What Are the Limits of Lagrange Projectors?

Carl de Boor

Abstract. The behavior of interpolants as interpolation sites coalesce is explored in the suitably restricted context of multivariate polynomial interpolation.

A **Lagrange projector** P_τ is, by definition, a linear map on some linear space X of (scalar-valued) functions on some domain T that associates each function $f \in X$ with the unique element $P_\tau f$ in $\operatorname{ran} P_\tau$ that agrees with f at a given (finite) set τ.

This note concerns the nature of limits of such Lagrange projectors, the limits taken in the (bounded) pointwise sense, with respect to some norm on X, and with the cardinality

$$n := \#\tau$$

of the set τ of **interpolation sites** kept fixed. Thus, this note does not deal with the convergence of an interpolation process as the interpolation sites become become dense. Rather, the interest focuses on what might or might not happen as τ approaches some set σ with $\#\sigma < \#\tau$.

1. Pointwise Limits of Linear Projectors of Finite Rank

In this section, some basic facts concerning linear projectors and bounded pointwise convergence are recalled for the reader's convenience.

Any linear projector P of finite rank on the linear space X over the commutative field $\mathbb{F}$ with **algebraic dual** X' can thought of as providing a linear interpolation scheme on X: For each $g \in X$, $f = Pg$ is the unique element of $\operatorname{ran} P := P(X)$ for which

$$\lambda f = \lambda g, \quad \forall \lambda \in \operatorname{ran} P' = \{\lambda \in X' : \lambda P = \lambda\},$$

with P' the **dual** of P, i.e., the linear map $X' \to X' : \lambda \mapsto \lambda P$. In other words, given that $\ker P := \{g \in X : Pg = 0\} = \operatorname{ran}(\mathrm{id} - P)$,

$$\operatorname{ran} P' = (\ker P)^\perp := \{\lambda \in X' : \ker P \subset \ker \lambda\}$$

is the set of **interpolation conditions** matched by P. Put into more practical terms, if the **column maps**

$$V : \mathbb{F}^n \to X : a \mapsto \sum_{j=1}^n v_j a(j) =: [v_1, \ldots, v_n]a$$

and

$$\Lambda : \mathbb{F}^n \to X' : a \mapsto \sum_{j=1}^n \lambda_j a(j) =: [\lambda_1, \ldots, \lambda_n]a,$$

into X and X' respectively, are such that, with

$$\Lambda^{\mathrm{t}} : X \to \mathbb{F}^n : f \mapsto (\lambda_i f : i = 1{:}n),$$

their **Gram** matrix

$$\Lambda^{\mathrm{t}} V := (\lambda_i v_j : i, j = 1{:}n)$$

is invertible, then, in particular, both V and Λ are 1-1, hence bases for their respective ranges and there is, for given $b \in \mathbb{F}^n$, exactly one element, call it Va, of $\operatorname{ran} V$ that satisfies the equation

$$\Lambda^{\mathrm{t}}(Va) = b,$$

thus giving rise to the map

$$P = V(\Lambda^{\mathrm{t}} V)^{-1}\Lambda^{\mathrm{t}}$$

on X, evidently a linear projector, that associates $g \in X$ with the unique element $f = Pg$ in $\operatorname{ran} V = \operatorname{ran} P$ for which $\Lambda^{\mathrm{t}} f$ agrees with $\Lambda^{\mathrm{t}} g$, hence $\lambda f = \lambda g$ for all $\lambda \in \operatorname{ran}\Lambda = \operatorname{ran} P'$.

Now assume, in addition, that X is a normed linear space. Then a finite-rank linear projector P on X is bounded iff $\operatorname{ran} P' \subset X^* :=$ the **continuous dual** of X. Also, if A is the pointwise limit of a bounded sequence (A_k) of linear maps on X (i.e., $\lim_k \|Ax - A_k x\| = 0$ for all $x \in X$ and $\sup_k \|A_k\| < \infty$), then A is also a bounded linear map; in fact, $\|A\| \le \liminf_k \|A_k\|$.

The following lemma is standard.

Lemma 1.1. *If Q is the pointwise limit of a bounded sequence (P_k) of linear projectors on the normed linear space Y, all of the same finite rank n, then also Q is a bounded linear projector, of rank $\le n$, and*

$$\operatorname{ran} Q = \lim_k \operatorname{ran} P_k, \quad \operatorname{ran} Q' = \lim_k \operatorname{ran} P_k'.$$

Proof: Let $f \in \operatorname{ran} Q$. Then, $f = \lim_k P_k g$ for some g, hence

$$Qf = \lim_j P_j \lim_k P_k g = \lim_j \lim_k (P_j P_j g + P_j(P_k - P_j)g),$$

with the last term no bigger than $(\sup_i \|P_i\|)\|P_k g - P_j g\|$, hence going to zero as $j, k \to \infty$ (since (P_j) is bounded), while the second last term is just $P_j g$. Therefore, altogether, $Qf = f$ for all $f \in \operatorname{ran} Q$, i.e., Q is a linear projector, and is bounded since (P_k) is bounded by assumption.

By the very definition of Q, $\operatorname{ran} Q = \lim_k \operatorname{ran} P_k$ in the sense that each $f \in \operatorname{ran} Q$ is the (norm) limit of a sequence $f_k \in \operatorname{ran} P_k$, all k. Since $\dim \operatorname{ran} P_k = n$ for all k, therefore (see the proof of Lemma 1.2(a) for details) $\dim \operatorname{ran} Q \le n$. Already the simple example, of piecewise linear interpolation at $0, 1/k, 1$ in $C([0\,..\,1])$, shows that we cannot, in general, expect equality here.

Further, any $\lambda \in \operatorname{ran} Q'$ is necessarily continuous, and, for any $f \in X$, $\lambda f = \lambda Q f = \lambda \lim_k P_k f = \lim_k \lambda P_k f$, showing that λ is the bounded pointwise limit of a sequence $\lambda_k \in \operatorname{ran} P_k'$, all k. It is in this sense that $\operatorname{ran} Q' = \lim_k \operatorname{ran} P_k'$. Already the simplest example, of interpolation at one site by constant polynomials to continuous functions, shows that we cannot expect any stronger convergence of the spaces of interpolation conditions than that. □

The following converse of Lemma 1.1 is also standard (or should be).

Lemma 1.2. *Let F_∞ be the limit (in norm) of some sequence (F_k) of linear subspaces of the normed linear space X, all of the same dimension n, and let M_∞ be the limit (under bounded pointwise convergence) of a sequence (M_k) of linear subspaces of X^*, all of dimension n. Then:*

(a) $\dim F_\infty, \dim \mathrm{M}_\infty \le n$*;*

(b) If $\dim F_\infty = n$ *and* $X = F_\infty \oplus \ker \mathrm{M}_\infty$*, with*

$$\ker \mathrm{M} := \cap_{\mu \in \mathrm{M}} \ker \mu,$$

then, from a certain k on, there is a bounded linear projector P_k (necessarily unique) for which $\operatorname{ran} P_k = F_k$ *and* $\operatorname{ran} P_k' = \mathrm{M}_k$*, and P_k converges boundedly pointwise to P_∞.*

Proof: Let $V =: [v_1, \ldots, v_r]$ be a basis for F_∞. By assumption, for each j, there is a sequence of elements $v_{j,k} \in \operatorname{ran} P_k$, all k, so that $\lim_k \|v_j - v_{j,k}\| = 0$. This implies that, from a certain k on, $[v_{1,k}, \ldots, v_{r,k}]$ is a 1-1 linear map and, since it maps into $\operatorname{ran} P_k$, therefore $\dim F_\infty = r \le \dim F_k = n$.

Let $\Lambda =: [\lambda_1, \ldots, \lambda_s]$ be a basis for M_∞. By assumption, for each j, there is a sequence of elements $\lambda_{j,k} \in \operatorname{ran} P_k'$, all k, converging pointwise to λ_j. Then, for the corresponding column maps

$$\Lambda_k := [\lambda_{1,k}, \ldots, \lambda_{s,k}],$$

${\Lambda_k}^{\rm t}$ converges pointwise to $\Lambda^{\rm t}$. On the other hand, since Λ is 1-1 (hence $\Lambda^{\rm t}$ is onto), there exists a column map $W =: [w_1, \ldots, w_s]$ into X **dual to** Λ in the sense that

$$\Lambda^{\rm t} W = \mathrm{id}.$$

Since $\lim_k {\Lambda_k}^{\rm t} W = \Lambda^{\rm t} W$, it follows that, from a certain k on, ${\Lambda_k}^{\rm t} W$ is invertible, hence Λ_k is a 1-1 linear map, and, since it maps into $\operatorname{ran} P_k'$, therefore $\dim \mathrm{M}_\infty = s \le \dim \mathrm{M}_k = n$. This finishes the proof of (a).

As to (b), the assumptions imply the existence of bases $V = [v_1, \ldots, v_n]$ and $\Lambda = [\lambda_1, \ldots, \lambda_n]$ for F_∞ and M_∞, respectively, and dual to each other. By assumption, there exist column maps V_k and Λ_k, into F_k and M_k respectively, so that V_k converges in norm to V and ${\Lambda_k}^{\rm t}$ converges boundedly pointwise to $\Lambda^{\rm t}$. This implies that, from a certain k on, $P_k := V_k({\Lambda_k}^{\rm t} V_k)^{-1} {\Lambda_k}^{\rm t}$ is well-defined, a linear projector with range F_k and interpolation conditions M_k, and its norm bounded independently of k, and converging pointwise to $V(\Lambda^{\rm t} V)^{-1} \Lambda^{\rm t} = P_\infty$. $\square$

Remark. In [4], I failed to stress the fact that, also there, I was concerned with *bounded* pointwise convergence.

2. Only the Convergence of $\operatorname{ran} P_\tau'$ Is of Interest Here

Lemmas 1.1 and 1.2 make clear that pointwise convergence of a sequence (P_k) of linear projectors involves, essentially, three parts: the convergence of $\operatorname{ran} P_k$, the convergence of $\operatorname{ran} P_k'$, and the correct interplay of the two limit spaces.

Here is an interesting example, from [12]. In its discussion (and throughout this note),

$$\delta_t : f \mapsto f(t)$$

denotes the linear functional of evaluation at the site t, and

$$D_j f$$

denotes the derivative of f with respect to its jth argument.

On the space Π of bivariate polynomials, consider the Lagrange projector P_τ with range the subspace $\Pi_{<3}$ of polynomials of degree < 3 and with τ the 6-set

$$\begin{array}{llll} & & (1,1{+}h) & \\ & & \ (1,1) & (1{+}h,1) \\ (0,h) & & & \\ (0,0) & (h,0) & & \end{array}$$

with $h \ne 0$. In that case, P_τ is well defined. However, P_τ fails to converge as $h \to 0$ since (in the pointwise sense)

$$\lim_{h\to 0} \operatorname{ran} P_\tau' = \mathrm{M}_0 := \operatorname{span}(\delta_{\sigma_i}, \delta_{\sigma_i} D_1, \delta_{\sigma_i} D_2 : i = 1, 2),$$

with $\sigma_1 := (0,0)$, $\sigma_2 := (1,1)$, and this space, though of dimension 6, fails to be of dimension 6 over $\Pi_{<3}$. Indeed, M_0 is also the pointwise limit, as $h \to 0$, of $\operatorname{span}(\delta_t : t \in \sigma)$ with σ the 6-set

$$\begin{array}{llll} & & (1{-}h,1) & (1,1) \\ & & & (1,1{-}h) \\ (0,h) & & & \\ (0,0) & (h,0) & & \end{array}$$

and, for $h > 0$, this set evidently lies on an ellipse, i.e., some nontrivial element of $\Pi_{<3}$ vanishes on σ, hence this must be true of the limiting interpolation conditions, too.

On the other hand, the limiting space, M_0, of interpolation conditions is 6-dimensional, hence it is easy to find polynomial spaces F for which $X = F \oplus \ker \mathrm{M}_0$, and for each such choice, Lagrange interpolation from F at τ (with $h \ne 0$) is well defined, at least for h close to zero, and converges, as $h \to 0$, to interpolation from F to the Hermite interpolation conditions M_0.

Since such failure of convergence is so easily fixed, I will avoid further consideration of it by concentrating entirely on the following

Question 2.1. *What is the limit, if any, of* $\operatorname{span}(\delta_t : t \in \tau)$ *as the finite set* τ *approaches the set* σ*?*

3. Only the Limit On Π Is of Interest Here

The answer to Question 2.1 depends crucially on the space of functions on which it is considered. If that space is, e.g., the space $C([a \,..\, b])$ of continuous functions on some (nontrivial) closed interval $[a \,..\, b]$, and $\tau = \{a, a+h\}$, then, as $h \searrow 0$, τ approaches the 1-set $\{a\}$ while the 2-dimensional space $\mathrm{span}(\delta_t : t \in \tau)$ approaches the 1-dimensional space spanned by δ_a. This is due to the fact that, on $C([a \,..\, b])$, $\lim_{h\to 0}(\delta_{a+h} - \delta_a)/h$ does not exist. But if we restrict attention to the smaller space $C^{(1)}([a \,..\, b])$ of continuously differentiable functions, then $\lim_{h\to 0} \mathrm{span}(\delta_a, \delta_{a+h}) = \mathrm{span}(\delta_a, \delta_a D)$.

To avoid such impediments to a full limit space, we consider Question 2.1 only on the space

$$\Pi \subset (\mathbb{F}^d \to \mathbb{F})$$

of all polynomials in d real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) variables, i.e., concentrate on the following

Question 3.1. *What is the limit, if any, of* $\mathrm{span}(\delta_t : t \in \tau)$ *as the finite set* τ *approaches the set* σ*, considered as linear functionals on polynomials of* d *(real or complex) variables?*

This seems sufficient in view of the fact that polynomials are dense in many function spaces of interest.

4. The Univariate Case

Question 3.1 is completely answered when $d = 1$. In that case, for an arbitrary infinite sequence τ in $\mathbb{F}$, there is uniquely associated with each $p \in \Pi$ its **Newton series**:

$$\begin{aligned} p \;&=:\; \sum_{j=1}^{\infty} c_j(p;\tau) \underbrace{(\cdot - \tau_1)\cdots(\cdot - \tau_{j-1})}_{=:\, w_{j-1,\tau}} \\ &=\; \sum_{j=1}^{n} c_j(p;\tau) w_{j-1,\tau} \;+\; \sum_{j>n} c_j(p;\tau) w_{j-1,\tau} \\ &=: \qquad\quad p_n \qquad\quad + \qquad w_{n,\tau}\, q_n \;. \end{aligned}$$

This identifies p_n as a polynomial of degree $< n$, necessarily unique, that matches p at $\tau_1, \ldots, \tau_n$ in the sense that $p - p_n$ is divisible by $w_{n,\tau} = (\cdot - \tau_1)\cdots(\cdot - \tau_n)$. Hence, p_n is the unique polynomial of degree $< n$ that **agrees with** p **at** $\tau_{1:n} := (\tau_1, \ldots, \tau_n)$ in the sense that

$$D^j p_n(z) \;=\; D^j p(z), \quad \forall\, 0 \le j < \#\{i \le n : \tau_i = z\},\ z \in \mathbb{F}.$$

In particular, $c_n(p;\tau)$ depends only on $\tau_{1:n}$, and depends linearly on p, and this is emphasized by writing it $\boldsymbol{\Delta}(\tau_{1:n})p$, with the *linear functional* $\boldsymbol{\Delta}(\tau_{1:n})$ called the *divided difference at* $\tau_{1:n}$ since, for the case that all the τ_j are distinct, it is formed from $\delta_{\tau_1}, \ldots, \delta_{\tau_n}$ by repeatedly forming divided differences. Further, since the $w_{j,\tau}$ depend continuously on τ, so must the linear functionals $\boldsymbol{\Delta}(\tau_{1:j})$. In particular, as the *sequence* $\tau_{1:n}$ approaches the n-sequence σ, $\mathrm{span}(\delta_t : t \in \tau_{1:n})$ approaches the n-dimensional space

$$\mathrm{span}(\delta_z D^j : 0 \le j < \#\{i \le n : \sigma_i = z\}, z \in \mathbb{F}).$$

5. A Multivariate Example

Before embarking on the discussion of the general case, here is a striking multivariate example, from [5]. In its discussion (and later), the following notational conventions are followed.

For lack of a standard notation, I use

$$()^\alpha : \mathbb{F}^d \to \mathbb{F} : x \mapsto x^\alpha \ := \ x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \qquad \alpha \in \mathbb{Z}_+^d,$$

for the monomial with exponent α, with $\mathbb{Z}_+^d$ all d-vectors with nonnegative integer entries, and use the standard notation

$$|\alpha| := \alpha_1 + \cdots + \alpha_d$$

for the **degree** of such a **multi-index** α.

Further, I use $\widehat{p}$ for the coefficients in the power form of a polynomial, i.e.,

$$p \ =: \sum_{\alpha \in \mathbb{Z}_+^d} \widehat{p}(\alpha)\, ()^\alpha, \quad \forall\, p \in \Pi,$$

and use the standard notation

$$p(D) \ := \ \sum_\alpha \widehat{p}(\alpha) D^\alpha$$

for the constant coefficient differential operator whose coefficients are the power coefficients of p. Here, $D^\alpha := D_1^{\alpha_1} \cdots D_d^{\alpha_d}$, with $D_j p$ the derivative of p with respect to its jth argument. Also, with

$$\deg p := \max\{|\alpha| : \widehat{p}(\alpha) \neq 0\},$$

I use the nonstandard notation

$$p_\uparrow := \sum_{|\alpha| = \deg p} \widehat{p}(\alpha)()^\alpha$$

for the **leading term** of the polynomial p and use, correspondingly,

$$f_\downarrow := \sum_{|\alpha| = \operatorname{ord} f} \widehat{f}(\alpha)()^\alpha$$

for the **least term** of the **formal power series**

$$f =: \sum_\alpha \widehat{f}(\alpha)()^\alpha,$$

with

$$\operatorname{ord} f := \min\{|\alpha| : \widehat{f}(\alpha) \neq 0\}.$$

To motivate these last two definitions, recall (e.g., from [6]) that the pairing

$$A_0 \times \Pi \to \mathbb{F} : (f, p) \mapsto f{*}p := \sum_\alpha \widehat{f}(\alpha)\alpha!\widehat{p}(\alpha) = p(D)f(0),$$

between the space A_0 of all formal power series and Π, provides a linear 1-1 correspondence between A_0 and the (algebraic) dual, Π', of Π. Thinking in this way of the formal power series f as a linear functional on Π, its **order** is the largest natural number k for which f vanishes on $\Pi_{<k}$.

Notice that, in this 1-1 correspondence, δ_t is represented by the **exponential with frequency** t, i.e., by the power series

$$\mathrm{e}_t := \sum_\alpha ()^\alpha t^\alpha / \alpha!.$$

Indeed, for any polynomial p,

$$\mathrm{e}_t{*}p = \sum_\alpha (t^\alpha/\alpha!)\ \alpha!\ \widehat{p}(\alpha) = \sum_\alpha t^\alpha\ \widehat{p}(\alpha) \ = \ p(t).$$

This may help to explain the appearance of the exponential space

$$E_\tau := \operatorname{span}(\mathrm{e}_t : t \in \tau)$$

in the following.

Proposition 5.1 ([5]). *Let v and T be a point, respectively a finite subset, in $\mathbb{Z}^d$. Then*

$$\lim_{h\to 0}\operatorname{span}(\delta_{v+ht} : t\in \mathrm{T}) \;=\; \delta_v\Pi_{\mathrm{T}}(D) := \{\delta_v q(D) : q\in\Pi_{\mathrm{T}}\}$$

in the pointwise sense, with

$$\Pi_{\mathrm{T}} \;:=\; \operatorname{span}(f_{\downarrow} : f\in E_{\mathrm{T}}).$$

For a detailed proof, see [4]. As that proof makes clear, the convergence is in the boundedly pointwise sense with respect to any norm on Π with respect to which the linear functionals $\delta_v\Pi_{\mathrm{T}}(D)$ are continuous, for example, in $C^{(k)}(T)$, with T any bounded domain in $\mathbb{F}^d$ containing v in its interior, and $k\ge \#\tau-1$.

What is the nature of the polynomial space Π_{T}? It is spanned by homogeneous polynomials, and this is equivalent to being **dilation-invariant**, meaning that, for any $s\in\mathbb{F}$ and any $p\in\Pi_{\mathrm{T}}$, also $p(s\cdot)$ is in Π_{T}. Further, since

$$(D_j f)_{\downarrow} = D_j(f_{\downarrow})$$

for any j and any formal power series f, Π_{T} is **D-invariant**, i.e., closed under differentiation. This raises the following question (first asked in [5]).

Question 5.2. *Is every finite-dimensional dilation- and D-invariant linear subspace of Π of the form Π_{T} for some finite set T?*

The answer to this question is negative (see Remark 7.2 below), but may well be seen to be negative because Π_{T} is not just dilation- and D-invariant but has the following property which implies such invariance but seems, offhand, stronger than that.

Proposition 5.3 ([6]). $\Pi_{\mathrm{T}} \;=\; \bigcap_{p|_{\mathrm{T}}=0}\ker p_{\uparrow}(D)$.

For a direct proof, see [2].

6. Some Help From Ideal Interpolation

Lagrange interpolation was, apparently, the inspiration for Birkhoff's [1] definition of **ideal interpolation** as any linear interpolation scheme whose errors form a polynomial ideal. For, certainly, the Lagrange projector P_τ on Π, whatever its range, has as its errors the set of all polynomials that vanish on τ, and this is an ideal, even a radical one.

Here is a slight strengthening of a characterization, from [3], of ideal projectors.

Lemma 6.1. *The linear map P on Π is an ideal projector if and only if*

$$P(pq) = P(pPq), \quad \forall\, p,q\in\Pi. \tag{6.2}$$

Indeed, for the choice $p : t\mapsto 1$, (6.2) states that P is a linear projector, hence $\ker P = \operatorname{ran}(\mathrm{id}-P)$ and, with that, (6.2) states that $\Pi\ker P\subset\ker P$.

Since this characterization of ideal projectors is pointwise, it is preserved under bounded pointwise convergence. In view of the general discussion in Section 1, we therefore have the following answer to Question 2.1.

Proposition 6.3. *Any bounded pointwise limit of $\operatorname{span}(\delta_t : t\in\tau)$ of dimension $\#\tau$ is necessarily of the form $\operatorname{ran}P'$ for some ideal projector of rank $\#\tau$.*

Recall from Section 1 that, for any linear projector of finite rank,

$$\operatorname{ran}P' = (\ker P)^{\perp}.$$

On the other hand, from results in Algebra that can be traced back via Groebner [7] to Macaulay's inverse systems [10] and well recalled and summarized in [11], any polynomial ideal $\mathcal{I}$ of finite codimension is necessarily of the form

$$\mathcal{I} \;=\; \bigcap_{v\in\mathcal{V}}\ker(\delta_v Q_v(D)),$$

with $\mathcal{V}$ the (necessarily finite) **variety** of the ideal, i.e., the set of common zeros of $p \in \mathcal{I}$, and each Q_v a finite-dimensional D-invariant polynomial space, the **multiplicity space at** v. As an aside, such a characterization is also available for ideals of infinite codimension, with the only difference being that the multiplicity spaces need not be finite-dimensional in that case; see [9].

This gives the following

Corollary 6.4. *If* $\mathrm{span}\{\delta_t : t \in \tau\}$ *converges pointwise to some space* M *of dimension* n *as the set* τ *of cardinality* n *approaches the set* σ*, then* M *is necessarily of the form*

$$\mathrm{M} \;=\; \sum_{s\in\sigma} \delta_s Q_s(D),$$

with each Q_s *a finite-dimensional* D*-invariant polynomial space.*

Now notice that, in contrast to the result in Proposition 5.1, there is no claim here that the Q_s are also dilation-invariant. And that is as it should be, as the following simple example, also from [5], makes clear.

Choose $d = 2$, $\mathbb{F} = \mathbb{R}$, and $\tau = \{-\xi, 0, \xi\}$, with $\xi := (h, h^2)$, and consider

$$\lim_{h\to 0} \mathrm{span}(\delta_t : t \in \tau).$$

It certainly contains δ_0 as well as $\delta_0 D_1 = \lim_{h\to 0}(\delta_\xi - \delta_{-\xi})/(2h)$. But, with $\zeta := (0, h^2)$, it also contains

$$\delta_0 q(D) := \delta_0(D_1^2 + 2D_2) = \lim_{h\to 0}((\delta_\xi - 2\delta_\zeta + \delta_{-\xi})/h^2 + (2\delta_\zeta - 2\delta_0)/h^2).$$

Since $[()^0, ()^{1,0}, q]$ is 1-1, the space $Q := \mathrm{span}(()^0, ()^{1,0}, q)$ is 3-dimensional, hence $\lim_{h\to 0} \mathrm{span}(\delta_t : t \in \tau) = \delta_0 Q(D)$. However, while Q is D-invariant (as it must be), it is not dilation-invariant since it does not have a basis consisting of homogeneous polynomials.

We are left with the question whether Corollary 6.4 is the *complete* answer, i.e., whether any finite sum $\sum_s \delta_s Q_s(D)$, with each Q_s a finite-dimensional D-invariant polynomial space, is the pointwise limit of spans of point evaluations. In view of the discussion in Section 1, this is equivalent to the following question.

Question 6.5. *Is every ideal projector of finite rank the pointwise limit of Lagrange projectors?*

I conjectured as much in 2003 (see [4]), but this conjecture was recently shown to be false for $d > 2$ by Boris Shekhtman [13], using a result of A. Iarrobino [8] pointed out to me by Geir Ellingsrud as the reason he thought that conjecture was false.

7. Shekhtman's Counterexample

The idea of the counterexample is to exhibit a polynomial subspace F for which 'most' ideal projectors having it as their range cannot be the pointwise limit of Lagrange projectors.

For this, pick the natural number m, and a nontrivial partition

$$V \cup W = \{()^\alpha : |\alpha| = m\}$$

of the set of monomials of degree m, and choose

$$F := \mathrm{span}(W) + \Pi_{<m}.$$

Now, as Iarrobino [8] observes, for this F and for an arbitrary matrix $C \in \mathbb{F}^{V\times W}$,

$$\Pi = F \oplus \mathcal{I}_C,$$

with

$$\mathcal{I}_C := \mathrm{span}(v - \sum_{w\in W} C(v,w)w : v \in V) + \mathrm{span}(()^\alpha : |\alpha| > m)$$

evidently an ideal since the first summand in its definition consists of polynomials homogeneous of degree m, hence multiplication by any nonconstant monomial produces an element of the second summand. This implies that each $C \in \mathbb{F}^{V\times W}$ gives rise to an ideal projector with range F, namely the linear projector P_C with range F and kernel $\mathcal{I}_C$.

Conversely, for an arbitrary linear projector P with $\operatorname{ran} P = F$, there is a unique matrix $C_P \in \mathbb{F}^{V\times W}$ so that

$$Pv \in \sum_{w\in W} C_P(v,w)w + \Pi_{<m}, \quad \forall\, v \in V,$$

and, evidently,

$$C_{P_C} = C.$$

Further, the resulting map $P \mapsto C_P$ is continuous. Hence, if every ideal projector were the limit of Lagrange projectors, then the image of the set of Lagrange projectors $P = P_\tau$ onto F under the map $P \mapsto C_P$ would have to be dense in $\mathbb{F}^{V\times W}$ and this, as Shekhtman shows, is not always the case.

Indeed, if $P = P_\tau$, hence $\#\tau = n := \dim F$ and, without loss, $\tau =: \{\tau_1, \dots, \tau_n\}$, then, by the definition of C_P, for each $v \in V$,

$$v(\tau_i) = \sum_{w\in W} C_P(v,w)w(\tau_i) + \sum_{|\alpha|<m} c(v,\alpha)\tau_i^\alpha, \quad i = 1{:}n. \tag{7.1}$$

View this as a linear system, for the unknowns $C_P(v,w)$, $w \in W$, and $c(v,\alpha)$, $|\alpha| < m$. Then we know, from the fact that P_τ is well defined, that the coefficient matrix, A_τ, of this linear system is invertible. Hence, Cramer's rule provides the formula

$$C_P(v,w) = \det A_{\tau,v,w} / \det A_\tau, \quad \forall\, w \in W,$$

with $A_{\tau,v,w}$ the matrix obtained from A_τ by replacing there w by v. Since A_τ is a matrix whose general entry is some monomial evaluated at one of the sites τ_i, therefore $\det A_\tau$, hence also each $\det A_{\tau,w,v}$, is a polynomial in the variables τ_i, $i = 1{:}n$, hence a polynomial in nd scalar variables. It follows that, for $P = P_\tau$, the corresponding matrix C_P lies in the range of the *polynomial* map

$$S : \mathbb{F}^{nd+1} \to \mathbb{F}^{V\times W} : (\tau_1, \dots, \tau_n, z) \mapsto (\det A_{\tau,v,w} : v \in V, w \in W)z.$$

In fact, $C_P = S(\tau_1, \dots, \tau_n, 1/\det A_\tau)$.

Now, by a standard theorem from Algebraic Geometry (see [13] for the reference and, in particular, Remark 2.2 there that fills in certain details) and under the assumption that $\mathbb{F} = \mathbb{C}$, the range of the polynomial map S lies in a proper hypersurface of its target, $\mathbb{C}^{V\times W}$, in case $\dim\operatorname{dom} S < \dim\operatorname{tar} S$, i.e., in case

$$nd + 1 < \#V \cdot \#W,$$

hence fails to be dense in $\mathbb{C}^{V\times W}$ in that case. But, as [8] already observed, this inequality holds, e.g., for $d = 3$, $m = 7$, and $\#V = \#W$.

Remark 7.2. Notice that $\mathcal{I}_C$ is a *homogeneous* ideal, i.e., is generated by homogeneous polynomials. Since it is also of finite codimension, this implies that its variety consists of the origin only, and that the corresponding multiplicity space, Q_0, is also spanned by homogeneous polynomials, hence is not only D-invariant but also dilation-invariant. Shekhtman's counterexample therefore also provides a negative answer to Question 5.2.

8. Hermite Interpolation vs. Ideal Interpolation

The fact that not every ideal projector is the limit of Lagrange projectors raises the question of just how to characterize those ideal projectors that are. So far, I have only a name for them, namely **Hermite projectors**, thus giving up on my agreement in [4] to follow Möller who, already in [11], i.e., well before [1], investigated what Birkhoff later called ideal interpolation, calling it 'Hermite interpolation'. For, I consider the fact that, in the univariate case, Hermite interpolation is the limit of Lagrange interpolation so important a property that I would like Hermite interpolation in the multivariate case to have that property also, even if that means that, at this point, I don't exactly know what I am talking about when I am discussing such Hermite interpolation.

Notice that Shekhtman's counterexample requires $d > 2$. In the same paper, [13], Shekhtman also gives a proof that, for $d = 2$, ideal interpolation is Hermite interpolation (in the newly minted sense). But, contrary to the claims I made in [4] (in particular, the proof outline I gave there for the Corollary to Proposition 7.4 does not seem to be realizable), his proof is not all that simple; it requires nontrivial facts from Algebraic Geometry. (Added 31oct11: A proof using only linear algebra can be found in [C. de Boor and B. Shekhtman, On the pointwise limits of bivariate Lagrange projectors; LAA; 429(1); 2008; 311–325;].)

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