Construction of compactly supported affine frames in $L_2(\mathbb{R}^d)$

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1. Wavelet frames: what and why?

Since the publication, less than ten years ago, of Mallat's paper on Multiresoltion Analysis [Ma], and Daubechies' paper on the construction of smooth compactly supported refinable functions [D], wavelets had gained enormous popularity in mathematics and in the application domains. It is sufficient to note that there are currently more than 10,000 subscribers to the monthly *Wavelet Digest*. At the same time, the construction of concrete wavelet systems that are be useful for applications still remains a challenge. Specifically, simple and feasible constructions of orthonormal and bi-orthogonal systems of wavelets with small support, high smoothness and many symmetries is hard in more than one dimension (both tensor product methods, or the methods suggested in [RiS] and [JRS] yields wavelets with relatively large supports.

In a series of recent articles [RS1-7] and [GR], a theory that changes the previous state-of-the-art had been developed. That theory makes wavelet constructions simple and feasible, and it is the intent of the present article to provide a brief glance into it, with an emphasis on particular examples of univariate and multivariate constructs.

We want to start with somewhat philosophical discussion: anyone who is familiar with wavelets knows that the simplest wavelet system is the Haar family. The Haar function is piecewise-constant, has a very small support, and the algorithms based on it are fast and simple. Had the Haar wavelet been found satisfactory, other wavelet constructions, together with the MRA framework, would have been superfluous. However, the frequency localization (read: the smoothness) of this wavelet is so bad, that improvements had been sought for at the outset. It is reasonable to argue that if piecewise-constants are rejected, then continuous piecewise-linears are next in line: this is exactly the line of development in *spline theory*. Indeed, even before MRA was introduced, Battle [B], and Lemarié [L], constructed (independently) a piecewise-linear continuous spline with orthonormal dilated shifts (and knots at the half-integers only). Alas, that spline is of *global* support, and even its exponential decay at ∞ did not attract the masses, who deserted it in favor of Daubechies' refinable functions and their bi-orthogonal off-springs (cf. [CDF]). The simplest function in Daubechies' family [D] of refinable functions (i.e., that with support $[0,3]$) is not piecewise-linear, but is related to piecewise-linears in some weak sense (its shifts reproduce all linear polynomials, just as the the shifts of the piecewise-linear hat function do); in any event, the question whether the corresponding wavelet is a 'natural' or 'unnatural' replacement for the Haar wavelet was not on the agenda anymore; rather, this wavelet is considered next in line the Haar's because it is the continuous orthonormal wavelet with *shortest support*.

Before we get to the main point of the present discussion, we need to introduce the notion of a *tight frame*. For that, we recall that, given any orthonormal system X for $L_2(\mathbb{R}^d)$, we have

$$f = \sum_{x\in X} \langle f, x\rangle x, \quad \text{all } f \in L_2(\mathbb{R}^d).$$

More concretely, the above identity states that we may use the same system X during the *decomposition process* $f \mapsto \{\langle f, x\rangle\}$, and during the *reconstruction process* $c \mapsto \sum_{x\in X} c(x)x$ (here, c is a any sequence defined on, and labeled by the elements of X). However, the property just expressed does not *characterize* orthonormality:

Definition: tight frames. A system $X \subset L_2(\mathbb{R}^d)$ is called a **tight frame** if the equality

$$f = \sum_{x\in X} \langle f, x\rangle x, \quad \text{all } f \in L_2(\mathbb{R}^d)$$

holds.

While piecewise-linear compactly supported orthonormal wavelet system (generated by a single mother wavelet) does not exist, the elements depicted in Figure 1 were shown in [RS3] to generate a tight frame (using dyadic dilations and integer translations) and may be viewed as a natural extension of the Haar wavelet. More importantly, it is the first in line in a wealth of constructions of affine (tight) frames. Examples of this class are given in §2 (univariate), and in §3 (multivariate). A glimpse into the theory that leads to such and other constructs is the goal of §4.

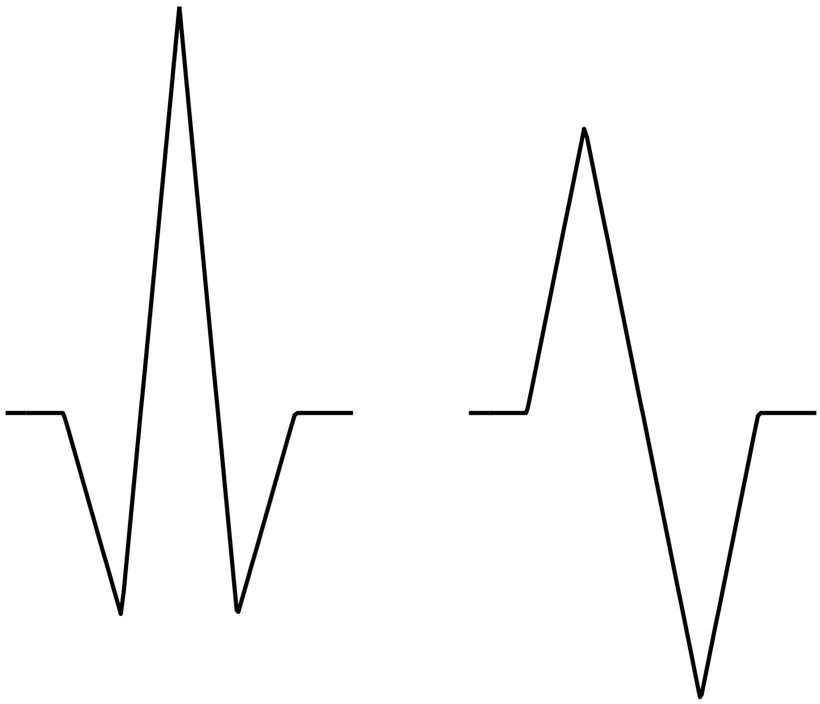

Figure 1. The generators of the piecewise-linear tight frame

We have explained so far what tight frames are. We 'almost' explained why they are needed: the main reason is that it is significantly simpler to construct tight wavelet frames (or, more, generally, bi-frames, a notion that is defined in §2) as compared to orthonormal wavelets systems or bi-orthogonal ones. This is largely due to the fact that the latter constructions require refinable functions with properties similar those desired of the sought-for wavelets: e.g., a refinable function with orthonormal shifts is required for the construction of an orthonormal wavelet system. In contrast, compactly supported tight wavelet frames can be derived from *any* refinable function, including splines in one dimension and box splines in higher dimensions. We do not even need to assume that the shifts of the refinable function form a Riesz basis! Of course, one should still keep in mind that tight frames are do not form an orthonormal system (they can be essentially regarded as 'redundant orthonormal systems'), and for certain applications (primarily data compression) the oversampling that is inherent in frames may be undesired. At the same time, other applications, such as noise reduction and/or feature detection may find the redundancy of frames a plus, and some other applications may find that a neutral feature.

2. Examples of univariate tight frames

As we said in the previous section, it is possible, at least in theory, to derive wavelet frames from *any* refinable function. We have defined in the previous section the notion of a *tight frame*, and explained that they should be considered as 'redundant orthonormal bases'. In a similar way, we define now the notion of *bi-frames*, which are the redundant analog of bi-orthogonal Riesz bases.

"biframes **Definition 2.1.** *Let X be a countable collection of functions in L_2. Let $\mathrm{R} : X \to L_2$ be some map. We call the pair $(X, \mathrm{R}X)$* **bi-frames** *if the following two conditions are satified:*

(1) The identity $\sum_{x\in X} \langle f, \mathrm{R}x\rangle x = f$, holds for every $f \in L_2$, and

(2) There exists a constant $C < \infty$, such that for every $f \in L_2$, the inequality

$$\sum_{x\in X} |\langle f, x\rangle|^2 + \sum_{x\in X} |\langle f, \mathrm{R}x\rangle|^2 \le C\|f\|_{L_2}$$

is valid.

In the above definition, the second property (which implies that X and RX are *Bessel systems*) is technical and mild. (Recall that a collection of functions X in L_2 is a **Bessel system** if there exists a constant $C < \infty$ such that, for every $f \in L_2$, the inequality $\sum_{x\in X} |\langle f, x\rangle|^2 \le C\|f\|_{L_2}$ holds.) The major property in the definition of bi-frames is the first one, (1). That property tells us that we may use the system $\mathrm{R}X$ for *decomposition* and then use the dual system X during the reconstruction.

We now provide various examples of univariate tight and bi- wavelet frames. All the constructs in the examples are derived from a *Multiresolution Analysis*. We recall in that context that a function $\phi \in L_2$ is called a (dyadic) **refinable function** or a **scaling function** or a **father wavelet** if there exists a **mask** $a_\phi : \mathbb{Z} \to \mathbb{C}$ such that

$$\phi = 2 \sum_{\alpha\in\mathbb{Z}} a_\phi(\alpha)\phi(2 \cdot -\alpha). \tag{2.2}$$

"defrefin

Sometimes, it is easier to express a_ϕ in terms of its symbol

$$\tau_\phi(\omega) := \sum_{\alpha\in\mathbb{Z}} a_\phi(\alpha) \exp(-i\alpha\omega).$$

In the examples we discuss, the mask a_ϕ is finite (which implies that ϕ is compactly supported), hence τ_ϕ is a trignometric polynomial. The refinement equation (2.2) can be written in Fourier domain as

$$\widehat{\phi}(2\cdot) = \tau_\phi\widehat{\phi}.$$

For notational convenience, when sequentially listing the entries of a sequence $a : \mathbb{Z} \to \mathbb{C}$, we put in **boldface** the entry $a(0)$, thus

$$a = (\ldots, 0, 1, 2, \mathbf{3}, 4, 0, \ldots)$$

means that $a(0) = 3$, $a(1) = 4$, $a(-1) = 2$, $a(-2) = 1$, and all other entries are 0.

In fact, in all our examples, the refinable function is chosen to be the B-spline of order k, with k varying from one example to another. Recall that the B-spline is a C^{k-2} piecewise-polynomial of local degree $k - 1$, which is supported in an interval of length k (that we choose to be $[-k/2, k/2]$, at least for *even* k) and has its knots at the integers only. The Fourier transform of that B-spline if given by

$$\widehat{\phi}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^k.$$

The B-spline is dyadically refinable with mask

$$\tau_\phi(\omega) = cos^k(\omega/2).$$

“exone **Example 2.3.** (Piecewise-linear tight frame) We choose ϕ to be the B-spline of order 2, i.e., the hat function. The generators of the tight frame are drawn in Figure 1. The refinement mask is

$$a_\phi = (\dots, 0, \frac{1}{4}, \boldsymbol{\frac{1}{2}}, \frac{1}{4}, 0, \dots).$$

The two wavelet masks are

$$a_{\psi_1} = (\dots, -\frac{\sqrt{2}}{4}, \mathbf{0}, \frac{\sqrt{2}}{4}, 0, \dots),$$

and

$$a_{\psi_2} = (\dots, -\frac{1}{4}, \boldsymbol{\frac{1}{2}}, -\frac{1}{4}, 0, \dots),$$

This example is the simplest in a general construction of tight spline wavelet frames that was described in [RS3]. In that construction, the number of wavelets is k (with k the order of the B-spline which is used as a refinable function). The details of the piecewise-cubic case are as follows.

“extwo **Example 2.4.** (Piecewise-cubic tight frame) We choose ϕ to be the B-spline of order 4. The generators of the tight frame are shown in Figure 2. The refinement mask is

$$a_\phi = (\dots, 0, \frac{1}{16}, \frac{1}{4}, \boldsymbol{\frac{3}{8}}, \frac{1}{4}, \frac{1}{16}, 0, \dots).$$

The four wavelets have masks as follows:

$$\begin{array}{lcccccccccccl}
a_{\psi_1} & = & (& \dots, & 0, & -\frac{1}{8}, & -\frac{1}{4}, & \mathbf{0}, & \frac{1}{4}, & \frac{1}{8}, & 0, & \dots &) \\
a_{\psi_2} & = & (& \dots, & 0, & \frac{1}{16}, & 0, & -\boldsymbol{\frac{1}{8}}, & 0, & \frac{1}{16}, & 0, & \dots &) \; *\sqrt{6} \\
a_{\psi_3} & = & (& \dots, & 0, & -\frac{1}{8}, & \frac{1}{4}, & \mathbf{0}, & -\frac{1}{4}, & \frac{1}{8}, & 0, & \dots &) \\
a_{\psi_4} & = & (& \dots, & 0, & \frac{1}{16}, & -\frac{1}{4}, & \boldsymbol{\frac{3}{8}}, & -\frac{1}{4}, & \frac{1}{16}, & 0, & \dots &)
\end{array}$$

It is also possible to construct bi-frames where the two frames involved are derived from B-splines of different orders. In the next example, we derive the frame X from cubic splines, while its dual is derived from piecewise-linear splines.

“exthree **Example 2.5.** (Bi-frames: cubics and linears mixed.) We choose one refinable function to be the B-spline of order 4 (whose mask is already listed in Example 2.4), and the other B-spline to be of order 2, (i.e., it is the hat function of Example 2.3). There are two sets of mother wavelets now: those that generate the wavelet system X, and those that generate the dual wavelet system $\mathrm{R}X$. The piecwise-linear wavelet (that can be used, say, during the decomposition step) are depicted in Figure 3. They are supported on the intervals $[.5, 3.5], [.5, 3], [1, 3.5]$ respectively. Note that, essentially, there are only two mother wavelets: the left-most one (together with its integer shifts) and the middle one (together with its *half* integer shifts). The masks of these three elements (ordered from left to right) are:

$$\begin{array}{cccccccccccccc}
(& \dots, & \mathbf{0}, & 1, & -4, & 6, & -4, & 1, & 0, & \dots &) & * & \frac{1}{16\sqrt{2}} \\
(& \dots, & \mathbf{0}, & 0, & -1, & -1, & 1, & 1, & 0, & \dots &) & * & \frac{\sqrt{3}}{8} \\
(& \dots, & \mathbf{0}, & -1, & -1, & 1, & 1, & 0, & 0, & \dots &) & * & \frac{\sqrt{3}}{8}
\end{array}$$

Figure 2. The four piecewise-cubic wavelets

The masks of the cubic dual frame are (in the same order)

$$\begin{array}{ccccccccccc}
(& \ldots, & \mathbf{0}, & \frac{1}{\sqrt{8}}, & -\frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{8}}, & 0, & \ldots &) & & \\
(& \ldots, & \mathbf{0}, & -\frac{1}{2}, & \frac{1}{2}, & 0, & 0, & \ldots &) & * & \frac{\sqrt{3}}{4} \\
(& \ldots, & \mathbf{0}, & 0, & -\frac{1}{2}, & \frac{1}{2}, & 0, & \ldots &) & * & \frac{\sqrt{3}}{4}
\end{array}$$

□

Note that in the last example two mother wavelets are used for creating the system (one is shifted along integer translations, while the other ones along the denser half-integer translations). Examples of that sort are the rule rather than the exception. For example, it is possible to derive from the B-spline of order k a tight compactly supported spline frame with similalrly two generators (however, the wavelets, in general, of those constructions are not symmetric.)

3. Examples of multivariate wavelet frames

Our examples of univariate wavelet frames in the previous section were derived from the multiresolution analysis whose 'father wavelet' is the B-spline. This ensured us, e.g., that the wavelets are smooth piecewise-polynomials. An attempt to extend this approach to the multivariate setup requires a multivariate analog of B-splines, i.e., smooth compactly supported refinable piecewise-polynomials. Fortunately, such functions exist and are known as 'box splines'. However, in contrast with the univariate cardinal B-splines that have only one 'degree of freedom', i.e., their order, a d-variate box spline is determined by a set of *directions*. Here, a **direction** is a non-zero vector in $\mathbb{Z}^d$. We stress that the 'sets' of directions below are not actually sets but multisets, i.e., a direction may appear several times in it. We do assume (without further notice) that each direction set to be considered spans the entire $\mathbb{R}^d$ space.

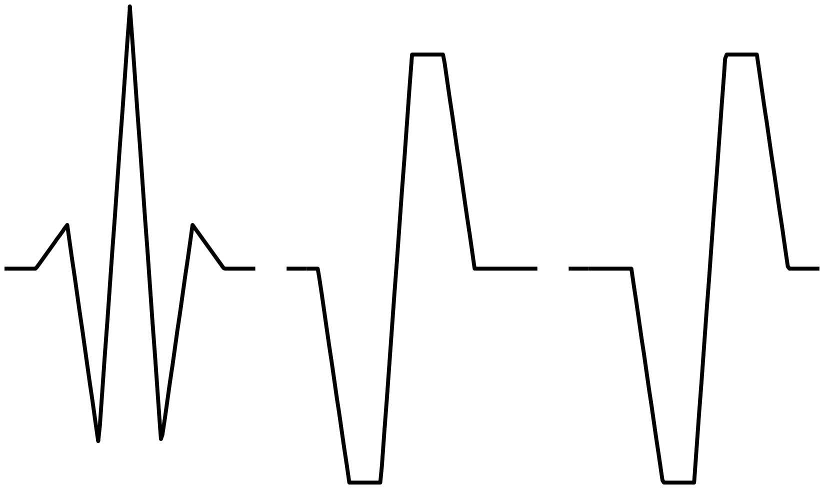

Figure 3. The generators of the piecewise-linear frame of Example 2.5

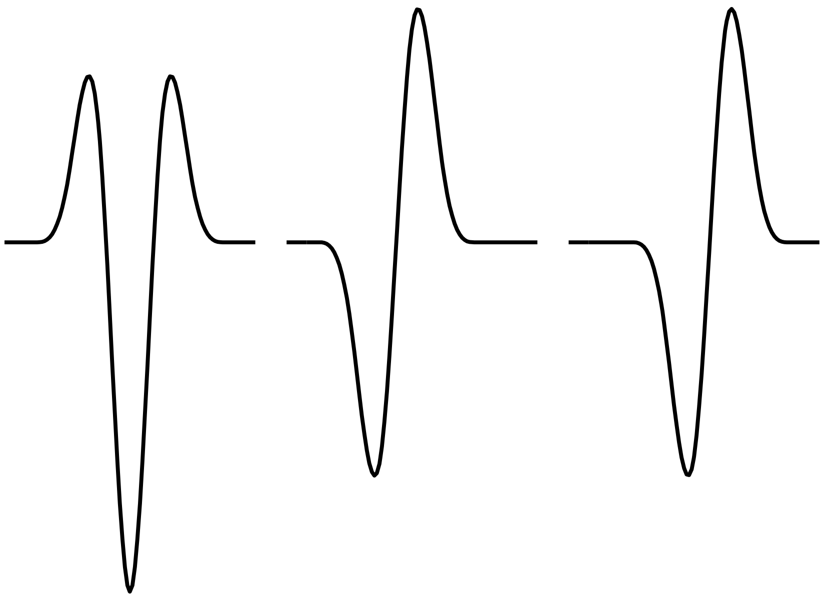

Figure 4. The generators of the piecewise-cubic frame of Example 2.5

“defbox **Definition 3.1**. Let $\Xi \subset \mathbb{Z}^d$ be a direction set. The **box spline** $\phi := \phi_\Xi$ is the function whose Fourier transform is

$$\widehat{\phi}(\omega) = \prod_{\xi \in \Xi} \frac{1 - e^{-i\xi\cdot\omega}}{i\xi \cdot \omega}.$$

The box spline ϕ is a piecewise-polynomial of local degree $n := \#\Xi - d$ (i.e., each of the polynomial pieces is of degree $\le n$). It lies in $C^k \backslash C^{k+1}$, with

$$k := \max\{\#Y :\ Y \subset \Xi,\ \operatorname{span}(\Xi \backslash Y) = \mathbb{R}^d\}.$$

Its support is the convex polyhedron

$$[0,1]^\Xi \Xi := \{\sum_{\xi\in\Xi} t_\xi \xi :\ t \in [0,1]^\Xi\}.$$

Much of the basic theory of box splines can be found in the book [BHR].

We will be interested primarily in the 4-*direction bivariate box splines.* These box splines correspond to a direction set Ξ which consists of the four vectors

$$(\xi_1, \xi_2, \xi_3, \xi_4) := \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

each appearing with a certain multiplicity. We set $m = (m_1, m_2, m_3, m_4) \in \mathbb{Z}_+^4$ for the vector of multiplicities (i.e., $\xi_1 = (1\ 0)'$ appears in Ξ m_1 times, etc.) The support of the 4-direction box spline is an octagon, four of its vertices are $(0,0), (m_1, 0), (m_1+m_3, m_3), (m_1+m_3, m_2+m_3), (m_1+m_3-m_4, m_2+m_3+m_4)$. Four direction box splines possess a wealth of symmmetries; nonetheless, prior to [RS3,5], there were hardly any wavelet constructions based on such splines. The reason for that is that the shifts (i.e., integer translates) of the 4-direction box spline are always *linearly dependent* (unless $m_3 m_4 = 0$, but then the box spline is not truly 4-directional); indeed, we always have that

“temp (3.2)
$$\sum_{\alpha \in \mathbb{Z}^2} (-1)^{\alpha_1 + \alpha_2} \phi(\cdot - \alpha) = 0,$$

for every 4-direction box spline; the major previous algorithms for deriving wavelets from multiresolution all required, at a minimum, that the shifts of the underlying refinable function form a Riesz basis or a frame for V_0 (the latter being the closed *shift invariant* space generated by the shifts of ϕ). However, the dependence relation (3.2) implies that the shifts of ϕ form neither a Riesz basis nor a frame for V_0. (The reader is warned that the last statement is more subtle than it may look like: first, the shifts of $\phi \in L_2$ can form a Riesz basis while being linearly dependent. However,in such a case, the coefficient sequence of each dependence relation is *unbounded.* Second, the elements of a frame can certainly be, and usually are, linearly dependent. However, a frame which consists of the shifts of a *single compactly supported* function is necessarily a Riesz basis, cf. [RS1]).

The box spline ϕ is dyadically refinable with mask whose symbol is

$$\prod_{j=1}^{4} e^{-i m_j \xi_j \cdot \omega/2} \cos^{m_j}(\xi_j \cdot \omega/2).$$

Moreover, if we restrict our attention to 4-direction box splines whose multiplicities satisfy $m_1 = m_3$, $m_2 = m_4$, then those box splines are also refinable with respect to the dilation matrix

“ttmat (3.3)
$$s = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and the symbol τ in this case is simpler:

$$\tau(\omega) = e^{-i(m_1, m_2)\cdot \omega/2} \cos^{m_1}(\omega_1/2) \cos^{m_2}(\omega_2/2).$$

Warning: the above τ is also the symbol of the *tensor product* B-spline. This of course is possible: it is the symbol of the 4-direction box spline, when we use the above dilation matrix, and it is the symbol of the tensor B-spline when we use the more standard dyadic dilation (another way to view that: the 4-direction box spline is the convolution product of the tensor B-spline with its s-dilate). This coincidence enables us to convert standard construction techniques of tensor-product wavelets to the 4-direction box spline setup.

In what follows we discuss masks of bivariate refinable functions and masks of the corresponding wavelets. Until further notice, the dilation matrix is always assumed to be

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We adopt the following convention concerning the mask discussed: given a finitely supported sequence on $\mathbb{Z}^2$, we simply display its non-zero values against the background of a (-n invisible) integer mesh. We mark with **boldface** the location of the origin, which is always displayed (even when its value is 0). For example, the notation

$$\begin{array}{cc} 4 & \\ \mathbf{0} & -1 \end{array}$$

stands for a sequence that takes the value 4 at $(0,1)$, the value -1 at $(1,0)$, and the value 0 anywhere else (on $\mathbb{Z}^2$).

“exfour **Example 3.4.** Let ϕ be the 4-direction box spline whose multiplicity vector is $(1,1,1,1)$. This box spline is known in the finite element literature as the *Powell-Zwart* element, and its graph is drawn in Figure 5.

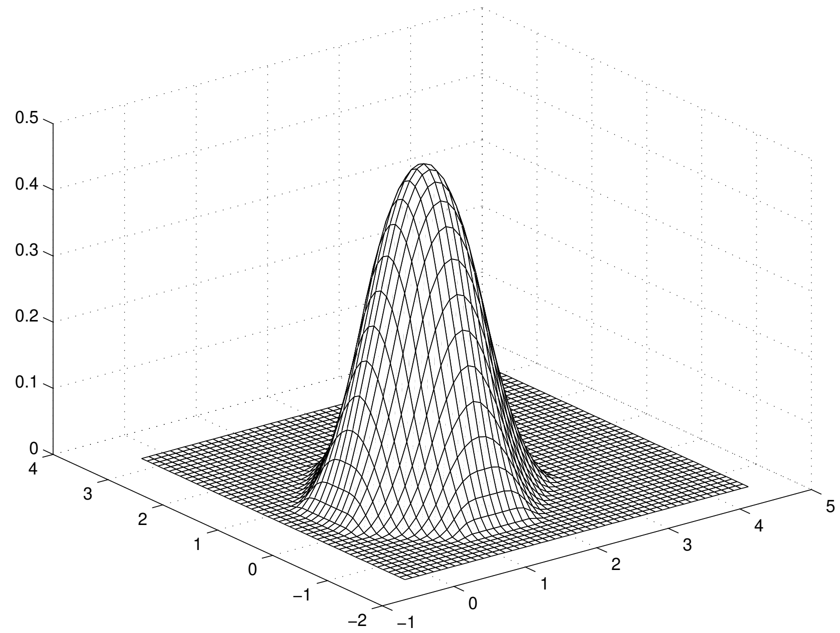

Figure 5. The Powell-Zwart element

The Powell-Zwart element is refinable with mask

$$a = \begin{array}{cc} .25 & .25 \\ \mathbf{.25} & .25 \end{array}.$$

It is a C^1 piecewise-quadratic spline, and its support is the smallest octagon with integer vertices (those vertices are $(.5,1.5)+(\pm1.5,\pm.5)$ and $(.5,1.5)+(\pm.5,\pm1.5)$. A tight frame that is generated by *three* wavelets can be derived from the multiresolution of the Powell-Zwart element. The three wavelet masks are

$$\begin{array}{cc} -.25 & -.25 \\ \mathbf{.25} & .25 \end{array}, \quad \begin{array}{cc} .25 & -.25 \\ \mathbf{.25} & -.25 \end{array}, \quad \begin{array}{cc} -.25 & .25 \\ \mathbf{.25} & -.25 \end{array}.$$

Note that these masks are identical to those used in the construction of the bivariate *dyadic* orthonormal Haar system. That latter system is derived from the multiresolution analysis of the support function χ of the unit square, and our refinable function here is indeed related to χ: the Powell-Zwart element is the convolution product of χ and $\chi(t_1+t_2, t_1-t_2)$. The graphs of the three wavelets are drawn in Figures 6-8. All the wavelets have the same octagonal support as that of the Powell-Zwart element.

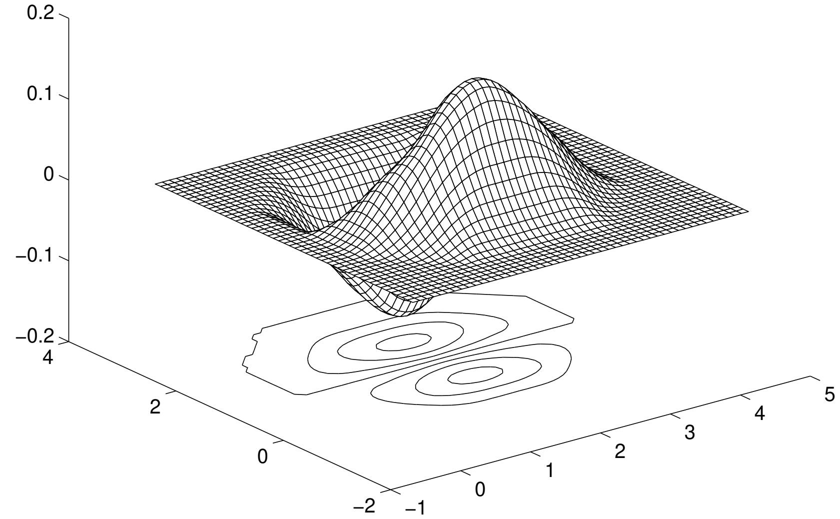

Figure 6. The first wavelet in Example 3.4

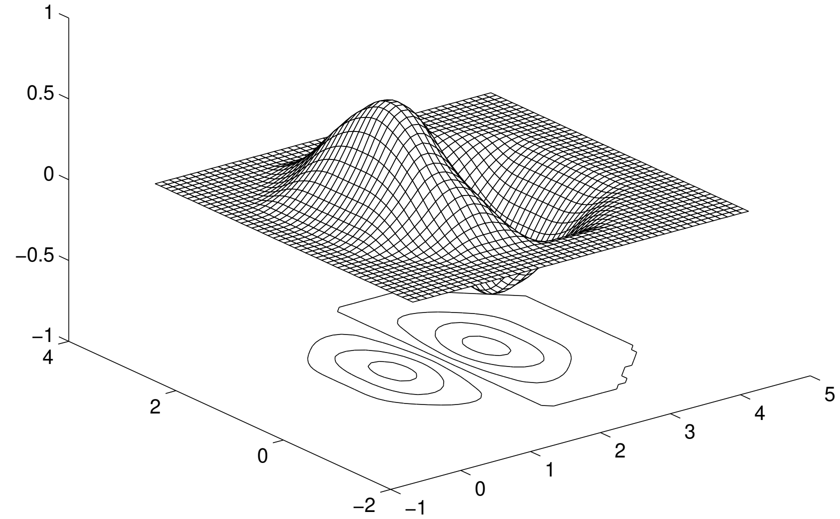

Figure 7. The second wavelet in Example 3.4

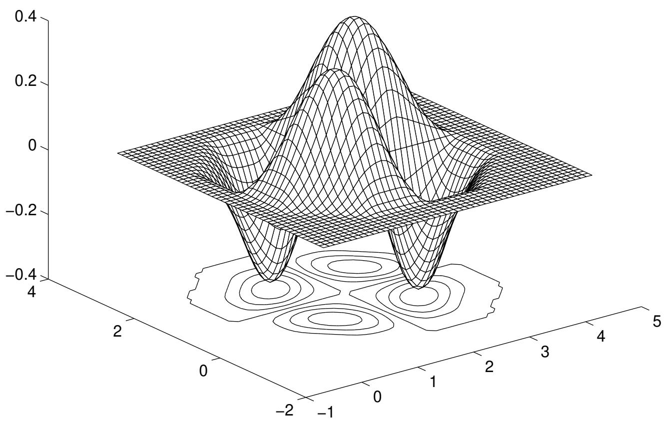

Figure 8. The third wavelet in Example 3.4

Since the dilation matrix s has determinant -2, one expects to use a single wavelet in the construction of irredundant wavelet systems (that are based on s). Since we used in Example 3.4 three mother wavelets, it seems reasonable to assert that the system there has 'a 3-fold rate of oversampling'. It is possible to modify the construction and to obtain a tight frame generated by *two* compactly supported wavelets. We refer to [RS5] for the details of that modified construction, but, for the reader convenience, list in the next example the corresponding masks.

“exfive **Example 3.5.** (C^1 piecewise-quadratic compactly supported tight frame generated by two wavelets) In this case the refinable function is slightly changed, and the refinement mask becomes:

$$\begin{array}{cc} & .25 \\ .25 & .25 \\ \mathbf{.25} & \end{array} .$$

The masks of the two wavelets are

$$\begin{array}{cc} & -.5 \\ .5 & -.5 \\ \mathbf{.5} & \end{array} , \qquad \begin{array}{c} \mathbf{1} \\ -1 \end{array} .$$

Note that the second wavelet has a smaller support than the first. Indeed, while in the previous example the each of the three mother wavelets is supported in a domain of area 7, the two wavelets here are supported in domains of areas 10 and 7 respectively.

Algorithms for constructing compactly supported tight spline frames from box splines of higher smoothness are detailed in [RS5]. These algorithms work, essentially, with *any* box spline (though they may require to modify somewhat the magnitude of the directions that define the box spline as was actually done in the last example). However, in all these algorithms the number of wavelets that are used increases with the increase of the smoothness of the box spline (the determining factor is the degree of the mask, viewed as a trigonometric polynomial, and that degree must increase together with the smoothness). In what follows, we describe a general algorithm that applies to 4-direction box splines whose multiplicity vector is of the form (m_1, m_2, m_1, m_2). Recall that that box spline is refinable with respect to the dilation matrix s of (3.3), and its mask, on the Fourier domain is

“maskPZ (3.6)
$$\tau(\omega) = e^{-i(m_1,m_1)\cdot\omega/2} \cos^{m_1}(\omega_1/2) \cos^{m_2}(\omega_2/2).$$

The algorithm can be extended to more general box splines (provided that those box splines are also refinable with respect a dilation matrix whose determinant is ± 2) and is new, i.e., appears here for the first time. In contrast with the previous constructions, it yields bi-frames rather than tight frames. On the other hand, the number of mother wavelets is 3 regardless of the values of m_1, m_2, in other wrods, regardless of the smoothness of the resulting wavelet system. We describe below the algorithm in general terms, and then provide the details of one of its special cases.

Algorithm: 4-directional compactly supported bi-frames of arbitrary smoothness generated by three mother wavelets. We need here *two* refinable functions, and assume both of them to be 4-direction box splines which are refinable with respect to the dilation matrix s, hence with mask of the form (3.6). We set ϕ for one of these functions, and ϕ^d for the other, set also τ and τ^d for their masks, and denote their multiplicity vectors by $m = (m_1, m_2, m_1, m_2)$ and $n = (n_1, n_2, n_1, n_2)$, respectively. We assume that all the entries of $r := (m + n)/2$ are (positive) *integers*; we also assume that $r_1 + r_2$ is even, since that assumption simplifies somewhat the presentation. Under these mere assumptions, we derive two wavelet systems that form a bi-frame in the following way. We first expand the expression

"temp (3.7) $$1 = (\cos^2(\omega_1/2) + \sin^2(\omega_1/2))^{r_1} (\cos^2(\omega_2/2) + \sin^2(\omega_2/2))^{r_2},$$

and group the various summands into four groups. The first two groups are the singletons $R_1(\omega) := \cos^{r_1}(\omega_1/2)\cos^{r_2}(\omega_2/2)$, and $R_2(\omega) := \sin^{r_1}(\omega_1/2)\sin^{r_2}(\omega_2/2)$. Since $R_2 = R_1(\cdot + (\pi, \pi))$, it is possible then to divide the other terms into two groups, R_3 and R_4, such that $R_4 = R_3(\cdot + (\pi, \pi))$. This can be done in many different ways, and the only condition we need is that R_3 is divisible by $\cos^2(\omega_1/2)\sin^2(\omega_2/2)$ (something that can be achieved by, e.g., putting all terms that are divisible by $\cos^{r_1}(\omega_1/2)$ into R_3 and all terms that are divisible by $\cos^{r_2}(\omega_2/2)$ into R_4). Observing that $R_1 = \tau\tau^d$, we factor R_3 into $\tau_1\tau_1^d$ in a way that both τ_1 and τ_1^d are divisible by $\sin(\omega_2/2)$. We then define two wavelet systems, each consists of three mother wavelets. In the first system, the three wavelets masks are

$$(t_1(\omega) := \exp(\omega_1)\overline{\tau^d(\omega + (\pi, \pi))},\ t_2(\omega) := \tau_1(\omega),\ t_3(\omega) := \exp(\omega_1)\overline{\tau_1^d(\omega + (\pi, \pi))}),$$

and in the second system the wavelet masks are

$$(t_1^d(\omega) := \exp(\omega_1)\overline{\tau(\omega + (\pi, \pi))},\ t_2^d(\omega) := \tau_1^d(\omega),\ t_3^d(\omega) := \exp(\omega_1)\overline{\tau_1(\omega + (\pi, \pi))}).$$

Since $t_j\overline{t_j^d} = R_{j+1}$, $j = 1, 2, 3$, we conclude that $\tau\overline{\tau^d} + \sum_{j=1}^3 t_j\overline{t_j^d} = 1$. At the same time, we have that $t_2\overline{t_2(\cdot + (\pi, \pi))} + t_3\overline{t_3(\cdot + (\pi, \pi))} = 0$, and also $\tau\overline{\tau(\cdot + (\pi, \pi))} + t_1\overline{t_1(\cdot + (\pi, \pi))} = 0$, and we thus conclude that the wavelets are constructed according to the mixed extension principle (see Theorem 4.9). Moreover, each of the mother wavelet in either system has a sin-factor in its mask, hence has a zero mean-value, which, together with its compact support assumption, implies that the wavelet system is Bessel. Altogether, the two wavelet systems generated as above are bi-frames.

"general **Example 3.8.** We let ϕ and ϕ^d be, both, the 4-direction box splines with multiplicity $(2, 2, 2, 2)$; the refinement masks (up to an exponential factor) are then $\tau(\omega) = \tau^d(\omega) = \cos^2(\omega_1/2)\cos^2(\omega_2/2)$. Also, $r = (2, 2, 2, 2)$, and the expression in (3.7) is

$$(\cos^2(\omega_1/2) + \sin^2(\omega_1/2))^2(\cos^2(\omega_2/2) + \sin^2(\omega_2/2))^2.$$

After defining $R_1(\omega) = \cos^4(\omega_1/2)\cos^4(\omega_2/2)$, and $R_2(\omega) = \sin^4(\omega_1/2)\sin^4(\omega_2/2)$, we are left with seven additional terms that should be split between R_3 and R_4. One possiblity is to define, with $b_j := \cos^2(\omega_j/2)$, $j = 1, 2$,

$$R_3(\omega) := b_1(1 - b_2)(b_1(1 + b_2) + 2(1 - b_2)),$$

and hence

$$R_4(\omega) := b_2(1-b_1)(b_2(1+b_1)+2(1-b_1)).$$

There are then many ways to construct the wavelets. For example, we can define the generators of the first system to be

$$t_1(\omega) = \exp(\omega_1/2)\sin^2(\omega_1/2)\sin^2(\omega_2/2), \quad t_2(\omega) = \exp(-\omega_1/2)\cos(\omega_1/2)\sin(\omega_2/2),$$
$$t_3(\omega) = \exp(\omega_1/2)\sin(\omega_1/2)\cos(\omega_2)(\sin(\omega_1/2)(1+\sin(\omega_2/2)+2\cos(\omega_2/2)),$$

and, correspondingly,

$$t_1^d(\omega) = \exp(\omega_1/2)\sin^2(\omega_1/2)\sin^2(\omega_2/2), \quad t_3^d(\omega) = \exp(\omega_1)\sin(\omega_1/2)\cos(\omega_2/2),$$
$$t_2^d(\omega) = \exp(-\omega_1/2)\cos(\omega_1/2)\sin(\omega_2/2)(\cos(\omega_1/2)(1+\cos(\omega_2/2)+2\sin(\omega_2/2)).$$

4. The theory of affine frames

In this section, we review the theory that led to the constructions detailed in the previous sections, and explained the basic principles behind the actual constructions.

The analysis of wavelet frames in [RS3] and [RS4] is based on the theory of *shift-invariant systems* that was developed in Approximation Theory (box splines, [BHR], form a special case of shift-invariant systems). A system $X \subset L_2$ is **shift-invariant** if there exists $F \subset X$ such that

$$X = (f(\cdot+\alpha) : f \in F,\ \alpha \in \mathbb{Z}^d).$$

A systematic study of the "frame properties" of a shift-invariant X can be found in [RS1], and the results there were subsequently applied in [RS2] to *Gabor systems* (which, indeed, are shift-invariant). Wavelet systems, on the other hand, are *not* shift-invariant (the negative dilation levels are invariant under translations that become sparser as the dilation level decreases). The main effort of [RS3] was devoted, indeed, to circumventing that obstacle, i.e., finding a way to apply the "shift-invariant methods" of [RS1] to the 'almost shift-invariant' wavelet systems.

This was achieved in [RS3] and [RS4] with the aid of the new notion of **quasi-affine system**, that we describe here (for the dyadic dilation case only; the development in [RS3] and [RS4] is valid with respect to general dilation matrices with integer entries). Let the affine system X be a wavelet system generated by a finite number of mother wavelets $\Psi \subset L_2(\mathbb{R}^d)$. The affine system X is the disjoint union of $D^k E(\Psi)$ where $E(\Psi) = \cup_{\psi\in\Psi} E(\psi)$ with $E(\psi) := \{\psi(\cdot-\alpha) : \alpha \in \mathbb{Z}^d\}$, the shift invariant set generated by ψ, and D is the dyadic dilation operator $D : f \mapsto 2^{d/2} f(2\cdot)$. That is

$$X = \bigcup_{k\in\mathbb{Z}} D^k E(\Psi).$$

The **quasi-affine** system associated with X (denoted by X^q) is, roughly speaking, the smallest shift-invariant set containing X. It is obtained from X by replacing, for each $k<0$, the set of the functions $2^{kd/2}\psi(2^k\cdot+j)$, $\psi \in \Psi$, $j \in \mathbb{Z}^d$ that appears in X, by the larger shift-invariant set of functions

$$2^{kd}\psi(2^k\cdot+j), \quad \psi\in\Psi, \quad k<0, \quad j \in 2^{-k}\mathbb{Z}^d.$$

Note that, while the affine system is dilation-invariant, the quasi-affine X^q is shift-invariant, but is not dilation invariant.

While the "basis properties" of X (such as the Riesz basis property) are not preserved when passing to X^q, the "frame properties" of X are preserved. The following result is a special case of Theorem 5.5 of [RS3].

“thmone **Theorem 4.1.** *An affine system X is a frame for $L_2(\mathbb{R}^d)$ if and only if its quasi-affine counterpart X^q is one. Furthermore, the two systems have the same frame bounds. In particular, the affine frame X is tight if and only if the corresponding quasi-affine system X^q is tight frame.*

The theorem allows one to analyse the ‘frame properties’ of the affine X via a study of its quasi-affine counterpart. The latter is more mathematically accessible, by virtue of its shift-invariance. Specifically, [RS3] employs the so-called “dual Gramian” analysis of [RS1] (which is a ‘shift-invariance method’) to this end. The result is a complete characterization of all wavelet frames that we now describe.

The characterization is in terms of certain bi-infinite matrices, dubbed ‘fibers’. The matrices and their entries are best desrcribed in terms of the following *affine product*:

$$\Psi[\omega,\omega'] := \sum_{\psi\in\Psi}\sum_{k=\kappa(\omega-\omega')}^{\infty} \widehat{\psi}(2^k\omega)\overline{\widehat{\psi}}(2^k\omega'), \quad \omega,\omega'\in\mathbb{R}^d,$$

where κ is the dyadic valuation:

$$\kappa : \mathbb{R}\to\mathbb{Z} : \omega\mapsto \inf\{k\in\mathbb{Z} : 2^k\omega\in 2\pi\mathbb{Z}^d\}.$$

(Thus, $\kappa(0)=-\infty$, and $\kappa(\omega)=\infty$ unless ω is 2π-dyadic.) Our convention is that $\Psi[\omega,\omega'] := \infty$ unless we have absolute convergence in the corresponding sum. We assume here that

“decaycond (4.2) $$|\widehat{\psi}(\omega)| = O(|\omega|^{-1/2-\delta}), \quad \text{near } \infty, \quad \text{for some } \delta>0,$$

for every wavelet $\psi\in\Psi$. This smoothness assumption on Ψ is mild, still the actual assumption in [RS3,4] is even milder (multivariate Haar wavelets do not satisfy the smoothness assumption here, but do satisfy the milder assumption of [RS3,4]). Theorem 4.1 is originally proved in [RS3] under this latter smoothness assumption; the subsequent proof in [CSS] avoids that assumption.

The fibers (i.e., matrices) in the ‘dual Gramian fiberization’ are indexed by $\omega\in\mathbb{R}^d$. Each fiber is a non-negative definite self-adjoint matrix $\widetilde{G}(\omega)$ whose rows and columns are indexed by $2\pi\mathbb{Z}^d$, and whose (α,β)-entry is

$$\widetilde{G}(\omega)(\alpha,\beta) = \Psi[\omega+\alpha,\omega+\beta].$$

The matrix $\widetilde{G}(\omega)$ is interpreted then as an endomorphism of $\ell_2(2\pi\mathbb{Z}^d)$ with norm denoted by $\mathcal{G}^*(\omega)$ and inverse norm $\mathcal{G}^{*-}(\omega)$. It is understood that $\mathcal{G}^*(\omega) := \infty$ whenever $\widetilde{G}(\omega)$ does not represent a bounded operator, and a similar remark applies to $\mathcal{G}^{*-}(\omega)$. Theorem 4.1 together with the general ‘shift-invariance tools’ of [RS1] lead to the following characterization of wavelet frames.

“thmtwo **Theorem 4.3.** *Let X be an affine system generated by the ‘mother wavelets’ Ψ. Let $\mathcal{G}^*$ and $\mathcal{G}^{*-}$ be the dual Gramian norm functions defined as above. Then X is a frame for $L_2(\mathbb{R}^d)$ if and only if $\mathcal{G}^*,\mathcal{G}^{*-}\in L_\infty$. Furthermore, the frame bounds of X are $\|\mathcal{G}^*\|_{L_\infty}$ and $1/\|\mathcal{G}^{*-}\|_{L_\infty}$.*

The theorem sheds new light on various previous studies of wavelet frames. For example, the estimates for the frames bounds of a wavelet frame (cf., e.g., [D1]) can be reviewed as an attempt to estimate the norm and/or inverse norm of a matrix (viz., $\widetilde{G}(\omega)$) in terms of its entries. ‘Oversampling principles’ (that extend that original work of Chui and Shi, cf. e.g., [CS]) are derived from the fact that the fibers of of the oversampled systems are submatrices of the fibers of the original system.

The above theorem leads to the following characterization of *tight wavelet frames* (cf. Corollary 5.7 of [RS3]. Part (a) of that result was independently established in [H]):

"thmthree **Corollary 4.4.**

(a) An affine system X generated by Ψ is a tight frame for $L_2(\mathbb{R}^d)$ with frame bound C if and only if

"one
$$\Psi[\omega,\omega] = C, \tag{4.5}$$

and

"two
$$\Psi[\omega,\omega+2\pi+4\pi j] = 0, \tag{4.6}$$

for a.e. $\omega \in \mathbb{R}$ and $j \in \mathbb{Z}^d$.

(b) An affine system X is an orthonormal basis of $L_2(\mathbb{R}^d)$ if only if (4.6) holds, (4.5) holds with $C = 1$, and Ψ lies on the unit sphere of L_2.

We now show how the above theory leads to concerte algorithms for constructing wavelet frames. Assume that ϕ is a compactly supported refinable function with $\widehat{\phi}(0) = 1$ (and satisfies (decaycond)). Note that, in contrast with most of the wavelet literature, we are not making *a-priori* any assumption on the shifts of the refinable function: these shifts may not be orthonormal, nor they need to form a Riesz basis, nor even a frame. (Furtheremore, we actually need only the condition $\widehat{\phi}(0) = 1$; the other assumptions are made here for convenience.)

We denote by

$$V_0$$

the closed linear span of the shifts of ϕ and by

$$V_j$$

the 2^j-dilate of V_0. The assumption that ϕ is **refinable** is defined here to merely mean that $V_0 \subset V_1$. We remark in passing that (cf. §4 of [BDR2]) $\cap_{j\in\mathbb{Z}} V_j = 0$ and that $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R}^d)$ (the latter follows from the compact support assumption on ϕ, while the former holds for *any* refinable function, compactly supported or not); however, we will need these two properties for the subsequent development.

In classical MRA constructions of orthogonal wavelets, prewavelets, biorthogonal wavelets, and frames, one starts with one or two refinable function(s) ϕ (and ϕ^d) that has certain properties (e.g., the shifts of ϕ are orthonormal, or form a Riesz basis; the shifts of ϕ^d are bi-orthogonal to those of ϕ, etc.) Then, one carefully selects set of mother wavelets Ψ from the space V_1 in a way maked the space W_0 which is spanned by $E(\Psi)$ complementary (in some suitable sense) to V_0 in V_1; for example, W_0 may be the orthogonal complement of V_0 in V_1. The cardinality of the wavelet set Ψ is always $2^d - 1$.

In these classical constructions, we encounter difficulties in one (or both) of the following two major steps: (i) finding refinable functions with desired properties (the main difficulty being the deduction of the properties of ϕ from its refinement mask), and (ii) constructioning of the corresponding wavlet masks when the masks of the refinable functions are given.

Our MRA constructions in [RS3-5] deviates from this classical approach in the following way: while still selecting the mother wavelets Ψ from V_1, we allow the cardinality of the mother wavelet set Ψ to exceed the traditional number $2^d - 1$. We use this acquired degrees of freedom to construct affine frames with desired properties without requiring the underlying scaling function(s) to satisfy any substantial property. The examples in the previous sections demonstrate this point.

All the construction of the wavelet systems in this paper are based on two closely related algorithms for the derivation of wavelet frames from MRA. The first is the *(rectangular) unitary extension principle*, [RS3], which is used in the construction of tight wavelet frames, and the other is the *mixed extension principle*, [RS4], which is used in the construction of wavelet bi-frames. The unitary extension principle (Theorem 4.8 below) is derived in[RS3] as follows: assuming that ϕ is refinable and that Ψ is any finite subset of V_1, one rewrites first the conditions in Corollary 4.4 in terms of the various masks and the scaling function ϕ only. This leads, [RS3] to a complete characterization of all tight wavelet frames which can be constructed from any MRA, in terms of of the underlying masks only. The following algorithm then follows easily from that general characterization. In its statement, we define the maks τ_ψ of $\psi \in \Psi' := \phi \cup \Psi$ as the 2π-periodic function in the relation

$$\widehat{\psi}(2\cdot) = \tau_\psi \widehat{\phi}.$$

We then construct a rectangular matrix Δ whose rows are indexed by Ψ', whose columns are indexed by $\mathcal{Z} := \{0, \pi\}^d$:

“maskmat (4.7)
$$\Delta := (E^\nu \tau_\psi)_{\psi \in \Psi', \nu \in \mathcal{Z}}.$$

“thmfour **Theorem 4.8 (the unitary extension principle).** *Let ϕ a refinable function corresponding to MRA $(V_j)_j$ and Ψ be a finite subset of V_1. Let Δ be the matrix (4.7) that corresponds to $\Psi' := \Psi \cup \phi$, and X the affine systems generated by Ψ. If $\Delta^* \Delta = I$, a.e., then X is a fundamental tight frame.*

In [RS4], the above algorithm algorithm was extended to include bi-frames.

“multidual **Theorem 4.9 (the mixed extension principle).** *Let ϕ and ϕ^d be two refinable functions corresponding to MRAs $(V_j)_j$ and $(V_j^d)_j$, respectively. Let Ψ be a finite subset of V_1, and let $\mathrm{R} : \Psi \to V_1^d$ be some map. Let Δ be the matrix (4.7) that corresponds to $\Psi' := \Psi \cup \phi$, and let Δ^d be the matrix of (4.7) that corresponds to $\Psi' := \mathrm{R}\Psi \cup \phi^d$. Finally, let X and $\mathrm{R}X$ be the affine systems generated by Ψ and $\mathrm{R}\Psi$, respectively. If*
(a) X and $\mathrm{R}X$ are Bessel, and
(b) $\Delta^* \Delta^d = I$, a.e.,
then X and $\mathrm{R}X$ are frames for L_2 that are dual one to the other.

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