How small can one make the derivatives of an interpolating function?

Carl de Boor

Dedicated to Professor G.G. Lorentz on the occasion of his sixty–fifth birthday

1. Introduction

In his pioneering paper [3], Favard considers the problem of minimizing $f^{(k)}$ over

$$F := \{f \in \mathbb{L}_\infty^{(k)} : f(t_i) = f_0(t_i), \quad i = 1, \ldots, n+k\},$$

for a given f_0 and a given strictly increasing sequence $\mathbf{t} = (t_i)_1^{n+k}$. Favard solves this problem in a rather ingenious way that is detailed and elaborated upon in [2]. Favard goes on to prove that, with

$$[t_i, \ldots, t_{i+k}]f_0$$

denoting the kth divided difference of f_0 on the points $t_i, \ldots, t_{i+k}$,

$$K(k) := \sup_{f_0, \mathbf{t}} \frac{\inf\{\|f^{(k)}\|_\infty : f \in \mathbb{L}_\infty^{(k)}, f(t_i) = f_0(t_i), \quad \text{all } t_i\}}{\max_i k!|[t_i, \ldots, t_{i+k}]f_0|},$$

is finite, and that $K(1) = 1$, $K(2) = 2$. For $k > 2$, Favard gives no quantitative information about $K(k)$.

An estimate for the supremum under the additional restriction that only uniform $\mathbf{t}$ be considered can be found in Jerome and Schumaker [5]. Their argument was extended by Golomb [4] as far as it will go, viz., to include nonuniform $\mathbf{t}$'s whose global mesh ratio $R_\mathbf{t} := \max_i \Delta t_i / \min_i \Delta t_i$ is bounded.

It is the purpose of the present paper to show how Favard's argument can be used to obtain upper bounds for $K(k)$. Further, an upper bound for $K(k)$ is also obtained by a completely different method which, incidentally, also provides a simple proof of a theorem concerning the existence of $H^{k,p}$–extensions, thereby simplifying and extending three theorems of Golomb [4]. A lower bound for $K(k)$ is also given.

The author's interest in the numbers $K(k)$ was sparked by a question about them from H–O. Kreiss, who apparently was looking for a shortcut in computing error bounds for a given finite difference approximation to the solution of an ordinary differential equation. A bound on $K(k)$ allows to bound the kth derivative (and therefore all lower derivatives) of *some* smooth interpolant f to given data $f(t_1), \ldots, f(t_{n+k})$ in terms of the *computable* absolutely biggest kth divided difference *without* actually constructing and then bounding such an interpolant and its derivatives.

2. Favard's argument

Favard's argument consists in showing that, with p_i the polynomial of degree $\le k$ that agrees with f_0 at $t_i, \ldots, t_{i+k}$, a function f in F could be constructed by blending $p_1, \ldots, p_n$ together without increasing the kth derivative too much. Because of some practical interest for small k, we describe Favard's construction in some detail.

Favard's construction

Given $k \ge 2$, the strictly increasing sequence $\mathbf{t} = (t_i)_1^{n+k}$, and the function f_0.

Step 1. For $i = 1, \ldots, n$, form $p_i :=$ the polynomial of degree $\le k$ that agrees with f_0 at $t_i, \ldots, t_{i+k}$, and set $f := p_1$, $i := 1$, $j(1) := 0$.

Step 2. At this point, f is in $\mathbb{L}_\infty^{(k)}$, agrees with f_0 at $t_1, \ldots, t_{k+i}$, and agrees with p_i on $t \ge t_{j(i)+1}$. If $i = n$, stop. Otherwise, increase i by 1 and continue.

Step 3. Pick $j := j(i)$ so that $j \ge j(i-1)$ and $I := (t_j \mathinner{.\,.} t_{j+1})$ is a largest among the $k-1$ intervals $(t_i \mathinner{.\,.} t_{i+1}), \ldots, (t_{i+k-2} \mathinner{.\,.} t_{i+k-1})$ and set $\psi_i(t) := (t - t_i) \cdots (t - t_{i+k-1})$.

Step 4. On I, add to f the function

$$h_i(t) := \alpha_i \int_{t_j}^t (t-s)^{k-1} g_i(s)\, \mathrm{d}s/(k-1)! \qquad (1)$$

with

$$\alpha_i := ([t_i, \dots, t_{i+k}] - [t_{i-1}, \dots, t_{i+k-1}])f_0$$

and g_i the piecewise constant function with jumps only at $t_j + (r/k)\Delta t_j$, $r = 1, \dots, k-1$, for which

$$h_i^{(r)}(t_{j+1}) = \alpha_i \psi_i^{(r)}(t_{j+1}) \quad (= (p_i - p_{i-1})^{(r)}(t_{j+1})), \quad r = 0, \dots, k-1. \tag{2}$$

Step 5. At this point, $f^{(r)}(t_{j+1}^-) = p_i^{(r)}(t_{j+1})$, $r = 0, \dots, k-1$. On $t > t_{j+1}$, redefine f to equal p_i, and go to Step 2.

For $k = 2$, this construction is particularly simple since then, for $i = 2, \dots, n$,

$$j(i) = i, \quad \psi_i(t) = (t - t_i)(t - t_{i+1}),$$

and, in terms of the piecewise constant

$$g_i(t) := \begin{cases} L, & t_i < t < t_{i+1/2}, \\ R, & t_{i+1/2} < t < t_{i+1} \end{cases}, \quad t_{i+1/2} := (t_i + t_{i+1})/2,$$

(1) and (2) become

$$-\frac{1}{2}\Big(\Big(\frac{\Delta t_i}{2}\Big)^2 - (\Delta t_i)^2\Big)L + \frac{1}{2}\Big(\frac{\Delta t_i}{2}\Big)^2 R = \psi_i(t_{i+1}) \qquad (= 0),$$

$$\frac{\Delta t_i}{2}L + \frac{\Delta t_i}{2}R = \psi_i^{(1)}(t_{i+1}) \qquad (= \Delta t_i).$$

Hence $L = -1$, $R = 3$, independently of i. Therefore, on $(t_i \,..\, t_{i+1})$,

$$f^{(2)} = p_{i-1}^{(2)} + \frac{1}{2}(p_i^{(2)} - p_{i-1}^{(2)})g_i = \frac{1}{2}\begin{cases} 3p_{i-1}^{(2)} - p_i^{(2)}, & t_i < t < t_{i+1/2}, \\ -p_{i-1}^{(2)} + 3p_i^{(2)}, & t_{i+1/2} < t < t_{i+1}, \end{cases}$$

$i = 2, \dots, n$, while $f^{(2)} = p_1^{(2)}$ on $t < t_2$, and $f^{(2)} = p_n^{(2)}$ on $t > t_{n+1}$. In particular, $K(2) \le 2$.

The crucial step in Favard's argument is the proof that

$$\|g_i\|_{\infty,I} \le \mathrm{const}_k \tag{3}$$

for some const_k depending only on k and not on $\mathbf{t}$ (or f_0). Once this is accepted, it then follows that, for the final f,

$$\|f^{(k)}\|_\infty \le \big(1 + 2\frac{\mathrm{const}_k}{(k-1)!}\big)k! \max_i |[t_i, \dots, t_{i+k}]f_0|,$$

since, on any given interval $(t_j \,..\, t_{j+1})$, $f^{(k)} = p_i^{(k)} + \alpha_{i+1}g_{i+1} + \dots + \alpha_{i+r}g_{i+r}$ for some i, and some $r \in [0 \,..\, k-1]$. But, rather than elaborating Favard's lapidary remarks in support of the bound (3), we prefer to discuss the following modification of Step 4 in Favard's construction: Let λ be the linear functional on $\mathbb{P}_k$ that satisfies

$$\lambda(t_{j+1} - \cdot)^{k-1-r}/(k-1-r)! = \psi_i^{(r)}(t_{j+1}), \quad r = 0, \dots, k-1. \tag{4}$$

Here, $\mathbb{P}_k :=$ the space of polynomials of degree $< k$, considered as a subspace of $\mathbb{L}_1(I)$. There is, clearly, one and only one such linear functional since the sequence $((t_{j+1} - \cdot)^{k-1-r})_{r=0}^{k-1}$ is a basis for $\mathbb{P}_k$. By the Hahn–Banach Theorem, we can now choose $g_i \in \mathbb{L}_\infty(I) \cong (\mathbb{L}_1(I))^*$ so that $\|g_i\|_\infty = \|\lambda\|$ while $\int_I pg_i = \lambda p$ for all $p \in \mathbb{P}_k$. For such g_i, h_i as given by (1) satisfies (2), while $\|h_i^{(k)}\|_{\infty,I} \le |\alpha_i|\|\lambda\|$.

It remains to bound $\|\lambda\|$. For this, observe that, for all $p \in \mathbb{P}_k$,

$$p = \sum_{r=0}^{k-1}(-)^{k-1-r}p^{(k-1-r)}(t_{j+1})(t_{j+1} - \cdot)^{k-1-r}/(k-1-r)!,$$

hence (4) implies that

$$\lambda p = \sum_{r=0}^{k-1}(-)^{k-1-r}p^{(k-1-r)}(t_{j+1})\psi_i^{(r)}(t_{j+1}), \quad \text{all } p \in \mathbb{P}_k. \tag{5}$$

From this, a bound for $\|\lambda\| = \sup_{p\in\mathbb{P}_k} |\lambda p| / \int_I |p|$ could be obtained much as in the proof of the next section's lemma.

3. Some estimates for Favard's Constants

There is no difficulty in considering the slightly more general case when $\mathbf{t} = (t_i)_1^{n+k}$ is merely nondecreasing, coincidences in the t_i's being interpreted as repeated or osculatory interpolation in the usual way. Precisely, with $\mathbf{t}$ nondecreasing and f sufficiently smooth, denote by

$$f|_{\mathbf{t}} := (f_i)$$

the corresponding sequence given by the rule

$$f_i := f^{(j)}(t_i) \quad \text{with} \quad j := j(i) := \max\{m : \mathbf{t}_{i-m} = t_i\}.$$

Assuming that $\operatorname{ran}\mathbf{t} \subseteq [a\,..\,b]$ and that $t_i < t_{i+k}$, all i, $f|_{\mathbf{t}}$ is defined for every f in the Sobolev space

$$\mathbb{L}_p^{(k)}[a\,..\,b] := \{f \in C^{(k-1)}[a\,..\,b] : f^{(k-1)}\text{abs.cont.};\ f^{(k)} \in \mathbb{L}_p[a\,..\,b]\}.$$

Consider the problem of minimizing $\|f^{(k)}\|_p$ over

$$F := F(\mathbf{t}, \boldsymbol{\alpha}, k, p, [a\,..\,b]) := \{f \in \mathbb{L}_p^{(k)}[a\,..\,b] : f|_{\mathbf{t}} = \boldsymbol{\alpha}\}$$

for some given $\boldsymbol{\alpha}$. F is certainly not empty; it is, e.g., well known that F contains exactly one polynomial of degree $< n+k$. Hence

$$F = \{f \in \mathbb{L}_p^{(k)}[a\,..\,b] : f|_{\mathbf{t}} = f_0|_{\mathbf{t}}\},$$

for some fixed function $f_0 \in F$. Favard already observes (without using the term "spline", of course) that

$$\inf_{f\in F} \|f^{(k)}\|_p = \inf_{g\in G} \|g\|_p, \tag{6}$$

with

$$G := G(\mathbf{t}, g_0, k, p, [a\,..\,b]) := \{g \in \mathbb{L}_p[a\,..\,b] : \int_a^b M_{i,k}(g - g_0) = 0 \quad \text{all } i\},$$

$$g_0 := f_0^{(k)},$$

and

$$M_{i,k}(t)/k! := [t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}/(k-1)! \tag{7}$$

a (polynomial) B–spline of order k having the knots $t_i, \ldots, t_{i+k}$. Equation (6) follows from the observations (i) that, with $P_1 f$ the polynomial of degree $< k$ for which

$$(P_1 f)|_{(t_i)_1^k} = f|_{(t_i)_1^k},$$

and

$$Vg := \int_a^b (\cdot - s)_+^{k-1} g(s)\,\mathrm{d}s/(k-1)!,$$

every $f \in \mathbb{L}_p^{(k)}[a\,..\,b]$ can be written in exactly one way as

$$f = p_1 + (1 - P_1)Vg,$$

with $p_1 \in \mathbb{P}_k$ (necessarily equal to $P_1 f$) and $g \in \mathbb{L}_p[a\,..\,b]$ (necessarily equal to $f^{(k)}$); and (ii) that

$$f|_{\mathbf{t}} = f_0|_{\mathbf{t}} \Longleftrightarrow P_1 f = P_1 f_0 \quad \text{and} \quad [t_i, \ldots, t_{i+k}](f - f_0) = 0, \quad \text{for all } i.$$

It follows that

$$K(k) = \sup_{g_0 \in \mathbb{L}_\infty, \mathbf{t}} \frac{\inf\{\|g\|_\infty : \int M_{i,k} g = \int M_{i,k} g_0, \text{all } i\}}{\max_i |\int M_{i,k} g_0|}.$$

The following lemma is therefore relevant to bounding $K(k)$.

Lemma. *If $t_i < t_{i+k}$, then, for every largest subinterval $I := (t_r \mathinner{..} t_{r+1})$ of $(t_i \mathinner{..} t_{i+k})$, there exists $h_i \in \mathbb{L}_\infty$ with support in I so that*

$$\int h_i M_{j,k} = \delta_{i,j}, \text{ all } j, \quad \text{and } \|h_i\|_p \le D_k((t_{i+k} - t_i)/k)/|I|^{1-1/p}, \quad 1 \le p \le \infty,$$

for some constant D_k depending only on k.

Proof: By [1], the linear functional λ_i given by the rule

$$\lambda_i f := \sum_{j<k} (-)^{k-1-j} \psi_{i,k}^{(k-1-j)}(\tau_i) f^{(j)}(\tau_i),$$

$$\psi_{i,k}(t) := (t_{i+1} - t) \cdots (t_{i+k-1} - t)/(k-1)!$$

satisfies

$$\lambda_i M_{j,k} = \delta_{j,k} k/(t_{i+k} - t_i),$$

provided $\tau_i \in (t_i \mathinner{..} t_{i+k})$. Let

$$\lambda := \lambda_i|_{\mathbb{P}_k},$$

with τ_i the midpoint of $I :=$ a largest among the k intervals $(t_i \mathinner{..} t_{i+1}), \dots, (t_{i+k-1} \mathinner{..} t_{i+k})$, and $\mathbb{P}_k :=$ the space of polynomials of degree $< k$ considered as a subspace of $\mathbb{L}_1(I)$. Then

$$|I| \ge (t_{i+k} - t_i)/k.$$

Also, by the Hahn–Banach theorem, there exists $h \in \mathbb{L}_\infty(I)$ such that $\|h\|_\infty = \|\lambda\|$ and $\int_I hg = \lambda g$ for all $g \in \mathbb{P}_k$. But then, since $g|_I \in \mathbb{P}_k$ for every g in $\mathbb{S}_{k,\mathbf{t}} := \operatorname{span}(M_{1,k}, \dots, M_{n,k})$, the function h_i defined by

$$h_i(t) := \begin{cases} h(t)((t_{i+k} - t_i)/k), & t \in I \\ 0, & t \notin I \end{cases}$$

satisfies

$$\int h_i g = ((t_{i+k} - t_i)/k)\lambda_i g, \quad \text{for all } g \in \mathbb{S}_{k,\mathbf{t}}$$

$$\|h_i\|_p \le (t_{i+k} - t_i)/k \, \|\lambda\| |I|^{1/p}.$$

It remains to show that $\|\lambda\| \le D_k/|I|$ for some constant D_k depending only on k. For this,

$$\psi_{i,k}^{(k-1-j)}(t) = \frac{(-)^{k-1-j}}{(k-1)!}(k-1-j)! \sum_{\substack{J \subseteq \{1,\dots,k-1\} \\ |J|=j}} \prod_{r \in J} (t_{i+r} - t),$$

hence, by choice of I, and of τ_i in I, we have

$$|\psi_{i,k}^{(k-1-j)}(\tau_i)| \le \binom{k-1}{j} |I|^j.$$

Also,

$$\sup_{g \in \mathbb{P}_k} |g^{(j)}(\tau_i)| \Big/ \int_I |g| = \mathrm{const}_{j,k} (2/|I|)^{j+1},$$

with

$$\mathrm{const}_{j,k} := \sup_{g \in \mathbb{P}_k} |g^{(j)}(0)| \Big/ \int_{-1}^{1} |g(t)| \, \mathrm{d}t \le (k-1)^j k(2k+1)/2.$$

Hence, the number

$$D_k := \sum_{j<k} \mathrm{const}_{j,k} 2^{j+1} \binom{k-1}{j} \le k(2k+1)(2k-1)^{k-1}$$

depends only on k, while

$$|\lambda g| = |\lambda_i g| \le D_k \int_I |g|/|I|, \quad \text{for all } g \in \mathbb{P}_k. \qquad |||$$

If now the numbers

$$c_j := k![t_j, \ldots, t_{j+k}]f_0, \quad j = 1, \ldots, n,$$

are given, then

$$g := \sum_{j=1}^{n} c_j h_j$$

satisfies

$$\int M_{i,k} g = c_i = \int M_{i,k} g_0, \quad i = 1, \ldots, n,$$

while

$$\|g\|_\infty \le \max_j |c_j| \| \sum_j |h_j| \|_\infty.$$

But since at most k of the h_j's can have any particular interval in their support, it follows that

$$K(k) \le \| \sum_j |h_j| \|_\infty \le k^2(2k+1)(2k-1)^{k-1}. \tag{8}$$

The construction of g is entirely *local*: On $(t_i \,..\, t_{i+1})$, g is the sum of all those terms $c_j h_j$ that have their support in that interval. For each such h_j, $(t_i \,..\, t_{i+1})$ must be a largest interval of that form in $(t_j \,..\, t_{j+k})$, hence in particular $j \in (i-k \,..\, i]$; i.e.,

$$\|g\|_{\infty,(t_i..t_{i+1})} \le k D_k \max_{i-k<j\le i} \Big| \int M_{j,k} g_0 \Big|.$$

In terms of the original problem of finding $f \in \mathbb{L}_\infty^{(k)}[a \,..\, b]$ that agrees with f_0 on $\mathbf{t}$ and has a "small" kth derivative, the above lemma has therefore the

Corollary. *For given $f_0 \in \mathbb{L}_\infty^{(k)}[a \,..\, b]$ and given $\mathbf{t} = (t_i)_1^{n+k}$ in $[a \,..\, b]$, nondecreasing with $t_i < t_{i+k}$,* all i, *there exists $f \in \mathbb{L}_\infty^{(k)}[a \,..\, b]$ such that $f|_{\mathbf{t}} = f_0|_{\mathbf{t}}$, and, for all i,*

$$\|f^{(k)}\|_{\infty,[t_i..t_{i+1}]} \le D'_k \max_{i-k<j\le i} k!|[t_j, \ldots, t_{j+k}]f_0|$$

with D'_k some constant depending only on k.

It seems likely that $K(k)$ is much closer to its lower bound

$$(\pi/2)^{k-1} \le K(k) \tag{9}$$

than to the rather fast growing upper bound (8). One obtains (9) with the aid of Schoenberg's Euler spline [6]: With $t_i = i$, all i, the kth degree Euler spline

$$\mathcal{E}_k(t) := \gamma_k \sum_i (-)^i M_{i,k+1}(t + (k+1)/2)$$

satisfies

$$\mathcal{E}_k(i) = (-)^i, \quad \text{all } i,$$

hence

$$k!|[i, \ldots, i+k]\mathcal{E}_k| = 2^k,$$

with

$$\gamma_k := 1\Big/\sum_j \left(\frac{\sin(2j+1)\pi/2}{(2j+1)\pi/2}\right)^{k+1} = (\pi/2)^{k+1}\Big/\sum_j ((-1)^j/(2j+1))^{k+1} \ge (\pi/2)^{k-1}.$$

In fact,

$$\lim_{k\to\infty} \gamma_k/(\pi/2)^{k+1} = 1/2.$$

We claim that $\gamma_k \le K(k)$, which then implies (9). Suppose, by way of contradiction, that $\gamma_k > K(k)$. Then there would exist, for $n = 1, 2, \ldots$, $f_n \in \mathbb{L}_\infty^{(k)}[1\,..\,k+n]$ so that $f_n(i) = (-)^i$, $i = 1, \ldots, n+k$, while

$$\|f_n^{(k)}\|_\infty \le K(k)2^k < \gamma_k 2^k = \|\mathcal{E}_k^{(k)}\|_\infty.$$

The function

$$e_n := \mathcal{E}_k^{(k)} - f_n^{(k)}$$

would then alternate in sign, changing sign only at the points $i + (k+1)/2$, and

$$\text{ess.}\inf |e_n| \ge -(K(k) - \gamma_k)2^k > 0,$$

while

$$\int M_{i,k} e_n = 0, \quad \text{for } i = 1, \ldots, n. \tag{10}$$

But then, using the fact that the scalar multiple

$$g_k(t) := \sum_i (-)^i M_{i,k}(t + k/2)$$

of $\mathcal{E}_{k-1}$ changes sign only at $(i + (k+1)/2)$, all i, we would have that

$$\begin{aligned} \Big|\int_1^{n+k} e_n g_k\Big| &\ge \text{ ess.}\inf |e_n| \|g_k\|_{1,[1..n+k]} \\ &\ge (\gamma_k - K(k))2^k(n+k)\|g_k\|_{1,[0..1]} \underset{n\to\infty}{\longrightarrow} \infty \end{aligned}$$

while also

$$\Big|\int_1^{n+k} e_n g_k\Big| = \Big|\int_1^{n+k} e_n \sum_{i\notin[1..n]} (-)^i M_{i,k}\Big| \le \|\mathcal{E}_k^{(k)}\|_\infty 2k < \infty,$$

a contradiction.

It is possible to compute better upper bounds for $K(k)$, at least for small values of k, simply by estimating the constant D_k in the lemma above more carefully, e.g., by computing explicitly a piecewise constant h (with appropriately placed jumps) that represents an extension of λ to all of $\mathbb{L}_1(I)$. To give an example, it is possible to show in this way that $D_3 < 12$, whereas the estimate in the lemma merely gives $D_3 < 525$. These and other such computations will be reported on elsewhere (cf. remark at paper's end).

For $k = 2$, $\gamma_k = 2$, hence $K(2) \ge 2$, therefore $K(2) = 2$, as we saw already in Section 2 that $K(2) \le 2$. This was already observed by Favard, using a variant of the Euler spline.

4. Existence of $H^{k,p}$–extensions

In this last section, we take advantage of the lemma just proved in the preceding section to give a very simple proof of a theorem that extends and unifies the three theorems in Section 3 of [4]. In that paper, Golomb discusses (among other things) the existence of $f \in H^{k,p} := \mathbb{L}_p^{(k)}(\mathbb{R})$ for which $f|_{\mathbf{t}} = \boldsymbol{\alpha}$ for given possibly biinfinite $\mathbf{t}$ with $t_i < t_{i+k}$, all i, and a corresponding real sequence $\boldsymbol{\alpha}$.

Denote by

$$[t_i, \ldots, t_{i+k}]\boldsymbol{\alpha}$$

the kth divided difference $[t_i, \ldots, t_{i+k}]g$ of any function g for which

$$g|_{(t_r)_j^{i+k}} = (\alpha_r)_j^{i+k},$$

with $t_{j-1} < t_j \le t_i$. While it is easy to see that $f \in \mathbb{L}_p^{(k)}(\mathbb{R})$ implies

$$\sum_i (t_{i+k} - t_i)|[t_i, \ldots, t_{i+k}]f|^p < \infty,$$

Golomb proves the converse statement, viz. that

(11) $\quad \|((t_{i+k} - t_i)^{1/p}[t_i, \ldots, t_{i+k}]\boldsymbol{\alpha})_i\|_p < \infty$ *implies the existence of* $f \in \mathbb{L}_p^{(k)}(\mathbb{R})$ *with* $f|_{\mathbf{t}} = \boldsymbol{\alpha}$

only in three special cases [4, Theorems 3.1, 3.2, 3.3] in which $\mathbf{t}$ satisfies some global mesh ratio restrictions. The lemma in the preceding section allows to prove (11) without any restriction on $\mathbf{t}$ (other than that $t_i < t_{i+k}$, all i, which quite reasonably prevents values of $f^{(k)}$ from being prescribed).

In view of the discussion in Section 3, (11) is equivalent to the statement

$\|((t_{i+k} - t_i)^{1/p}[t_i, \ldots, t_{i+k}]\boldsymbol{\alpha})_i\|_p < \infty$ *implies the existence of* $g \in \mathbb{L}_p(\mathbb{R})$ *such that*

$$\int M_{i,k} g = k![t_i, \ldots, t_{i+k}]\boldsymbol{\alpha}, \quad \text{all } i. \tag{12}$$

For all i, let now h_i be the $\mathbb{L}_\infty$–function constructed for the lemma. Since h_i has support in some subinterval $(t_r \,..\, t_{r+1})$ of $(t_i \,..\, t_{i+k})$, no more than k of the h_j's are nonzero at any particular point. Hence, the sum

$$\sum_i c_i h_i$$

makes sense as a pointwise sum for arbitrary (c_i). Since

$$\int h_i M_{j,k} = \delta_{i,j},$$

it follows that the function

$$g := k! \sum_i ([t_i, \ldots, t_{i+k}]\boldsymbol{\alpha}) h_i$$

satisfies (12). It remains to bound g. For $1 \le p < \infty$,

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} \Big|\sum_j c_j h_j\Big|^p &\le \int_{t_i}^{t_{i+1}} \Big(\sum_{\operatorname{supp} h_j \subseteq [t_i..t_{i+1}]} |c_j| D_k \frac{t_{j+k} - t_j}{k \Delta t_i} \Big)^p \\
&= \Big(\sum_{\operatorname{supp} h_j \subseteq [t_i..t_{i+1}]} |c_j| \Big(\frac{t_{j+k} - t_j}{k}\Big)^{1/p} \Big(\frac{t_{j+k} - t_j}{k \Delta t_i}\Big)^{1-1/p} \Big)^p {D_k}^p \\
&\le \Big(\sum_{\operatorname{supp} h_j \subseteq [t_i..t_{i+1}]} |c_j|^p \frac{t_{j+k} - t_j}{k} \Big) k^{p-1} {D_k}^p.
\end{aligned}$$

Hence

$$\|\sum_j c_j h_j\|_p^p \le k^{p-1} {D_k}^p \sum_j |c_j|^p (t_{j+k} - t_j)/k,$$

i.e.,

$$\|g\|_p \le k! k^{1-1/p} D_k \|\Big(\Big(\frac{t_{j+k} - t_j}{k}\Big)^{1/p} [t_j, \ldots, t_{j+k}]\boldsymbol{\alpha}\Big)_j\|_p$$

and this holds for $p = \infty$, too, as one checks directly.

Theorem. *For given nondecreasing* $\mathbf{t}$ *(finite, infinite or biinfinite) with* $t_i < t_{i+k}$, all i, *and given corresponding real sequence* $\boldsymbol{\alpha}$, *and given* p *with* $1 \le p \le \infty$, *there exists* $f \in \mathbb{L}_p^{(k)}(\mathbb{R})$ *such that* $f|_{\mathbf{t}} = \boldsymbol{\alpha}$ *if and only if* $\|(((t_{j+k} - t_j)/k)^{1/p}[t_j, \ldots, t_{j+k}]\boldsymbol{\alpha})_j\|_p < \infty$.

We note that the above argument (as well as the argument for (8)) is based on the linear projector $P := \sum_i h_i \otimes M_{i,k}$ given on $\mathbb{L}_p$ by the rule

$$Pf := \sum_i \Big(\int M_{i,k} f\Big) h_i, \quad \text{all } f \in \mathbb{L}_p,$$

and shows this projector to satisfy

$$\|Pf\|_{p,(t_i..t_{i+1})} \le D_k k^{1-1/p} \Big(\sum_{\operatorname{supp} h_j \subseteq [t_i..t_{i+1}]} \Big| \int M_{j,k} f \Big|^p \frac{t_{j+k} - t_j}{k} \Big)^{1/p}.$$

This implies the local bound

$$\|Pf\|_{p,(t_i..t_{i+1})} \le k D_k \|f\|_{p,(t_{i+1-k}..t_{i+k})} \tag{13}$$

as well as the global bound $\|P\| \le kD_k$. The dual map for P, i.e., the linear projector $P^* := \sum_i M_{i,k} \otimes h_i$ on $\mathbb{L}_q$ (with $1/p + 1/q = 1$) with range equal to $\mathbb{S}_{k,\mathbf{t}}$, is therefore also bounded by kD_k. In addition, direct application of the Lemma in Section 3 gives the local bound

$$\|P^* f\|_{q,(t_i..t_{i+1})} \le k^{1/q} D_k \|f\|_{q,(t_{i+1-k}..t_{i+k})}. \tag{14}$$

Note added in proof. The computations alluded to in Section 3 have been reported on in [C. de Boor, A smooth and local interpolant with "small" k–th derivative, MRC TSR#1466; to appear in "Numerical Solutions of Boundary Problems for Ordinary Differential Equations," (A.K. Aziz, Ed.), Academic Press, New York, 1974], and show that $K(k)$ grows "initially" no faster than 2^k. The same reference contains a proof that $K(k) \le (k-1)9^k$ for all k.

References

1. C. de Boor and G. J. Fix, Spline approximation by quasi-interpolants, *J. Approx. Theory* **8** (1973), 19–45.
2. C. de Boor, On 'best' interpolation, *J. Approx. Theory* **16** (1976), 28–42.
3. J. Favard, Sur l'interpolation, *J. Math. Pures Appl.* **19** (1940), 281–306.
4. M. Golomb, $H^{m,p}$-extensions by $H^{m,p}$-splines, *J. Approx. Theory* **5** (1972), 238–275.
5. J. Jerome and L. L. Schumaker, Characterization of functions with higher order derivatives in L^p, *Trans. Amer. Math. Soc.* **143** (1969), 363–371.
6. I. J. Schoenberg, "Cardinal Spline Interpolation", CBMS, SIAM, Philadelphia, 1973.