcal{T}_\Phi \mathcal{T}_{\mathrm{R}\Phi}^*$ is an orthogonal projector. On the Fourier domain, this operator is represented by $J_\Phi J_{\mathrm{R}\Phi}^*$ whose matrix representation is

$$J_\Phi J_{\mathrm{R}\Phi}^* = \Big(\sum_{\phi \in \Phi} \widehat{\phi}(\cdot + \alpha)\overline{\widehat{\mathrm{R}\phi}(\cdot + \alpha')}\Big)_{\alpha, \alpha' \in 2\pi \mathbb{Z}^d}.$$

The sum above that defines the entries of $J_\Phi J_{\mathrm{R}\Phi}^*$ can be shown to converge absolutely for every $\alpha, \alpha' \in 2\pi\mathbb{Z}^d$, and for almost every $w \in \mathbb{T}^d$ (Schwartz' inequality followed by an application of Proposition 3.3.1). Corollary 1.3.9 also implies that the operator $\mathcal{T}_\Phi^* \mathcal{T}_{\mathrm{R}\Phi}$ is an orthogonal projector. Here, the Fourier transform analogue is $J_\Phi^* J_{\mathrm{R}\Phi}$, whose matrix representation is

$$J_\Phi^* J_{\mathrm{R}\Phi} = ([\widehat{\mathrm{R}\phi'}, \widehat{\phi}])_{\phi, \phi' \in \Phi}.$$

The entries of this latter matrix are certainly well-defined (a.e.).

Lemma 4.1. *With Φ and $\mathrm{R}\Phi$ as above,*

(a) *$\mathcal{T}_\Phi \mathcal{T}^*_{\mathrm{R}\Phi}$ is an orthogonal projector if and only if, for almost every $w \in \mathbb{T}^d$, $J_\Phi(w) J^*_{\mathrm{R}\Phi}(w)$ is an orthogonal projector (on $\ell_2(2\pi\mathbb{Z}^d)$).*

(b) *$\mathcal{T}^*_\Phi \mathcal{T}_{\mathrm{R}\Phi}$ is an orthogonal projector if and only if, for almost every $w \in \mathbb{T}^d$, $J^*_\Phi(w) J_{\mathrm{R}\Phi}(w)$ is an orthogonal projector (on $\ell_2(\Phi) = \ell_2(\mathrm{R}\Phi)$).*

Proof: The arguments for proving (a) and (b) are essentially the same, hence we prove only (b).

Since the Fourier transformation is an isometry, we may replace in the proof the operator $\mathcal{T}^*_\Phi \mathcal{T}_{\mathrm{R}\Phi}$ by its Fourier transform analogue $J^*_\Phi J_{\mathrm{R}\Phi}$. Also, for the sake of notational simplicity, we set $G := J^*_\Phi J_{\mathrm{R}\Phi}$, though, of course, this G is the Gramian of neither Φ nor $\mathrm{R}\Phi$.

First, one checks that G is non-negative self-adjoint if and only if almost every $G(w)$ is so.

Assume that $G(w)$ is an orthogonal projector for a.e.w. In particular, each $G(w)$ is self-adjoint, hence, by the above, G is self-adjoint, too. To show that G is an orthogonal projector, we need to prove that $G\tau = \tau$ for every $\tau \in (\ker G)^\perp$. Let, therefore, $\tau \in (\ker G)^\perp$. By a proof identical to that of Lemma 3.4.2, for a.e. $w \in \mathbb{T}^d$, $\tau(w) \in (\ker G(w))^\perp$. Since $G(w)$ is assumed to be an orthogonal projector (a.e.), we conclude that $G(w)\tau(w) = \tau(w)$ (a.e.), implying that $G\tau = \tau$. This proves that G is an orthogonal projector, as needed.

Now assume that G is an orthogonal projector. We want to invoke here Theorem 3.4.4, hence need to verify its assumptions. The basic relation $(G\tau)(w) = G(w)\tau(w)$ is straightforward. The fact that each $G(w)$ is non-negative self-adjoint follows from the fact that G is assumed to be so. Finally, analogously to the derivation of (a) in Theorem 3.2.3, one proves the relation $\|G\| = \|\Lambda\|_{L_\infty(\mathbb{T}^d)}$, with $\Lambda(w) := \|G(w)\|$. Since $\|G\| = 1$ here, we conclude that, for a.e. $w \in \mathbb{T}^d$, $\Sigma(G(w)) \subset [0, 1]$.

Now, we invoke Theorem 3.4.4. Since G is partially invertible (being an orthogonal projector), and $\|G_|{}^{-1}\| = 1$, that theorem tells us that $\lambda^+(w) \geq 1$, for almost all w that satisfy $G(w) \neq 0$. This implies that, a.e., $\Sigma(G(w)) \subset \{0\} \cup [1, \infty)$. Combining that with the result of the previous paragraph, we conclude that, a.e., $\Sigma(G(w)) \subset \{0, 1\}$. Each such $G(w)$ is also known to be self-adjoint, hence must be an orthogonal projector. □

In case E_Φ is fundamental in $L_2(\mathbb{R}^d)$, $J_\Phi J^*_{\mathrm{R}\Phi}$ is the identity operator, and this immediately implies that almost every operator $J_\Phi(w) J^*_{\mathrm{R}\Phi}(w)$ is the identity. Thus, we get the following:

Corollary 4.2. *Let E_Φ be a frame and let $E_{\mathrm{R}\Phi}$ be its dual. Then:*

(a) *For every $\alpha, \alpha' \in 2\pi\mathbb{Z}^d$, and for almost every $w \in \mathbb{T}^d$,*

$$\sum_{\phi \in \Phi} \widehat{\phi}(w+\alpha) \overline{\widehat{\mathrm{R}\phi}(w+\alpha')} = \sum_{\phi \in \Phi} \widehat{\mathrm{R}\phi}(w+\alpha) \overline{\widehat{\phi}(w+\alpha')}.$$

(b) *If E_Φ is fundamental in $L_2(\mathbb{R}^d)$, then, for every $\alpha, \alpha' \in 2\pi\mathbb{Z}^d$ and for almost every $w \in \mathbb{T}^d$,*

$$\sum_{\phi \in \Phi} \widehat{\phi}(w+\alpha) \overline{\widehat{\mathrm{R}\phi}(w+\alpha')} = \delta_{\alpha,\alpha'}.$$

Proof: The first claim follows from the self-adjointness of the $J_\Phi(w) J^*_{\mathrm{R}\Phi}(w)$-matrices. The second claim follows from Lemma 4.1 and also directly from the remarks preceding the present corollary. □

Corollary 1.3.9 provides us also with a sufficient condition for $E_{\mathrm{R}\Phi}$ to be the dual frame of the frame E_Φ. In the shift-invariant case, that corollary, combined with Lemma 4.1, leads to the following conclusion:

Corollary 4.3. *Let H be a closed subspace of $L_2(\mathbb{R}^d)$, and let E_Φ be a frame for H. Let $E_{\mathrm{R}\Phi}$ be a Bessel set which is fundamental in H. Then $E_{\mathrm{R}\Phi}$ is the dual of E_Φ if and only if for almost every $w \in \mathbb{T}^d$ each of the operators $J^*_{\mathrm{R}\Phi}(w)J_\Phi(w)$, $J^*_\Phi(w)J_{\mathrm{R}\Phi}(w)$, $J_{\mathrm{R}\Phi}(w)J^*_\Phi(w)$, and $J_\Phi(w)J^*_{\mathrm{R}\Phi}(w)$ is an orthogonal projector.*

We have stated Corollary 4.3 primarily for proving our following final result. That result, though might look very special, will play a crucial role in the development of *the duality principle of Weyl-Heisenberg frames* in [RS1].

Corollary 4.4. *Let E_Φ be a frame for $H \subset L_2(\mathbb{R}^d)$, with a dual $E_{\mathrm{R}\Phi}$. Let E_Ψ be a frame for $H' \subset L_2(\mathbb{R}^d)$, and let $\mathrm{R}' : \Psi \to L_2(\mathbb{R}^d)$. Assume that, for almost every $w \in \mathbb{T}^d$,*

$$J_\Phi(w) = J^*_\Psi(w), \quad J_{\mathrm{R}\Phi}(w) = J^*_{\mathrm{R}'\Psi}(w). \tag{4.5}$$

(That is, for some indexing $\Phi = (\phi_\alpha)_{\alpha\in 2\pi\mathbb{Z}^d}$, and $\Psi = (\psi_\alpha)_{\alpha\in 2\pi\mathbb{Z}^d}$, $\widehat{\phi_\alpha}(w+\beta) = \overline{\widehat{\psi_\beta}(w+\alpha)}$, etc.) Then $E_{\mathrm{R}'\Psi}$ is the dual frame of E_Ψ.

Proof: Since $E_{\mathrm{R}\Phi}$ is a frame, the equality $J_{\mathrm{R}\Phi}(w) = J^*_{\mathrm{R}'\Psi}(w)$ easily implies (by Theorems 3.2.3, 3.3.5, and 3.4.1) that $E_{\mathrm{R}'\Psi}$ is a frame, as well.

Since $E_{\mathrm{R}\Phi}$ is the dual frame of E_Φ, then, by Corollary 4.3, for almost every $w \in \mathbb{T}^d$ each of the operators $J^*_{\mathrm{R}\Phi}(w)J_\Phi(w)$, $J^*_\Phi(w)J_{\mathrm{R}\Phi}(w)$, $J_{\mathrm{R}\Phi}(w)J^*_\Phi(w)$, and $J_\Phi(w)J^*_{\mathrm{R}\Phi}(w)$ is an orthogonal projector. By virtue of (4.5), we get that for almost every $w \in \mathbb{T}^d$ each of the operators $J^*_{\mathrm{R}'\Psi}(w)J_\Psi(w)$, $J^*_\Psi(w)J_{\mathrm{R}'\Psi}(w)$, $J_{\mathrm{R}'\Psi}(w)J^*_\Psi(w)$, and $J_\Psi(w)J^*_{\mathrm{R}'\Psi}(w)$ is an orthogonal projector. Therefore, Corollary 4.3 would imply that $E_{\mathrm{R}'\Psi}$ is a frame dual to E_Ψ as soon as we show that $E_{\mathrm{R}'\Psi}$ is a fundamental set of H'.

Let H'' be the closure of the algebraic span of $E_{\mathrm{R}'\Psi}$. If $H'' \neq H'$, then, since E_Ψ is fundamental in H', there exists, say, some $f \in L_2(\mathbb{R}^d)$ such that $\mathcal{T}^*_\Psi f = 0$, but $\mathcal{T}^*_{\mathrm{R}'\Psi} f \neq 0$. (Otherwise, there exists f such that $\mathcal{T}^*_\Psi f \neq 0$, but $\mathcal{T}^*_{\mathrm{R}'\Psi} f = 0$, and the argument below can be adapted to this case, as well). By Lemma 1.4.8, this implies that, while

$$J^*_\Psi(w)\widehat{f}_{|w} = 0, \quad \text{a.e. } w,$$

$$J^*_{\mathrm{R}'\Psi}(w)\widehat{f}_{|w} \neq 0, \quad \text{on a set of positive measure.}$$

On the other hand, since $E_{\mathrm{R}\Phi}$ is the dual frame of E_Φ, Proposition 1.3.7 implies that $\ker \mathcal{T}_\Phi = \ker \mathcal{T}_{\mathrm{R}\Phi}$, and hence that, for a.e. w, $\ker J^*_\Psi(w) = \ker J_\Phi(w) = \ker J_{\mathrm{R}\Phi}(w) = \ker J^*_{\mathrm{R}'\Psi}(w)$, and we have reached a contradiction. □

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