Generalized shift invariant systems

Amos Ron*
Computer Sciences Department
University of Wisconsin-Madison
1210 West Dayton
Madison, WI 53706
amos@cs.wisc.edu

Zuowei Shen
Department of Mathematics
National University of Singapore
10 Kent Ridge Crescent
Singapore 119260
matzuows@math.nus.edu.sg

ABSTRACT

A countable collection X of functions in $L_2(\mathbb{R}^d)$ is said to be a Bessel system if the associated analysis operator

$$T_X^*: L_2(\mathbb{R}^d) \to \ell_2(X) \ : \ f \mapsto (\langle f, x\rangle)_{x\in X}$$

is well-defined and bounded. A Bessel system is a fundamental frame if T_X^* is injective and its range is closed.

This paper considers the above two properties for a generalized shift-invariant system X. By definition, such a system has the form

$$X = \cup_{j\in J} Y_j,$$

where each Y_j is a shift-invariant system (i.e., is comprised of lattice translates of some function(s)) and J is a countable (or finite) index set. The definition is general enough to include wavelet systems, shift-invariant systems, Gabor systems, and many variations of wavelet systems such as quasi-affine ones and non-stationary ones.

The main theme of this paper is the 'fiberization' of T_X^*, which allows one to study the frame and Bessel properties of X via the spectral properties of a collection of finite-order Hermitian non-negative matrices.

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1. Introduction

Our primary goal in this paper is to introduce the notion of a *generalized shift-invariant system* and to study the Bessel and the frame properties of such system. This will provide, inter alia, a uniform theory that covers shift-invariant systems, wavelet systems, and several other types of representation systems. Let us begin this introduction by defining the various notions that were just mentioned.

A **system** X is merely a countable collection of functions in $L_2(\mathbb{R}^d)$. The system is used either in order to approximate functions in $L_2(\mathbb{R}^d)$ or in order to represent such functions (usually as a convergent series). Alternatively, the system can be used in order to decompose other functions. Then, the elements in X are considered as linear functionals, and the relevant operator is then the **analysis operator**

$$T^* := T_X^* : L_2(\mathbb{R}^d) \to \ell_2(X) \ : \ f \mapsto (\langle f, x\rangle)_{x\in X}.$$

Here, the inner product $\langle \cdot, \cdot \rangle$ is the usual one in $L_2(\mathbb{R}^d)$. The notation we use, T_X^*, indicates that the analysis operator is the adjoint of another useful operator, T_X, known as the *synthesis operator* (see e.g. [RS1]). We skip the explicit definition of this latter operator since it does not play a role in this article.

The system X is **Bessel** if T_X^* is well-defined and bounded. A Bessel system is a **fundamental frame** (or, "a frame of $L_2(\mathbb{R}^d)$") if T_X^* has also a bounded inverse. I.e., X is a fundamental frame whenever there exist two positive constants A, B, such that, for every $f \in L_2(\mathbb{R}^d)$,

$$B\|f\|^2 \le \sum_{x\in X} |\langle f, x\rangle|^2 \le A\|f\|^2, \tag{1.1}$$

where the norm $\|\cdot\|$ is the $L_2(\mathbb{R}^d)$-one. (We also use the symbol $\|\cdot\|$ to denote other norms, e.g. the $\ell_2(X)$-one; the relevant norm should always be clear from the context.) The sharpest possible A (B) in the above is the **upper (lower) frame bound**. The upper frame bound is also known as the **Bessel bound**. Obviously, $A = \|T^*\|^2$. If T^* is unitary (i.e., if (1.1) holds for $A = B = 1$), one says that X is a **fundamental tight frame**. We refer to [D2], [DS], [HW] and [RS1] for frames basics.

A system is **shift-invariant** (SI) if it is invariant under lattice translations:

$$f \in X \quad \Longleftrightarrow \quad f(\cdot + k), \ \ \forall k \in L,$$

with L a **lattice** i.e., $L = R\,\mathbb{Z}^d$ with R a linear bijection of $\mathbb{R}^d$. SI systems are extensively used in the area of approximation theory (see e.g. [BDR1], [BDR2], [BHR]). They also play an essential role in wavelet theory (see e.g. [BDR3], [D1], [HW], [JM], [M], [Me], [RS3], [RS4]), as well as in some other areas such as sampling theory, subdivision schemes, Gabor systems and more. A general overview of SI systems can be found in the recent surveys [R1] and [JP].

Given an SI system X, we studied in [RS1] the Bessel and frame properties of X via integral representations, on the Fourier domain, of the operators T^*T (Gramian analysis) and TT^* (dual Gramian analysis). In the language that we introduce and explain carefully in the current article, the above-mentioned representation of TT^* is block-diagonal. Each block is a *fiber*, while the technique of connecting properties of the fibers to properties of TT^* is *fiberization*. We will review, revisit and generalize this approach in the current paper. The fiberization approach was employed in [RS2] in the analysis of Weyl-Heisenberg (aka "Gabor") systems.

In order to develop further this preliminary discussion, we need to introduce the notion of a *wavelet system*. A simplified, special, case of this notion is defined below (see (3.20) for the general definition).

(1.2) A wavelet system. Given a **dilation parameter** $a > 1$, a wavelet system X takes the form

$$X = \cup_{j=-\infty}^{\infty} X_j$$

where

$$X_j := \{a^{\frac{jd}{2}}\psi(a^j \cdot +k) : \ \psi \in \Psi, \quad k \in \mathbb{Z}^d\},$$

and where $\Psi \subset L_2(\mathbb{R}^d)$ is a finite set of **mother wavelets**. □

The relevance of the shift-invariance notion to wavelets stems from the fact that each X_j above is SI. However, the underlying lattice depends on j, viz., X_j is invariant under $a^{-j}\,\mathbb{Z}^d$-translations, implying that the whole wavelet system is never shift-invariant. As a result, the SI fiberization techniques of [RS1] do not apply directly to wavelets. This lack of shift-invariance in wavelet systems was overcome in [RS3] and [RS4] via the introduction of *quasi-affine systems*: systems that are SI on the one hand, but are proved to share some important properties with an underlying wavelet system, on the other hand. The resulting theory, which applies to *integer* dilation parameters, includes, among other things, a complete characterization of all wavelet frames. We refer to [RS3] for more details.

Our goal in the present article is to establish a new, general, theory that will include the SI systems of [RS1], as well as the wavelet systems of (1.2) as special cases. Let us begin by describing the setup for this more general theory, and discussing briefly the main ingredients of it. For the sake of clarity, we describe in the introduction a simplified variant of the actual setup that is studied in the current paper.

(1.3) Definition: Generalized Shift-Invariant Systems. A generalized shift-invariant (GSI) system is a union

$$X := \cup_{j\in J} Y_j$$

of SI systems $(Y_j)_{j\in J}$. Here, J is a countable (or finite) index set, and each Y_j has the form

$$Y_j := \{\phi_j(\cdot + a_j k), \quad k \in \mathbb{Z}^d\},$$

where a_j is positive, and $\phi_j \in L_2(\mathbb{R}^d)$. We refer to each Y_j as a **layer** of X. A **subsystem** of X is obtained by replacing, in the above definition, the index set J by any subset of it. □

The definition of a GSI system is more general than its SI counterpart since we allow the different layers Y_j of the system to be invariant under different lattice translations. It is useful to note that, in contrast with SI systems and with wavelet systems, a countable (or finite) union of GSI systems is always GSI. Also, one recognises that the wavelet systems discussed in (1.2) are always GSI: since GSI systems are closed under finite unions, we need consider only the case when Ψ in (1.2) is a singleton $\{\psi\}$. For that case, we can choose in the GSI definition $J := \mathbb{Z}$, $a_j := a^{-j}$, and $\phi_j := a^{\frac{jd}{2}}\psi(a^j\cdot)$, to obtain a realization of the wavelet system as a GSI one.

As our subsequent analysis shows, GSI systems are *not* fiberizable, i.e., their corresponding TT^* does not have the alluded-to block-diagonal structure (known to exist for SI systems). Also, the innovative trick of quasi-affine systems does not extend from wavelet systems to GSI ones. In fact, we were somewhat surprised to find out that there exists a cohesive analysis of GSI systems which enables one to characterize their Bessel and frame properties. This article details those findings (Section 2), and elaborates on the application of the general theory to a variety of special cases (Section 3). In the remainder of the introduction we discuss some of the highlights of the new theory.

The main tool in our study is *dual Gramian representation and fiberization* of the GSI system X. We begin the discussion in this direction by introducing the dual Gramian *kernel*. We use the same notation as in (1.3).

The Dual Gramian of a GSI system. The dual Gramian $\widetilde{G} := \widetilde{G}_X$ is a kernel

$$\widetilde{G} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \ : \ (\omega, \tau) \mapsto \sum_{j \in \kappa(\omega-\tau)} a_j^{-d} \widehat{\phi}_j(\omega)\overline{\widehat{\phi}_j(\tau)},$$

where

$$\kappa(\omega) := \{j \in J : \ \omega \in 2\pi\, \mathbb{Z}^d / a_j\}.$$

□

Note that the kernel $\widetilde{G}$ is very sparse: for a fixed $\omega \in \mathbb{R}^d$, $\widetilde{G}(\omega, \tau) = 0$, unless τ lies in the countable set $\omega + \cup_{j \in J}(2\pi\, \mathbb{Z}^d / a_j)$. This means that, at least formally, we are allowed to consider, for $f : \mathbb{R}^d \to \mathbb{C}$, a product $\widetilde{G}f$ that is defined as

$$(\widetilde{G}f)(\omega) := \sum_{\tau \in \mathbb{R}^d} \widetilde{G}(\omega, \tau) f(\tau).$$

The dual Gramian provides the following formal quadratic form representation of the norm of the analysis operator T_X^* of the GSI system X:

$$(2\pi)^d \|T_X^* f\|^2_{\ell_2(X)} = \langle \widehat{f}, \widetilde{G}\widehat{f} \rangle, \qquad (1.4)$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $L_2(\mathbb{R}^d)$, and where $\widetilde{G}\widehat{f}$ is defined as above. There are several reasons why we label the representation (1.4) "formal": not only that we need some conditions on X to guarantee a meaningful convergence in the definition of $\widetilde{G}\widehat{f}$, but, more importantly, the two sides of (1.4) can be shown to be the result of two different orderings of a certain series, and one needs some caution before claiming that both expressions are meaningful and identical. We treat these questions with utmost rigorousness in the body of this article.

Our goal is to analyse X via the inspection of some "accessible" properties of $\widetilde{G}$. In this regard, the representation (1.4) is useful only to a limited extent: it is not straightforward to connect between the properties of the quadratic form in the right hand side of (1.4) and the concrete pointwise values of the kernel $\widetilde{G}$. It is far more tempting to attempt interpreting $\widetilde{G}$ as a *discrete* operator as follows. Once again, we simplify the discussion at this point by ignoring the problems that arise due to the fact that $\widetilde{G}$ is only defined a.e.

(1.5) The discrete dual Gramian of a GSI system. With X and $\widetilde{G}$ as before, let P be any finite set in $\mathbb{R}^d$, and let $\widetilde{G}(P)$ be the finite matrix whose rows and columns are indexed by P, and whose (p,q)-entry is $\widetilde{G}(p,q)$. We may regard $\widetilde{G}(P)$ as a linear endomorphism on $\ell_2(P)$ with norm $\mathcal{G}(P) := \mathcal{G}_X(P)$, and with inverse norm $\mathcal{G}^-(P) := \mathcal{G}^-_X(P)$. We obtain in this way two maps $\mathcal{G}$ and $\mathcal{G}^-$, each defined from the domain of all finite subsets of $\mathbb{R}^d$ into $[0,\infty]$. We say that $\mathcal{G}$ is **bounded** if there exists $A \ge 0$, such that, for every finite $P \subset \mathbb{R}^d$, $\|\mathcal{G}(\cdot + P)\|_{L_\infty} \le A$. The smallest such A is the **norm** of $\mathcal{G}$. Similarly for $\mathcal{G}^-$. $\square$

The discrete dual Gramian is convenient and, perhaps, neat. The question is whether the information gathered from it is useful in the analysis of the corresponding GSI system X. Our major objective in this paper is to establish two basic connections of this type:

Objective 1.6. *Prove, under suitable conditions on the GSI X, the following: "X is Bessel with Bessel bound A if and only if $\mathcal{G}$ is bounded by A."*

Objective 1.7. *Let X be a Bessel GSI system. Prove, under suitable conditions, the following: "X is a fundamental frame with lower frame bound B if and only if $\mathcal{G}^-$ is bounded by $1/B$."*

As an illustration, we describe in this introduction the nature of the results we obtain with respect to the special GSI system from (1.3). For the Bessel property, we have:

Theorem 1.8. *Let X be a GSI system as in (1.3), associated with a norm function $\mathcal{G}$. Then:*

(i) If X is Bessel with Bessel bound A, then $\mathcal{G}$ is bounded with norm $\le A$.

(ii) Assume that each of the numbers a_j in (1.3) is rational. Then X is Bessel with Bessel bound A if and only if $\mathcal{G}$ is bounded with norm A.

Some mild decay conditions on the Fourier transform of the generators are needed in order to analyse the frame property.

Theorem 1.9. *Let X be a GSI Bessel system as in (1.3) associated with an inverse-norm function $\mathcal{G}^-$. Assume that the set $\{a_j\}$ does not have an accumulation point in $(0,\infty)$.*

(i) If X is a fundamental frame with lower frame bound B, and if, for every compact Ω which does not intersect the origin, there exists $t > 0$ such that

$$\sum_{a_j > t} \|\widehat{\phi}_j\|^2_{L_2(\Omega)} < \infty, \tag{1.10}$$

then $\mathcal{G}^-$ is bounded, and its norm is $\le 1/B$.

(ii) Define, for every $t \in \mathbb{R}$, and for every $\Omega \subset \mathbb{R}^d$,

$$(1.11) \qquad F_{\Omega,t} := \sum_{a_j > t} \| \sum_{k \in \mathbb{Z}^d} \frac{|(\widehat{\phi}_j \chi_\Omega)(\cdot + 2\pi k/a_j)|^2}{a_j^d} \|_{L_\infty(\mathbb{R}^d)}.$$

(Here, χ_Ω is the support function of Ω.)
Assume that, for every compact Ω which is disjoint of the origin, there exists $t > 0$ such that $F_{\Omega,t} < \infty$. If, in addition, every a_j is rational, then X is a fundamental frame with lower frame bound B if and only if $\mathcal{G}^-$ is bounded by $1/B$.

Next, we provide a few clarifications to and illustrations of the nature of the theorems above. We need first the following

Definition 1.12: the diagonal function of a GSI system. Let X be a GSI system with dual Gramian $\widetilde{G}_X$. We denote by $\widetilde{g} := \widetilde{g}_X$ the diagonal of $\widetilde{G}_X$, i.e.,

$$(1.13) \qquad \widetilde{g}_X : \mathbb{R}^d \to [0, \infty] : \omega \mapsto \widetilde{G}_X(\omega, \omega).$$

The function $\widetilde{g}$ is the *diagonal function of X*.

Definition. We say that X is **a scalar system** if there is a null set $\mathcal{N}$ of $\mathbb{R}^d$, such that $\widetilde{G}(\omega, \tau) = \delta_{\omega,\tau}$ for every $\omega, \tau \in \mathbb{R}^d \backslash \mathcal{N}$. □

Corollary 1.14. *Let X be a GSI Bessel system with Bessel bound ≤ 1. Let $\widetilde{G}$ be the associated dual Gramian, and $\widetilde{g}$ the associated diagonal function. Assume that the set $\{a_j\}$ does not have an accumulation point in $(0, \infty)$. Then:*

(i) If $\widetilde{g} \geq 1$ a.e., then $\widetilde{G}(\omega, \tau) = 0$, for a.e. ω and every τ, $\tau \neq \omega$.
(ii) If X is a fundamental tight frame that satisfies (1.10) then X is a scalar system.

Proof: (i) follows directly from (i) of Theorem 1.8 when applied to the case $P = \{\omega, \tau\}$ there. For (ii), we invoke (i) of Theorems 1.8 and 1.9 with respect to $P := \{\omega\}$ to conclude that $\widetilde{g} = 1$ (a.e.), and then invoke the first part of the current corollary. □

Let us explain briefly our eventual use of the conditions in (ii) of Theorem 1.9. The technical condition (1.11) there allows us to approximate the GSI X by a subsystem X' of it whose dual Gramian $\widetilde{G}_{X'}$ is sparser (hence is simpler for analysis) than $\widetilde{G}_X$. The rationality assumption then grants us a structure on the non-zero entries of $\widetilde{G}_{X'}$ that yields to our techniques. This entire analysis can be simplified if "magic cancellations" occur in $\widetilde{G}_X$, i.e., if certain entries $\widetilde{G}_X(\omega, \tau)$ vanish despite of the fact that $\kappa(\omega - \tau)$ is not empty. Such cancellation assumptions may be used in lieu of (1.11) and/or the rationality assumption.

We illustrate this possibility by considering **diagonal GSI systems**, which is the case when $\widetilde{G}_X(\omega, \tau) \neq 0$ only if $\omega = \tau$. In this case the quadratic form representation is greatly simplified:

$$\langle \widehat{f}, \widetilde{G}\widehat{f} \rangle = \| \widetilde{g} |\widehat{f}|^2 \|_{L_1(\mathbb{R}^d)}.$$

The entire analysis in this case is reduced to the mere verification of the identity (1.4). We show later that (1.4) is valid under assumption (1.10) (cf. Lemma 4.6 for the precise statement) and this leads to the following:

Corollary 1.15. *Let X be a diagonal GSI system, associated with a diagonal function $\widetilde{g}$. Assume that (1.10) holds and that the set $\{a_j\}$ does not have an accumulation point in $(0, \infty)$. Then X is a fundamental frame if and only if the functions $\widetilde{g}$ and $1/\widetilde{g}$ are essentially bounded. The upper (lower) frame bound of X is then the essential supremum (infimum) of $\widetilde{g}$.*

As an immediate corollary of Corollary 1.15 and Corollary 1.14, we obtain the following.

Corollary 1.16. *Let X be a GSI system associated with a dual Gramian $\widetilde{G}$. Assume that (1.10) holds and that the set $\{a_j\}$ does not have an accumulation point in $(0, \infty)$. Then X is a tight frame if and only if it is a scalar system.*

Finally, we look closer at the technical conditions (1.10) and (1.11). They definitely look complicated, but they are actually very mild. Let us illustrate this point by considering a wavelet system (cf. (1.2)). Then, $\widehat{\phi}_j = a^{\frac{jd}{2}} \widehat{\psi}(a^j \cdot)$. Therefore,

$$\|\widehat{\phi}_j\|^2_{L_2(\Omega)} = \|\widehat{\psi}\|^2_{L_2(a^j\Omega)}.$$

Condition (1.10) then requires that

$$\sum_{\psi \in \Psi} \sum_{j=j_0}^{\infty} \|\widehat{\psi}\|^2_{L_2(a^j\Omega)} < \infty,$$

which is *always* satisfied (since each mother wavelet ψ lies in $L_2(\mathbb{R}^d)$, and since Ω is disjoint of the origin.) Consequently, (i) of Theorem 1.9 applies to *every* wavelet system. As to the condition we assume in (ii) of that theorem (viz., (1.11)), it is easy to see that the expression in (1.11) coincides, for a wavelet system, with

$$(1.17) \qquad \sum_{\psi \in \Psi} \sum_{a^j > t} \| \sum_{k \in \mathbb{Z}^d} |(\chi_{a^j\Omega}\widehat{\psi})(\cdot + 2\pi k)|^2 \|^2_{L_\infty(\mathbb{R}^d)}.$$

Requiring (1.17) to be finite is tantamount to imposing a mild smoothness assumption on each mother wavelet ψ. (A truly mild one: even the d-dimensional Haar wavelets satisfy it). Thus, (ii) of Theorem 1.9 applies to all wavelet systems whose dilation parameter a is rational, and whose mother wavelets satisfy a mild smoothness condition.

The above discussion reveals that Corollaries 1.15 and 1.16 apply to every wavelet system, since wavelet systems automatically satisfy the requisite (1.10). We note that the wavelet case of Corollary 1.16 is due to [CS1] (univariate case) and [CCMW] (multivariate case). It was actually the reading of [CCMW] that provided us with the motivation to look for a theory that is more general than that of [RS1] and [RS3].

The paper is laid out as follows. The analysis of the Bessel property and the frame property of a GSI system occupies §2. In §3, we discuss several special cases such as nested GSI systems, block-diagonal and local block-diagonal GSI systems, wavelet systems, Müntz systems, quasi-affine systems, and more. In addition, we study, as by-products, GSI tight and bi-frames, as well as the possible oversampling of GSI systems. §4, which may be considered as an appendix, improves slightly some of the results of §2 by introducing the notions of *temperateness* and *roundedness* in GSI systems.

During the preparation of the current article, we became aware of the work [HLW] by Eugenio Hernández, Demetrio Labate and Guido Weiss, in which the tight frame property of a GSI system is extensively studied. Most of the results we obtain in the current paper on tight and bi- GSI frames (but none of the results we obtain on more general frames) can also be found in that reference. One should consider our effort here and that of [HLW] as "concurrent and independent".

2. Analysis of GSI systems

Our goal in this section is to employ dual Gramian analysis in order to reveal the structure of GSI systems. In doing so, we obtain characterizations of the Bessel property and the frame property of these systems.

2.1. GSI systems introduced

Let $\mathcal{L} := (L_j \subset \mathbb{R}^d)_{j\in J}$ be a multiset of d-dimensional lattices. Here, J is a finite or countable index set. Writing $L_j = R_j\,\mathbb{Z}^d$, with R_j a linear bijection, we recall that the determinant $|L_j| := |\det R_j|$ of the lattice L_j depends only on the lattice and not on the choice of the linear map R_j.

We associate each $j \in J$ with a function $\phi_j \in L_2(\mathbb{R}^d)$, referred to hereafter as the **generator** of the SI system Y_j:

$$Y_j := \{\phi_j(\cdot + l) : \ l \in L_j\}.$$

The union

$$X := \cup_{j\in J} Y_j$$

is termed a **generalized shift-invariant** (GSI) system. Each Y_j is a **layer** of X. If J is finite, we call the system X a **finitely-generated GSI (FGSI) system**. We call $(\phi_j)_{j\in J}$ a **generating set** of X. Sometimes, we will write $J(X)$ for the index set used in the definition of X, and $\mathcal{L}(X)$ for the corresponding sequence of lattices.

As we explained before, our goal is to analyse the structure of the generalized shift-invariant system X via the inspection of an associated system of quadratic forms $\widetilde{G} := \widetilde{G}_X$, which we refer to here and hereafter collectively as the **dual Gramian of** X. To this end, we first recall that the **dual lattice** $\widetilde{L}$ of a lattice L is defined by

$$\widetilde{L} := \{l \in \mathbb{R}^d : l \cdot t \in 2\pi\mathbb{Z}, \ \forall t \in L\}.$$

Also, a **fundamental domain** O_j of $\widetilde{L}_j$ is any measurable set whose $\widetilde{L}_j$-shifts partition $\mathbb{R}^d$:

$$\sum_{l\in\widetilde{L}_j} \chi_{O_j}(\cdot + l) = 1.$$

Note that the measure $|O_j|$ of O_j is $\frac{2\pi}{|L_j|}$.

In passing, we note that we use the symbol

$$|\cdot|$$

for several different purposes: for $\omega \in \mathbb{R}^d$, $|\omega|$ stands for the ℓ_2-norm of ω. For a lattice L, $|L|$ stands for the determinant of the lattice (as above), while for a measurable set O, $|O|$ stands for the measure of O.

The **dual Gramian** of X, $\widetilde{G} := \widetilde{G}_X : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is defined as follows:

$$\widetilde{G}_X(\omega, \tau) := \sum_{j \in \kappa(\omega-\tau)} \frac{\widehat{\phi}_j(\omega)\,\overline{\widehat{\phi}_j(\tau)}}{|L_j|}, \quad \omega, \tau \in \mathbb{R}^d,$$

where

$$\kappa(\omega) := \kappa_X(\omega) := \{j \in J(X) : \ \omega \in \widetilde{L}_j\}.$$

The dual Gramian $\widetilde{G}$ is *sparse* in the sense that, for each $\omega \in \mathbb{R}^d$, $\widetilde{G}(\omega, \tau) = 0$ unless $\omega - \tau$ lies in the countable set*

$$\widetilde{\mathcal{L}}(X) := \cup_{j \in J(X)} \widetilde{L}_j. \tag{2.1}$$

This means that, at least formally, we can define, for a given function $f : \mathbb{R}^d \to \mathbb{C}$, the product

$$(\widetilde{G}f)(\omega) := \sum_{\tau \in \mathbb{R}^d} \widetilde{G}(\omega, \tau) f(\tau).$$

Note also that $\widetilde{G}$ is *additive*: if $X = X_1 \cup X_2$ (disjoint union), then $\widetilde{G}_X = \widetilde{G}_{X_1} + \widetilde{G}_{X_2}$.

We treat $\widetilde{G}$ sometimes as a kernel, and sometimes as a matrix. Particularly, we will refer to the value $\widetilde{G}(\omega, \tau)$ as the **(ω, τ)-entry** of $\widetilde{G}$. Along the same lines, given any measurable subset $P \subset \mathbb{R}^d$, we denote by

$$\widetilde{G}(P) := \widetilde{G}_X(P)$$

the (sub)matrix of $\widetilde{G}$ whose rows and columns are indexed by P and whose (p, q)-entry is $\widetilde{G}(p, q)$. We refer to $\widetilde{G}(P)$ as the **P-submatrix of $\widetilde{G}$**. (The similarity of the notation $\widetilde{G}(p, q)$ for the (p, q)-entry of $\widetilde{G}$ to the notation $\widetilde{G}(P)$ for the P-submatrix of $\widetilde{G}$ should not cause any confusion).

As we explained before, we regard, given a finite (or countable) $P \subset \mathbb{R}^d$, the matrix $\widetilde{G}(P)$ as an endomorphism of $\ell_2(P)$, and denote by

$$\mathcal{G}(P)$$

* Note that we are slightly inconsistent with our notations: while $\mathcal{L}(X)$ is a family of lattices, $\widetilde{\mathcal{L}}(X)$ is a union of lattices.

the norm of this map. If any of the entries of $\widetilde{G}(P)$ is not well-defined or is not finite, we define $\mathcal{G}(P) := \infty$. We say that $\mathcal{G}$ (or, $\widetilde{G}$) is **bounded by** $A > 0$, if, for every finite $P \subset \mathbb{R}^d$, **

$$\|\mathcal{G}(\cdot + P)\|_{L_\infty(\mathbb{R}^d)} \le A.$$

The **norm** of $\mathcal{G}$ ($\widetilde{G}$), denoted by $\|\mathcal{G}\|_\infty$, is its least bound, i.e.

$$\|\mathcal{G}\|_\infty := \sup_{P \in \mathcal{P}} \mathcal{G}(P), \tag{2.2}$$

where $\mathcal{P}$ is the collection of all finite subsets of $\mathbb{R}^d$.

We also use a *local* version of the above: given a subset $\Omega \subset \mathbb{R}^d$, we may consider only those matrices $\widetilde{G}(P)$ for which $P \subset \Omega$. Thus, $\widetilde{G}$ **is bounded** on Ω by $A > 0$, if, for every finite $P \subset \mathbb{R}^d$ we have that

$$\operatorname{ess\,sup}\{\mathcal{G}(\omega + P) : \ \omega + P \subset \Omega\} \le A.$$

We refer to $\mathcal{G}$ as **the norm function** of X. In a similar way, we associate X with an **inverse norm function** $\mathcal{G}^-$: given P as before, $\mathcal{G}^-(P)$ is the *inverse* norm of $\widetilde{G}(P)$ (and as before, $\mathcal{G}^-(P) := \infty$, if $\widetilde{G}(P)$ is not well-defined, or is not invertible). The definition of the boundedness of $\mathcal{G}^-$ and the norm of $\mathcal{G}^-$ are analogous to their $\mathcal{G}$ counterparts, and we will use local versions for the inverse norm function, too. We say that $\widetilde{G}$ is **bounded below** (on Ω) by $B > 0$ if $\mathcal{G}^-$ is bounded (on Ω) by $1/B$.

Finally, given any lattice $L_j \in \mathcal{L}(X)$, we use the symbol $[\cdot,\cdot]_j$ to denote the **bracket product** associated with L_j, i.e., given $f, g \in L_2(\mathbb{R}^d)$,

$$[f, g]_j := \sum_{l \in \widetilde{L}_j} (f\overline{g})(\cdot + l). \tag{2.3}$$

Note that $[f, g]_j \in L_1(O_j)$.

2.2. Technical lemmata

We collect in this subsection the technical backbone of our treatment of GSI systems. The key result is Lemma 2.11, that provides, under suitable conditions, a connection between T_X^* and the discrete version of the dual Gramian of X. In the proofs here, as well as in several other locations in this paper, we use the notation, for any measurable $\Omega \subset \mathbb{R}^d$,

$$H_\Omega := \{f \in L_2(\mathbb{R}^d) : \ \operatorname{supp} \widehat{f} \subset \Omega\}. \tag{2.4}$$

** It is implicit in our treatment that the norm function $\omega \mapsto \mathcal{G}(\omega + P)$ is measurable. Indeed, $\widetilde{G}(\omega + P)$ is self-adjoint for every $\omega \in \mathbb{R}^d$, hence (cf. [RS1: Lemma 2.3.5]) the measurability of the afore-mentioned map is implied by the measurability of $\omega \mapsto \widetilde{G}(\omega + p, \omega + p')$, $p, p' \in P$, a property that follows directly from the definition of $\widetilde{G}$.

Lemma 2.5. *Let X be an FGSI system, with dual Gramian $\widetilde{G}_X$, and diagonal function $\widetilde{g}_X$. Then:*

(i) The entries of the dual Gramian $\widetilde{G}_X$ are well-defined a.e. (i.e., well-defined for every $\omega, \tau \in \mathbb{R}^d \backslash \mathcal{N}$, for some null-set $\mathcal{N}$).

(ii) For every $f \in L_2(\mathbb{R}^d)$, and every $j \in J(X)$, we have

$$(2.6) \qquad (2\pi)^d \|T^*_{Y_j} f\|^2 = |L_j|^{-1} \|[\widehat{f}, \widehat{\phi}_j]_j\|^2_{L_2(O_j)} = \langle \widehat{f}, \widetilde{G}_{Y_j} \widehat{f} \rangle,$$

where O_j is a fundamental domain of $\widetilde{L}_j$. For the right-most equality we require that Y_j be Bessel, or that $\widehat{f}$ be compactly supported, or that $[|\widehat{f}|, |\widehat{\phi}_j|]_j \in L_2(O_j)$.

(iii) If X is Bessel, or if the Fourier transform of $f \in L_2(\mathbb{R}^d)$ is compactly supported, we have

$$(2.7) \qquad (2\pi)^d \|T^*_X f\|^2 = \langle \widehat{f}, \widetilde{G}_X \widehat{f} \rangle.$$

(iv) The representation (2.7) is also valid for a general GSI system X, provided that the number of different lattices in $\mathcal{L}(X)$ is finite, that $\widetilde{g}_X$ is finite a.e., and that supp $\widehat{f}$ *is compact.*

Proof: Assertion (i) is obvious since, for an FGSI system, $\widetilde{G}_X(\omega, \tau)$ is a *finite* sum, and each summand is well-defined a.e. Assertion (iii) follows from assertion (ii) by summing (2.6) over all j in the finite $J(X)$. Moreover, [RS1] establishes (2.7) for a shift-invariant system (i.e., a system X whose $\mathcal{L}(X)$ is a singleton), provided that $\widetilde{g}_X$ is finite a.e. (and $\widehat{f}$ is compactly supported). Thus, (iv) follows from this afore-mentioned result (since the systems considered in (iv) are finite unions of shift-invariant ones).

We prove, then, (ii). The left-most identity in (ii) is known (cf. [RS1]). For the right-most equality, we set $\widetilde{G} := \widetilde{G}_{Y_j}$, $\phi := \phi_j$, $O := O_j$, and $L := L_j$. Then, from the definitions of the dual Gramian and the bracket product,

$$\langle \widehat{f}, \widetilde{G} \widehat{f} \rangle = \frac{1}{|L|} \int_{\mathbb{R}^d} \widehat{f}(\omega) \sum_{l \in \widetilde{L}} \overline{\widehat{\phi}(\omega)} \widehat{\phi}(\omega + l) \overline{\widehat{f}(\omega + l)} \, d\omega = \frac{1}{|L|} \int_{\mathbb{R}^d} \widehat{f}(\omega) \overline{\widehat{\phi}(\omega)} [\widehat{\phi}, \widehat{f}]_j(\omega) \, d\omega.$$

Writing $\mathbb{R}^d$ as the disjoint union of the set $l + O$, $l \in \widetilde{L}$, and using the assumption that $[|\widehat{f}|, |\widehat{\phi}|]_j \in L_2(O)$, we obtain from the dominated convergence theorem that

$$\langle \widehat{f}, \widetilde{G} \widehat{f} \rangle = |L|^{-1} \int_O |[\widehat{f}, \widehat{\phi}_j]_j|^2 \, d\omega,$$

as claimed.

Alternatively, assume that Y_j is Bessel, and let $f \in L_2(\mathbb{R}^d)$. Let $g \in L_2(\mathbb{R}^d)$ be such that, everywhere, $\widehat{g}\overline{\widehat{\phi}} = |\widehat{f}\widehat{\phi}|$. Then $[\widehat{g}, \widehat{\phi}]_j = [|\widehat{f}|, |\widehat{\phi}|]_j$. Hence, by the left-most equality in (2.6), $\|T^*_{Y_j} g\| = (2\pi)^{-d/2} |L_j|^{-1/2} \|[|\widehat{f}|, |\widehat{\phi}|]_j\|_{L_2(O)}$, which, together with the Bessel assumption on Y_j, shows that $[|\widehat{f}|, |\widehat{\phi}|]_j \in L_2(O)$, and our argument in the previous paragraph applies, then, to this case, as well.

Finally, the validity of (ii) under the compact support assumption on $\widehat{f}$ follows from the results of [RS1], as we already indicated in the proof for (iv) here. $\square$

Lemma 2.8. *Let X be a GSI system, and assume that one of the following conditions holds:*

(a) X is Bessel with Bessel bound $\le A$.

(b) $\|\widetilde{g}_X\|_{L_\infty(\mathbb{R}^d)} \le A$.

Then the following is valid:

(i) Almost each entry $\widetilde{G}_X(\omega,\tau)$, $\omega,\tau \in \mathbb{R}^d$, is an absolutely convergent series, whose absolute sum is $\le A$.

(ii) For every finite $P \subset \mathbb{R}^d$, the matrix $\widetilde{G}_X(\omega+P)$ is non-negative definite for almost every ω. Moreover, if X' is a subsystem of X, then $\widetilde{G}_{X'}(\cdot+P) \le \widetilde{G}_X(\cdot+P)$ a.e.

Proof: We first show that condition (b) here is implied by condition (a). For that, assume (a) and let X' be an FGSI subsystem of X. Then X' is also Bessel with Bessel bound $\le A$. Since $\mathcal{L}(X')$ consists of only finitely many lattices, there exists a small neighborhood $V \subset \mathbb{R}^d$ of the origin such that $V \cap \widetilde{L}_j = 0$, for every $j \in J(X')$. Let U be neighborhood of the origin such that $U - U \subset V$. Let $t \in \mathbb{R}^d$, and $f \in H_{t+U}$. Invoking Lemma 2.5, we obtain that

$$(2\pi)^d \|T^*_{X'} f\|^2 = \langle \widehat{f}, \widetilde{G}_{X'} \widehat{f} \rangle.$$

Now, assume that $\widetilde{G}(\omega,\tau) \neq 0$, and that $\omega,\tau \in \operatorname{supp} \widehat{f} \subset t+U$. Then, due to the latter assumption, $\omega-\tau \in V$, while due to the former assumption, $\omega-\tau \in \widetilde{L}_j$, for some $j \in J(X')$. We then conclude that $\omega = \tau$, and consequently,

$$\langle \widehat{f}, \widetilde{G}_{X'} \widehat{f} \rangle = \int_{\mathbb{R}^d} \widehat{f}(\omega) \widetilde{G}_{X'}(\omega,\omega) \overline{\widehat{f}(\omega)}\, d\omega.$$

Thus, we obtain that

$$(2\pi)^d \|T^*_{X'} f\|^2 = \|\sqrt{\widetilde{g}_{X'}}\,\widehat{f}\|^2_{L_2(\mathbb{R}^d)},$$

with $\widetilde{g}_{X'}$ the diagonal function of X'. Since X' has a Bessel bound $\le A$, we get that

$$(2\pi)^d \|T^*_{X'} f\|^2 \le (2\pi)^d A \|f\|^2 = A \|\widehat{f}\|^2,$$

and, consequently, for every $f \in H_{t+U}$,

$$\|\sqrt{\widetilde{g}_{X'}}\,\widehat{f}\|^2 \le A \|\widehat{f}\|^2.$$

This easily implies that $\|\widetilde{g}_{X'}\|_{L_\infty(t+U)} \le A$, and by varying t over $\mathbb{Q}^d$ (i.e., the rationals), we obtain that $\|\widetilde{g}_{X'}\|_{L_\infty(\mathbb{R}^d)} \le A$. Writing X as the union of nested FGSI systems, we obtain (b).

So, it suffices to assume (b), and to prove (i) and (ii), as we do. Assertion (i) follows directly from (b) by an application of Cauchy-Schwarz Inequality. For the proof of (ii), let $P \subset \mathbb{R}^d$ be finite. For each Y_j, $j \in J(X)$, and for a.e. $\omega \in \mathbb{R}^d$, it is easy to see that $\widetilde{G}_{Y_j}(\omega+P)$ is block-diagonal, with each block being a rank-1 non-negative definite matrix. Now, by (i) above, the sum

$$\widetilde{G}_X(\omega+P) = \sum_{j \in J(X)} \widetilde{G}_{Y_j}(\omega+P) \tag{2.9}$$

absolutely converges a.e., hence the limit is non-negative definite a.e. Finally, the inequality $\widetilde{G}_{X'} \le \widetilde{G}_X$ is immediate from (2.9). $\square$

Definition 2.10. Let X be a GSI system with dual Gramian $\widetilde{G}$. Let U be some measurable set and let $P \subset \mathbb{R}^d$ be finite or countable. We say that X is **P-fiberizable with fibers in U**, if the $(U+P)$-submatrix of $\widetilde{G}$ is block-diagonal with blocks indexed by P: for a.e. $u \in U$ and for every $p \in P$, the condition $\widetilde{G}(u+p,\tau) \neq 0$, $\tau \in U+P$, implies that $\tau \in u+P$. □

Lemma 2.11. *Let X be a GSI system which is P-fiberizable with fibers in U for some U and P. Let $\widetilde{G}$ be the dual Gramian of X. Assume that $\{U+p\}_{p\in P}$ are pairwise disjoint. Assume also that P is finite, or that it is a subset of some lattice L.*

(i) If $f \in H_{U+P}$ satisfies

$$(2\pi)^d \|T_X^* f\|^2 = \langle \widehat{f}, \widetilde{G}\widehat{f}\rangle, \tag{2.12}$$

then

$$(2\pi)^d \|T_X^* f\|^2 = \int_U \langle \widehat{f}_u, \widetilde{G}(u+P)\widehat{f}_u\rangle_{\ell_2(P)}\, du,$$

where

$$\widehat{f}_u : P \to \mathbb{C} : p \mapsto \widehat{f}(u+p),$$

and $\langle\cdot,\cdot\rangle_{\ell_2(P)}$ is the usual discrete ℓ_2-product.

(ii) If (2.12) is valid for a dense subset of H_{U+P}, then, the following sharp inequalities:

$$\frac{\|f\|^2}{\|\mathcal{G}^-(\cdot+P)\|_{L_\infty(U)}} \leq \|T_X^* f\|^2 \leq \|\mathcal{G}(\cdot+P)\|_{L_\infty(U)}\|f\|^2,$$

hold for all $f \in H_{U+P}$.

Proof: Let $f \in H_{U+P}$. By assumption, the $(U+P)$-submatrix of $\widetilde{G}$ is block-diagonal, with each block of the form $\widetilde{G}(u+P)$, $u \in U$. Therefore,

$$\langle \widehat{f}, \widetilde{G}\widehat{f}\rangle = \int_U \langle \widehat{f}_u, \widetilde{G}(u+P)\widehat{f}_u\rangle_{\ell_2(P)}\, du.$$

This proves (i). Assertion (ii) follows now from (i) by a standard argument (cf. [RS1:Lemma 2.3.6 and Proposition 3.3.4]). □

Lemma 2.13. *Let X be a GSI system. Suppose that the set*

$$\widetilde{\mathcal{L}} := \cup_{j\in J(X)} \widetilde{L}_j$$

has no accumulation points. Let $P \subset \mathbb{R}^d$ be finite. Then there exists a neighborhood U of the origin such that, for every $t \in \mathbb{R}^d$, X is P-fiberizable with fibers in $t+U$.

Proof: Fix $t \in \mathbb{R}^d$. Let Q be the set $(t+P)-(t+P) = P-P$, and let $Q_0 := Q \cap \widetilde{\mathcal{L}}$. Then Q is finite since P is finite. By assumption, each $q \in Q$ is not an accumulation point of $\widetilde{\mathcal{L}}$. Since Q is finite, we can find thus a sufficiently small ball U around the origin such that (i) the set of balls $q+2U$, $q \in Q$, are pairwise disjoint, and (ii) $(Q+2U)\backslash Q_0$ is disjoint of $\widetilde{L}$. Let $q+u \in (Q+2U)\cap\widetilde{L}$. By the first condition on U, $(q+2U)\cap Q = \{q\}$, while by the second condition on U, $q+u \in (q+2U)\cap\widetilde{L} \subset Q_0$, and thus $u=0$.

Since $Q+2U = (t+P+U)-(t+P+U)$, the above argument establishes that, for $p,p' \in P$ and $u,u' \in U$, the condition $(t+p+u)-(t+p'+u') \in \widetilde{L}$ forces the equality $u=u'$. This shows that if $\omega,\tau \in t+U+P$, and $\widetilde{G}(\omega,\tau)\neq 0$, then $\omega = t+p+u$, $\tau = t+p'+u$, $p,p' \in P$, $u \in U$, which is exactly the required block-diagonal structure. □

2.3. Bessel systems

We start our analysis of GSI systems with a study of the Bessel property.

Theorem 2.14. *Let X be a GSI system associated with a norm function $\mathcal{G}$.*

(i) Let $\Omega \subset \mathbb{R}^d$. If the restriction of T_X^* to H_Ω is bounded by A, then $\mathcal{G}$ is bounded on Ω by A.

(ii) If X is Bessel with Bessel bound A, then $\mathcal{G}$ is bounded with norm $\le A$. In particular, $\|\widetilde{g}_X\|_{L_\infty(\mathbb{R}^d)} \le A$.

Proof: Fix a finite $P \subset \mathbb{R}^d$, and let X' be an FGSI subsystem of X. Since we assume that T_X^* is bounded by A on H_Ω, so does $T_{X'}^*$. Also, since $\mathcal{L}(X')$ involves only finitely many lattices, then, trivially, the union $\widetilde{\mathcal{L}}(X') = \cup_{j \in J(X')} \widetilde{L}_j$ has no accumulation points. We can thus invoke Lemma 2.13 to conclude that, for some neighborhood U of the origin and for every $t \in \mathbb{R}^d$, X' is P-fiberizable with fibers in $t+U$. Setting

$$U_t := \{u \in t+U : \ u+P \subset \Omega\},$$

we still have that X' is P-fiberizable with fibers in U_t. Moreover, (iii) of Lemma 2.5 ensures that, on H_{P+U_t}, the representation (2.12) is valid. Part (ii) of Lemma 2.11 can thus be invoked to yield that, since $T_{X'}^*$ is bounded by A on H_{P+U_t}, $\mathcal{G}_{X'}(u+P) \le A$, for a.e. $u \in t+U$ that satisfies $u+P \subset \Omega$. Varying t over $\mathbb{Q}^d$, we conclude that $\mathcal{G}_{X'}(u+P) \le A$, for almost every $u \in \Omega_P := \{u \in \mathbb{R}^d : \ u+P \subset \Omega\}$.

To finish the proof of (i), we take $(X_k)_{k=1}^\infty$ to be an FGSI filtration of X, i.e., each X_k is an FGSI subsystem of X, $X_k \subset X_{k+1}$, $\forall k$, and $X = \cup_k X_k$. By (i) of Lemma 2.8,

$$\widetilde{G}_{X_k}(\cdot + P) \underset{k \to \infty}{\longrightarrow} \widetilde{G}_X(\cdot + P),$$

pointwise a.e. on Ω_P. Since $\mathcal{G}_{X_k}(\cdot + P) \le A$ on Ω_P, for every k, a.e., (by virtue of the first part of the proof), it follows that $\mathcal{G}_X(\cdot + P) \le A$, a.e. on Ω_P, as claimed.

Part (ii) of the theorem is a trivial consequence of part (i). □

For the converse of Theorem 2.14, we impose on X the following condition:

The Finite Intersection (FI) Condition. Given a GSI system X, we say that X satisfies the FI condition if the intersection of any *finitely many* lattices from $\mathcal{L}(X)$ is a d-dimensional lattice.

Discussion. If each lattice $L_j \in \mathcal{L}(X)$ is of the form $a_j \mathbb{Z}^d$, $a_j > 0$, then the FI condition simply says that there exists some $a > 0$ such that a_j/a is rational for every $j \in J(X)$. For a general X, the FI condition holds if, e.g., every lattice L_j, $j \in J(X)$, is *rational.* □

Theorem 1.8 is a special case of our next result:

Theorem 2.15. *Let X be a GSI system that satisfies the FI property.*
(i) Let $\Omega \subset \mathbb{R}^d$ be compact. Then the norm of the restriction of T_X^* to H_Ω coincides with the norm of $\mathcal{G}$ on Ω.
(ii) X is Bessel with Bessel bound A if and only if $\mathcal{G}$ is bounded by A.

Proof: We recall that the "only if" part is already proved in Proposition 2.14.

When proving the "if" assertion, we assume first that X is FGSI. Set $V := \Omega - \Omega$. By the FI property, the intersection $L := \cap_{j \in J(X)} L_j$ is a d-dimensional lattice. Now, every $\widetilde{L}_j$, $j \in J(X)$, lies in $\widetilde{L}$, proving that, for $\omega, \tau \in \Omega$, $\widetilde{G}_X(\omega, \tau) = 0$, unless $\omega - \tau \in V \cap \widetilde{L} =: P$. Since P is additive (i.e., $(P + P) \cap V = P$), we conclude that the Ω-submatrix of $\widetilde{G}_X$ is block-diagonal, with each block indexed by $\omega + P$, $\omega \in \Omega$, i.e., X is P-fiberizable with fibers in Ω. Combining our assumption that $\mathcal{G}_X \leq A$ on Ω together with the right-most inequality in (ii) of Lemma 2.11, we obtain that the restriction of T_X^* to H_Ω is bounded above by A. This proves (i) for an FGSI system X. The proof of (ii) for an FGSI system is analogous.

We now extend (ii) from the FGSI case to the general GSI case. The extension for the local case (i) is similar. To this end, we let $(X_k)_{k=1}^\infty$ be an FGSI filtration of X, i.e.,

$$X_1 \subset X_2 \subset ..., \tag{2.16}$$

and $\cup_k X_k = X$. The boundedness assumption on $\mathcal{G}_X$ implies, in particular, that (b) of Lemma 2.8 holds. From that lemma, we conclude then that the entries of $\widetilde{G}_X$ are absolutely summable (a.e.), hence that, for every finite $P \subset \mathbb{R}^d$, $\mathcal{G}_{X_k}(\cdot + P)$ converges a.e. to $\mathcal{G}_X(\cdot + P)$. Moreover, the lemma implies that, a.e., $\mathcal{G}_{X_k}(\cdot + P) \leq \mathcal{G}_X(\cdot + P)$.

Fix now k. Since we assume that $\|\mathcal{G}_X(\cdot + P)\|_{L_\infty(\mathbb{R}^d)} \leq A$, for every finite P, it follows from the above that $\|\mathcal{G}_{X_k}(\cdot + P)\|_{L_\infty(\mathbb{R}^d)} \leq A$, for every finite P. Since X_k is FGSI, we conclude from the first part of the proof that X_k is Bessel with Bessel bound $\leq A$. This being true for every k, we conclude that X is Bessel with Bessel bound $\leq A$, as asserted. □

2.4. Fundamental frames: tailless systems

In this section we analyse a special type of GSI systems: tailless ones. The attraction in the analysis is that the "side-conditions' we need to impose (taillessness, and the FI condition) are purely in terms of the lattices $\mathcal{L}(X)$ of the system, and do not involve the generators of the system.

Definition 2.17: Tailless GSI systems. Let X be a GSI system, associated with lattices $\mathcal{L}(X) = (L_j)_{j \in J(X)}$. We say that X is **tailless** if, for every compact Ω that excludes the origin, the number of *different* lattices $L_j \in \mathcal{L}(X)$ that satisfy $\widetilde{L}_j \cap \Omega \neq \emptyset$ is finite. □

Note that every FGSI system is trivially tailless.

Discussion 2.18. As an illustration for the above property, let us consider the univariate case $d = 1$. Then, $L_j = a_j\mathbb{Z}$ for some $a_j > 0$, and $\widetilde{L}_j = 2\pi\mathbb{Z}/a_j$. It is then easy to see that X is tailless exactly when the set of all *different* numbers in the multiset $(a_j)_{j\in J(X)}$ is bounded and has no accumulation points other than 0.

The above discussion illustrates the essence of the taillessness condition: it prohibits the inclusion in $\mathcal{L}$ of lattices that are increasingly sparse.

We will need now the following lemma, which is proved (in greater generality) as Lemma 3.2:

Lemma 2.19. *Assume that the GSI system X is tailless, and let $\widetilde{G}$ be its dual Gramian kernel. Assume that the diagonal function $\widetilde{g}$ of X (cf. (1.13)) is in $L_\infty(\mathbb{R}^d)$. Then, the equality*

$$(2\pi)^d \|T_X^* f\|^2 = \langle \widehat{f}, \widetilde{G}_X \widehat{f}\rangle$$

holds for every $f \in L_2(\mathbb{R}^d)$, provided that $\operatorname{supp} \widehat{f}$ *is compact and does not contain the origin.*

Proposition 2.20. *Let X be a tailless GSI system, associated with a dual Gramian $\widetilde{G}$, and corresponding norm functions $\mathcal{G}$ and $\mathcal{G}^-$. Assume that X is a fundamental frame with frame bounds A, B. Then, for every finite $P \subset \mathbb{R}^d$, and for almost every $\omega \in \mathbb{R}^d$, $\mathcal{G}^-(\omega + P) \le 1/B$.*

Proof: Let $P \subset \mathbb{R}^d$ be finite. In order to prove the claim, we let $t \in \mathbb{Q}^d$. By Lemma 2.13 (which we are allowed to invoke thanks to the taillessness assumption) there exists a neighborhood U of the origin such $\widetilde{G}$ is P-fiberizable with fibers in $t + U$. Furthermore, U depends on P, but not on t. By our assumption here, we have

$$B\|f\|^2 \le \|T_X^* f\|^2, \quad \forall f \in H_{t+U+P}. \tag{2.21}$$

On the other hand, the above fiberizability of X allows us to invoke Lemma 2.11: the sharpness of the left-most inequality in (ii) of that lemma, when combined with (2.21), implies that $\mathcal{G}_X^-(\cdot + P) \le 1/B$, a.e. on $t + U$. (In order to invoke Lemma 2.11, we need the validity of (2.12), which is guaranteed by Lemma 2.19.) Since t is arbitrary, the result follows. □

The main result of this section is as follows:

Theorem 2.22. *Let X be a GSI tailless Bessel system, associated with a dual Gramian kernel $\widetilde{G}$ and norm functions $\mathcal{G}$ and $\mathcal{G}^-$. Assume further that X satisfies the FI condition. Then X is a fundamental frame with lower frame bound B if and only if $\mathcal{G}^-$ is bounded with bound $1/B$.*

Proof: The "only if" assertion is already established in Proposition 2.20, without appealing to the FI property. The proof of the "if" assertion is very similar to the proof of Theorem 2.15, hence is only sketched here. We first let Ω be a compact set that excludes the origin. Then with P as in the proof of Theorem 2.15, we first need to show that P is additive. For that, we easily conclude from the taillessness assumption that 0 is an isolated point in $\widetilde{\mathcal{L}}$, hence that $P\backslash 0$ is the intersection of $\widetilde{\mathcal{L}}$ with some compact set V' that excludes the origin (viz., a set that is obtained after removing from $\Omega - \Omega$ a small neighborhood of the origin.) Then, the taillessness assumption grants us that only finitely many dual lattices have an intersection with V', while the FI property ensures us that the union of these finitely many dual lattices is a subset of some lattice, i.e., $P\backslash 0$ is a subset of a d-dimensional lattice M. Thus, X is M-fiberizable with fibers in Ω.

The rest of the proof then follows *verbatim* that of Theorem 2.15: we invoke the left-most inequality in (ii) of Lemma 2.11 together with the assumption that $\mathcal{G}^-(\cdot + P) \leq 1/B$ a.e., to conclude that the restriction of T_X^* to H_Ω is bounded below by B. Again, since the compact Ω is arbitrary, we obtain that T_X^* is bounded below by B on the entire $L_2(\mathbb{R}^d)$ space. $\square$

Remark 2.23. The proof of Proposition 2.20 shows that, in analogy to Theorems 2.14 and 2.15, it admits a local version: if X is a frame for H_Ω, for some $\Omega \subset \mathbb{R}^d$, with lower frame bound B, then $\mathcal{G}^-$ is bounded on Ω by $1/B$. We note that Theorem 2.22 admits a local version, too: under the conditions there, X is a frame for H_Ω with lower frame bound B, if and only if $\widetilde{G}$ is bounded below on Ω by B. $\square$

2.5. Fundamental frames: small tail systems

In the previous section, we investigated the validity of the assertions in (1.6) and (1.7) for tailless systems. An attraction in the analysis there is that the conditions we imposed on the GSI system X are purely in terms of the underlying lattices in $\mathcal{L}(X)$: the taillessness condition is of this nature, and the additional FI condition is of this nature, too.

Unfortunately, once the taillessness assumption is abandoned (and unless we are merely interested in the Bessel property, §2.2), we will need to impose restrictions on the generators ϕ_j, $j \in J(X)$: we decompose the GSI system into a tailless part and its complement (=:the "tail"), and impose some constraints on the "size" of the tail. To this end, we introduce the following notion.

Definition 2.24. Let X be a GSI system. We say that X has a **small tail**, if for every compact Ω that excludes the origin, there exists a decomposition $J(X) = J_1 \cup J_2$ of $J(X)$ such that:

(i) $X_1 := \cup_{j \in J_1} Y_j$ is tailless.

(ii) $\sum_{j \in J_2} \|T_{Y_j,\Omega}^*\|^2 < \infty$, with

$$T_{Y,\Omega}^* \tag{2.25}$$

the restriction of T_Y^* to H_Ω. $\square$

Remark 2.26. Note that the small tail condition implies the following: given any compact Ω that excludes the origin, and given any $\varepsilon > 0$, the exists a decomposition $J(X) = J_1 \cup J_2$ such that (i) X_1 (defined as above) is tailless, and (ii) with $X_2 := \cup_{j\in J_2} Y_j$, $\|T^*_{X_2,\Omega}\| < \varepsilon$. We use in the sequel this observation without further notice.

Furthermore, the theory of shift-invariant spaces allows us to explicitly write the small tail condition in terms of the generators of the tail J_2. Indeed, since the orthogonal projection into H_Ω is a self-adjoint convolution operator, it is easy to see that, for every shift-invariant

$$Y := \{\phi(\cdot + l),\ l \in L\}$$

(with L some lattice), we have that $\|T^*_{Y,\Omega}\| = \|T^*_{Y_\Omega}\|$, where

$$Y_\Omega := \{\phi_\Omega(\cdot + l),\ l \in L\},$$

and ϕ_Ω is the projection of ϕ into H_Ω. Invoking then [RS1] (with respect to systems of the form Y_Ω) we obtain that

$$\|T^*_{Y_j,\Omega}\|^2 = \frac{\|\,[\chi_\Omega \widehat{\phi}_j, \widehat{\phi}_j]_j\|_{L_\infty(\mathbb{R}^d)}}{|L_j|}.$$

Thus, (ii) of Definition 2.24 is equivalent to

$$\sum_{j\in J_2} \frac{\|\,[\chi_\Omega \widehat{\phi}_j, \widehat{\phi}_j]_j\|_{L_\infty(\mathbb{R}^d)}}{|L_j|} < \infty.$$

$\square$

Corollary 2.27. *The conclusions of Proposition 2.20 as well as of Theorem 2.22 are valid if the taillessness assumption on X there is replaced by the assumption that X has a small tail.*

Proof: The proofs are straightforward, hence are only sketched. For example, let us illustrate the extension of the "if" statement in Theorem 2.22. Assume, thus, that X is Bessel with small tail, that it satisfies the FI property, and that $\mathcal{G}^-_X$ is bounded by $1/B$. We want to show that X is a fundamental frame with lower frame bound $\geq B$. It suffices for that to show that the restriction of T^*_X to H_Ω is bounded below by $B - \varepsilon$, for every compact Ω that excludes the origin, and for every $\varepsilon > 0$. Let Ω and ε be such. Then, since X has a small tail, we can find a subsystem X_1 of X, such that X_1 is tailless, and the norm of the restriction of $T^*_{X\setminus X_1}$ to H_Ω is $\leq \varepsilon$. From Theorem 2.14 we know that $\mathcal{G}_{X\setminus X_1}$ is bounded above by ε on Ω. At the same time, we have, by assumption, that $\mathcal{G}_X$ is bounded below by B on Ω. This readily implies that $\widetilde{G}_{X_1}$ is bounded below on Ω by $B - \varepsilon$. Invoking Theorem 2.22 (or, more precisely, its local version, see Remark 2.23) we conclude that $T^*_{X_1}$ is bounded below on H_Ω by $B - \varepsilon$, *a fortiori* T^*_X is bounded below on that space by that constant.

The extension of the 'only if' part is similar: assume that X is a fundamental frame with lower frame bound B. With Ω as before, we choose $X_1 \subset X$ so that the norm of the restriction of $T^*_{X\backslash X_1}$ to H_Ω is $\le \varepsilon$. Thus, $T^*_{X_1}$ is bounded below on H_Ω by $B-\varepsilon$. By the local version of Proposition 2.20, this implies that $\widetilde{G}_{X_1}$ is bounded below on Ω by $B-\varepsilon$. Invoking Lemma 2.8, we conclude that $\widetilde{G}_X$ is bounded below on Ω by $B-\varepsilon$. Since Ω and ε are arbitrary, the desired result follows. □

2.6. Summary

All the main results we proved in the section assumed no more, and sometimes less, than the small tail property and the FI condition. Thus, we have:

Corollary 2.28. *Let X be a GSI system with a small tail that satisfies the FI condition. Let $\mathcal{G}$ and $\mathcal{G}^-$ be associated norm functions. Then,*
(i) X is Bessel with Bessel bound A if and only if $\mathcal{G}$ is bounded by A.
(ii) Assume that X is Bessel. Then X is a fundamental frame with lower frame bound B if and only if $\mathcal{G}^-$ is bounded by $1/B$. □

Theorem 1.9 (ii) follows from this general result and Remark 2.26. Also, note that if X is a shift-invariant system, then it is tailless and it satisfies the FI condition. Hence, Corollary 2.28 recovers the corresponding results for SI systems given in [RS1].

3. Special types of GSI systems

The variety of different systems that are put under the roof of the GSI notion is somewhat overwhelming. Our attempt in this section is, thus, to provide some classification of GSI systems. We will discuss nested, diagonal, block-diagonal, and wavelet systems as well as a multitude of possible variations of the wavelet notion. We will also establish methods for comparing two systems via the idea of dominance, and will consider two oversampling procedures: uniform and oblique, the latter will lead us to the notion of quasi-affine systems. Finally, we will obtain suitable characterizations of GSI tight frames and GSI bi-frames.

3.1. Block-diagonal GSI systems: FI, diagonal and scalar systems. Tight frames.

Let X be a GSI system with dual Gramian $\widetilde{G}$. We say that X is **block-diagonal** if, for some lattice M, X is M-fiberizable, i.e., $\widetilde{G}(\omega,\tau)=0$, a.e., unless $\omega-\tau \in M$.

Example: SI systems. The block-diagonality of X may be implied by some structural assumptions on the lattices $\mathcal{L}(X)$ associated with the system X: if we assume that $L_0 := \cap\{L : L \in L(X)\}$ is a d-dimensional lattice, then X is block-diagonal with respect to the lattice $\widetilde{L}_0$. In this case we say that X is *shift-invariant*. Theorem 2.22 applies to this case (since X is tailless and satisfies the FI condition); moreover, our analysis from [RS1] covers this case as well. □

A block-diagonal system lends itself to our analysis (Lemma 2.11) provided that it satisfies the integral identity (1.4). We will return to this issue shortly. But, first, we would like to extend slightly the notion of block-diagonality.

Definition 3.1: locally block-diagonal systems. A system X is *locally block-diagonal* if, for each compact set Ω, there is a lattice M_Ω such that the dual Gramian $\widetilde{G}$ of X is M_Ω-fiberizable on Ω, i.e., there exists an Ω-dependent nullset $\mathcal{N}$ such that, if $\omega, \tau \in \Omega \backslash \mathcal{N}$ and $\widetilde{G}(\omega, \tau) \neq 0$, then $\omega - \tau \in M_\Omega$. □

Example: tailless FI systems. One checks that every tailless GSI system that satisfies the FI condition is locally block-diagonal. □

For our first result, we need the following lemma, whose proof is given at §3.6:

Lemma 3.2. *Assume that the GSI system X has a small tail. Let $\widetilde{G}$ be the associated dual Gramian, and $\widetilde{g}$ the corresponding diagonal function. If $\widetilde{g} \in L_\infty(\mathbb{R}^d)$, then the equality*

$$(2\pi)^d \|T_X^* f\|^2 = \langle \widehat{f}, \widetilde{G}_X \widehat{f} \rangle \tag{3.3}$$

holds for every $f \in H_\Omega$, with Ω any compact set that excludes the origin.

Theorem 3.4. *Let X be a locally block-diagonal GSI system that has a small tail. Let $\mathcal{G}$ and $\mathcal{G}^-$ be the associated norm functions. Then,*

(i) X is Bessel with Bessel bound A if and only if $\mathcal{G}$ is bounded by A.

(ii) Assume that X is Bessel. Then X is a fundamental frame with lower frame bound B if and only if $\mathcal{G}^-$ is bounded by $1/B$.

Proof: Let $\widetilde{g}$ be the diagonal function of X. If $\|\widetilde{g}\|_{L_\infty(\mathbb{R}^d)} = \infty$, then, by Theorem 2.14, X is not a Bessel system. Thus, we may assume without loss that $\|\widetilde{g}\|_{L_\infty(\mathbb{R}^d)}$ is finite. By Cauchy-Swartz, this implies that the entries of $\widetilde{G}$ are well defined a.e. Let Ω be a compact set and let M_Ω be a lattice that fits Definition 3.1. Let O_Ω be a fundamental domain for M_Ω, and $\widetilde{G}_\Omega$ the Ω-submatrix of $\widetilde{G}_X$. The local block-diagonality implies then that $\widetilde{G}_\Omega$ is M_Ω-fiberizable with fibers in O_Ω. Moreover, the small tail assumption we make here, when combined with Lemma 3.2, implies the validity of the representation (3.3). The desired result then follows from (ii) of Lemma 2.11, and from the fact that we can write $\mathbb{R}^d$ as a countable union of compact sets. □

Discussion. The core of our analysis in §2 was based on structural assumptions in terms of $\mathcal{L}(X)$. The current discussion takes a different direction: we impose directly a desired structure on $\widetilde{G}_X$. In particular, we do not assume any more the FI condition: the mere virtue of the FI condition is the local block-diagonality it imposes on $\widetilde{G}_X$; currently, we assume this latter condition directly. □

A GSI system is a **diagonal system** if there is a null set $\mathcal{N}$ of $\mathbb{R}^d$, such that $\widetilde{G}(\omega,\tau) = 0$ for every ω, τ in $\mathbb{R}^d \backslash \mathcal{N}$ with $\tau \neq \omega$. Clearly, a diagonal system is block-diagonal. Moreover, for a diagonal system, we have the following immediate connections

$$\|\mathcal{G}_X\|_{L_\infty} = \|\widetilde{g}_X\|_{L_\infty}, \quad \|\mathcal{G}_X^-\|_{L_\infty} = \|1/\widetilde{g}_X\|_{L_\infty}.$$

Therefore, we have:

Corollary 3.5. *Let X be a diagonal system that has a small tail. Set $\widetilde{g}$ for the diagonal function of X. Then:*
(i) X is Bessel with Bessel bound A if and only if $\|\widetilde{g}\|_{L_\infty} = A$.
(ii) For a Bessel X, X is a fundamental frame with lower frame bound B if and only if $\|1/\widetilde{g}\|_{L_\infty} = 1/B$.
In particular, if X is scalar then it is a fundamental tight frame. □

The last part of the above corollary asserts that if $\widetilde{G}$ is the identity a.e., then the system X is a fundamental tight frame. We will show next that the converse is also true for GSI systems with small tail. For this we need the following:

Lemma 3.6. *Let X be a GSI Bessel system with Bessel bound A. Let $\widetilde{g}$ and $\mathcal{G}^-$ be the diagonal function and the inverse norm function of X, respectively.*
(i) If $\widetilde{g} \geq A$ a.e., then X is diagonal.
(ii) If $A = 1$, and $\|\mathcal{G}^-\|_{L_\infty} \leq 1$, then X is scalar.

Proof: For (i), we need to show that, for some null set $\mathcal{N}$, and for every $\omega, \tau \in \mathbb{R}^d \backslash \mathcal{N}$, $\widetilde{G}(\omega,\tau) = 0$, unless $\omega = \tau$. Setting $t := \omega - \tau$, we first note that for almost all ω, $\widetilde{G}_X(\omega, \omega + t) = 0$ whenever $t \notin \widetilde{\mathcal{L}}(X)$. Assuming thus that $t \in \widetilde{\mathcal{L}}(X)$, we obtain from Theorem 2.14 that, with $P_t := \{0,t\}$, $\mathcal{G}_X(\cdot + P_t) \leq A$, everywhere on the complement of some nullset $\mathcal{N}_t$. However, the diagonal elements of $\widetilde{G}_X(\cdot + P_t)$ are assumed to be no smaller than A, and this implies thus that the off-diagonal elements of $\widetilde{G}(\cdot + P_t)$ are 0, i.e., $\widetilde{G}(\omega, \omega + t) = 0$ for $\omega \in \mathbb{R}^d \backslash \mathcal{N}_t$. Since $\widetilde{\mathcal{L}}(X)$ is countable, we conclude that $\mathcal{N} := \cup\{\mathcal{N}_t : t \in \widetilde{L}(X)\}$ is a nullset, and that $\widetilde{G}(\omega,\tau) = 0$ on $\mathbb{R}^d \backslash \mathcal{N}$, unless $\omega = \tau$.

For (ii), we note that $\|\mathcal{G}^-\|_{L_\infty} \geq \|1/\widetilde{g}\|_{L_\infty}$. Thus, the assumption in (ii) implies that $1/\widetilde{g} \leq 1$ a.e. Hence (i) applies to show that X is diagonal. □

Corollary 3.7. *Let X be a GSI system with small tail. Then*
(i) X is a fundamental tight frame if and only if X is scalar.
(ii) X is an orthonormal basis of $L_2(\mathbb{R}^d)$ if and only if (a) each generator ϕ_j, $j \in J(X)$ has L_2-norm 1, (b) X is scalar.

Proof: (i): the "if" implication was proved in Corollary 3.5. For the "only if" implication, upon assuming X to be tight, we invoke Corollary 2.27 to conclude that $\mathcal{G}^-$ is bounded here by 1. This, in view of (ii) of Lemma 3.6, proves that X is scalar.

Part (ii) follows from (i) by a standard argument (cf. [RS3]). □

Remark. Note that, in contrast with the general analysis of fundamental frames (cf. e.g., Corollary 2.28), our analysis of the tight frame case does not require the FI condition. In fact, a closer scrutiny of our argument for the tight frame case reveals that all the relevant results rely only on the verification of the quadratic form representation (2.12) (for a dense subspace of $L_2(\mathbb{R}^d)$).

3.2. Dominance in uniform and oblique oversampling. Nested systems

"Oversampling" is a general procedure that extends a given system to a larger one by adding more elements and, when necessary, modifying the norms of existing ones. In [RS3], in the context of wavelet systems, we considered two different versions of oversampling: the first one, which was originated in [CS1], will be referred to hereafter as *uniform*. While [RS3] investigated in detail the uniform oversampling of *wavelet systems with integer dilation matrix*, the recent article [CCMW] obtained interesting results on the uniform oversampling of tight wavelet frames that correspond to general dilation matrices. We will recapture all these results as special cases of the uniform oversampling of GSI systems.

Another type of oversampling, which was introduced in [RS3], and which led us to the introduction of quasi-affine systems, will be labeled later on as *oblique*. We will approach this oversampling class via the notion of *dominance*, and will establish basic oblique oversampling results for *nested* systems. *Dominance* is a useful way for comparing two GSI systems via the inspection of the entries of their dual Gramians. Of course, we will define each and every of the above-mentioned notions in the sequel.

Definition 3.8: Oversampling. *Let $X = \cup_{j\in J} Y_j$ and $X^o = \cup_{j\in J} Y^o_j$ be two GSI systems, whose layers are indexed by the same index set J. We say that X^o is an oversampling of X if, for every $j \in J$, the following holds:*
(i) The lattice L^o_j (of the layer Y^o_j) is a superlattice of L_j.
(ii) With ϕ_j and ϕ^o_j the generators of Y_j and Y^o_j, we have the connection

$$\phi^o_j = \left(\frac{|L^o_j|}{|L_j|}\right)^{\frac{1}{2}} \phi_j.$$

□

The key observation concerning oversampling is the following:

Proposition 3.9. *Let X^o be an oversampling of a GSI system X. Let κ and κ^o be the valuation functions of X and X^o respectively (cf. the display above (2.1)). Then:*
(i) $\widetilde{G}_X(\omega,\tau) = \widetilde{G}_{X^o}(\omega,\tau)$ whenever $\kappa(\omega-\tau) = \kappa^o(\omega-\tau)$.
(ii) The two systems have identical diagonal functions:

$$\widetilde{g}_X = \widetilde{g}_{X^o}.$$

Proof: (i) is follows directly from the definitions of oversampling and dual Gramians. (ii) is a special case of (i) since $\kappa(0)$ always equals $\kappa^o(0)$. □

In order to connect between properties of a GSI system and its oversampling, we will need to know the relation between the off-diagonal entries of their dual Gramians. We will investigate this issue shortly. But first, we would like to define *uniform oversampling*:

Definition 3.10: uniform oversampling. Writing each lattice L_j as $L_j = R_j\,\mathbb{Z}^d$, with R_j a linear bijection, the uniform oversampling is obtained by choosing a matrix S whose inverse is integral, and defining

$$L_j^o := R_j S\,\mathbb{Z}^d, \quad j \in J(X).$$

Since $S\,\mathbb{Z}^d \supset \mathbb{Z}^d$, we have that $L_j^o \supset L_j$. Also, $\phi_j^o = |\det S|^{\frac{1}{2}}\phi_j$. The adjective "uniform" refers here to the fact that S is chosen independently of $j \in J(X)$. □

The critical notion that allows one to connect between a system X and its oversampling X^o is *compatibility*, which is defined as follows:

Definition: compatible oversampling. Let X and X^o be as in Definition 3.8. Let κ and κ^o be the valuation functions of X and X^o respectively. We say that the oversampling X^o is compatible (with X), if, for every $\omega \in \mathbb{R}^d$, the assumption $\kappa^o(\omega) \neq \emptyset$ implies that $\kappa^o(\omega) = \kappa(\omega)$. □

For the case of compatible oversampling, we have the following (immediate) strengthening of Proposition 3.9:

Proposition 3.11. *Let X^o be a compatible oversampling of a GSI system X. Then, for every $\omega, \tau \in \mathbb{R}^d$, one of the following two conditions must hold:*
(i) $\widetilde{G}_{X^o}(\omega,\tau) = \widetilde{G}_X(\omega,\tau)$, or
(ii) $\kappa_{X^o}(\omega-\tau) = \emptyset$ (and in particular, $\widetilde{G}_{X^o}(\omega,\tau) = 0$). □

Discussion. The notion of compatibility is purely in terms of the lattices in $\mathcal{L}(X)$ and $\mathcal{L}(X^o)$ and is not related to the actual selection of generators of the systems. In order to illustrate this notion, let us consider the example where each lattice L_j is scalar: $L_j = a_j\,\mathbb{Z}^d$, for a positive a_j, and where the oversampling is uniform and scalar, i.e., $L_j^o = L_j/n$, for some fixed integer n. A simple computation then shows that compatibility means here the following: for every $j, j' \in J(X)$, if $a_j/a_{j'}$ is a rational p/q with $g.c.d(p,q) = 1$, then $g.c.d(p,n) = 1$.

More generally, suppose that X is associated with lattices $L_j = R_j\,\mathbb{Z}^d$, $j \in J(X)$. Let X^o be a uniform oversampling of X with respect to the oversampling matrix S (see Example 3.10). Then, $\widetilde{L}_j = 2\pi R^{*-1}\,\mathbb{Z}^d$ and $\widetilde{L}_j^o = 2\pi R^{*-1}S^{*-1}\,\mathbb{Z}^d$, $j \in J$. The compatibility assumption then reads as

$$R_{j'}^{*-1}\,\mathbb{Z}^d \cap R_j^{*-1}S^{*-1}\,\mathbb{Z}^d \subset R_{j'}^{*-1}S^{*-1}\,\mathbb{Z}^d, \quad \forall j, j' \in J. \tag{3.12}$$

□

Our first result shows that compatible oversampling of diagonal systems yields a new diagonal system with the same frame bounds.

Theorem 3.13. *Let X be a diagonal GSI system with small tail, and X^o be a compatible oversampling system of it.*

(i) X is a Bessel system if and only if X^o is. The two systems have then the same Bessel bound.

(ii) X is a fundamental frame if and only if X^o is a fundamental frame. The two systems have then the same frame bounds.

(iii) If X is a fundamental tight frame if and only if X^o is.

Proof: (iii) follows from (ii), and (i) requires a subset of the arguments we use for (ii). Hence we prove only (ii). We first note that, since X has a small tail, so does X^o, as follows easily from Remark 2.26. Now, since we assume that X is diagonal, Proposition 3.11 can be invoked to yield that X^o is diagonal, too. Since the two systems have the same diagonal function (Proposition 3.9), we obtain the desired results by appealing to Corollary 3.5. □

When we oversample a system which is not diagonal, we need to assume a bit more than compatibility. We use to this end the following notion of dominance:

Definition 3.14. *Let X^o and X be two GSI systems. We say that X* **dominates** *X^o if*

(i) X^o is locally block-diagonal: for every compact Ω, $\widetilde{G}_{X^o}(\Omega)$ is block-diagonal with respect to some lattice M_Ω.

(ii) Let M be the union of all the lattices of the form M_Ω from (i). Then $\widetilde{G}_X(\omega,\tau) = \widetilde{G}_{X^o}(\omega,\tau)$ whenever $\omega-\tau\in M$ and $\omega\in \mathbb{R}^d\setminus\mathcal{N}$. Here, $\mathcal{N}$ is a fixed nullset. □

The essence of dominance is that $\widetilde{G}_X$ coincides with $\widetilde{G}_{X^o}$ at all the entries that "matter", i.e., all those that need to be used to order to determine the Bessel and frame properties of X^o. Indeed, we have:

Proposition 3.15. *Let X^o and X be GSI systems with small tails. Assume that X dominates X^o.*

(i) If X is Bessel then X^o is Bessel, too and its Bessel bound is no larger than that of X.

(ii) If X is a fundamental frame, so is X^o and its lower frame bound is no smaller than that of X.

(iii) In particular, X^o is a fundamental tight frame whenever X is.

Proof: Since X^o is locally block-diagonal,

$$\|\mathcal{G}_{X^o}\| = \sup_\Omega \|\mathcal{G}_{X^o}(\cdot + M_\Omega)\|_{L_\infty(\mathbb{R}^d)},$$

where Ω varies over all compact subsets of $\mathbb{R}^d$. On the other hand, the inequality

$$\|\mathcal{G}_X\| \ge \sup_\Omega \|\mathcal{G}_X(\cdot + M_\Omega)\|_{L_\infty(\mathbb{R}^d)}$$

is valid for every GSI system X including the current one (directly from the definition of the norm function). Since the dominance assumption shows that $\mathcal{G}_X(\cdot + M_\Omega) = \mathcal{G}_{X^o}(\cdot + M_\Omega)$, (a.e.) for every compact Ω, we conclude that $\|\mathcal{G}_{X^o}\| \le \|\mathcal{G}_X\|$. A similar argument yields an analogous relation on the inverse norm functions of X and X^o.

Now, Theorem 3.4 ascertains that, with $A^o \in [0,\infty]$ the Bessel bound of X^o, $\|\mathcal{G}_{X^o}\| = A^o$. At the same time, Theorem 2.14 shows that, with $A \in [0,\infty]$ the Bessel bound of X, $\|\mathcal{G}_X\| \le A$. Altogether we obtain that $A^o \le A$, as claimed.

The corresponding inequality on the lower frame bound ($B \le B^o$) is proved analogously, by using this time Corollary 2.27, instead of Theorem 2.14. □

Discussion: dominance vs. compatibility. The definition of dominance does not stipulate any conditions that lead to the requisite similarities between the dual Gramians of X and X^o. It is possible that those similarities are due to relations between the lattices in $\mathcal{L}(X)$ and their counterparts in $\mathcal{L}(X^o)$. This type of dominance is closely related to compatibility. In fact, once we assume that X^o is block-diagonal, then (with M as in Definition 3.14) compatibility *implies* dominance, provided that

$$M \subset \cup_{j \in J(X)} \widetilde{L}_j^o.$$

However, dominance may also be the result of a more intricate relation that involves the generators of the two systems. We provide now two examples for the former type of dominance. Theorem 3.13 already provides an illustration to the latter type of dominance. □

Definition 3.16: Nested systems. Let X be a GSI system, and assume that $\le$ is some full ordering of $J(X)$. We say that X is **nested** if, for every $j, j' \in J(X)$,

$$j \le j' \Longleftrightarrow L_j \subset L_{j'}.$$

An example of a nested system is a wavelet system whose dilation parameter is integral. □

Definition 3.17: Oblique oversampling. Let X be a nested system. Let X^o be an oversampling of X. We say that the oversampling is *oblique* if there exists $j_0 \in J(X)$ such that, for every $j \in J(X)$,

$$L_j^o = \begin{cases} L_j, & j > j_0, \\ L_{j_0}, & j \le j_0. \end{cases}$$

We note that the oblique oversampling system X^o is always shift-invariant (with respect to L_{j_0}-shifts). The most interesting example of oblique oversampling is that of a *quasi-affine* system. This case is defined and discussed in §3.4. □

Corollary 3.18. *Let X be a nested GSI system with small tail, and let X^o be an oblique oversampling of it. Then X^o has a small tail and is dominated by X (and hence Proposition 3.15 applies to this case).*

Proof: Since X^o is shift-invariant, it trivially has a small tail (it is actually tail-less). It is also block-diagonal with respect to the lattice $\widetilde{L}_{j_0}$ (with j_0 as in the definition of oblique oversampling.) Thus, to prove the dominance, we need to show that $\kappa^o = \kappa$ on $\widetilde{L}_{j_0}$ (since that, in view of Proposition 3.9, implies that the two dual Gramians coincide on the fibers (=blocks) of $\widetilde{G}_{X^o}$).

So, we show that, for $l \in \widetilde{L}_{j_0}$, and for $j \in J(X)$, $j \in \kappa(l) \Longleftrightarrow j \in \kappa^o(l)$, i.e.,

$$l \in \widetilde{L}_j \Longleftrightarrow l \in \widetilde{L}^o_j. \tag{3.19}$$

For $j \geq j_0$, the above is a triviality, since $L_j = L^o_j$ in that case. Otherwise, $j < j_0$, and then $\widetilde{L}^o_j = \widetilde{L}_{j_0} \subset \widetilde{L}_j$ (with the equality follows from the obliqueness of the oversampling, while the inclusion follows from the nestedness assumption). Since we assume $l \in \widetilde{L}_{j_0}$, (3.19) follows. $\square$

3.3. Wavelet systems

We apply in this subsection our theory of GSI systems to one of its most important special cases, viz., wavelet systems. There were many contributions, during the last decade, to the study of the Bessel, frame and other related properties of wavelet systems. Examples of univariate wavelet frames could already be found in [DGM]; necessary and sufficient conditions for mother wavelets to generate frames were discussed (implicitly) in [Me] and [D2]. Characterizations of univariate tight wavelet frames associated with integer dilation were established in [FGWW] and [HW], with the multivariate counterparts of these results appearing in [H]. More recently, a characterization of wavelet tight frames and bi-frames for non-integer dilation matrices were obtained in [CS2] (the univariate case) and [CCMW] (the multivariate case). Independently of all these, we provided in [RS3] a general characterization of all wavelet frames whose dilation matrix is integral (via dual Gramian analysis) and derived from it a special characterization of tight wavelet frames. The article [RS3] then continues to define the notion of MRA-based wavelet constructions and to provide a complete characterization of all MRA-based tight wavelet frames. The theory of [RS3] led several authors to developing various interesting constructions of compactly supported tight wavelet systems (see e.g. [CHS], [DHRS], [GR], [RS3], [RS4], [RS5]). Finally, band-limited wavelet tight frames were constructed earlier through multiresolution analysis in [BL].

Definition 3.20: wavelet systems. *Let $\Psi \subset L_2(\mathbb{R}^d)$ be a finite set. Let s be a $d \times d$ expansive matrix, i.e., a matrix whose spectrum lies outside the closed unit disc. We say that the GSI system X is a wavelet system associated with the mother wavelet set Ψ and the dilation matrix s if it can be written as a union*

$$X = \cup_{j=-\infty}^{\infty} X_j,$$

where

$$X_j := \{(\mathcal{D}^j\psi)(\cdot + k) :\ \psi \in \Psi,\quad k \in s^{-j}\,\mathbb{Z}^d\},$$

and

$$\mathcal{D}^j : f \mapsto |\det s|^{\frac{j}{2}} f(s^j\cdot). \tag{3.21}$$

□

We can also write each X_j above as a union of layers of the form $Y_{j,\mathcal{D}^j\psi}$, $\psi \in \Psi$, with

$$Y_{j,\mathcal{D}^j\psi} := \{(\mathcal{D}^j\psi)(\cdot + k),\quad k \in s^{-j}\,\mathbb{Z}^d\}.$$

The representation

$$X = \cup\{Y_{j,\mathcal{D}^j\psi} :\ j \in \mathbb{Z},\ \psi \in \Psi\}$$

evidently shows that, indeed, every wavelet system is a GSI one.

A straightforward computation shows that, given a wavelet system X, its dual Gramian can be written directly in terms of the mother wavelets Ψ and the dilation matrix s as follows:

$$\widetilde{G}_X(\omega,\tau) = \sum_{\psi\in\Psi}\sum_{j\in\kappa(\omega-\tau)} \widehat{\psi}(s^{*j}\omega)\overline{\widehat{\psi}(s^{*j}\tau)},\quad \omega,\tau \in \mathbb{R}^d,$$

where

$$\kappa(\omega) := \{j \in \mathbb{Z} : \omega \in 2\pi s^{*-j}\,\mathbb{Z}^d\}. \tag{3.22}$$

Thus, the diagonal function $\widetilde{g}$ of a wavelet system has the form

$$\widetilde{g} : \omega \mapsto \sum_{\psi\in\Psi, j\in\mathbb{Z}} |\widehat{\psi}(s^{*j}\omega)|^2. \tag{3.23}$$

One of the two main results of this subsection concerns the analysis of diagonal wavelet systems (and in particular tight wavelet frames). The result is nothing but a rewrite of Corollaries 3.5 and 3.7 for the special case of a wavelet system, using the above explicit formulæ for the entries of the dual Gramian of a wavelet system:

Corollary 3.24. *Let X be a wavelet system with small tail.*

(a) Assume that X is diagonal, i.e., that, a.e.,

$$\sum_{\psi\in\Psi}\sum_{j\in\kappa(\omega-\tau)} \widehat{\psi}(s^{*j}\omega)\overline{\widehat{\psi}}(s^{*j}\tau) = 0, \tag{3.25}$$

unless $\omega = \tau$. Then:
(i) X is Bessel with Bessel bound A if and only if $\|\widetilde{g}\|_{L_\infty} \le A$, with $\widetilde{g}$ the diagonal function from (3.23).
(ii) For a Bessel X, X is a fundamental frame with lower frame bound B if and only if $\|1/\widetilde{g}\|_{L_\infty} \le 1/B$.

(b) X is a fundamental tight frame if and only if it is scalar, i.e., (3.25) holds, and $\widetilde{g} = 1$ a.e. □

The tight frame characterization in (b) is due to [CCMW]. In fact, the result there is proved without assuming X to have a small tail. We will revisit this issue in §4.

Our other results in §2, §3.1 and §3.2 can be specialized to wavelet systems, too. As an illustration, we discuss the case when the dilation matrix s is rational-valued. Note that, for a wavelet system X, the lattices in $\mathcal{L}(X)$ are all of the form $s^j\,\mathbb{Z}^d$, $j \in \mathbb{Z}$, and, per our assumption here, they are all rational. Consequently, the corresponding wavelet system X satisfies the FI condition. Applying, then, Corollary 2.28, we have the following characterizations:

Theorem 3.26. *Let X be a wavelet system with small tail, and assume that the underlying dilation matrix is rational. Let $\mathcal{G}$ and $\mathcal{G}^-$ be the associated norm functions. Then,*

(i) X is Bessel with Bessel bound A if and only if $\mathcal{G}$ is bounded by A.

(ii) Assume that X is Bessel. Then X is a fundamental frame with lower frame bound B if and only if $\mathcal{G}^-$ is bounded with bound $1/B$. □

The special case of an *integral* s in the above characterization can be found in [RS3].

The above results require X to have a small tail. We would like to emphasize that, for a wavelet system, this is a very mild assumption, made on the smoothness of the mother wavelets Ψ (cf. Remark 2.26), or, equivalently, on the decay of their Fourier transform. Even the famous Haar wavelet system (in any number of dimensions) satisfies this condition. In passing, we recall a similar assumption that we adopted in [RS3], and which implies the small tail condition. To describe it, set, for every $j \in \mathbb{Z}_+$,

$$A_j := \{\alpha \in 2\pi\,\mathbb{Z}^d : |\alpha| > 2^j\}, \tag{3.27}$$

and

$$c(\psi, j) := \|\sum_{\alpha \in A_j} |\widehat{\psi}(\cdot + \alpha)|^2\|_{L_\infty([-\pi,\pi]^d)}.$$

The condition we have just alluded to is:

$$\sum_{\psi \in \Psi} \sum_{j=0}^{\infty} c(\psi, j) < \infty. \tag{3.28}$$

Lemma 3.29. *Let X be a wavelet system associated with mother wavelets Ψ, and a dilation matrix s. Assume that Ψ satisfies (3.28). Then X has a small tail.*

Proof: We prove the lemma for a singleton Ψ. The extension of the argument to a general Ψ is purely notational. For a singleton $\Psi = \{\psi\}$, the wavelet system X can be indexed by $J = \mathbb{Z}$, with each layer Y_j consisting of the $s^{-j}\,\mathbb{Z}^d$-shifts of the function $\phi_j := \mathcal{D}^j\psi$. Note that, with $L_j := s^{-j}\,\mathbb{Z}^d$, we have that $\widetilde{L}_j = 2\pi s^{*j}\,\mathbb{Z}^d$. Also, $|L_j|^{-1}|\widehat{\phi}_j|^2 = |\widehat{\psi}(s^{*-j}\cdot)|^2$.

Since the dilation matrix s is expansive, it follows that, for any $j' \in \mathbb{Z}$, the subsystem $\cup_{j>j'} Y_j$ is tailless. Thus, in view of Remark 2.26 (together with the observations in the previous paragraph), it remains to prove that for an arbitrary given Ω that excludes the origin, there exists j' such that

$$\sum_{j<j'} \| \sum_{\alpha \in 2\pi s^{*j} \mathbb{Z}^d} |(\chi_\Omega \widehat{\psi})(s^{*-j}(\cdot + \alpha))|^2 \|_{L_\infty(\mathbb{R}^d)} < \infty.$$

The last condition is the same as

$$\sum_{j<j'} \| \sum_{\alpha \in 2\pi \mathbb{Z}^d} |(\chi_{s^{*-j}\Omega} \widehat{\psi})(\cdot + \alpha)|^2 \|_{L_\infty(\mathbb{R}^d)} < \infty.$$

Since that $L_\infty(\mathbb{R}^d)$-norm above is applied to 2π-periodic functions, we can replace it by an $L_\infty([-\pi, \pi]^d)$-norm, as we do. Without loss, we may now assume that Ω is the complement of some ball centered at the origin of radius δ. Since s^* is expansive, there exists $C > 0$ and $\lambda > 1$ such that $s^{*j}\Omega$ is contained in the complement Ω_j of a ball centered at the origin of radius $C\lambda^j\delta$. Thus, we may prove that

$$\sum_{j<j'} \| \sum_{\alpha \in 2\pi \mathbb{Z}^d} |(\chi_{\Omega_{-j}} \widehat{\psi})(\cdot + \alpha)|^2 \|_{L_\infty(\mathbb{R}^d)} < \infty.$$

Finally, since the above series is monotonic in j, we may prove, instead, that for a given fixed positive integer m, there exists an integer j' such that

$$\sum_{j<j'} \| \sum_{\alpha \in 2\pi \mathbb{Z}^d} |(\chi_{\Omega_{-jm}} \widehat{\psi})(\cdot + \alpha)|^2 \|_{L_\infty(\mathbb{R}^d)} < \infty. \tag{3.30}$$

We choose m large enough so that $C\lambda^{jm}\delta \geq 2^{j+1}$, for every positive integer j. Then, for every sufficiently large j,

$$\Omega_{jm} \cap (\alpha + [\pi, \pi]^d) = \emptyset, \quad \forall \alpha \in (2\pi \mathbb{Z}^d) \backslash A_j,$$

(with A_j defined as in (3.27)). This shows that the left-hand-side of (3.30) is bounded, up to a constant that depends only on Ω and s (and m), by

$$\sum_{k>-j'} \| \sum_{\alpha \in A_k} |\widehat{\psi}(\cdot + \alpha)|^2 \|_{L_\infty([\pi,\pi]^d)}.$$

Thanks to (3.28), this last expression is bounded. $\square$

3.4. Variations of wavelet systems

The theory of GSI systems allows us to analyse many different systems that are obtained as a variation of the wavelet theme. We discuss some of these possibilities in the current section. Let us list first a few options in this direction:

Tailless wavelets. Such systems are obtained by removing first from the wavelet system X all its negative dilation layers, to obtain the so-called *truncated wavelet system*. The truncated wavelet system (which is always tailless) is then supplemented by a shift-invariant system. We have two major examples to offer in this regard: *MRA tailless systems*, and *quasi-affine systems*. In the first case the shift-invariant supplement is generated by the underlying refinable function(s), while in the second case the shift-invariant supplement is created by an oblique oversampling of the dilation layers that were removed. We provide more details on these variations in the sequel.

Non-stationary wavelets. In this case we still assume that each layer X_j is invariant under $s^{-j}\,\mathbb{Z}^d$-shifts (as in the definition of wavelet systems), but do not make any specific assumption on the generators used at each level $j \in \mathbb{Z}$. Thus,

$$X_j := \{(\mathcal{D}^j\psi)(\cdot + k) : \; k \in s^{-j}\,\mathbb{Z}^d, \psi \in \Psi_j\},$$

(with $\mathcal{D}$ as in (3.21) and) with the generating set Ψ_j depending now on j (as opposed to the wavelet case, where the same set of mother wavelets is employed at all layers). The analysis of non-stationary wavelets is identical to that of their wavelet counterparts, with two major exceptions. First, the dual Gramian now has the form

$$\widetilde{G}(\omega,\tau) = \sum_{j\in\kappa(\omega-\tau)} \sum_{\psi\in\Psi_j} \widehat{\psi}(s^{*j}\omega)\overline{\widehat{\psi}(s^{*j}\tau)}, \quad \omega,\tau \in \mathbb{R}^d,$$

(with κ defined as in the wavelet case). Second, the counterparts of conditions like (3.28) (that are aimed at securing the small tail property) are more involved, and it is impossible to assess any more whether these conditions are "mild". We leave it to the reader to restate results like Corollary 3.24 and Theorem 3.26 for this non-stationary setup. $\square$

Müntz wavelets. Let us assume that the dilation matrix s in the definition of a wavelet system is symmetric and positive-definite: $s = P^{-1}\Lambda P$, where Λ is diagonal (and its entries are > 1), and P is orthogonal. In this case, we define

$$s^t := P^{-1}\Lambda^t P, \quad t \in \mathbb{R}.$$

With that, the dilation process in the definition of a wavelet system need not be restricted to integer powers of s. Indeed, we may, for each $j \in \mathbb{Z}$, select $t_j \in \mathbb{R}$, and define

$$X_j := \{(\mathcal{D}^{t_j}\psi)(\cdot + k) : \; k \in s^{-t_j}\,\mathbb{Z}^d\},$$

with

$$\mathcal{D}^t : f \mapsto |\det s|^{t/2} f(s^t\cdot).$$

One calculates that the dual Gramian entries of such system have the form (cf. (3.22) for a definition of κ)

$$\sum_{\psi\in\Psi}\sum_{j\in\kappa(\omega-\tau)}\widehat{\psi}(s^{*t_j}\omega)\overline{\widehat{\psi}(s^{*t_j}\tau)}.$$

The analysis of diagonal Müntz wavelet systems as well as tight Müntz wavelet systems is entirely analogous to their wavelet counterpart, and results like Corollary 3.24 and Theorem 3.26 extend *verbatim* to this case. However, the analysis of non-diagonal Müntz systems is harder than that of their wavelet counterparts: our main tool in that analysis is the fiberizability of the system, viz., the fact that, at least locally, the dual Gramian of the system is block-diagonal. Many wavelet systems satisfy that condition; e.g., all the systems whose dilation matrix is rational. In contrast, Müntz wavelet systems, even those that are based on rational dilation, are not, in general, fiberizable. □

Multiple wavelet systems: In general, the union of two or more wavelet systems is not a wavelet system any more. For example, this may be due to the fact that the different systems employ different dilation processes (i.e., different dilation matrices s). However, each wavelet system is a GSI one, and, as said, GSI systems *are* closed under (countable) unions. We refer to a finite union of wavelet systems as a *multiple wavelet system*. Obviously, the dual Gramian $\widetilde{G}$ of a multiple wavelet system is the sum of the individual dual Gramians:

$$\widetilde{G}=\sum_{i=1}^{n}\widetilde{G}_i.$$

If we denote by Ψ^i the mother wavelets of the ith wavelet system, and by s_i the corresponding dilation matrix, we get that (cf. the display above (3.22))

$$\widetilde{G}(\omega,\tau)=\sum_{i=1}^{n}\sum_{\psi\in\Psi^i}\sum_{j\in\kappa_i(\omega-\tau)}\widehat{\psi}(s_i^{*j}\omega)\overline{\widehat{\psi}(s_i^{*j}\tau)}, \tag{3.31}$$

with κ_i the valuation function associated with the ith system (cf. (3.22)). With this we have:

Corollary 3.32. *The conclusions of Corollary 3.24 and Theorem 3.26 hold for the multiple wavelet systems, with the left-hand-side of (3.25) replaced by (3.31).*

We note that the multiple wavelet system has a small tail exactly when each of its components has a small tail. □

One can also mix the above generalizations, and talk about MRA tailless Müntz wavelets, non-stationary quasi-affine systems and the like. The above discussion should provide interested readers with sufficient guidance in order to explore those extensions on their own.

In the rest of this subsection, we analyse further the case of tailless wavelet systems.

(3.33) Analysis of tailless wavelet systems: Given a wavelet system X associated with a mother wavelet set Ψ and dilation matrix s, we define, as before, for $j \in \mathbb{Z}$,

$$X_j := \{(\mathcal{D}^j \psi)(\cdot + k) : \ \psi \in \Psi, \quad k \in s^{-j}\, \mathbb{Z}^d\}. \tag{3.34}$$

The **truncated wavelet system** X_+ is then defined as

$$X_+ := \cup_{j=0}^{\infty} X_j. \tag{3.35}$$

The truncated system has a much simpler structure than the original X: for example, it is always tailless (and is even shift-invariant, provided that s is an integer matrix). At the same time, the system X_+ is usually *deficient* (see Corollary 4.12 of [RS3] for a concrete statement in this regard), and should be "compensated" for the removal of the negative dilation levels. The "compensation" is carried out by appending to X_+ a shift-invariant system, as follows:

One selects a (finite or countable) set $\Phi \subset L_2(\mathbb{R}^d)$, and defines the corresponding SI system:

$$E(\Phi) := E_{\mathbb{Z}^d}(\Phi), \quad E_L(\Phi) := \{\phi(\cdot + k) : \ \phi \in \Phi, \ k \in L\}. \tag{3.36}$$

The augmented system

$$X^t := X_+ \cup E(\Phi) \tag{3.37}$$

is a **tailless wavelet system**. Of course, the above description is too generic: one should expect the set Φ to be related in some meaningful way to the original wavelet system X. We discuss this point in the sequel.

Since the dilation matrix s (in the definition of a wavelet system) is assumed to be expansive, one easily checks that the truncated system X_+ is indeed *tailless*. Since every SI system is automatically tailless, we have that X^t is tailless (as is already indicated in its name). This allows us to invoke the previous analysis of tailless systems. For example, Corollaries 3.5 and 3.7 read here as follows:

Corollary 3.38. *Let X be a tailless wavelet system as defined in (3.37). Let $\widetilde{G}$ be the associated dual Gramian kernel, and $\widetilde{g}$ the diagonal function. Then:*

(a) If X is diagonal, then:

(i) X is Bessel with Bessel bound A if and only if $\|\widetilde{g}\|_{L_\infty} = A$.

(ii) For a Bessel X, X is a fundamental frame with lower frame bound B if and only if $\|1/\widetilde{g}\|_{L_\infty} = 1/B$.

(b) X is a fundamental tight frame if and only if X is scalar.

(c) X is an orthonormal basis of $L_2(\mathbb{R}^d)$ if and only if (1) each $\phi \in \Phi$, as well as each mother wavelet $\psi \in \Psi$ has L_2-norm 1, (2) X is scalar. □

In order to assist the reader with the digestion of a result like the above, it is worthwhile to write explicitly the dual Gramian of a tailless wavelet system. This dual Gramian is the sum

$$\widetilde{G}_{X^t} = \widetilde{G}_{X_+} + \widetilde{G}_{E(\Phi)}.$$

The dual Gramian of the truncated wavelet X_+ is analogous to that of its wavelet counterpart:

$$\widetilde{G}_{X_+}(\omega,\tau) := \sum_{\psi\in\Psi} \sum_{j\in\kappa_+(\omega-\tau)} \widehat{\psi}(s^{*j}\omega)\overline{\widehat{\psi}(s^{*j}\tau)}, \quad \omega,\tau \in \mathbb{R}^d,$$

where

$$\kappa_+(\omega) := \{j \in \mathbb{Z}_+ : \ \omega \in 2\pi s^{*-j}\, \mathbb{Z}^d\}.$$

The dual Gramian of $E(\Phi)$ has the form

$$\widetilde{G}_{E(\Phi)}(\omega,\tau) := \sum_{\phi\in\Phi} \delta(\omega-\tau)\widehat{\phi}(\omega)\overline{\widehat{\phi}(\tau)}, \quad \omega,\tau \in \mathbb{R}^d,$$

where

$$\delta(\omega) := \begin{cases} 1, & \omega \in 2\pi\, \mathbb{Z}^d, \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 3.38 covers diagonal tailless wavelet systems. As to the non-diagonal case, several of our general results from §2 and §3.1 apply well here. For example, if we assume the dilation matrix s to be rational, then the tailless system X^t will satisfy the FI property and will be covered by Corollary 2.28. The result then reads as follows:

Corollary 3.39. *Let X be a tailless wavelet system as defined in (3.37). Let $\mathcal{G}$ and $\mathcal{G}^-$ be the corresponding norm functions. Assume that the dilation matrix s is rational. Then,*
(i) X is Bessel with Bessel bound A if and only if $\mathcal{G}$ is bounded by A.
(ii) Assume that X is Bessel. Then, X is a fundamental frame with lower frame bound B if and only if $\mathcal{G}^-$ is bounded with bound $1/B$. □

Remark. Tailless wavelet systems can be derived from a wavelet system by truncating the latter at any level j (instead of level 0). The proper way to approach this generalization is as follows: first, one constructs a tailless wavelet system, as outlined and analysed above. Then, for a suitably selected integer j, one replaces X^t by its dilated version $\mathcal{D}^j X^t$. Since $\mathcal{D}$ is unitary, our analysis above applies *verbatim* to this more general case.

We also note that tailless wavelet systems are common in applications: if the original wavelet system is constructed via the MRA tool, it is common to append to the truncated system the integer shifts of the *refinable function* that generates the multiresolution analysis. (And then, if needed, to dilate the entire system, cf. the discussion in the preceding paragraph.) □

Quasi-affine systems. A quasi-affine system is a special case of a tailless wavelet one. Given a wavelet system X whose mother wavelet set is Ψ and whose dilation matrix is s, its quasi-affine counterpart, X^q, is defined as

$$X^q := X_+ \cup \ E_L(\Psi^q) =: X_+ \cup X_-^q,$$

where L is some lattice, E_L is defined as in (3.36), and

$$\Psi^q := \{\sqrt{|\det s|^j |L|}\, \mathcal{D}^j \psi : j < 0,\, \psi \in \Psi\}.$$

The most common choice for the lattice L is $L := \mathbb{Z}^d$. We note that if the dilation matrix s is integral (and, say, $L := \mathbb{Z}^d$), the quasi-affine system is shift-invariant, and is the result of an oblique oversampling of the corresponding wavelet system.

There are two different reasons for our interest in quasi-affine systems. The oblique oversampling procedure may create a system with favorable properties for applications (e.g., denoising. Indeed, one of the prevailing denoising techniques, the so-called *wavelet cyclospinning*, is based, implicitly, on a decomposition using quasi-affine systems).

The original reason for the introduction of quasi-affine systems, [RS3], is the surprising rigid connection between the Bessel and frame properties of the wavelet system X and its quasi-affine X^q counterpart. [RS3] established that connection, and used it in order to analyse wavelet systems via the fiberization of their associated quasi-affine one. Our effort in this paper provides an alternative to that original approach: a direct fiberization of the wavelet system, without a recourse to quasi-affine ones.

In the language introduced in this paper, the basic result of [RS3] is the following (the result in [RS3] is established under a small tail condition. The elimination of that condition is due to [CSS]).

Result 3.40. *Let X be a wavelet system associated with an integer dilation matrix s. Let X^q be its quasi-affine counterpart, corresponding to $L := \mathbb{Z}^d$. Let A, B be two positive numbers. Then:*

(i) X is a Bessel system with Bessel bound A if and only if X^q is a Bessel system with Bessel bound A.

(ii) Assume that X is Bessel. Then X is a fundamental frame with lower frame bound B if and only if X^q is a fundamental frame with lower frame bound B. □

For a dilation matrix which is not necessarily integral, we have the following:

Theorem 3.41. *Let X be the wavelet system as in (3.20), and assume that the mother wavelet set Ψ satisfies (3.28). Then:*

(i) If X^q is a Bessel system, then X is a Bessel system, too. Furthermore, the Bessel bound of X is no larger than that of X^q.

(ii) If X^q is a fundamental frame, X is a fundamental frame, too. Furthermore, the lower bound of X is no smaller than that of X^q.

In particular, if X^q is a fundamental tight frame, X is a fundamental tight frame as well.

Proof: The argument here follows the original argument from [RS3]. As such, it does not require any of the new tools developed in the current paper.

Part (i) is actually trivial: if X^q is Bessel with Bessel bound A, then its subsystem X_+ is Bessel with Bessel bound $\le A$. Now, if X is not Bessel, or its Bessel bound is larger than A, then there exists $f \in L_2(\mathbb{R}^d)$ and an integer j such that

$$\|T^*_{X_+}(\mathcal{D}^{-j}f)\|^2 = \|T^*_{\mathcal{D}^j(X_+)}f\|^2 > A\|f\|^2 = A\|\mathcal{D}^{-j}f\|^2,$$

which is a contradiction.

For (ii), we note that since Ψ satisfies (3.28) one obtains, by the same proof as Lemma 5.4 of [RS3], that for an arbitrary $\varepsilon > 0$, there exists r such that

$$\|T^*_{X^q_-,r}\| \le \varepsilon, \quad X^q_- := X^q \backslash X_+, \tag{3.42}$$

where $T^*_{X^q_-,r}$ is the map $T^*_{X^q_-}$ restricted to

$$H_r := \{f \in L_2(\mathbb{R}^d) : \widehat{f}(\omega) = 0, \ |\omega| \le r\}.$$

If we now assume that X^q is a fundamental frame with frame bound B, then, since, for every f,

$$\|T^*_{X_+}f\|^2 = \|T^*_{X^q}f\|^2 - \|T^*_{X^q_-}f\|^2,$$

we get that, on H_r, $T^*_{X_+}$ is bounded below by $\sqrt{B-\varepsilon^2}$, *a fortiori* T^*_X is bounded below on H_r by that constant.

Now, assume that $f \in L_2(\mathbb{R}^d)$, and that $\widehat{f}$ vanishes on some neighborhood of the origin. Then, for some j, $\mathcal{D}^j f \in H_r$. Since $\mathcal{D}^j X = X$, this implies that

$$\|T^*_X f\|^2 = \|T^*_X(\mathcal{D}^j f)\|^2 \ge (B-\varepsilon^2)\|\mathcal{D}^j f\|^2 = (B-\varepsilon^2)\|f\|^2.$$

Using a density argument, this shows that T^*_X is bounded below on $L_2(\mathbb{R}^d)$ by $\sqrt{B-\varepsilon^2}$, hence by $\sqrt{B}$. $\square$

Remark: the converse implication in Theorem 3.41. If the dilation matrix s is integer, and if $L := \mathbb{Z}^d$, then X^q is an oblique oversampling of X, and Corollary 3.18 (viz., Proposition 3.15) applies. This recaptures Result 3.40. $\square$

3.5. Bi-frames

Let X be a system in $L_2(\mathbb{R}^d)$. Let $\mathrm{R} : X \to L_2(\mathbb{R}^d)$ be some assignment, and assume that both X and $\mathrm{R}(X)$ are Bessel systems. If, with T_X the adjoint operator of T^*_X,

$$\langle f, T_X T^*_{\mathrm{R}X} g\rangle = \langle f, g\rangle$$

for every f, $g \in L_2(\mathbb{R}^d)$, then both X and $\mathrm{R}X$ are fundamental frames and $Sf = f$ for every $f \in L_2(\mathbb{R}^d)$, where $S := T_X T^*_{\mathrm{R}X}$, i.e.,

$$Sf := \sum_{x \in X} \langle f, \mathrm{R}x\rangle\, x, \quad f \in L_2(\mathbb{R}^d). \tag{3.43}$$

We refer to the pair $(X, \mathrm{R}X)$ as a **bi-frame**.

As is shown in the current article, the analysis of GSI fundamental frames is significantly more involved than that of the special tight frame case. In contrast, it is quite well-understood (see e.g. [RS4], [Ha] and [CCMW]), that the analysis of bi-frames is almost completely analogous to that of tight frames, once the two underlying systems are assumed to be Bessel. Taking into account that the Bessel property of a GSI system is analysed in §2, we may focus here on the analysis of the bi-frame property, under the assumption that the two underlying systems were already shown to be Bessel.

In the context of GSI systems, one expects the map R above to respect the GSI structure: given a GSI X with generators (ϕ_j) and lattices (L_j), $j \in J(X)$, we assume that

$$(\mathrm{R}\phi_j)(\cdot + l) = \mathrm{R}(\phi_j(\cdot + l)), \quad \forall j \in J(X),\ l \in L_j.$$

We refer then to $(X, \mathrm{R}X)$ as a **GSI dual pair**.

As we explained before, our goal is to analyse the bi-frame property of the GSI systems X and $\mathrm{R}X$ via the dual Gramian analysis as we did for the (tight) frame case. To this end, we first introduce the *mixed* dual Gramian. For given Bessel systems X and $\mathrm{R}X$, the **mixed dual Gramian** of $(X, \mathrm{R}X)$, $\widetilde{G}_{X,\mathrm{R}X} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is defined as follows:

$$\widetilde{G}_{X,\mathrm{R}X}(\omega, \tau) := \sum_{j \in \kappa(\omega - \tau)} \frac{\widehat{\phi}_j(\omega)\, \overline{\widehat{\mathrm{R}\phi_j}(\tau)}}{|L_j|}, \quad \omega, \tau \in \mathbb{R}^d,$$

where

$$\kappa(\omega) := \{j \in J :\ \omega \in \widetilde{L}_j\}.$$

Several properties of the mixed dual Gramian can be proved by carefully modifying the proofs of corresponding results of the dual Gramian. For example, one can prove that the identity

$$(2\pi)^d \langle f, T_X T^*_{\mathrm{R}X} g\rangle = \langle \widehat{f}, \widetilde{G}_{X,\mathrm{R}X} \widehat{g}\rangle$$

holds for, say, band-limited functions (i.e., for functions whose Fourier transform is compactly supported) f and g which are in $L_2(\mathbb{R}^d)$. Such a result is valid for Bessel systems X, $\mathrm{R}X$ which either have small tails, or are both tempered and round, the latter being the subject of §4.

More generally, one has:

Proposition 3.44. *Let $(X, \mathrm{R}X)$ be a GSI dual pair of Bessel systems, associated with the mixed dual Gramian $\widetilde{G}_{X,\mathrm{R}X}$. Assume that both X and $\mathrm{R}X$ have small tails (alternatively, are tempered and round). Then, the pair $(X, \mathrm{R}X)$ is a bi-frame if and only if the associated mixed dual Gramian $\widetilde{G}_{X,\mathrm{R}X}$ is the identity a.e.* □

Applying this result to wavelet systems and their variations is straightforward, and we omit those discussions.

3.6. Proof of Lemma 3.2

Let Ω be as in the lemma, and assume (without loss) that $\Omega-\Omega$ contains some ball U centered at the origin. Let $J(X) = J_1 \cup J_2$ be the decomposition that appears in Definition 2.24. Set $V := (\Omega-\Omega)\backslash(U/2)$, and

$$J_1' := \{j \in J_1 : \widetilde{L}_j \cap V \neq \emptyset\}, \quad J_1'' := J_1 \backslash J_1'.$$

Note first that $(\Omega-\Omega)\cap \widetilde{L}_j = 0$, for every $j \in J_1''$. (Otherwise, $\widetilde{L}_j$ contains points from $(U/2)\backslash 0$, hence must have some intersection with $U\backslash(U/2)$, which is a contradiction since the latter is a subset of V.) This implies that $\widetilde{G}_{Y_j}(\Omega)$, $j \in J_1''$, is diagonal, hence (2.6) implies that, for every $f \in H_\Omega$,

$$(2\pi)^d \|T_{Y_j}^* f\|^2 = \int_{\mathbb{R}^d} |\widehat{f}|^2 \widetilde{g}_{Y_j}.$$

Summing the last identity over $j \in J_1''$, we obtain that

$$(2\pi)^d \|T_{X_1''}^* f\|^2 = \int_{\mathbb{R}^d} |\widehat{f}|^2 \widetilde{g}_{X_1''} = \langle \widehat{f}, \widetilde{G}_{X_1''} \widehat{f}\rangle,$$

where $X_1'' := \cup_{j\in J_1''} Y_j$. Set now

$$X_1' := \cup_{j \in J_1'} Y_j.$$

Since X_1 is tailless, $\mathcal{L}(X_1')$ contains finitely many different lattices, and hence part (iv) of Lemma 2.5 implies that

$$(2\pi)^d \|T_{X_1'}^* f\|^2 = \langle \widehat{f}, \widetilde{G}_{X_1'} \widehat{f}\rangle.$$

It remains to deal with the tail J_2. Let $f \in H_\Omega$. Then, by Cauchy-Swartz,

$$\begin{aligned}\sum_{j\in J_2} \frac{1}{|L_j|}\int_{O_j} [|\widehat{f}|, |\widehat{\phi}_j|]_j^2 &\leq \sum_{j\in J_2} \frac{1}{|L_j|}\int_{O_j} [\widehat{f},\widehat{f}]_j\, [\widehat{\phi}_j \chi_\Omega, \widehat{\phi}_j]_j \\ &\leq \|\widehat{f}\|^2_{L_2(\Omega)} \sum_{j\in J_2} \frac{\|[\chi_\Omega \widehat{\phi}_j, \widehat{\phi}_j]_j\|_{L_\infty(\mathbb{R}^d)}}{|L_j|},\end{aligned} \tag{3.45}$$

with the last expression finite by the small tail assumption (cf. Remark 2.26). Here, as before, O_j is a fundamental domain of the lattice $\widetilde{L}_j$.

Setting $X_2 := \cup_{j\in J_2} Y_j$, we invoke now the left-most equality in (2.6) to obtain that

$$\begin{aligned}(2\pi)^d \|T_{X_2}^* f\|^2 &= \sum_{j\in J_2} |L_j|^{-1} \|[\widehat{f}, \widehat{\phi}_j]_j\|^2_{L_2(O_j)} \\ &= \sum_{j\in J_2} \frac{1}{|L_j|} \int_{\mathbb{R}^d} \sum_{l\in \widetilde{L}_j} \widehat{f}(\omega)\overline{\widehat{\phi}_j(\omega)}\widehat{\phi}_j(\omega+l)\overline{\widehat{f}(\omega+l)}\,d\omega \\ &= \int_{\mathbb{R}^d} \sum_{l\in\widetilde{L}_j} \widehat{f}(\omega) \left(\sum_{j\in J_2} \frac{1}{|L_j|}\overline{\widehat{\phi}_j(\omega)}\widehat{\phi}_j(\omega+l)\right) \overline{\widehat{f}(\omega+l)}\,d\omega \\ &= \langle \widehat{f}, \widetilde{G}_{X_2}\widehat{f}\rangle.\end{aligned}$$

Here, we have used, in the second right-most equality, the absolute convergence that was established in (3.45). Also, a simple (and fairly standard) periodization trick is used in order to obtain the second left-most equality.

In conclusion, we decomposed X into three parts and proved the identity for each of these three. $\square$

4. GSI systems that are tempered and round

Our most general results in this paper are derived under the small tail condition. For the case of a wavelet system, this condition is translated to a mild smoothness requirement on the mother wavelet set Ψ. For example, Corollary 3.24 confirms that a wavelet system, whose generating set Ψ satisfies (3.28), is a fundamental tight frame if and only if its associated dual Gramian $\widetilde{G}$ is scalar, i.e., equals the identity a.e. As we already noted before, this particular result is not new: it was first established in [CCMW], and, moreover, that earlier result was established *without* assuming (3.28). One may argue that (3.28) is a very reasonable assumption: on the one hand, it is truly hard to construct a wavelet frame that does not satisfy it (and we are not aware of any such example), and, on the other hand, the smoothness of the mother wavelets of such construct is so abysmal that such a system lacks any practical value.

That said, a close scrutiny of the argument used in [CCMW] shows that our small tail assumption (and its (3.28) off-spring) can be exchanged for another, closely related condition, which we label in this section as "temperateness". While the small tail condition and the temperateness conditions are very close one to the other, the latter is automatically satisfied by all wavelet systems, leading thereby to cleaner statements and neater theory.

Following, thus, the approach that was used in [CCMW] (for analysing tight wavelet frames), we provide in this section an alternative analysis of GSI frames where the small tail assumption is replaced by a temperateness one.

This section is organized as follows: In the first subsection, we provide the definitions of temperateness and roundedness together with a couple of lemmata. The subsequent subsection includes several results about frames, and in particular a characterization of fundamental tight frames for tempered and round GSI systems. In the last subsection, we illustrate again the efficacy of the dual Gramian kernel characterizations by confirming a conjecture of G. Weiss.

4.1. Definitions and lemmata

We start with the definition of temperateness.

Definition 4.1. Let X be a GSI system associated with lattices L_j, $j \in J(X)$. We say that X is **tempered** if, for every compact Ω that excludes the origin, there exists a decomposition $J(X) = J_1 \cup J_2$, such that

(i) $X_1 := \cup_{j \in J_1} Y_j$ is tailless, and

(ii)

$$\sum_{j \in J_2} \|\widehat{\phi}_j\|^2_{L_2(\Omega)} < \infty.$$

$\square$

We also need the following concept of roundedness.

Definition 4.2. Let X be a GSI system. We say that X is **round** if there exists a constant $c(X) > 0$, such that, for every ball $V \subset \mathbb{R}^d$ and every $j \in J(X)$,

$$\| \sum_{l \in \widetilde{L}_j} \chi_V(\cdot + l) \|_{L_\infty(\mathbb{R}^d)} \leq 1 + c(X) \, |L_j| |V|.$$

Here, χ_V is the support function of V. $\square$

Discussion. The temperateness condition holds for every wavelet system of the form (1.2) (cf. Lemma 4.14 below). The roundedness assumption provides a bound on the number of lattice points in a ball in terms of the determinant of the lattice. For example, if all the lattices $L_j \in \mathcal{L}(X)$ are of the form $a_j R_j \mathbb{Z}^d$, with R_j unitary and $a_j > 0$, then X is round (and $c(X)$ then depends only on the spatial dimension d.) On the other hand, for $d = 2$, if we choose $\mathcal{L}(X)$ to consist of all the lattices of the form

$$L_{k,m} := \begin{pmatrix} 2^k & 0 \\ 0 & 2^m \end{pmatrix} \mathbb{Z}^2, \quad k, m \in \mathbb{Z},$$

then X is not round: $|L_{k,-k}| = 1$ for every k, but a ball V of radius ε whose center is $(2^k, 2^{-k})$, intersects $\widetilde{L}_{k,-k}$ at approximately $2^k \varepsilon^2$ points rather than at $\sim \varepsilon^2$ points. $\square$

In addition to the above definitions, we provide in this subsection two results that we will need later.

Lemma 4.3. *Let X be a round GSI system. Let Ω be a given compact set, $P \subset \mathbb{R}^d$ a finite set, and $f \in H_{(P+\Omega)}$ such that $\widehat{f} \in L_\infty(\mathbb{R}^d)$. Let $\widetilde{G}$ be the dual Gramian of X, and let $\widetilde{g}$ be the diagonal function of X.*

(i) If V is a ball of $\mathbb{R}^d$ that contains Ω, then

(4.4)
$$(2\pi)^d \|T_X^* f\|^2 \leq |P| \, |V| \, \|\widehat{f}\|_{L_\infty(\mathbb{R}^d)} \left(|P| \, \|\widetilde{g}\|_{L_\infty(P+\Omega)} + c(X) \sum_{j \in J(X)} \|\widehat{\phi}_j\|^2_{L_2(P+\Omega)} \right).$$

Here, $c(X)$ is the constant given in Definition 4.2.

(ii) Assume that the system X further satisfies

$$\|\widetilde{g}\|_{L_\infty(P+\Omega)} < \infty \quad \text{and} \quad \sum_{j \in J(X)} \|\widehat{\phi}_j\|^2_{L_2(P+\Omega)} < \infty.$$

Then,

$$\sum_{j \in J(X)} \sum_{l \in \widetilde{L}_j} \int_{\mathbb{R}^d} \frac{1}{|L_j|} \left| \widehat{f}(\omega) \overline{\widehat{\phi}_j(\omega)} \widehat{\phi}_j(\omega + l) \overline{\widehat{f}(\omega + l)} \right| d\omega < \infty.$$

Proof: From Lemma 2.5 (and in the notations of that lemma), we know that, for $f \in H_{P+\Omega}$ with $\widehat{f} \in L_\infty(\mathbb{R}^d)$,

$$(2\pi)^d \|T^*_{Y_j} f\|^2 = |L_j|^{-1} \|[\widehat{f}, \widehat{\phi}_j]_j\|^2_{L_2(O_j)}.$$

We estimate the right-hand side of the above as follows:

$$\|[|\widehat{f}|, |\widehat{\phi}_j|]_j\|^2_{L_2(O_j)} \le \|[\widehat{f}, \widehat{f}]_j [\widehat{\phi}_j \chi_{P+\Omega}, \widehat{\phi}_j]_j\|_{L_1(O_j)} = \|[\widehat{f}, \widehat{f}]_j |\widehat{\phi}_j|^2\|_{L_1(P+\Omega)}, \tag{4.5}$$

with the inequality by Cauchy-Swartz, and the equality by a simple periodization argument.

Now,

$$[\widehat{f}, \widehat{f}]_j \le \|\widehat{f}\|^2_{L_\infty(\mathbb{R}^d)} \| \sum_{l \in \widetilde{L}_j} \chi_{P+\Omega}(\cdot + l)\|_{L_\infty(\mathbb{R}^d)} \le \|\widehat{f}\|^2_{L_\infty(\mathbb{R}^d)} |P| \,(1 + c(X)|V||L_j|),$$

where we invoked the roundedness assumption in the right-most inequality. We conclude from the last three displays that

$$(2\pi)^d \|T^*_{Y_j} f\|^2 \le \|\widehat{f}\|^2_{L_\infty(\mathbb{R}^d)} \,|P|\,(|L_j|^{-1} + c(X)|V|) \|\widehat{\phi}^2_j\|_{L_1(P+\Omega)}.$$

Summing the last inequality over $j \in J$, and using then the trivial inequality

$$\| \sum_{j \in J(X)} \frac{|\widehat{\phi}_j|^2}{|L_j|} \|_{L_1(P+\Omega)} = \|\widetilde{g}\|_{L_1(P+\Omega)} \le |V|\,|P|\,\|\widetilde{g}\|_{L_\infty(P+\Omega)},$$

we get (i).

In order to prove (ii), we first note that, under the assumptions made in (ii), the right-hand side of (4.4) is finite. Next, we multiply the left-hand side expression in (4.5) by $|L_j|^{-1}$, and sum it over all $j \in J(X)$. Then, as a simple periodization argument shows, the resulting expression is nothing but the expression we try to estimate in (ii). Thus, the argument we used for the proof of (i) shows that the expression in (ii) is bounded by the right-hand side of (4.4), hence must be finite. □

Lemma 4.6. *Assume that the GSI system X is tempered and round and that the diagonal function $\widetilde{g}$ of X is in $L_\infty(\mathbb{R}^d)$. Given a compact Ω which excludes the origin, and a function $f \in H_\Omega$ for which $\widehat{f} \in L_\infty(\Omega)$, we have that*

$$(2\pi)^d \|T^*_X f\|^2 = \langle \widehat{f}, \widetilde{G}_X \widehat{f} \rangle.$$

Proof: The proof is very similar to that of Lemma 3.2. Using the temperateness of the system, we split X into a tailless system X_1 and its complement X_2 that satisfies (ii) of Definition 4.1. Once we show that we can change the order of summation in the series

$$\sum_{j \in J_2} \frac{1}{|L_j|} \int_{\mathbb{R}^d} \sum_{l \in \widetilde{L}_j} \widehat{f}(\omega) \overline{\widehat{\phi}_j(\omega)} \widehat{\phi}_j(\omega + l) \overline{\widehat{f}(\omega + l)} d\omega,$$

we will be able to use *verbatim* the proof of Lemma 3.2. In order to do the above change in the order of summation, it suffices that we show that

$$\sum_{j\in J_2} \frac{1}{|L_j|} \int_{\mathbb{R}^d} \sum_{l\in \widetilde{L}_j} \left| \widehat{f}(\omega)\overline{\widehat{\phi}_j(\omega)}\widehat{\phi}_j(\omega+l)\overline{\widehat{f}(\omega+l)}\right| d\omega < \infty.$$

However, this latter condition follows from (ii) of Lemma 4.3, with $J(X)$ there replaced by J_2 here. $\square$

4.2. Results for tempered, round GSI systems

In Corollary 2.27, we extended Proposition 2.20 to systems with small tail. We now extend that proposition to systems that are temperate and round. The proof of the next proposition follows an argument we found in [CCMW].

Proposition 4.7. *Let X be a GSI system which is temperate and round, associated with a dual Gramian $\widetilde{G}$, and norm functions $\mathcal{G}$ and $\mathcal{G}^-$. Assume that X is a fundamental frame with lower frame bound B. Then, for every finite $P \subset \mathbb{R}^d$, and for almost every $\omega \in \mathbb{R}^d$, $\mathcal{G}^-(\omega+P) \le 1/B$.*

Proof: We need to show that for every finite $P \subset \mathbb{R}^d$, $\mathcal{G}^-(\cdot + P) \le 1/B$, a.e. under the assumption that X is a fundamental frame with upper and lower bounds A and B respectively. We prove it by contradiction. Suppose, otherwise, that there exist an $\varepsilon > 0$, a finite set $P \subset \mathbb{R}^d$ and a set Ω of positive measure, such that for every $\omega \in \Omega$ $\mathcal{G}^-(\omega+P) > 1/(B-\varepsilon)$. Without loss, we may choose Ω to be compact, and assume that $0 \notin P + \Omega$.

Now, since X is tempered, we can decompose $J(X)$ into $J_1 \cup J_2$ as in Definition 4.1, and define X_1 as there, $X_2 := X\backslash X_1$. Then:

(i) Since X is Bessel, the diagonal function $\widetilde{g}_X$ is in L_∞ (see Lemma 2.8). By passing, if necessary, to a subset of Ω, and by moving from J_2 to J_1 finitely many indices, we may assume that

$$\|\widetilde{g}_{X_2}\|_{L_\infty(P+\Omega)} \le \varepsilon_1, \tag{4.8}$$

with ε_1 any (fixed) positive number.

(ii) Lemma 2.8 also implies that $\widetilde{G}_X(\omega+P)$ is positive definite and $\widetilde{G}_X(\omega+P) \ge \widetilde{G}_{X_1}(\omega+P)$, almost everywhere on Ω. This in particular says that for almost every $\omega \in \Omega$, $\mathcal{G}^-_{X_1}(\omega+P) > 1/(B-\varepsilon)$.

(iii) Given any (small) ball U centered at the origin, we can always find some $a \in \mathbb{R}^d$ such that $|U| \le c_1|(a+U)\cap\Omega|$, for some constant c_1 which is independent of the radius of U.

Thanks to (ii) above (and by Lemma 2.13), we can find a sufficiently small ball centered at the origin, U, such that $\widetilde{G}_{X_1}$ is P-fiberizable with fibers in $a+U$, for every $a \in \mathbb{R}^d$. We choose a so as to satisfy (iii) above (with respect to the current U). Then, by the sharpness of the left-hand-side inequality in (ii) of Lemma 2.11, we can find $f \in H_{P+(a+U)}$ such that

$$(B-\varepsilon)\|f\|^2 \ge \|T^*_{X_1} f\|^2. \tag{4.9}$$

Moreover, the actual construction of such f can be done (cf. [RS1]) so that

$$\|\widehat{f}\|_{L_\infty(\mathbb{R}^d)} \le \frac{\|\widehat{f}\|_{L_2(\mathbb{R}^d)}}{|(a+U)\cap\Omega|}. \tag{4.10}$$

We now assume that ε_1 in (4.8) equals $\varepsilon/(4c_1|P|^2)$, i.e.,

$$\widetilde{g}_{X_2} = \sum_{j\in J_2} |L_j|^{-1}|\widehat{\phi}_j|^2 \le \frac{\varepsilon}{4c_1|P|^2}, \tag{4.11}$$

everywhere on $P+\Omega$. Moreover, since X is tempered, and by moving more indices from J_2 to J_1, we may assume, with $c(X)$ the constant that appears in the definition of roundedness, that

$$\sum_{j\in J(X_2)} \|\widehat{\phi}_j\|^2_{L_2(P+\Omega)} < \frac{\varepsilon}{4c_1 c(X)|P|}. \tag{4.12}$$

We invoke now Lemma 4.3 (with Ω there replaced by $(a+U)\cap\Omega$ here, and X there replaced X_2 here). The lemma, when combined with (4.11), (4.12) and (4.10) yields that

$$(2\pi)^d\|T^*_{X_2}f\|^2 \le \frac{\varepsilon}{2c_1}|a+U|\|\widehat{f}\|^2_{L_\infty(\mathbb{R}^d)} \le \frac{\varepsilon}{2c_1}\frac{|U|}{|(a+U)\cap\Omega|}\|\widehat{f}\|^2_{L_2(\mathbb{R}^d)}.$$

By our construction, $|U| \le c_1|(a+U)\cap\Omega|$, hence the last inequality implies that

$$\|T^*_{X_2}f\|^2 \le \frac{\varepsilon}{2}\|f\|^2.$$

This, together with (4.9), leads to

$$\|T^*_X f\|^2 \le (B-\varepsilon/2)\|f\|^2,$$

which is a contradiction. □

The above proposition allows us to replace in several of the results of §3.1 the small tail assumption by temperateness and roundedness. For example, here is the counterpart of (i) in Corollary 3.7:

Corollary 4.13. *Let X be a tempered and round GSI system. Then, X is a fundamental tight frame if and only if X is scalar, i.e., $\widetilde{G}$ is the identity.*

Proof: If X is a fundamental tight frame, then Proposition 4.7 shows that its inverse norm function $\mathcal{G}^-$ is bounded by 1. Part (ii) of Lemma 3.6 then implies that X is scalar.

For the converse, we invoke Lemma 4.6. The lemma implies that the identity

$$(2\pi)^d\|T^*_X f\|^2 = \langle \widehat{f}, \widetilde{G}_X \widehat{f}\rangle$$

holds for a dense subset of $L_2(\mathbb{R}^d)$. The assumption that X is scalar trivially implies that

$$\langle \widehat{f}, \widetilde{G}_X \widehat{f}\rangle = \|\widehat{f}\|^2_{L_2(\mathbb{R}^d)}, \quad \forall f\in L_2(\mathbb{R}^d).$$

Altogether, we have established the isometry condition $\|T^*_X f\|^2 = \|f\|^2$ for a dense subset of $L_2(\mathbb{R}^d)$, proving thereby that X is a fundamental tight frame, as asserted. □

Next, we show the relevance of the above development to wavelet systems. The key here is the following simple observation that *wavelet systems are always temperate*:

Lemma 4.14. *Wavelet systems (as defined in (3.20)) are temperate.*

Proof: Let Ω be an arbitrary compact set of $\mathbb{R}^d$ that excludes the origin. We will show that

$$\sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}}\int_\Omega |\det s|^j|\widehat{\psi}(s^{*j}\cdot)|^2 = \sum_{j\in J(X)} \|\widehat{\phi}_j\|^2_{L_2(\Omega)} < \infty.$$

(The equality here is trivial from (3.20). It is the inequality that we aim at proving). Such result will certainly imply that we can satisfy Definition 4.1 by choosing $J_1 := \emptyset$ there.

Now,

$$\sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}}\int_\Omega |\det s|^j|\widehat{\psi}(s^{*j}\omega)|^2 d\,\omega = \sum_{\psi\in\Psi}\int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}}\chi_{s^{*-j}\Omega}(\omega)|\widehat{\psi}(\omega)|^2 d\,\omega.$$

Since Ω is a compact set that excludes the origin, and since s is expansive, it is easy to see that the sum $\sum_{j\in\mathbb{Z}}\chi_{s^{*-j}\Omega}$ is bounded, say by M. Then,

$$\sum_{\psi\in\Psi}\int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}}\chi_{s^{*-j}\Omega}(\omega)|\widehat{\psi}(\omega)|^2 d\,\omega \le M\sum_{\psi\in\Psi}\int_{\mathbb{R}^d}|\widehat{\psi}(\omega)|^2 d\,\omega < \infty.$$

□

As a consequence, we obtain that all the results of this section apply to wavelet systems that are round. We note that *all univariate wavelet systems are round.* In more than one dimension, the roundedness of the system X depends on the selection of the expansive matrix s. For example, if s is isotropic (i.e., it is a scalar multiple of a unitary matrix) then it is round. With the roundedness assumption, Corollary 4.13 leads to the following result which was first proved in [CCMW] (and where the roundedness assumption was used implicitly). The univariate version of this result was proved earlier by [CS2].

Corollary 4.15 ([CCMW]). *Assume that the wavelet system X is round. Then,*
(i) X is a fundamental tight frame if and only if it is scalar.
(ii) X is an orthonormal basis if and only if (a) $\|\psi\| = 1$, $\psi\in\Psi$, (b) X is scalar. □

4.3. A conjecture of G. Weiss

While looking at the recent paper [R], we came across the following conjecture of Guido Weiss:

Conjecture 4.16. *Let X be the wavelet system associated with a mother wavelet set Ψ and an expansive dilation matrix s. Assume that it is orthonormal. Then, it is fundamental (i.e., complete in $L_2(\mathbb{R}^d)$) if and only if*

$$\sum_{\psi\in\Psi}\sum_{k=-\infty}^{\infty}|\widehat{\psi}(s^{*k}\omega)|^2 = 1 \tag{4.17}$$

holds for a.e. $\omega \in \mathbb{R}^d$.

The proof below confirms Conjecture 4.16 for (i) wavelet systems generated by an arbitrary expansive dilation matrix with the generating set Ψ satisfying (3.28), and (ii) any wavelet system which is round (including thus all univariate systems). We note that the proof extends *verbatim* to systems that, instead of being orthonormal are merely Bessel with Bessel bound ≤ 1. Moreover, if s is integer, then the original analysis of [RS3] (which does not need (3.28), see [CSS]) can be applied to prove the conjecture without any further assumption on X. □

Proof: We first note that, for a wavelet system X, the left-hand-side of identity (4.17) coincides with $\widetilde{g}(\omega)$, with $\widetilde{g}$ the diagonal function of X.

Since X is orthonormal, it is Bessel with bound 1. Now, if we assume (4.17) to be valid, then, by Lemma 3.6, X is scalar.

Now, if (3.28) holds, then by Lemma 3.29 X has a small tail, hence, by (i) of Corollary 3.7, X is fundamental. Alternatively, if X is round, then the fundamentality is implied by Corollary 4.13 when combined with Lemma 4.14.

The converse implication is similar: once we know X to be a fundamental orthonormal system (hence in particular a fundamental tight frame) we can invoke either Corollary 3.7 or Corollary 4.13 (depending on the assumption we made) to conclude that X is scalar, *a fortiori* its diagonal function is identically 1. □

The above conjecture was proved true before for an integer dilation matrix: in [WW] for the dyadic dilation system (i.e., $s = 2\,\mathrm{Id}$), and in [Bo] and in [R] for a general integer expansive matrix s.

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