Hierarchical zonotopal spaces

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Abstract

Zonotopal algebra interweaves algebraic, geometric and combinatorial properties of a given linear map X. Of basic significance in this theory is the fact that the algebraic structures are derived from the geometry (via a non-linear procedure known as "the least map"), and that the statistics of the algebraic structures (e.g., the Hilbert series of various polynomial ideals) are combinatorial, i.e., computable using a simple discrete algorithm known as "the valuation function". On the other hand, the theory is somewhat rigid since it deals, for the given X, with exactly two pairs each of which is made of a nested sequence of three ideals: an external ideal (the smallest), a central ideal (the middle), and an internal ideal (the largest).

In this paper we show that the fundamental principles of zonotopal algebra as described in the previous paragraph extend far beyond the setup of external, central and internal ideals by building a whole hierarchy of new combinatorially defined zonotopal spaces.

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1 Introduction

1.1 Motivation

We are interested in this article in the study of algebraic structures, most notably in terms of homogeneous zero-dimensional polynomial ideals, over hyperplane arrangements, and, by duality, over zonotopes. We start by describing the pertinent setup.

Let X be a $n \times N$ matrix, which can be also viewed as a multiset of its columns: the theory, indeed, is invariant of the order of the columns (some of the algorithms will require us to order the columns, but will produce output that is independent of the order). The **zonotope** $Z(X)$ associated with X is the polytope

$$Z(X) := \{\sum_{x\in X} t_x x : \ t \in [0,1]^N\}.$$

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In other words, the zonotope $Z(X)$ is the image of the unit cube $[0,1]^N$ under X viewed as a linear map from $\mathbb{R}^N$ to $\mathbb{R}^n$ or, equivalently, $Z(X)$ is a Minkowski sum of segments $[0,x]$ over all column vectors $x \in X$.

The theory of **Zonotopal Algebra** (see [11]) is built around three pairs of zero-dimensional homogeneous polynomial ideals that are associated with X: an external pair $(\mathcal{I}_+(X),\ \mathcal{J}_+(X))$, a central pair $(\mathcal{I}(X), \mathcal{J}(X))$, and an internal pair $(\mathcal{I}_-(X), \mathcal{J}_-(X))$. These polynomial ideals play a role in several different areas of mathematics. In Approximation Theory, these ideals provide important information about multivariate splines on regular grids (box splines, see [4]). In Algebra, these ideals appear, for example, in the context of group representations and also in the context of particular types of orthogonal polynomials. In Combinatorics, these ideals are pertinent to algebraic graph theory, and are intimately connected with the Tutte polynomial. The list goes on, with the most direct connection being particular topics within convex geometry such as zonotopes, zonotope tilings, lattice points in zonotopes, and various aspects of hyperplane arrangements. For additional connections, see [2, 12, 14].

A centerpiece in the theory of Zonotopal Algebra are formulæ that capture the codimensions of the above-mentioned ideals, and more generally, their Hilbert series in terms of the basic statistics of an associated matroid. The other pillar of the theory is the connection between the aforementioned ideals and the geometric structures, viz., the zonotope $Z(X)$, and the hyperplane arrangement $\mathcal{H}(X, \lambda)$.

We pause in order to illustrate some of these aspects. First, we let

$$\Pi := \mathbb{C}[t_1, \ldots, t_n]$$

be the space of polynomials in n variables, and let

$$\Pi_k^0$$

be the subspace of Π that contains all homogeneous polynomials of exact degree k. Also, for any homogeneous ideal $I \subset \Pi$, we denote

$$\ker I := \{p \in \Pi : \quad q(D)p = 0 \quad \text{for all} \quad q \in I\}.$$

Here, $q(D)$ is the counterpart of $q \in \Pi$ in the ring $\mathbb{C}[\partial/\partial t_1, \ldots, \partial/\partial t_n]$. Given a zero-dimensional polynomial ideal $I \subset \Pi$, we denote by

$$\operatorname{codim} I$$

the dimension of the quotient space

$$\Pi/I,$$

which (is always finite and) is equal to $\dim \ker I$.

We next associate the matrix X with a suitable **hyperplane arrangement** $\mathcal{H}$: we consider each column $x \in X$ as a linear functional p_x in $(\mathbb{R}^n)^*$ (using the standard inner product in $\mathbb{R}^n$), and denote by H_{x,λ_x} the zero set of the affine polynomial

$$\mathbb{R}^n \ni t \mapsto p_x(t) - \lambda_x,$$

with $\lambda_x \in \mathbb{R}$ (arbitrary, but fixed). The hyperplane arrangement $\mathcal{H}(X, \lambda)$ is the union of the hyperplanes H_{x,λ_x}, $x \in X$. We assume that $\mathcal{H}(X, \lambda)$ is **generic** which means that every subcollection of $n + 1$ hyperplanes has an empty intersection. Note that for any given X, the set of all $\lambda \in \mathbb{R}^X$ for which $\mathcal{H}(X, \lambda)$ is generic form an open dense subset of $\mathbb{R}^X$, [13].

Next, we describe the three $\mathcal{I}$ ideals. To this end, we consider submatrices $Y \subset X$ (that are obtained from X by removing some of its columns) that are of rank $n-1$. The column span of such a submatrix is a **facet hyperplane** of X, and we denote by

$$\mathcal{F}(X)$$

the set of all facet hyperplanes of X. The normal (in any direction) to a facet hyperplane F is denoted by

$$\eta_F,$$

and the **multiplicity** (in X) of a facet hyperplane $F \in \mathcal{F}(X)$ is the cardinality

$$m(F) := m_X(F) := \#\{x \in X : x \notin F\}.$$

The three $\mathcal{I}$ ideals are generated, each, by suitable powers of the normal polynomials:

$$\{p_{\eta_F}^{m(F)+\epsilon} : \ F \in \mathcal{F}(X)\}.$$

The internal ideal $\mathcal{I}_-(X)$ corresponds to the choice $\epsilon = -1$, the central ideal $\mathcal{I}(X)$ corresponds to the choice $\epsilon = 0$, while the external ideal $\mathcal{I}_+(X)$ corresponds to the choice $\epsilon = +1$. We now state a result that connects these three ideals to the hyperplane arrangement $\mathcal{H}(X,\lambda)$. In the sequel, we will also show the connection of these three ideals to the zonotope $Z(X)$, and to several other constructs.

Theorem 1.1

1. $\operatorname{codim}\mathcal{I}(X)$ *equals the number of vertices in the generic arrangement* $\mathcal{H}(X,\lambda)$.
2. $\operatorname{codim}\mathcal{I}_+(X)$ *equals the number of connected components in* $\mathbb{R}^n\backslash\mathcal{H}(X,\lambda)$.
3. $\operatorname{codim}\mathcal{I}_-(X)$ *equals the number of bounded connected components in* $\mathbb{R}^n\backslash\mathcal{H}(X,\lambda)$.

An important highlight of the $\mathcal{I}$-ideals is that their associated kernels can be described cleanly and explicitly in terms of the columns of X. Here, we discuss this point in the context of the central zonotopal space. Given $Y \subset X$,[1] we say that Y is **short** if $\operatorname{rank}(X\backslash Y) = n$. A **short polynomial** is a product

$$p_Y := \prod_{y\in Y} p_y, \tag{1}$$

over a short subset Y. We let

$$\mathcal{P}(X) := \operatorname{span}\{p_Y : \operatorname{rank}(X\backslash Y) = n\},$$

be the span of the short polynomials. A subset $Y \subset X$ that is not short is called **long**. Let $\mathcal{J}(X)$ denote the ideal generated by the **long polynomials**:

$$\mathcal{J}(X) := \operatorname{Ideal}\{p_Y : Y \subset X,\ \operatorname{rank}(X\backslash Y) < n\},$$

and set

$$\mathcal{D}(X) := \ker \mathcal{J}(X).$$

Below we collect some of main results of [1, 5, 7, 8, 9, 10]; see also [11, Theorem 3.8], where this summary appears in its present form. We use the notation

$$\mathbb{B}(X) := \{B \subset X : B \text{ is a basis for } \mathbb{R}^n\},$$

as well as the pairing

$$\Pi \to \Pi' : p \mapsto \langle p, \cdot\rangle, \quad \langle p, q\rangle := (p(D)q)(0) = (q(D)p)(0).$$

The space $\Pi(V)$ that appears in Theorem 1.2 is defined in Section 1.2.

[1] Recall that we refer to X as the multiset of its columns, hence Y is obtained by removing some columns of X.

Theorem 1.2

1. *$\dim \mathcal{P}(X) = \dim \mathcal{D}(X) = \#\mathbb{B}(X)$.*

2. *The map $p \mapsto \langle p, \cdot \rangle$ is a bijection between $\mathcal{P}(X)$ and $\mathcal{D}(X)'$.*

3. *$\mathcal{D}(X) = \Pi(V)$, with V the vertex set of the hyperplane arrangement $\mathcal{H}(X, \lambda)$.*

4. *$\mathcal{P}(X) = \ker \mathcal{I}(X)$.*

5. *$\mathcal{P}(X) \bigoplus \mathcal{J}(X) = \Pi$.*

1.2 The least map $V \mapsto \Pi(V)$

Definition 1.3 *Let V be a finite pointset in $\mathbb{R}^n$. Given $v \in \mathbb{R}^n$, let*

$$e_v : t \mapsto e^{v \cdot t}$$

be the exponential with frequency v, and define

$$\mathrm{Exp}(V) := \mathrm{span}\,\{e_v : \ v \in V\}.$$

Given $f \in \mathrm{Exp}(V)$, let

$$f = \sum_{j=j_f}^{\infty} f_j, \quad j_f \geq 0,$$

be its homogeneous power expansion, i.e., $f_j \in \Pi_j^0$, for all j, and $f_{j_f} \neq 0$. Define

$$f_{\downarrow} := f_{j_f}, \quad \Pi(V) := \mathrm{span}\,\{f_{\downarrow} : f \in \mathrm{Exp}(V) \backslash 0\}.$$

Our interest in this paper is focused on point sets V that are either subsets of the set of integer points in the zonotope $Z(X)$, or subsets of the vertex set $V(X, \lambda)$ of the hyperplane arrangement $\mathcal{H}(X, \lambda)$. Note that in the latter case, since we assume $\mathcal{H}(X, \lambda)$ to be generic, we have that $\#V(X, \lambda) = \#\mathbb{B}(X)$, and there is a set bijection

$$V \ni v \mapsto B_v \in \mathbb{B}(X),$$

that sends each vertex v to the set of columns of X whose hyperplanes contain v. Thus, genericity means in this context that the latter set is always a basis, and never larger than that. Now, given $V' \subset V(X, \lambda)$, let $\mathbb{B}' := \mathbb{B}'(X) := \{B_v : \ v \in V'\}$, and define the ideal

$$\mathcal{J}_{\mathbb{B}'}(X) := \mathrm{Ideal}\{p_Y : \ \ Y \subset X, \ \ Y \cap B \neq \emptyset, \ \text{for all } B \in \mathbb{B}'\}.$$

Theorem 1.4 ([6, 7])

1. *For every finite $V \subset \mathbb{R}^n$, the restriction map*

$$\Pi(V) \ni f \mapsto f_{|V}$$

is a bijection between $\Pi(V)$ and $\mathbb{C}^V$. In particular, $\dim \Pi(V) = \#V$.

2. *With V', $\mathbb{B}'$ and $\mathcal{J}_{\mathbb{B}'}(X)$ as above, we have that $\Pi(V') \subset \ker \mathcal{J}_{\mathbb{B}'}(X)$. In particular,*

$$\mathrm{codim}\, \mathcal{J}_{\mathbb{B}'}(X) \geq \dim \Pi(V') = \#V' = \#\mathbb{B}'.$$

Note that the choice $V' := V(X, \lambda)$ leads to $\mathbb{B}' = \mathbb{B}(X)$, and to $\mathcal{J}_{\mathbb{B}'}(X) = \mathcal{J}(X)$. Hence the above result shows that $\Pi(V(X, \lambda)) \subset \ker \mathcal{J}(X)$. Theorem 1.2 asserts that for this particular choice of V' equality holds: $\Pi(V(X, \lambda)) = \ker \mathcal{J}(X)$. For general $\mathbb{B}'$, though, the inclusion is proper.

The least map, thus, connects between the vertices of $\mathcal{H}(X, \lambda)$ and the $\mathcal{J}$-ideals. It also connects the integers points in the zonotope $Z(X)$ to the $\mathcal{I}$-ideal. To this end, recall that X is **unimodular** if $X \subset \mathbb{Z}^n$ and $|\det B| = 1$, for every $B \in \mathbb{B}(X)$. Furthermore, let $\mathbb{I}(X)$ be the collection of (linearly) **independent subsets** of X:

$$\mathbb{I}(X) := \{I \subset X : I \text{ is independent in } \mathbb{R}^n\}.$$

Theorem 1.5 ([11])

1. $\ker \mathcal{I}_+(X) = \mathcal{P}_+(X) := \operatorname{span}\{p_Y : \ Y \subset X\}$.
2. $\dim \mathcal{P}_+(X) = \#\mathbb{I}(X)$.
3. *Assume X is unimodular. Then* $\mathcal{P}_+(X) = \Pi(Z(X) \cap \mathbb{Z}^n)$.

1.3 Hilbert series

The Hilbert series of the three $\mathcal{I}$-ideals are closely associated with the external activity of the Tutte polynomial of the given multiset X. Let $\prec$ be any order on X. Given a set $Y \subset X$, we define the valuation of Y (per the given order) by

$$\operatorname{val}(Y) := \#X(Y), \quad X(Y) := \{x \in X \backslash Y : \ x \notin \operatorname{span}\{y \in Y : \ y \prec x\}\}.$$

This valuation function determines the Hilbert series of the $\mathcal{I}$-ideals as follows.

Theorem 1.6 ([10, 11])

1. *The polynomials*
$$Q_B := p_{X(B)}, \quad B \in \mathbb{B}(X)$$
form a basis for $\mathcal{P}(X)$. In particular, for every positive integer j,
$$\dim(\mathcal{P}(X) \cap \Pi_j^0) = \#\{B \in \mathbb{B}(X) : \operatorname{val}(B) = j\}.$$
2. *The polynomials*
$$Q_I := p_{X(I)}, \quad I \in \mathbb{I}(X)$$
form a basis for $\mathcal{P}_+(X)$. In particular, for every positive integer j,
$$\dim(\mathcal{P}_+(X) \cap \Pi_j^0) = \#\{I \in \mathbb{I}(X) : \operatorname{val}(I) = j\}.$$

1.4 Intermediate setups

Zonotopal algebra interweaves, thus, algebraic, geometric and combinatorial properties of the linear map X. Of basic significance of this theory is the fact that the algebraic structures are derived from the geometry (via the least map), and that the statistics of the algebraic structures (i.e., the various Hilbert series) are combinatorial, i.e., computable using the valuation function. On the other hand, the theory is somewhat rigid since it deals with exactly three sets of ideals for each given X, their three Hilbert series and so on.

In this paper we show that the fundamental principles of zonotopal algebra as described in the previous paragraph extend far beyond the rigid setup of external, central and internal ideals. For example, let $\mathbb{I}' \subset \mathbb{I}(X)$, and let us define $Y \subset X$ to be $\mathbb{I}'$**-short** if $X\backslash Y$ contains an element of $\mathbb{I}'$. Define

$$\mathcal{P}_+(X, \mathbb{I}')$$

to be the span of all $\mathbb{I}'$-short polynomials. The central case $\mathcal{P}(X)$ corresponds to the choice $\mathbb{I}' = \mathbb{B}(X)$, and the external case $\mathcal{P}_+(X)$ corresponds to the choice $\mathbb{I}' = \mathbb{I}(X)$. In either of these extreme cases, the dimension of the polynomial space $\mathcal{P}_+(X, \mathbb{I}')$ coincides with the cardinality of $\mathbb{I}'$.

The following questions now arise naturally:

- Do we have $\dim \mathcal{P}_+(X, \mathbb{I}') = \#\mathbb{I}'$?
- Is the ideal $\mathcal{I}_+(X, \mathbb{I}')$ of all differential operators that annihilate $\mathcal{P}_+(X, \mathbb{I}')$ still generated by powers of the normals p_{η_F} to the facets?
- Do the polynomials Q_I, $I \in \mathbb{I}'$ form a basis for $\mathcal{P}_+(X, \mathbb{I}')$ or, at least, does the valuation of the set $X(I)$, $I \in \mathbb{I}'$ determine the Hilbert series of $\mathcal{P}_+(X, \mathbb{I}')$?
- Do we have a dual setup in terms of an ideal $\mathcal{J}_+(X, \mathbb{I}')$ that is generated by a suitable notion of long polynomials, so that its kernel is connected to a suitable set of vertices of some hyperplane arrangement?

It is somewhat surprising (at least to us) that the questions above can all be answered in the affirmative for a large class of sets $\mathbb{I}'$. We deal in this paper with two different setups.

In the first one, which we refer to as semi-external, we select an arbitrary subset $\mathbb{I}'$ of $\mathbb{I}(X)$ and impose only one condition on it, viz., that $\mathbb{I}'$ should be closed under (subspace) inclusion in the sense that if $\operatorname{span} I \subset \operatorname{span} I'$, for $I \in \mathbb{I}'$ and $I' \in \mathbb{I}(X)$ then $I' \in \mathbb{I}'$. We introduce then a suitable $\mathcal{J}$-ideal, $\mathcal{J}_+(X, \mathbb{I}')$, and a suitable $\mathcal{I}$-ideal $\mathcal{I}_+(X, \mathbb{I}')$, provide an explicit description and bases for $\ker \mathcal{I}_+(X, \mathbb{I}')$, develop an algorithm for computing the Hilbert series of these ideals (both share the same Hilbert series) in terms of the aforementioned valuation, and describe a suitable geometric derivation of $\ker \mathcal{J}_+(X, \mathbb{I}')$ (by acting on vertices of hyperplane arrangements). The ideal $\mathcal{I}_+(X, \mathbb{I}')$ is not generated, however, by powers of the normal polynomials p_{η_F}, $F \in \mathcal{F}(X)$. That said, by assuming slightly more on $\mathbb{I}'$, that property can be guaranteed as well.

In the second setup, which we refer to as semi-internal, we select and fix an $I \in \mathbb{I}(X)$, and develop a theory where the focus is on the space

$$\mathcal{P}_-(X, I) := \cap_{b \in I} \mathcal{P}(X\backslash b).$$

When I is a basis, the above space coincides with the kernel of $\mathcal{I}_-(X)$. Once again, this case gives rise to a theory that parallels in its ingredients the one outlined for the semi-external case. In particular, we prove that the corresponding $\mathcal{I}$-ideal is generated by powers of the normals, and we further identify a subset $\mathbb{B}_-(X, I) \subset \mathbb{B}(X)$ whose valuation determines the Hilbert series of $\mathcal{P}_-(X, I)$.

2 Semi-external zonotopal spaces

2.1 A review of external zonotopal spaces

Recall that we consider our $n \times N$ matrix X of full rank n as a finite multiset $X \subset \mathbb{R}^n\backslash\{0\}$ of size $N = \#X$. Also recall from Section 1.1 that $\mathbb{B}(X)$ denotes the multiset of all (linear) bases of X, and $\mathbb{I}(X)$ denotes the multiset of all (linearly) independent subsets of X.

The definition of the external ideal $\mathcal{J}_+(X)$ requires to choose an additional basis B_0 for $\mathbb{R}^n$, and to order its elements. There is no restriction on the choice of B_0 or on the chosen order $\prec$, but the definition of the ideal $\mathcal{J}_+(X)$ depends on the choice of B_0 and the choice of the order. We augment X by B_0 and define $X' := X \sqcup B_0$, where $\sqcup$ denote the union of two multisets, i.e., a collection obtained by listing all vectors in X and all vectors in B_0. We extend the order $\prec$ to a full order on X', with the only requirement on this extension that $x \prec b$, for every $x \in X$ and $b \in B_0$. We use B_0 to extend each independent subset $I \in \mathbb{I}(X)$ to a basis $\mathrm{ex}(I) \in \mathbb{B}(X')$ by a greedy completion, i.e., $b \in \mathrm{ex}(I)$ if and only if $b \in I$ or else $b \in B_0$ and

$$b \notin \mathrm{span}\,\{I \cup \{b' \in B_0 : b' \prec b\}\}.$$

For each $I \in \mathbb{I}(X)$, we define

$$X(I) := \{x \in X \backslash I : x \notin \mathrm{span}\,\{b \in I : b \prec x\}\}.$$

We associate each independent subset I with the polynomial

$$Q_I := p_{X(I)}.$$

Before proceeding with introduction and analysis of the semi-external zonotopal spaces, we pause momentarily in order to discuss the (full) external case, whose theory was developed in [11]. To this end, we recall the definition of the ideal $\mathcal{I}_+(X)$ and the space $\mathcal{P}_+(X)$ from the introduction, and add the following definitions:

Definition 2.1

$$\begin{aligned} \mathcal{J}_+(X) &:= \mathrm{Ideal}\{p_Y : Y \subset X',\ Y \cap \mathrm{ex}(I) \neq \emptyset, \quad \textit{for all } I \in \mathbb{I}(X)\}, \\ \mathcal{D}_+(X) &:= \ker \mathcal{J}_+(X). \end{aligned}$$

Let $\mathcal{H}(X', \lambda)$ be a generic hyperplane arrangement associated with the extended set X'. Let $V' := V(X')$ be the vertex set of this arrangement. Since $\mathcal{H}(X', \lambda)$ is assumed to be generic, there is a bijection

$$\mathcal{V} : \mathbb{B}(X') \to V'$$

in which each basis B is mapped to the intersection of the hyperplanes $\{H_{b,\lambda_b} : b \in B\}$. We denote

$$V_+ := \{\mathcal{V}(\mathrm{ex}(I)) : I \in \mathbb{I}(X)\}.$$

Note that V_+ depends on B_0, on the order $\prec$ imposed on B_0, and the parameter vector λ. (It does not depend however on the way $\prec$ is extended to X.)

Before describing the main result from [11] regarding the full external case, we recall the pairing from Section 1.1 that plays an important role in the underlying duality between the spaces $\mathcal{P}_+(X)$ and $\mathcal{D}_+(X)$: Given two polynomials p and q, the pairing $\langle p, q\rangle$ is defined by

$$\langle p, q\rangle := (p(D)q)(0).$$

We now describe the pertinent result from [11]:

Theorem 2.2 ([11, Theorem 4.10])

1. $\dim \mathcal{P}_+(X) = \dim \mathcal{D}_+(X) = \#\mathbb{I}(X)$.
2. $\mathcal{D}_+(X) = \Pi(V_+)$, *with* V_+ *as above.*
3. $\mathcal{J}_+(X) \oplus \mathcal{P}_+(X) = \Pi$.
4. $\mathcal{P}_+(X) = \ker \mathcal{I}_+(X)$.
5. *The pairing* $\langle \cdot, \cdot \rangle$ *defines a bijection between* $\mathcal{P}_+(X)$ *and the dual* $\mathcal{D}_+(X)'$ *of* $\mathcal{D}_+(X)$.

6. *The polynomials* $\{Q_I : I \in \mathbb{I}(X)\}$ *(that depend on the order* $\prec$*) form a basis for* $\mathcal{P}_+(X)$ *(whose definition is independent of that order). In particular, the Hilbert series of* $\mathcal{P}_+(X)$ *is determined by*
$$h_+(j) := h_{+,X}(j) := \#\{I \in \mathbb{I}(X) : \mathrm{val}\,(I) = j\}.$$

Under an additional assumption that X is unimodular, the results in [11] also draw a connection between the integer points $\mathcal{Z}(X)$ in the closed zonotope $Z(X)$, and the external zonotopal spaces, viz.
$$\mathcal{P}_+(X) \;=\; \Pi(\mathcal{Z}(X)).$$

2.2 Introduction and analysis of semi-external zonotopal spaces

The central zonotopal spaces are defined with respect to the set of all bases $\mathbb{B}(X)$. The external ones are defined with respect to the independent sets $\mathbb{I}(X)$. The semi-external spaces we now introduce are defined by selecting a set $\mathbb{I}'$ in between:
$$\mathbb{B}(X) \subset \mathbb{I}' \subset \mathbb{I}(X).$$

Definition 2.3 *Let* $\mathbb{I}' \subset \mathbb{I}(X)$*. We say that* $\mathbb{I}'$ *is* **semi-external** *if the following holds:*
$$\textit{If } I \in \mathbb{I}',\ I' \in \mathbb{I}(X) \textit{ and } \mathrm{span}\, I \subset \mathrm{span}\, I', \textit{ then } I' \in \mathbb{I}'.$$

Note that a (non-empty) semi-external set $\mathbb{I}'$ must contain $\mathbb{B}(X)$.

Our goal is to define and analyse the zonotopal ideals and the zonotopal spaces that are associated with a semi-external collection $\mathbb{I}'$. To this end, we denote
$$S(X, \mathbb{I}') := \{\mathrm{span}\, I : I \in \mathbb{I}(X) \backslash \mathbb{I}'\},$$
and define

Definition 2.4
$$\begin{aligned}
\mathcal{I}_+(X, \mathbb{I}') &:= \mathcal{I}_+(X) + \mathrm{Ideal}\{\Pi^0_{\#(X\backslash S)}(S\perp) : S \in S(X, \mathbb{I}')\},\\
\mathcal{J}_+(X, \mathbb{I}') &:= \mathrm{Ideal}\{p_Y : Y \subset X',\ Y \cap \mathrm{ex}(I) \neq \emptyset,\ \forall I \in \mathbb{I}'\},\\
\mathcal{P}_+(X, \mathbb{I}') &:= \mathrm{span}\,\{p_Y :\ Y \subset X,\ Y \cap I = \emptyset \textit{ for some } I \in \mathbb{I}'\},\\
\mathcal{D}_+(X, \mathbb{I}') &:= \ker \mathcal{J}_+(X, \mathbb{I}').
\end{aligned}$$

To be sure, the polynomial space $\Pi^0_j(S\perp)$ consists of homogeneous polynomials of degree j on the orthogonal complement of S in $\mathbb{R}^n$. Equivalently, $\Pi(S\perp) := \{p \in \Pi \ :\ D_\eta p = 0 \text{ for all } \eta \in S\}$.

We note that the semi-external spaces capture the external ones and the central ones as special cases. In the case $\mathbb{I}' = \mathbb{I}(X)$, the $\mathbb{I}'$-ideals $(\mathcal{I}_+(X, \mathbb{I}'), \mathcal{J}_+(X, \mathbb{I}'))$ coincide with the external ideals. In the case $\mathbb{I}' = \mathbb{B}(X)$, they coincide with the central ideals.

Here is a simple example showing how the space $\mathcal{P}_+(X, \mathbb{I}')$ and the ideal $\mathcal{I}_+(X, \mathbb{I}')$ change depending on the choice of our special collection $\mathbb{I}'$ of independent subsets.

Example 2.5
$$X \;=\; \begin{bmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1 \end{bmatrix} =: [x_1, x_2, x_3, x_4]$$

If
$$\mathbb{I}' \;=\; \{[x_1, x_2], [x_1, x_3], [x_1, x_4]\} \cup \mathbb{B}(X),$$

then

$$\begin{aligned}\mathcal{P}_+(X,\mathbb{I}') &= \operatorname{span}\{1, p_{x_1}, p_{x_2}, p_{x_3}, p_{x_2}p_{x_3}, p_{x_3}p_{x_4}, p_{x_2}p_{x_4}\},\\ \mathcal{I}_+(X,\mathbb{I}') &= \operatorname{Ideal}\{p_{x_3}^3, p_{x_2}^3, p_{x_2-x_3}^3, p_{x_1}^2, p_{x_1-x_3}^2, p_{x_1-x_2}^2, p_{x_3}^2 p_{x_2}\}.\end{aligned}$$

If

$$\mathbb{I}' = \{[x_1], [x_1,x_2], [x_1,x_3], [x_1,x_4]\} \cup \mathbb{B}(X),$$

then

$$\begin{aligned}\mathcal{P}_+(X,\mathbb{I}') &= \operatorname{span}\{1, p_{x_1}, p_{x_2}, p_{x_3}, p_{x_2}p_{x_3}, p_{x_3}p_{x_4}, p_{x_2}p_{x_4}, p_{x_2}p_{x_3}p_{x_4}\},\\ \mathcal{I}_+(X,\mathbb{I}') &= \operatorname{Ideal}\{p_{x_3}^3, p_{x_2}^3, p_{x_2-x_3}^3, p_{x_1}^2, p_{x_1-x_3}^2, p_{x_1-x_2}^2\}.\end{aligned}$$

Theorem 2.6

1. *The polynomials $(Q_I)_{I\in\mathbb{I}'}$ form a basis for $\mathcal{P}_+(X,\mathbb{I}')$. In particular, $\dim \mathcal{P}_+(X,\mathbb{I}') = \#\mathbb{I}'$. The Hilbert series of $\mathcal{P}_+(X,\mathbb{I}')$ is determined by these polynomials:*

$$h_{+,\mathbb{I}'}(j) := \dim(\mathcal{P}_+(X,\mathbb{I}') \cap \Pi_j^0) = \#\{I \in \mathbb{I}' : \operatorname{val}(I) = j\}.$$

2. *Let $V(\mathbb{I}')$ be the vertices in the arrangement $\mathcal{H}(X',\lambda)$ that correspond to $\operatorname{ex}(\mathbb{I}')$. Then $\mathcal{D}_+(X,\mathbb{I}') = \Pi(V(\mathbb{I}'))$. In particular, $\dim \mathcal{D}_+(X,\mathbb{I}') = \#\mathbb{I}'$.*
3. *$\mathcal{J}_+(X,\mathbb{I}') \oplus \mathcal{P}_+(X,\mathbb{I}') = \Pi$.*
4. *$\mathcal{P}_+(X,\mathbb{I}') = \ker \mathcal{I}_+(X,\mathbb{I}')$.*
5. *The map $p \mapsto \langle p, \cdot\rangle$ is a bijection between $\mathcal{P}_+(X,\mathbb{I}')$ and $\mathcal{D}_+(X,\mathbb{I}')'$.*

Proof. 1. We first prove that $\{Q_I : I \in \mathbb{I}'\}$ is a basis for $\mathcal{P}_+(X,\mathbb{I}')$.

The fact that $Q_I \in \mathcal{P}_+(X,\mathbb{I}')$ for every $I \in \mathbb{I}'$ is trivial since $Q_I = p_{X(I)}$, with $X(I) \cap I = \emptyset$. Now, we choose $I' \in \mathbb{I}'$, and $Y \subset X\backslash I'$. According to the definition of $\mathbb{I}'$, since $\operatorname{span} I' \subset S := \operatorname{span}(X\backslash Y)$, any basis $I \subset X\backslash Y$ for S must lie in $\mathbb{I}'$. We set $\tilde{X} := X \cap S$. Now, with $\{\tilde{Q}_B := p_{\tilde{X}(B)} : B \in \mathbb{B}(\tilde{X})\}$ the homogeneous basis for $\mathcal{P}(\tilde{X})$ (per the fixed order we chose for X) [10], we have

$$\begin{aligned}p_Y \in p_{X\backslash S}\cdot \mathcal{P}(\tilde{X}) &= p_{X\backslash S}\cdot \operatorname{span}\{\tilde{Q}_B : B \in \mathbb{B}(\tilde{X})\}\\ &= \operatorname{span}\{Q_I : I \in \mathbb{B}(\tilde{X})\} \subset \operatorname{span}\{Q_I : I \in \mathbb{I}'\},\end{aligned}$$

with the last inclusion following from the fact that every basis for S from X must lie in $\mathbb{I}'$, since $\mathbb{I}'$ is semi-external. Thus,

$$\mathcal{P}_+(X,\mathbb{I}') \subset \operatorname{span}\{Q_I : I \in \mathbb{I}'\}.$$

We conclude that the two spaces coincide and

$$\dim \mathcal{P}_+(X,\mathbb{I}') = \#\mathbb{I}'.$$

2. and 3. : Note that $\{\operatorname{ex}(I) : I \in \mathbb{I}'\} \subset \mathbb{B}(X')$. We first apply Theorem 1.4 to this case (i.e., with X there replaced by X' and $\mathbb{B}'$ being our $\{\operatorname{ex}(I) : I \in \mathbb{I}'\}$). The theorem thus tells us that

$$\Pi(\mathcal{V}(\operatorname{ex}(\mathbb{I}'))) \subset \mathcal{D}_+(X,\mathbb{I}'),$$

and that $\dim \mathcal{D}_+(X,\mathbb{I}') \geq \#\mathbb{I}'$.

We next claim that $\mathcal{P}_+(X,\mathbb{I}') + \mathcal{J}_+(X,\mathbb{I}') = \Pi$. Once we prove this claim, we will have that $\dim \mathcal{D}_+(X,\mathbb{I}') = \operatorname{codim} \mathcal{J}_+(X,\mathbb{I}') \leq \dim \mathcal{P}_+(X,\mathbb{I}') = \#\mathbb{I}'$. This will yield that $\dim \mathcal{D}_+(X,\mathbb{I}') = \#\mathbb{I}'$, hence that $\mathcal{D}_+(X,\mathbb{I}') = \Pi(V(\mathbb{I}'))$. The same will also yield that $\operatorname{codim} \mathcal{J}_+(X,\mathbb{I}') = \#\mathbb{I}'$, hence that the sum $\mathcal{P}_+(X,\mathbb{I}') + \mathcal{J}_+(X,\mathbb{I}')$ is direct (since $\dim \mathcal{P}_+(X,\mathbb{I}') = \#\mathbb{I}'$, too). In summary, the proofs of (2) and (3) will be completed once we show that $\mathcal{P}_+(X,\mathbb{I}') + \mathcal{J}_+(X,\mathbb{I}') = \Pi$, as we do now.

Since we know that $\mathcal{P}(X) \oplus \mathcal{J}(X) = \Pi$ (see [11]; the result was first proved in [1, 10]), we conclude from the fact that $\mathcal{P}(X) \subset \mathcal{P}_+(X,\mathbb{I}')$, that $\mathcal{P}_+(X,\mathbb{I}') + \mathcal{J}(X) = \Pi$. We prove now that

$$\mathcal{J}(X) \subset \mathcal{P}_+(X,\mathbb{I}') + \mathcal{J}_+(X,\mathbb{I}').$$

A polynomial in $\mathcal{J}(X)$ is a linear combination of polynomials of the form $p_Y f$, where $f \in \Pi$, $Y \subset X$ and $\operatorname{rank}(X\backslash Y) < n$. Let $I \subset X\backslash Y$ be a basis for $\operatorname{span} X\backslash Y$. If $I \notin \mathbb{I}'$, then, since $\mathbb{I}'$ is semi-external, $X\backslash Y$ contains no element in $\mathbb{I}'$, hence $p_Y \in \mathcal{J}_+(X,\mathbb{I}')$, *a fortiori* $p_Y f \in \mathcal{J}_+(X,\mathbb{I}')$, and we are done. Otherwise, $I \in \mathbb{I}'$. We prove this case by induction on $\#(X\backslash Y)$. Thus, we assume that $p_{Y'} f \in \mathcal{P}_+(X,\mathbb{I}')+\mathcal{J}_+(X,\mathbb{I}')$ whenever $\#(X\backslash Y') = k \geq -1$ and consider $p_Y f$ where $\#(X\backslash Y) = k+1$ (the initial case $X = Y$ corresponds to the choice $k = -1$).

The proof goes as follows: we denote $B := \operatorname{ex}(I)$, and, using the fact that

$$\Pi = \Pi_0^0 + \operatorname{Ideal}\{p_b : b \in B\},$$

write

$$f = c + \sum_{b\in B} c_b p_b f_b,$$

with c and $(c_b)_{b\in B}$ some scalars, and $(f_b)_{b\in B}$ some polynomials. Then

$$p_Y f = c p_Y + \sum_{b\in B} c_b p_{Y\sqcup b} f_b.$$

We show that each of the terms on the right-hand-side lies in $\mathcal{P}_+(X,\mathbb{I}') + \mathcal{J}_+(X,\mathbb{I}')$.

Starting with p_Y, we note that $Y \cap I = \emptyset$ and that $I \in \mathbb{I}'$, hence that $p_Y \in \mathcal{P}_+(X,\mathbb{I}')$. We then consider the term $p_{Y\sqcup b} f_b$ under the assumption that $b \in I$. In this case, $Y' := Y \sqcup b \subset X$ and $\#(X\backslash Y') = k$, and hence the induction hypothesis applies to yield that $p_{Y\sqcup b} f_b \in \mathcal{P}_+(X,\mathbb{I}') + \mathcal{J}_+(X,\mathbb{I}')$.

Finally, we treat the summand $p_{Y\sqcup b} f_b$ under the assumption that $b \in B_0$. Let $I' \in \mathbb{I}(X)$, and assume that $Y \cap I' = \emptyset$. Then $I' \subset X\backslash Y$, hence $\operatorname{span} I' \subset \operatorname{span} I$, and therefore $\operatorname{ex}(I) \cap B_0 \subset \operatorname{ex}(I') \cap B_0$. Consequently, $b \in \operatorname{ex}(I')$. In summary, the set $Y \sqcup b$ intersects every extended basis $\operatorname{ex}(I')$, $I' \in \mathbb{I}'$, and this means that $p_{Y\sqcup b} f_b$ lies in the ideal $\mathcal{J}_+(X,\mathbb{I}')$.

4. We now prove that $\mathcal{P}_+(X,\mathbb{I}') = \ker \mathcal{I}_+(X,\mathbb{I}')$. We first show that $\mathcal{P}_+(X,\mathbb{I}') \subset \ker \mathcal{I}_+(X,\mathbb{I}')$. To this end, choose $Y \subset X$ such that $X\backslash Y$ contains a set $I \in \mathbb{I}'$. We need to show that p_Y is annihilated by each of the generators of the ideal $\mathcal{I}_+(X,\mathbb{I}')$. The fact that p_Y is annihilated by $\mathcal{I}_+(X)$ follows from the fact that $\ker \mathcal{I}_+(X) = \mathcal{P}_+(X) \supset \mathcal{P}_+(X,\mathbb{I}')$ by Theorem 2.2.

We now deal with a differential operator $q(D)$, $q \in \Pi^0_{\#(X\backslash S)}(S\perp)$, $S \in S(X,\mathbb{I}')$. Note that

$$q(D)p_Y = p_{Y\cap S}\, q(D) p_{Y\backslash S}.$$

Now, $\#(X\backslash S) = \deg q$, and $Y\backslash S \subset X\backslash S$. Therefore, we just need to rule out the possibility that $Y\backslash S = X\backslash S$. Indeed, if $Y\backslash S = X\backslash S$ then $X\backslash Y \subset S$, and hence $I \subset S$. Since $I \in \mathbb{I}'$ and $\mathbb{I}'$ is semi-external, this implies that every basis for S from X is in $\mathbb{I}'$, contradicting thereby the assumption that $S \in S(X,\mathbb{I}')$. Hence $\#(Y\backslash S) < \deg q$, and we obtain that $q(D)p_Y = 0$. Consequently, $\mathcal{P}_+(X,\mathbb{I}') \subset \ker \mathcal{I}_+(X,\mathbb{I}')$.

Proving the converse inclusion is somewhat harder. First, since $\mathcal{I}_+(X) \subset \mathcal{I}_+(X,\mathbb{I}')$ directly from the definition of $\mathcal{I}_+(X,\mathbb{I}')$, we have that $\ker \mathcal{I}_+(X,\mathbb{I}') \subset \ker \mathcal{I}_+(X) = \mathcal{P}_+(X)$ (cf. Theorem 2.2). In addition, since $(Q_I)_{I\in\mathbb{I}'}$ is a basis for $\mathcal{P}_+(X,\mathbb{I}')$, while $(Q_I)_{I\in\mathbb{I}(X)}$ is a basis for $\mathcal{P}_+(X)$, we have that

$$\mathcal{P}_+(X,\mathbb{I}') \subset \ker \mathcal{I}_+(X,\mathbb{I}') \subset \ker \mathcal{I}_+(X) = \mathcal{P}_+(X) = \mathcal{P}_+(X,\mathbb{I}') + \mathbb{Q}_{\mathbb{I}'},$$

with

$$\mathbb{Q}_{\mathbb{I}'} := \operatorname{span}\{Q_I : I \in \mathbb{I}(X)\backslash\mathbb{I}'\}.$$

We show now that $\mathbb{Q}_{\mathbb{I}'} \cap \ker \mathcal{I}_+(X, \mathbb{I}') = 0$, and this will complete the proof.

To this end, let

$$f := \sum_{I\in\mathbb{I}(X)\backslash\mathbb{I}'} c(I) Q_I \in \ker \mathcal{I}_+(X, \mathbb{I}').$$

We will show that $c(I) = 0$ for all $I \in \mathbb{I}(X)\backslash\mathbb{I}'$. Assume, to the contrary, that $c(I') \neq 0$ for some $I' \in \mathbb{I}(X)\backslash\mathbb{I}'$, and assume, without loss of generality, that $c(I) = 0$ for every $I \in \mathbb{I}(X)\backslash\mathbb{I}'$ of smaller cardinality. I' cannot be a basis in $\mathbb{B}(X)$, since it is not in $\mathbb{I}'$. We therefore extend I' to a basis $B' \in \mathbb{B}(X)$ using the vectors $(b_1, \ldots, b_k) \subset X$, $k \geq 1$, and denote

$$S_0 := \operatorname{span} I', \quad S_i := \operatorname{span}\left(I' \cup \{b_1, \ldots, b_i\}\right), \quad i = 1, \ldots, k.$$

For each $1 \leq i \leq k$, we choose vector $0 \neq \eta_i \in S_i$, such that $\eta_i \perp S_{i-1} \supset S_0$. Setting $m_i := \#((X \cap S_i)\backslash S_{i-1})$, $i = 1, \ldots, k$, we define

$$q := \prod_{i=1}^{k} p_{\eta_i}^{m_i}.$$

Since $\eta_i \perp S_0$ for every i, and since $\sum_{i=1}^{k} m_i = \#(X\backslash S_0)$, we conclude that $q \in \Pi^0_{\#(X\backslash S_0)}(S_0\perp)$, which implies that $q \in \mathcal{I}_+(X, \mathbb{I}')$ and that $q(D)f = 0$.

Now, for $Y \subset X$,

$$q(D)p_Y = p_{\eta_1}^{m_1}(D)p_{(Y\cap S_1)\backslash S_0} p_{\eta_2}^{m_2}(D)p_{(Y\cap S_2)\backslash S_1} \cdots p_{\eta_{k-1}}^{m_{k-1}}(D)p_{(Y\cap S_{k-1})\backslash S_{k-2}} p_{\eta_k}^{m_k}(D)p_{Y\backslash S_{k-1}}.$$

Since $\#(X \cap S_j)\backslash S_{j-1} = m_j$, $1 \leq j \leq k$, we have $q(D)Q_I \neq 0$ only if $X\backslash S_0 = X(I)\backslash S_0$, which is possible only if $I \subset S_0$ (since $I \cap X(I) = \emptyset$), in which case

$$q(D)Q_I = a p_{X(I)\cap S_0},$$

for some $a \neq 0$. Thus, with

$$\operatorname{Sub}(S_0) := \{I \in \mathbb{I}(X)\backslash\mathbb{I}' : I \subset S_0\},$$

we have

$$0 = q(D)f = a \sum_{I\in \operatorname{Sub}(S_0)} c(I) p_{X(I)\cap S_0}.$$

Per our assumption on the minimal cardinality of I', we have that $c(I) = 0$ whenever $I \in \operatorname{Sub}(S_0)$ and $\#I < \#I'$. However, the polynomial set $\{p_{X(I)\cap S_0} : I \in \mathbb{I}(X), \operatorname{span} I = S_0\}$ is a basis for the central space $\mathcal{P}(X \cap S_0)$, hence we conclude that $c(I) = 0$, for every $I \in \operatorname{Sub}(S_0)$. In particular, $c(I') = 0$.

5. Pick $q \in \mathcal{D}_+(X, \mathbb{I}')\backslash\{0\}$. Since $\mathcal{J}_+(X, \mathbb{I}') \oplus \mathcal{P}_+(X, \mathbb{I}') = \Pi$, q can be written in the form of $q = f + p$ where $f \in \mathcal{J}_+(X, \mathbb{I}')$ and $p \in \mathcal{P}_+(X, \mathbb{I}')$. $\mathcal{D}_+(X, \mathbb{I}') = \ker \mathcal{J}_+(X, \mathbb{I}')$ implies that $\langle q, f\rangle = 0$. We conclude that $\langle q, p\rangle = \langle q, q\rangle \neq 0$, since $q \neq 0$. This means that there exists no $q \in \mathcal{D}_+(X, \mathbb{I}')\backslash\{0\}$ that satisfies

$$\langle q, p\rangle = 0, \quad \forall p \in \mathcal{P}_+(X, \mathbb{I}').$$

The result follows from the fact that $\dim \mathcal{P}_+(X, \mathbb{I}') = \dim \mathcal{D}_+(X, \mathbb{I}')$. □

One observes that the definition of the ideal $\mathcal{I}_+(X, \mathbb{I}')$ involves more than the powers of the normals to the facet hyperplanes. We now investigate a special situation where the set $\mathbb{I}'$ satisfies an additional condition, and show that $\mathcal{I}_+(X, \mathbb{I}')$ is generated then by powers of the normals to the hyperplanes and by nothing else. Precisely, we will assume that an independent subset I_0 necessarily lies in $\mathbb{I}'$ whenever all its extensions to a set of rank $n-1$ lie in $\mathbb{I}'$. We will use, for $I \in \mathbb{I}(X)\backslash\mathbb{B}(X)$, the notation

$$\mathcal{M}(I) := \{I' \in \mathbb{I}(X) : I \subset \mathrm{span}\,(I') \in \mathcal{F}(X)\}.$$

Also,

$$\mathcal{I}_\varepsilon(X, \mathbb{I}') := \mathrm{Ideal}\{p_{\eta_F}^{m(F)+\varepsilon(F)} : F \in \mathcal{F}(X)\},$$

where $\varepsilon(F) := 1$ if $F = \mathrm{span}\,(I)$ for some $I \in \mathbb{I}'$, or else $\varepsilon(F) := 0$.

For the proof of our Theorem 2.8 below, we will need the following proposition:

Proposition 2.7 ([11, Proposition 4.8]) *Let $\mathcal{I}$ be a polynomial ideal and let V be a subspace of $\mathbb{R}^n$ of dimension $d \geq 2$. Let $V_1, \ldots, V_k$ be distinct subspaces of V, each of dimension $d-1$. Suppose that, for $n_1, \ldots, n_k \in \mathbb{N}$, the ideal $\mathcal{I}$ contains all homogeneous polynomials defined on V_i of degree n_i:*

$$\Pi^0_{n_i}(V_i) \subset \mathcal{I}.$$

Then

$$\Pi^0_N(V) \subset \mathcal{I} \quad \textit{whenever} \quad (N+1)(k-1) \geq \sum_{i=1}^k n_i.$$

Theorem 2.8 *Suppose that the semi-external set $\mathbb{I}'$ satisfies the following additional condition:*

for any $I \in \mathbb{I}(X)\backslash\mathbb{B}(X)$, $\mathcal{M}(I) \subset \mathbb{I}'$ implies $I \in \mathbb{I}'$.

Then

(1) $\mathcal{I}_+(X, \mathbb{I}') = \mathcal{I}_\varepsilon(X, \mathbb{I}')$.

(2) $\mathcal{P}_+(X, \mathbb{I}') = \sum_{I\, is\ minimal\ in\ \mathbb{I}'} \left(\bigcap_{Z \in \mathrm{Comp}_I(X)} \mathcal{P}(X \sqcup Z)\right)$. *Here, I is minimal in $\mathbb{I}'$ provided $I \in \mathbb{I}'$ and $(\mathbb{I}' \cap 2^I)\backslash I = \emptyset$, and* $\mathrm{Comp}_I(X)$ *denotes all completions of I to a basis:* $\mathrm{Comp}_I(X) := \{Z \subset X : I \cup Z \in \mathbb{B}(X)\}$.

Proof. (1) Every generator of $\mathcal{I}_\varepsilon(X, \mathbb{I}')$ is in $\mathcal{I}_+(X, \mathbb{I}')$ directly from the definition of these ideals, hence

$$\mathcal{I}_\varepsilon(X, \mathbb{I}') \subset \mathcal{I}_+(X, \mathbb{I}').$$

Therefore, we only need to prove that

$$\Pi^0_{\#(X\backslash S)}(S\perp) \ \subset\ \mathcal{I}_\varepsilon(X, \mathbb{I}'), \ \text{ for all } S \in S(X, \mathbb{I}').$$

We run the proof by induction on $n - \dim S$. When $n - \dim S = 1$, i.e., $S \in \mathcal{F}(X)$, we have

$$\Pi^0_{\#(X\backslash S)}(S\perp) \quad \subset \quad \mathcal{I}_\varepsilon(X, \mathbb{I}'), \quad S \in S(X, \mathbb{I}'), \tag{2}$$

$$\Pi^0_{\#(X\backslash S)+1}(S\perp) \quad \subset \quad \mathcal{I}_\varepsilon(X, \mathbb{I}'), \quad S \in \{\mathrm{span}\,(I) : I \in \mathbb{I}'\}. \tag{3}$$

We will extend now (2) and (3) to sets S of lower dimension. For the inductive step, we suppose that (2) and (3) hold when $n - \dim S = d > 0$. We now consider the case where $S = \mathrm{span}\, I$ for some independent $I \in \mathbb{I}(X)$, and $n - \dim S = d + 1$. Consider all possible linear spaces obtained as the span

$$\mathrm{span}\,\{S \cup x\}, \qquad x \in X\backslash S.$$

Assume that j of them are distinct and label them S_1 through S_j (note that $j > 1$ since $d+1 > 1$). By our induction hypothesis, $\Pi^0_{m_i}(S_i\perp) \subset \mathcal{I}_\varepsilon(X, \mathbb{I}')$ for each $i = 1, \ldots, j$, where $m_i := \#(X\backslash S_i) + \varepsilon_i$ and $\varepsilon_i := 1$ if $S_i \in \mathbb{I}'$ and $\varepsilon_i := 0$ otherwise, i.e., if $S_i \notin \mathbb{I}'$. By Proposition 2.7, we conclude that

$$\Pi^0_N(S\perp) \ \subset\ \mathcal{I}_\varepsilon(X)$$

whenever $(N+1)(j-1)$ is at least

$$\sum_{i=1}^{j}(\#(X\backslash S_i)+\varepsilon_i) = j\#(X\backslash S) - \#(X\backslash S) + \sum_{i=1}^{j}\varepsilon_i \leq (j-1)\#(X\backslash S) + j.$$

If S is spanned by a set that is in $\mathbb{I}'$, then we can take N to be $\#(X\backslash S)+1$ since N so chosen satisfies

$$N \geq \#(X\backslash S) + \frac{1}{j-1}.$$

If $S \in S(X,\mathbb{I}')$, then at least one of the extensions S_i is not in $\mathbb{I}'$ (otherwise, it will be easy to see that we violate the extra condition that is now assumed on $\mathbb{I}'$), and therefore $\sum_{i=1}^{j}\varepsilon_i \leq j-1$, hence the value $N = \#(X\backslash S)$ is already large enough for our purposes. This completes the inductive step, hence completes the proof of this part.

(2) We first prove that

$$\mathcal{P}_+(X,\mathbb{I}') \subset \sum_{I \text{ is minimal in } \mathbb{I}'} \left(\bigcap_{Z\in \mathrm{Comp}_I(X)} \mathcal{P}(X \sqcup Z) \right).$$

Pick $I' \in \mathbb{I}'$. There exists a minimal set I from $\mathbb{I}'$ such that $I \subset I'$. The definition of $p_{X(I')}$ shows that $p_{X(I')} \in \mathcal{P}(X \sqcup Z)$ for all $Z \in \mathrm{Comp}_I(X)$. We conclude that

$$p_{X(I')} \in \bigcap_{Z\in \mathrm{Comp}_I(X)} \mathcal{P}(X \sqcup Z),$$

which implies that

$$\mathcal{P}_+(X,\mathbb{I}') \subset \sum_{I \text{ is minimal in } \mathbb{I}'} \left(\bigcap_{Z\in \mathrm{Comp}_I(X)} \mathcal{P}(X \sqcup Z) \right).$$

We complete the proof by showing that every polynomial p in $\bigcap_{Z\in \mathrm{Comp}_I(X)} \mathcal{P}(X \sqcup Z)$ lies in $\ker \mathcal{I}_\varepsilon(X,\mathbb{I}')$. Let p be such a polynomial, and let $F \in S(X,\mathbb{I}') \cap \mathcal{F}(X)$ (we need to check only this case since the other case, $F \notin S(X,\mathbb{I}')$, is straightforward). We need to show that $D_{\eta_F}^{m(F)} p = 0$, where $\eta_F \perp F$ and $m(F) := \#(X\backslash F)$. Incidentally, note that $I = \emptyset$ is possible only if $\mathbb{I}' = \mathbb{I}(X)$, i.e., when $\mathcal{P}_+(X,\mathbb{I}') = \mathcal{P}_+(X)$, the case when all multiplicities are increased by 1, which causes no problem. Thus, I contains at least one element whenever at least one "problematic" multiplicity exists. We now select $Z \subset F \cap X$ so that $Z \in \mathrm{Comp}_I(X)$ (such a Z exists, since $\mathrm{span}\, I \not\subset F$ and $\mathrm{codim}\, F = 1$). Then, $p \in \mathcal{P}(X \sqcup Z)$, hence p is annihilated by $D_{\eta_F}^{\#(X\sqcup Z)\backslash F}$. Since all vectors of Z lie inside F, we conclude that $\#(X \sqcup Z)\backslash F = m(F)$ and the result follows. $\square$

3 Semi-internal zonotopal spaces

3.1 A review of internal zonotopal spaces

In this section we recall pertinent results from [11] concerning the (full) internal zonotopal spaces and their associated ideals. We impose an (arbitrary but fixed) ordering $\prec$ on X. Let $B \in \mathbb{B}(X)$, and $b \in B$. We say that b **is internally active** in B, if

$$b = \max\{X\backslash F\}, \quad F := \mathrm{span}\,\{B\backslash b\} \in \mathcal{F}(X).$$

A basis b that contains no internally active vectors is called an **internal basis**. We denote the set of all internal bases by $\mathbb{B}_-(X)$. We now recall the definition of the ideal $\mathcal{I}_-(X)$ from the introduction, and add the following definitions:

Definition 3.1

$$\begin{array}{rcl}\mathcal{J}_-(X) & := & \text{Ideal}\{p_Y : Y \subset X,\ Y \cap B \neq \emptyset,\ \forall B \in \mathbb{B}_-(X)\},\\ \mathcal{P}_-(X) & := & \cap_{x\in X}\mathcal{P}(X\backslash x),\\ \mathcal{D}_-(X) & := & \ker \mathcal{J}_-(X).\end{array}$$

We denote

$$V_- := \{\mathcal{V}(B) : B \in \mathbb{B}_-(X)\}.$$

The following theorem is taken from [11]:

Theorem 3.2

1. $\dim \mathcal{P}_-(X) = \dim \mathcal{D}_-(X) = \#\mathbb{B}_-(X)$.
2. *The map* $p \mapsto \langle p, \cdot\rangle$ *is a bijection between* $\mathcal{P}(X)$ *and* $\mathcal{D}(X)'$.
3. $\mathcal{D}_-(X) = \Pi(V_-)$.
4. $\mathcal{P}_-(X) = \ker \mathcal{I}_-(X)$.
5. $\mathcal{P}_-(X) \bigoplus \mathcal{J}_-(X) = \Pi$.

In contrast with the external and semi-external cases, the polynomials $\{Q_B : B \in \mathbb{B}_-(X)\}$ do not in general form a basis for $\mathcal{P}_-(X)$ [11]. However, the Hilbert series of $\mathcal{P}_-(X)$ can be still determined in the usual way using the valuation function for internal bases, namely, via the sequence (see [11])

$$h_-(j) := h_{-,X}(j) := \dim(\mathcal{P}_-(X) \cap \Pi_j^0) = \#\{B \in \mathbb{B}_-(X) : \text{val}\,(B) = j\}.$$

3.2 Introduction and analysis of semi-internal zonotopal spaces

Given the set X, the internal space $\mathcal{P}_-(X)$ can be also defined as follows [11]:

$$\mathcal{P}_-(X) := \cap_{b\in B}\mathcal{P}(X\backslash b),$$

where $B \in \mathbb{B}(X)$ is arbitrary. In particular, this implies that the dimension of the above intersection is still expressed in terms of the matroidal statistics of X, i.e., the number of internally inactive bases of X. Suppose, instead, that we choose $I \in \mathbb{I}(X)$ and define, similarly

$$\mathcal{P}_-(X, I) := \cap_{b\in I}\mathcal{P}(X\backslash b).$$

Then, a few questions arise naturally:

- Does the space $\mathcal{P}_-(X, I)$ depend only on $\text{span}\, I$ and not on I itself?
- Is there a simple formula that expresses $\dim \mathcal{P}_-(X, I)$ in terms of the cardinality of a suitable subset of $\mathbb{B}(X)$?
- Is the ideal $\mathcal{I}_-(X, I)$ of differential operators that annihilate $\mathcal{P}_-(X, I)$ generated by powers of the normals to the facets of X?
- Is there a dual construction (on the hyperplane arrangement) of an ideal of the $\mathcal{J}$-class?

The answer to all the questions above turns out to be affirmative. In order to present our results for the above setup, we select a full order $\prec$ on X, and put the vectors in I to be the last ones in this order, i.e.,

$$x \prec y \quad \text{for all } y \in I, \quad x \in X\backslash I.$$

Further, we select the following subset of ***I*-facets:**

$$\mathcal{F}(X, I) := \{F \in \mathcal{F}(X) : I \not\subset F\}.$$

We then single out the following subset of $\mathbb{B}(X)$ of ***I*-internal bases**:

Definition 3.3 *Let $B \in \mathbb{B}(X)$. We say that $b \in B$ is I-internally active (in B) if $b = \max\{X\backslash \operatorname{span}(B\backslash b)\}$ and $b \in I$. We say that B is* **I-internal** *if no vector b in B is I-internally active in B. We denote the set of all I-internal bases by*

$$\mathbb{B}_-(X, I).$$

Note that if $I \in \mathbb{B}(X)$ then $\mathbb{B}_-(X, I) = \mathbb{B}_-(X)$, while if $I = \emptyset$ then $\mathbb{B}_-(X, I) = \mathbb{B}(X)$. Further, note that, while $\mathcal{P}_-(X, I)$ does not depend on the order $\prec$, the set of I-internal bases does depend on that order. Note also that every internal basis is I-internal:

$$\mathbb{B}_-(X) \subset \mathbb{B}_-(X, I).$$

In addition to the I-internal bases, we need the following ideal:

$$\mathcal{I}_-(X, I) := \operatorname{Ideal}\{\mathcal{I}(X) \cup \{\eta_F^{m(F)-1} : \ F \in \mathcal{F}(X, I), \ 0 \neq \eta_F \perp F\}\}.$$

Theorem 3.4

1. $\dim \mathcal{P}_-(X, I) = \#\mathbb{B}_-(X, I)$.
2. $\mathcal{P}_-(X, I) = \ker \mathcal{I}_-(X, I)$.

The result $\mathcal{P}_-(X, I) = \ker \mathcal{I}_-(X, I)$ implies that $\mathcal{P}_-(X, I)$ depends only on $\operatorname{span} I$ and not on I itself. We prove this theorem in the sequel. Let us next introduce the dual setup, that goes as follows: first, we say that $Y \subset X$ is **I-long** if $Y \cap B \neq \emptyset$ for each $B \in B_-(X, I)$. Set

$$\mathcal{J}_-(X, I) := \operatorname{Ideal}\{p_Y : \ Y \subset X \text{ is } I\text{-long}\}.$$

Note that the ideal $\mathcal{J}_-(X, I)$ depends on the order we choose, since the set $B_-(X, I)$ depends on that order.

Let now $\mathcal{H}(X, \lambda)$ be a generic hyperplane arrangement as in Section 1.1. Then there is a natural bijection $B \mapsto v_B$ from $\mathbb{B}(X)$ onto the vertex set $V(X, \lambda)$ of the hyperplane arrangement. Denote

$$V_-(X, \lambda, I) := \{v_B : \ B \in \mathbb{B}_-(X, I)\}, \quad \mathcal{D}_-(X, I) := \ker \mathcal{J}_-(X, I).$$

Theorem 3.5

1. $\mathcal{D}_-(X, I) = \Pi(V_-(X, \lambda, I))$, *in particular*

$$\dim \mathcal{D}_-(X, I) = \#\mathbb{B}_-(X, I).$$

2. $\mathcal{J}_-(X, I) \oplus \mathcal{P}_-(X, I) = \Pi$.
3. *The map $p \mapsto \langle p, \cdot\rangle$ is a bijection from $\mathcal{P}_-(X, I)$ onto $\mathcal{D}_-(X, I)'$.*

Proof. (Theorems 3.4 and 3.5). The last assertion in Theorem 3.5 is a direct consequence of the second assertion in that theorem and the fact that $\mathcal{D}_-(X, I) = \ker \mathcal{J}_-(X, I)$: the argument is identical to the one used to prove **5.** of Theorem 2.6. We divide the rest of the proof into six parts as follows.

Part I: $\mathcal{P}_-(X, I) = \ker \mathcal{I}_-(X, I)$.
Since, for every $x \in X$, $\mathcal{P}(X\backslash x) = \ker \mathcal{I}(X\backslash x)$, we may prove the stated result by showing that (i) $\mathcal{I}(X\backslash x) \subset \mathcal{I}_-(X, I)$ for every $x \in I$, and (ii) $\mathcal{I}_-(X, I) \subset \operatorname{Ideal}\{\cup_{x \in I} \mathcal{I}(X\backslash x)\}$.

For the proof of (i), fix $x \in I$, and denote $X' := X\backslash x$. A generator Q in the ideal $\mathcal{I}(X')$ is of the form $Q := p_{\eta_F}^{m_{X'}(F)}$, with $F \in \mathcal{F}(X')$. Then F is also a facet hyperplane of X. Now, if $I \subset F$, then $x \in F$. Therefore $m_{X'}(F) = m_X(F)$ and Q above lies in $\mathcal{I}(X)$ hence also in $\mathcal{I}_-(X, I)$. If, on the other hand, $I \not\subset F$, then the polynomial $p_{\eta_F}^{m_X(F)-1}$ lies in $\mathcal{I}_-(X,I)$. This implies that Q lies in that ideal, too, since $m_{X'}(F)) \geq m_X(F) - 1$.

For (ii), we first note that $\mathcal{I}(X)$ lies in each ideal of the form $\mathcal{I}(X\backslash x)$. Thus, we may simply show that every generator of $\mathcal{I}_-(X,I)$ of the form $Q = p_{\eta_F}^{m(F)-1}$ lies in one of the ideals $\mathcal{I}(X\backslash x)$, $x \in I$. Here, F is a facet hyperplane of X, and $I \not\subset F$. Let $x \in I\backslash F$. Denote $X' := X\backslash x$. Since $x \notin F$, it is clear that $F \in \mathcal{F}(X')$, and then $m_{X'}(F) = m_X(F) - 1$. Thus the polynomial Q lies in $\mathcal{I}(X')$, and (ii) follows.

Part II: $\Pi(V_-(X,\lambda,I)) \subset \mathcal{D}_-(X,I)$, $\dim \mathcal{D}_-(X,I) \geq \#\mathbb{B}_-(X,I)$. Both claims are obtained by a standard argument; see Theorem 1.4.

Part III: $\dim \mathcal{D}_-(X,I) = \#\mathbb{B}_-(X,I)$. In view of Part II, we only need to prove the $\leq$ inequality.

To this end, we note that $\mathcal{J}(X) + \mathcal{P}(X) = \Pi$ by Theorem 1.2 and since $\mathcal{J}(X) \subset \mathcal{J}_-(X,I)$, we have that $\mathcal{J}_-(X,I) + \mathcal{P}(X) = \Pi$. Now, let $(Q_B)_{B\in\mathbb{B}(X)}$ be the homogeneous basis for $\mathcal{P}(X)$ (per our chosen order for X; see Theorem 1.6). Set

$$\mathcal{P}_{ex} := \text{span}\,\{Q_B : \ B \in \mathbb{B}(X)\backslash\mathbb{B}_-(X,I)\}.$$

We show that $\mathcal{P}_{ex} \subset \mathcal{J}_-(X,I)$. This will imply that

$$\Pi = \mathcal{J}_-(X,I) + \mathcal{P}_{in}, \quad \mathcal{P}_{in} := \text{span}\,\{Q_B : \ B \in \mathbb{B}_-(X,I)\},$$

hence that

$$\dim \mathcal{D}_-(X,I) = \dim \Pi/\mathcal{J}_-(X,I) \leq \dim \mathcal{P}_{in} = \#\mathbb{B}_-(X,I),$$

which is the desired result.

So, we need to show that each Q_B, $B \in \mathbb{B}(X)\backslash\mathbb{B}_-(X,I)$ lies in $\mathcal{J}_-(X,I)$. Now, $Q_B = p_Y$, with

$$Y := \{x \in X\backslash B : \ x \notin \text{span}\,\{b \in B : b \prec x\}\}.$$

Since $B \notin \mathbb{B}_-(X,I)$, there exists $b \in B \cap I$ such that, with $F := \text{span}\,(B\backslash b)$, $b = \max\{X\backslash F\}$. This shows that

$$X\backslash Y \subset F \sqcup \{b\}.$$

However, every basis $B \subset F \sqcup \{b\}$ is a basis for F augmented by b, hence is not I-internal. Consequently, Y is I-long hence lies in $\mathcal{J}_-(X,I)$.

Part IV: $\mathcal{D}_-(X,I) = \Pi(V_-(X,\lambda,I))$. This follows directly from Parts II and III, since $\dim \Pi(V_-(X,\lambda,I)) = \#\mathbb{B}_-(X,I)$.

Part V: $\dim \mathcal{P}_-(X,I) \leq \#\mathbb{B}_-(X,I)$. The proof of this assertion follows from the fact that

$$\mathcal{P}_-(X,I) \cap \mathcal{P}_{ex} = \{0\}. \tag{4}$$

Indeed, once (4) is proved, we conclude that, since $\mathcal{P}_-(X,I), \mathcal{P}_{ex} \subset \mathcal{P}(X)$,

$$\dim \mathcal{P}_-(X,I) \leq \dim \mathcal{P}(X) - \dim \mathcal{P}_{ex} = \dim \mathcal{P}_{in} = \#\mathbb{B}_-(X,I).$$

The actual proof of (4) follows almost verbatim the proof of the special case $I \in \mathbb{B}(X)$ from [11, Theorem 5.8]. We briefly outline the proof there, and add an additional argument that is required in our more general setup.

We start by writing down an arbitrary element in $\mathcal{P}_{ex}\backslash 0$:

$$\sum_{B\in\mathbb{B}(X)\backslash\mathbb{B}_-(X,I)} a(B)Q_B.$$

We then select a summand $a(B')Q_{B'}$ in the above sum such that $a(B') \neq 0$, and such that B' is minimal (among all summands with non-zero coefficients) with respect to the valuation

$$\alpha(B) := \#M(B), \quad M(B) := \{b \in B \cap I : \ b = \max(X\backslash\text{span}\,(B\backslash b))\}.$$

By the definition of $\mathcal{P}_{ex}$, $\alpha(B') > 1$. Let $b' \in M(B')$, and set $F' := \operatorname{span}(B' \backslash b')$. The argument in [11, Theorem 5.8] then reduces the proof of the fact that $a(B') = 0$ to showing that, if $B \in \mathbb{B}(X) \backslash \mathbb{B}_-(X, I)$, if $B \cap F' = B' \cap F'$, and if $B \neq B'$, then $\alpha(B) < \alpha(B')$. So, we pick now such B, and prove that $M(B) \subset M(B')$. This proves that $\alpha(B) < \alpha(B')$, since $b' \in M(B') \backslash M(B)$ (if $b' \in M(B)$ it follows that $B = B'$).

Thus, we pick $x \in M(B)$ and prove that it lies in $M(B')$, too. To this end, we denote $A := B' \backslash b'$. Then A is a basis for F', and $B = A \sqcup b$, for some $b \in X$. Necessarily, $x \in A$. Set $S := A \backslash x$. Note that $\operatorname{rank} S = n - 2$. Assume that $x \notin M(B')$. Since $x \in I \cap B'$, we conclude that there exists $y \succ x$ such that $y \notin \operatorname{span}\{S \cup b'\}$. Assume y to be maximal element outside $\operatorname{span}\{S \cup b'\}$. We get the contradiction to the existence of such y by showing that it is impossible to have $y \succ b'$, and it is also impossible to have $y \prec b'$. Note that $y \in I$, since $x \in I$.

If $y \succ b'$, then, since b' is maximal outside $\operatorname{span}\{B \backslash b'\} = \operatorname{span} A = \operatorname{span}\{S \cup x\}$, we have that $y \in \operatorname{span}\{S \cup x\}$. Also, since $y \succ x$, and x is maximal outside $\operatorname{span}\{B \backslash x\} = \operatorname{span}\{S \cup b\}$, we have $y \in \operatorname{span}\{S \cup b\}$. But $S \cup b \cup x = B$, and B is independent, hence $y \in \operatorname{span} S$, which is impossible since we assume y to be outside $\operatorname{span}\{S \cup b'\}$.

Otherwise, $y \prec b'$. The maximality of y then implies that $x \prec y \prec b'$. The maximality of x outside $\operatorname{span}\{S \cup b\}$ implies that $b' \in \operatorname{span}\{S \cup b\}$. Since $b' \notin S$, we obtain that $\operatorname{span}\{S \cup b\} = \operatorname{span}\{S \cup b'\}$, which is impossible since y lies in exactly one of these two spaces.

Part VI: $\dim \mathcal{P}_-(X, I) = \#\mathbb{B}_-(X, I)$**, and** $\mathcal{J}_-(X, I) \oplus \mathcal{P}_-(X, I) = \Pi$**.** We prove in Lemma 3.6 below that

$$\mathcal{J}_-(X, I) + \mathcal{P}_-(X, I) = \Pi.$$

This implies that

$$\dim \mathcal{P}_-(X, I) \geq \dim \ker \mathcal{J}_-(X, I) = \#\mathbb{B}_-(X, I),$$

with the equality by Part III. This, together with Part V shows that $\dim \mathcal{P}_-(X, I) = \#\mathbb{B}_-(X, I)$. Thus, $\dim \mathcal{P}_-(X, I) = \dim \Pi / \mathcal{J}_-(X, I)$, hence the sum $\mathcal{P}_-(X, I) + \mathcal{J}_-(X, I) = \Pi$ must be direct. □

Lemma 3.6

$$\mathcal{J}_-(X, I) + \mathcal{P}_-(X, I) = \Pi.$$

Proof. The special case of this result for the choice $I \in \mathbb{B}(X)$ was proved in [11, Theorem 5.7]. While most of the proof here parallels the one in [11], there is a significant difference in one of the details which requires us to provide here a complete self-contained proof.

The proof of the previous theorem reduces the proof here to showing that, for each Q_B, $B \in \mathbb{B}_-(X, I)$, $Q_B \in \mathcal{J}_-(X, I) + \mathcal{P}_-(X, I)$. Fixing $B \in \mathbb{B}_-(X, I)$, we know that $Q_B = p_{X(B)}$, for suitable $X(B) \subset X$. We decompose $X(B)$ in a certain way $X(B) = Z \sqcup W$. Thus

$$Q_B = p_Z p_W.$$

We then replace each $w \in W$ by a vector w' (not necessarily from X), to obtain a new polynomial

$$\widetilde{Q}_B := p_Z p_{W'},$$

and prove that (i) $\widetilde{Q}_B \in \mathcal{P}_-(X, I)$, and (ii) $Q_B - \widetilde{Q}_B \in \mathcal{J}_-(X, I)$.

So, let $Q_B = p_{X(B)}$ be given. If $Q_B \in \ker \mathcal{I}_-(X, I) = \mathcal{P}_-(X, I)$, there is nothing to prove. Otherwise, let $\mathbf{F} \subset \mathcal{F}(X)$ be the collection of *all* facet hyperplanes F for which $D_{\eta_F}^{m(F)-1} Q_B \neq 0$, and $\max(X \backslash F) \in I$. The set $\mathbf{F}$ is not empty, since otherwise $Q_B \in \ker \mathcal{I}_-(X, I)$. Given $F \in \mathbf{F}$, we conclude that $\#(X(B) \backslash F) \geq m(F) - 1$, hence that, with $Y := X \backslash X(B)$, $\#(Y \backslash F) \leq 1$. Since $B \subset Y$, the set $Y \backslash F$ must be a singleton $x_F \in B$. We denote

$$X_{\mathbf{F}} := \{x_F : F \in \mathbf{F}\}.$$

Define

$$W{:=}\{\max\{X\backslash F\}:\ F\in\mathbf{F}\}.$$

Then $W\subset I$, by the definition of $\mathbf{F}$. We index the vectors in W according to their order in X: $W=\{w_1\prec w_2\prec\ldots\prec w_k\}$. For each $1\le i\le k$, we define

$$X_i{:=}\{x_F:\ F\in\mathbf{F},\ \max\{X\backslash F\}=w_i\},\quad \mathbf{F}_i{:=}\{F\in\mathbf{F}:\ x_F\in X_i\}.$$

Thus, $X_{\mathbf{F}}=\bigcup_{i=1}^k X_i$.

Setting all these notations, we first observe that $W\cap X_{\mathbf{F}}=\emptyset$, i.e., w_i does not lie in X_i. Indeed, the set $X_{\mathbf{F}}$ is a subset of every $B'\in\mathbb{B}(Y)$, with $\mathrm{span}\,(B'\backslash x_F)=F$ for each $x_F\in X_{\mathbf{F}}$. If some x_F is $\max\{X\backslash F\}$, it will be I-internally active in every $B'\in\mathbb{B}(Y)$, which would imply that $\mathbb{B}(Y)$ does not contain I-internal bases, which is impossible since $B\in\mathbb{B}(Y)$. Thus, $W\subset X(B)$, and we define $Z{:=}X(B)\backslash W$, to obtain

$$Q_B=p_Zp_W.$$

Define further:

$$S_i{:=}\cap\{F: F\in\cup_{j=1}^i\mathbf{F}_j\},\quad S_0{:=}\mathbb{R}^n.$$

Then, for $i=1,\ldots,k$, $S_{i-1}=S_i\oplus\mathrm{span}\ X_i$, and $w_i\in S_{i-1}\backslash S_i$. Thus, for $i=1,\ldots k$, the vector w_i admits a unique representation of the form

$$w_i=w_i'+\sum_{x\in X_i}a_xx,\quad w_i'\in S_i,\ a_x\in\mathbb{R}\backslash\{0\}. \tag{5}$$

Define

$$W'=\{w_1',\ldots,w_k'\},\ \text{and}\ \widetilde{Q}_B{:=}p_Zp_{W'}.$$

We prove first that

$$\widetilde{Q}_B-Q_B=p_Z(p_{W'}-p_W)$$

lies in $\mathcal{J}_-(X,I)$. To this end, we multiply out the product

$$p_{W'}=\prod_{i=1}^k p_{w_i'}=\prod_{i=1}^k(p_{w_i}-\sum_{x\in X_i}a_xp_x). \tag{6}$$

Every summand in the above expansion is of the form p_Ξ, with Ξ a suitable mix of W-vectors and $X_{\mathbf{F}}$-vectors. The summand p_W in the above expansion in canceled when we subtract Q_B. Any other Ξ is obtained from W by replacing at least once a w_i vector by some vector in X_i, which we denote by x_i. Let $w_{i_1}\prec w_{i_2}\prec\ldots\prec w_{i_j}$ be all the w-vectors in $W\backslash\Xi$, and let F_1 be the facet hyperplane that corresponds to x_{i_1} ($F_1{:=}\mathrm{span}\,(B\backslash x_{i_1})$.) Then, we have that $w_{i_1}\in X\backslash(Z\sqcup\Xi)=:Y'$, and we claim that $Y'\backslash w_{i_1}\subset F_1$. To this end, we write $Y'\backslash F_1=((Y'\cap Y)\backslash F_1)\sqcup(Y'\backslash Y)\backslash F_1$. Now, $Y\backslash F_1=x_{i_1}$, and since $x_{i_1}\notin Y'$ (as it was replaced by w_{i_1}), the term $(Y'\cap Y)\backslash F_1$ is empty. The second term consists of $(w_{i_m})_{m=1}^j\backslash F_1$. However, $w_{i_m}\in S_{i_m-1}\subset S_{i_1}\subset F_1$, for every $m\ge 2$. Thus, w_{i_1} is the only vector in $Y'\backslash F_1$. Being also the last vector in $X\backslash F_1$, we conclude that w_{i_1} is I-internally active in every $B\in\mathbb{B}(Y')$, hence that $p_{Z\sqcup\Xi}\in\mathcal{J}_-(X,I)$. This being true for every summand in $\widetilde{Q}_B-Q_B$, we conclude that this latter polynomial lies in $\mathcal{J}_-(X,I)$.

We now prove that $\widetilde{Q}_B=p_{Z\sqcup W'}\in\ker\mathcal{I}_-(X,I)$. To this end, we need to show that, for every $F\in\mathcal{F}(X)$, $\#((Z\sqcup W')\backslash F)<m(F)-\epsilon(F)$, with $\epsilon(F)=1$ if $I\not\subset F$, and $\epsilon(F)=0$ otherwise. We divide the discussion here to three cases. As before, $Y{:=}X\backslash X(B)$.

Assume first that $F\in\mathbf{F}_i$ for some $1\le i\le k$. In this case, $\epsilon(F)=1$. Now, for $X(B)=Z\sqcup W$ we had that $\#((Z\sqcup W)\backslash F)=m(F)-1$. Also, x_F is the only vector in $Y\backslash F$, and $x_F\in X_i$. Thus, the subset $X_j\subset Y$, must lie in F for every $j\neq i$, which means that we conclude that, $w_j\in F$ iff

$w_j' \in F$ (since $w_j - w_j' \in \operatorname{span} X_j \subset F$). Finally, while $w_i \notin F$, $w_i' \in S_i \subset F$, hence, altogether, $\#(W'\backslash F) < \#(W\backslash F)$, and we reach the final conclusion that

$$\#((Z \sqcup W')\backslash F) < \#((Z \sqcup W)\backslash F) = m(F) - 1.$$

Secondly, we assume $F \in \mathcal{F}(X)\backslash \mathbf{F}$, but still that $S_k \subset F$. Let $j \geq 1$ be the minimal index i for which $S_i \subset F$. Define:

$$m_1 {:=} \#\{w \in W\backslash w_j : w \in F \text{ and } w' \notin F\}, \text{ and } m_2 {:=} \#(\sqcup_{i \neq j}(X_i\backslash F)).$$

Note that (since $w_j' \in F$) $\#((Z \sqcup W')\backslash F) \leq m(F) + m_1 - m_2 - \#((X_j \sqcup w_j)\backslash F)$. Note further that for $i \neq j$, if $w_i' \notin F$, while $w_i \in F$, then, since $w_i - w_i' \in \operatorname{span} X_i$, we have that $\#(X_i\backslash F) > 0$. Thus, $m_1 \leq m_2$. In addition, for $i = j$,

$$w_j - w_j' \in \operatorname{span} X_j.$$

We know *a priori* that $S_j \oplus \operatorname{span} X_j = S_{j-1}$. Since $S_j \subset F$, while $S_{j-1} \not\subset F$, we must have that $X_j\backslash F \neq \emptyset$. But, $w_j' \in F$, hence $\#((X_j \sqcup w_j)\backslash F) \geq 2$. Thus,

$$\#((Z \sqcup W')\backslash F) \leq m(F) + m_1 - m_2 - \#((X_j \sqcup w_j)\backslash F) < m(F) - 1.$$

Lastly, assume that $S' {:=} S_k \cap F \neq S_k$. We define now, similarly,

$$m_1 {:=} \#\{w \in W : w \in F \wedge w' \notin F\}, \text{ and } m_2 {:=} \#(\sqcup_{i=1}^{k}(X_i\backslash F)).$$

Then, as before, $m_1 \leq m_2$. Hence, $\#((Z \sqcup W')\backslash F) \leq m(F) - \#U$, with $U {:=} (Y \cap S_k)\backslash S'$. Note that all the vectors of $Y\backslash U$ lie in the rank deficient set $(Y \cap S') \sqcup (Y\backslash S_k))$, hence U is not empty. If $\#U \geq 2$, or if $m_1 < m_2$, we are done since it follows that $\#((Z \sqcup W')\backslash F) \leq m(F) - 2$. However, if U is a singleton and $m_1 = m_2$, our analysis only shows that $\#((Z \sqcup W')\backslash F) \leq m(F) - 1$. That means that, for this case, we either need to furnish a finer estimate, or show that $I \subset F$. We prove the latter. The argument below uses the following approach: after realizing that the singleton U lies in the basis B, we define $F' := \operatorname{span}(B\backslash U)$, and conclude that F' must contain I. We then invoke the condition $m_1 = m_2$ in order to obtain a spanning set for F by removing from $F' \cap Y$ all the vectors in $F' \cap X_{\mathbf{F}}$ and adding instead vectors from $W \subset I$. In this way, we guarantee that $I \subset F$, too.

Here are the details. Let F' be the hyperplane spanned by $X\backslash(X(B) \sqcup U)$. This hyperplane is spanned by elements of B. Moreover, $(X\backslash X(B))\backslash F' = U$, and U is a singleton. At the same time, F' is not listed in $\mathbf{F}$ (since $U \in S_k$ and S_k is disjoint of $X_{\mathbf{F}}$). Then, necessarily, $I \subset F'$, hence also $W \subset F'$.

Next, let $J \subset \{1, \ldots, k\}$ be defined by

$$j \in J \iff (w_j \in F \text{ and } w_j' \notin F).$$

The equality $m_1 = m_2$ implies that $X_j\backslash F$ is a singleton x_j for $j \in J$ and is empty otherwise. Now, we know that $Y' := Y\backslash U$ spans F'. $Y'' := Y'\backslash\{x_j : j \in J\}$ is a subset of $F \cap F'$ of rank $n - \#J - 1$. Since $W \subset F$, we have that $Y''' := Y'' \sqcup \{w_j : j \in J\} \subset F$. Since each w_j, $j \in J$, is independent of $(Y\backslash x_j) \sqcup \{w_i : i > j\}$, we conclude that $\operatorname{rank} Y' = \operatorname{rank} Y'''$, hence[2] $I \subset \operatorname{span} Y'''$. Thus, $I \subset F$, and our proof is complete. □

We now consider the Hilbert series of $\mathcal{P}_-(X, I)$, i.e.,

$$h_{-,I}(j) := \dim(\mathcal{P}_-(X, I) \cap \Pi_j^0).$$

In general, it is not true that the polynomials $Q_B := p_{X(B)}$, $B \in \mathbb{B}_-(X, I)$, form a basis for $\mathcal{P}_-(X, I)$. However, they *can* be used for computing $h_{-,I}(j)$. In fact, we have

$$h_{-,I}(j) = \#\{B \in \mathbb{B}_-(X, I) : \operatorname{val}(B) = \deg Q_B = j\}.$$

[2] If a set A in a vector space V contains a subspace V' in its span, and if $A' = v' \sqcup (A\backslash a)$ for some $a \in A$ and $v' \in V'$, then either $V' \subset \operatorname{span} A$, or else $\operatorname{rank} A' < \operatorname{rank} A$.

We observe this fact from the proof of Lemma 3.6: Every Q_B there was proved to be writable as

$$Q_B = \tilde{Q}_B + f_B$$

with $\tilde{Q}_B \in \mathcal{P}_-(X, I)$ and $f_B \in \mathcal{J}_-(X, I)$. The fact $\tilde{Q}_B, B \in \mathbb{B}_-(X, I)$, are independent follows directly from the independence of $Q_B, B \in \mathbb{B}_-(X, I)$, and the fact that the sum $\mathcal{J}_-(X, I) + \operatorname{span}\{Q_B : B \in \mathbb{B}_-(X, I)\}$ is direct (from Part III of the proof of Theorems 3.4 and 3.5, and the fact $\dim\ker \mathcal{J}_-(X, I) = \#\mathbb{B}_-(X, I)$), which implies that $\{\tilde{Q}_B : B \in \mathbb{B}_-(X, I)\}$ is a basis for $\mathcal{P}_-(X, I)$. Note that each $\tilde{Q}_B$ is obtained by replacing some of the factors $p_\omega, \omega \in X$ of Q_B, by polynomials $p_{\omega'}, \omega' \in \mathbb{R}^n\backslash 0$. Thus, $\deg \tilde{Q}_B = \deg Q_B = \operatorname{val}(B)$, hence we may indeed compute $h_{-,I}(j)$ via the polynomials $Q_B, B \in \mathbb{B}_-(X, I)$.

Remark 3.7 *We note that, if $\#I \leq 2$, then*

$$\mathcal{P}_-(X, I) = \mathcal{P}_-(X) + \operatorname{span}\{Q_B : B \in \mathbb{B}_-(X, I)\backslash\mathbb{B}_-(X)\}. \tag{7}$$

In general, however, (7) is not valid for $\#I \geq 3$.

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