STABILITY AND INDEPENDENCE OF THE SHIFTS OF FINITELY MANY REFINABLE FUNCTIONS

Thomas A. Hogan

27 February 1997

ABSTRACT. Typical constructions of wavelets depend on the stability of the shifts of an underlying refinable function. Unfortunately, several desirable properties are not available with compactly supported orthogonal wavelets, e.g. symmetry and piecewise polynomial structure. Presently, multiwavelets seem to offer a satisfactory alternative. The study of multiwavelets involves the consideration of the properties of *several* (simultaneously) refinable functions. In Section 2 of this paper, we characterize stability and linear independence of the shifts of a finite refinable function set in terms of the refinement mask. Several illustrative examples are provided.

The characterizations given in Section 2 actually require that the refinable functions be minimal in some sense. This notion of minimality is made clear in Section 3, where we provide sufficient conditions on the mask to ensure minimality. The conditions are shown to be also necessary under further assumptions on the refinement mask. An example is provided illustrating how the software package `MAPLE` can be used to investigate at least the case of two simultaneously refinable functions.

1. Introduction

In this paper, we provide a characterization of the stability and linear independence of the shifts of finitely many compactly supported refinable functions $\Phi \subset L^2(\mathbb{R})$. The characterization is in terms of the refinement mask. The nature of these results leads us to a question involving the "length" of the space generated by these functions. Some notes on this topic are also provided.

We begin with some notation used throughout this paper. The symbols $\mathbb{N}$ and $\mathbb{Z}_+$ denote the set of natural numbers and non-negative integers respectively. We also use

$$\mathbb{T} := [0 \,..\, 2\pi)$$

frequently, but with the understanding that any reference to a function f defined on the half-open interval $\mathbb{T}$ also refers to the 2π-periodization of f,

$$f(x + 2j\pi) := f(x) \text{ for every } x \in \mathbb{T},\ j \in \mathbb{Z}$$

AMS (MOS) Subject Classification. 42C15.

Key words and phrases. Multiresolution, (global) linear independence, stability, Riesz basis, matrix refinement mask, shift-invariant spaces, multiwavelets.

and *vice versa.* For any set S, we denote by $\ell^2(S)$, the set of all $a \in \mathbb{C}^S$ satisfying

$$\|a\|_{\ell^2} := \left\{ \sum_{s\in S} |a(s)|^2 \right\}^{1/2} < \infty.$$

And we denote by $L^2(\mathbb{R})$ the set of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\|f\|_{L^2} := \left\{ \int_{\mathbb{R}} |f(x)|^2 \, dx \right\}^{1/2} < \infty,$$

and equipped with the usual inner product

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx.$$

A finite set $\Phi \subset L^2(\mathbb{R})$ is said to have **stable shifts** if there exist positive constants c_1 and c_2 such that

$$c_1\|a\|_{\ell^2} \le \| \sum_{\phi\in\Phi} \sum_{j\in\mathbb{Z}} a_\phi(j)\phi(\cdot - j)\|_{L^2} \le c_2\|a\|_{\ell^2}$$

for any sequence $a \in \ell^2(\Phi \times \mathbb{Z})$ (it is often said that Φ provides a **Riesz basis** (for its closed linear span) in this case); a *compactly supported* finite set $\Phi \subset L^2(\mathbb{R})$ is said to have **linearly independent shifts** if the map

$$a \mapsto \sum_{\phi\in\Phi} \sum_{j\in\mathbb{Z}} a_\phi(j)\phi(\cdot - j)$$

is one-to-one on $\mathbb{C}^{\Phi\times\mathbb{Z}}$.

In [JM1], Jia and Micchelli gave necessary and sufficient conditions for the linear independence of the shifts of a finite compactly supported set $\Phi \subset L^2(\mathbb{R})$. They showed that Φ has linearly independent shifts if and only if the sequences

$$\left(\widehat{\phi}(\xi + 2j\pi)\right)_{j\in\mathbb{Z}} \qquad (\phi \in \Phi) \tag{1.1}$$

are linearly independent for every $\xi \in \mathbb{C}$. In [JM2], moreover, they showed that a finite compactly supported set $\Phi \subset L^2(\mathbb{R})$ has stable shifts if and only if the sequences in (1.1) are linearly independent for every $\xi \in \mathbb{R}$. In this paper, we are concerned with the stability and independence of the shifts of refinable functions.

A compactly supported finite set $\Phi \subset L^2(\mathbb{R})$ is said to be **refinable** if (Φ is not identically zero and) there exists a finitely supported sequence

$$a : \mathbb{Z} \to \mathbb{C}^{\Phi\times\Phi}$$

satisfying

$$\Phi = \sum_{j\in\mathbb{Z}} a(j)\Phi(2\cdot -j) = \left(\sum_{j\in\mathbb{Z}}\sum_{\psi\in\Phi} a_{\phi,\psi}(j)\psi(2\cdot -j)\right)_{\phi\in\Phi}.$$

Equivalently, Φ is refinable if

$$\widehat{\Phi}(2\omega) = A(e^{-i\omega})\widehat{\Phi}(\omega) \text{ for all } \omega\in\mathbb{C},$$

where $A := (A_{\phi,\psi})_{\phi,\psi\in\Phi}$ and

$$A_{\phi,\psi}(z) := \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{\phi,\psi}(j)z^j \qquad (z\in\mathbb{C}\backslash 0).$$

Unless stated otherwise, whenever we refer to a refinable Φ with **(refinement) mask** A, we mean that

$$A = (\, a_{\phi,\psi}\,)_{\phi,\psi\in\Phi}$$

is a matrix of trigonometric polynomials; that 1 is an eigenvalue of $A(0)$ and a right 1-eigenvector v is specified, i.e.,

$$A(0)v = v \neq 0;$$

and that $\Phi\subset L^2(\mathbb{R})$ is a compactly supported solution to the **refinement equation**

$$\widehat{\Phi}(2\cdot) = A\widehat{\Phi}; \qquad \widehat{\Phi}(0) = v. \tag{1.2}$$

The notions of stability and linear independence of the shifts of refinable functions are important in many areas including subdivision, finite element analyses, approximation from shift-invariant spaces, and wavelets. And most recently, they have proved to be important in the study of what are now being referred to as *multiwavelets*. Multiwavelets have been studied by Donovan, Geronimo, Hardin, Kessler, and Massopust beginning with [HKM], in which continuous *scaling functions* Φ were provided. Then in [GHM], arbitrarily smooth scaling functions were constructed including a pair with orthonormal shifts, and these scaling functions were then used to construct compactly supported symmetric/anti-symmetric orthonormal multiwavelets in [DGHM]. Multiwavelets have also been constructed independently by Goodman and Lee. In [GL], they dealt with Riesz bases and dual bases for a set S of spline functions on $\mathbb{R}$ with multiple knots on $\mathbb{Z}$. They further provided a Riesz basis for the orthogonal complement of S in the set $\{\, s(2\cdot) \,:\, s\in S\,\}$ (see also [GLT]). [SS] provides an alternative pair of piecewise linear multiwavelets.

As with wavelets, it seems important to understand the properties of multiwavelets in terms of the refinement mask of a refinable function set Φ. Already much work has been done in this area. Approximation orders of refinable function sets have been characterized in [HSS]. This characterization appeared independently in [P], which also featured a factorization of the mask. This factorization has already been put to use in [CDP] to study the regularity or smoothness of refinable function sets. All of these results were arrived at under the assumption that the shifts of the refinable set be stable or independent.

Yet none of these papers provided a means for testing the mask to determine stability or independence.

In Section 2, we provide a characterisation of stability and linear independence in terms of the mask. This however brings up another issue which does not appear when dealing with a single refinable function. The issue concerns the notion which de Boor, DeVore, and Ron refer to in [BDR] as *length.* A necessary condition for a set Φ to have stable shifts is that Φ be a *minimally generating* set. This unfortunately is not a condition which is easily characterised in terms of the mask. For this reason, Section 3 is devoted to *notes* on the length of a refinable space and how it depends on the mask.

2. Stability and independence

2.1. General

This section deals with $\Phi \subset L^2(\mathbb{R})$ which satisfy $\operatorname{len} S(\Phi) = \#\Phi$, where

$$\begin{aligned}
\operatorname{len} S(\Phi) &:= \min\{\, \#\Psi \;:\; S(\Psi) = S(\Phi)\,\}; \\
S(\Phi) &:= \text{the } L^2\text{-closure of } S_0(\Phi);\ \text{and} \\
S_0(\Phi) &:= \text{the set of all finite linear combinations} \\
&\qquad\qquad \text{of integer translates of all the } \phi \in \Phi.
\end{aligned}$$

The results of [BDR] imply that for any compactly supported finite set $\Phi \subset L^2(\mathbb{R})$ with $\operatorname{len} S(\Phi) = \#\Phi$, the sequences in (1.1) are linearly independent for almost every $\xi \in \mathbb{C}$. If $\operatorname{len} S(\Phi) < \#\Phi$, on the other hand, the sequences were shown to be linearly dependent for almost every $\xi \in \mathbb{C}$. Evidently, if $\operatorname{len} \Phi < \#\Phi$, then Φ does not have stable shifts. The main results of this section characterize stability and linear independence of a refinable set Φ in terms of the refinement mask A under the assumption that $\operatorname{len} S(\Phi) = \#\Phi$. Then, we complete the analysis by providing, in Section 3, necessary and sufficient conditions for $\operatorname{len} S(\Phi) = \#\Phi$ in terms of the mask A.

Before stating our main results, we provide some additional definitions. For a given integer $m \geq 2$, we say that a point $\omega \in \mathbb{R}$ is **m-cyclic in $\mathbb{T}$** if $2^m\omega = \omega \neq 0$ in $\mathbb{T}$; i.e., if $\omega \notin 2\mathbb{Z}\pi$, while $2^m\omega - \omega \in 2\mathbb{Z}\pi$. Equivalently, $\omega \in \mathbb{R}$ is m-cyclic in $\mathbb{T}$ if and only if

$$\omega = \frac{2\mu\pi}{2^m - 1}$$

for some $\mu \in \mathbb{Z}\backslash(2^m-1)\mathbb{Z}$. It follows that $e^{-i\omega} \in \mathbb{C}$ is a primitive odd root of unity if and only if ω is **cyclic**, i.e. is m-cyclic in $\mathbb{T}$ for some m. We point out that if ω is m-cyclic in $\mathbb{T}$, then so is $2^\ell\omega$ for any $\ell \in \mathbb{Z}_+$. Also, if ω is cyclic, then $\omega + \pi$ is **acyclic**, i.e. is not m-cyclic in $\mathbb{T}$ for any integer m. Finally, it was shown in [H, pp. 10–11] that, if ω is m-cyclic in $\mathbb{T}$ for some integer $m \geq 2$, then for any $j \in \mathbb{Z}$,

$$\omega + 2j\pi = 2^{mn}\omega + 2^{mn-k}\nu\pi \tag{2.1}$$

for some $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, m-1\}$, and $\nu \in \mathbb{Z}\backslash 2\mathbb{Z}$ (see also [JW, p. 1123]).

As in [H], we say that a 2π-periodic function $f : \mathbb{C} \to \mathbb{C}$ has a **π-periodic zero in** $\mathbb{R}$ (resp. $\mathbb{C}$) if there exists $z \in \mathbb{R}$ (resp. $\mathbb{C}$) such that $f(z) = f(z+\pi) = 0$; and a **contaminating zero** if there exists an integer $m \geq 2$ and ω which is m-cyclic in $\mathbb{T}$, such that

$$f(2^k\omega + \pi) = 0 \text{ for all } k \in \{0,\, 1,\, \ldots,\, m-1\}.$$

We also define, for any $k,\, n \in \mathbb{Z}$,

$$\mathcal{A}_{n,k} := \prod_{n>\ell>k} A(2^\ell \cdot) := A(2^{n-1}\cdot)A(2^{n-2}\cdot)\cdots A(2^{k+1}\cdot),$$

with the understanding that the (empty) product represents the $\Phi \times \Phi$ identity matrix when $n-1 < k+1$.

Lastly, for $\lambda \in \mathbb{C}^\Phi$, we define

$$\lambda\Phi := \sum_{\phi\in\Phi} \lambda_\phi \phi \quad \text{and} \quad \lambda A := \left(\sum_{\phi\in\Phi} \lambda_\phi a_{\phi,\psi} \right)_{\psi\in\Phi}.$$

2.2. Statement of main results

Theorem 2.1. *Suppose Φ is a solution to the refinement equation (1.2) and* $\operatorname{len} S(\Phi) = \#\Phi$. *Then Φ has stable (resp. linearly independent) shifts if and only if for every $\lambda \in \mathbb{C}^\Phi \backslash 0$,*

(i) if $\lambda\widehat{\Phi}(0) = 0$, then there exists $n \in \mathbb{Z}_+$ so that $\lambda A^n(0)A(\pi) \neq 0$;

(ii) if $\lambda A(\omega) = 0$ for some $\omega \in \mathbb{R}$ (resp. $\omega \in \mathbb{C}$), then $\lambda A(\omega+\pi) \neq 0$; and

(iii) for any integer $m \geq 2$ and any $\omega \in \mathbb{R}$ which is m-cyclic in $\mathbb{T}$, there exists $n \in \mathbb{N}$ and $k \in \{0,\, 1,\, \ldots,\, m-1\}$ so that

$$\lambda\mathcal{A}_{mn,k}(\omega)A(2^k\omega+\pi) \neq 0.$$

As stated in Section 2.1, if Φ has stable shifts then $\operatorname{len} S(\Phi) = \#\Phi$. This provides the following necessary conditions.

Corollary 2.2. *Suppose Φ is a solution to the refinement equation (1.2). If Φ has stable (resp. linearly independent) shifts, then for every $\lambda \in \mathbb{C}^\Phi \backslash 0$,*

(i) if $\lambda\widehat{\Phi}(0) = 0$, then there exists $n \in \mathbb{Z}_+$ so that $\lambda A^n(0)A(\pi) \neq 0$;

(ii) if $\lambda A(\omega) = 0$ for some $\omega \in \mathbb{R}$ (resp. $\omega \in \mathbb{C}$), then $\lambda A(\omega+\pi) \neq 0$; and

(iii) for any integer $m \geq 2$ and any $\omega \in \mathbb{R}$ which is m-cyclic in $\mathbb{T}$, there exists $n \in \mathbb{N}$ and $k \in \{0,\, 1,\, \ldots,\, m-1\}$ so that

$$\lambda\mathcal{A}_{mn,k}(\omega)A(2^k\omega+\pi) \neq 0.$$

It should be noted that, in both of the conditions (i) and (iii) of the above theorem and corollary, n can be bounded. Specifically, one need only verify condition (i) for $n = 0,\, 1,\, \ldots,\, \#\Phi - 1$; and one need only verify condition (iii) for $n = 1,\, 2,\, \ldots,\, \#\Phi$. As regards condition (i), this is obvious since A is a $\Phi\times\Phi$ matrix. To see that n can also be bounded in condition (iii), we introduce, for given integer $m\ge 2$ and ω which is m-cyclic in $\mathbb{T}$, the matrix

$$M(\omega) := \mathcal{A}_{nm+m,nm-1}(\omega).$$

Since

$$2^{nm+\ell}\omega - 2^{m+\ell}\omega = 2^{\ell}\frac{2^{nm}-2^{m}}{2^{m}-1}(2^{m}\omega-\omega)\in 2\mathbb{Z}\pi$$

for $n\in\mathbb{N}$ and $\ell = 0, 1, \cdots, m-1$, $M(\omega)$ is independent of $n\in\mathbb{N}$. Moreover,

$$\mathcal{A}_{mn,k}(\omega) = M^{n-1}(\omega)\mathcal{A}_{m,k}(\omega),$$

which are spanned already by the cases $n = 1, 2, \cdots, \#\Phi$ for any particular k. Note also that if $\det A$ is not identically zero, then m could also be bounded in terms of the number (and locations) of zeros of $\det A$.

We also provide the following sufficient conditions.

Corollary 2.3. *Suppose Φ is a solution to the refinement equation (1.2) and $\operatorname{len} S(\Phi) = \#\Phi$. If A satisfies the following three conditions, then the shifts of Φ are stable (resp. linearly independent):*
(i) the matrix $\begin{bmatrix}\widehat{\Phi}(0) & A(\pi)\end{bmatrix}$ has full rank;
(ii) $\det A$ has no π-periodic zeros in $\mathbb{R}$ (resp. $\mathbb{C}$); and
(iii) $\det A$ has no contaminating zeros.

At first, the condition (iii) appearing in Theorem 2.1 and Corollary 2.2 may seem to be nearly inaccessible. However, if one first tries to apply Corollary 2.3, and finds that the mask does not satisfy the easily checked condition (iii) of Corollary 2.3, then one should begin to analyze the (left-)nullspaces of the matrix A at the contaminating zero of $\det A$. Verifying condition (iii) of Theorem 2.1 at these points, is then straight-forward.

2.3. Lemmata

For a single function $\phi\in L^2(\mathbb{R})$, we refer to $S(\phi) := S(\{\phi\})$ as the **principal shift-invariant**, or **PSI** space, generated by ϕ. This terminology is used in the first of the following results both of which can be found in [BDR].

Result 2.4. *For any compactly supported finite set $\Phi\subset L^2(\mathbb{R})$, $S(\Phi)$ is the orthogonal sum of finitely many PSI spaces, each generated by a compactly supported function having linearly independent shifts.*

Result 2.5. *If the compactly supported finite set $\Phi\subset L^2(\mathbb{R})$ has linearly independent shifts and $\psi\in S(\Phi)$ is compactly supported, then*

$$\psi = \sum_{\phi\in\Phi}\sum_{j\in\mathbb{Z}}\phi(\cdot - j)a_{\phi}(j)$$

for some finitely supported $a\in\mathbb{C}^{\Phi\times\mathbb{Z}}$.

Result 2.4 guarantees, for any compactly supported finite set $\Phi\subset L^2(\mathbb{R})$, the existence of a compactly supported set $\Psi\subset L^2(\mathbb{R})$ having linearly independent shifts for which

$S(\Phi) = S(\Psi)$. Then Result 2.5 implies that each $\phi \in \Phi$ is a finite linear combination of the shifts of the elements of Ψ. In the Fourier domain, this implies the existence of a $\Phi \times \Psi$ matrix U of trigonometric polynomials satisfying

$$\widehat{\Phi} = U\widehat{\Psi}.$$

If $\operatorname{len} S(\Phi) = \#\Phi$, then U is a square matrix. Moreover, since Ψ has linearly independent shifts, the sequences (1.1) are linearly dependent at ξ if and only if $U(\xi)$ is singular. More specifically, for $\lambda \in \mathbb{C}^{\Phi}$,

$$\left(\lambda\widehat{\Phi}(\xi + 2j\pi)\right)_{j\in\mathbb{Z}} = 0 \Longleftrightarrow \lambda U(\xi) = 0. \tag{2.2}$$

Assuming that $\operatorname{len} S(\Phi) = \#\Phi$, the sequences (1.1) are linearly independent for almost every $\xi \in \mathbb{C}$. This implies that $\det U$ is a non-trivial trigonometric polynomial, which in turn implies that the matrix B, of *rational* trigonometric polynomials, is well-defined by the relation

$$U(2\cdot)B := AU. \tag{2.3}$$

This definition leads to the relation

$$U(2\cdot)\widehat{\Psi}(2\cdot) = \widehat{\Phi}(2\cdot) = A\widehat{\Phi} = AU\widehat{\Psi} = U(2\cdot)B\widehat{\Psi},$$

which implies that

$$\widehat{\Psi}(2\cdot) = B\widehat{\Psi} \tag{2.4}$$

almost everywhere. The characterization of $S(\Psi)$ given in [BDR] then implies that $\Psi(\cdot/2) \in S(\Psi)$. Since the shifts of Ψ are linearly independent, Result 2.5 implies that the entries of the matrix B are trigonometric polynomials. Now, an immediate consequence of (2.3) is that

$$\mathcal{A}_{n,k}(\omega)U(2^{k+1}\omega) = U(2^n\omega)\mathcal{B}_{n,k}(\omega) \tag{2.5}$$

for any $\omega \in \mathbb{C}$ and integers $n \geq k+1$.

2.4. Proof of main results

Proof of Theorem 2.1. To see that condition (i) is necessary for Φ to have independent shifts, note that every element of $2\mathbb{Z}\pi\backslash 0$ has the form $2^{n+1}(2k+1)\pi$ for some $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, and that

$$\widehat{\Phi}(2^{n+1}(2k+1)\pi) = A^n(0)A(\pi)\widehat{\Phi}((2k+1)\pi).$$

It is then clear that if A does not satisfy condition (i), then $\left(\lambda\widehat{\Phi}(2j\pi)\right)_{j\in\mathbb{Z}} = 0$. Also, if A does not satisfy condition (ii), then

$$\lambda\widehat{\Phi}(2\omega + 4j\pi) = \lambda A(\omega)\widehat{\Phi}(\omega + 2j\pi) = 0 \text{ and}$$
$$\lambda\widehat{\Phi}(2\omega + 2\pi + 4j\pi) = \lambda A(\omega+\pi)\widehat{\Phi}(\omega + \pi + 2j\pi) = 0 \text{ for all } j \in \mathbb{Z}.$$

So the sequences (1.1) are linearly dependent at $\xi = 2\omega$. Of course $\omega \in \mathbb{R} \Longrightarrow \xi \in \mathbb{R}$.

Now, if A does not satisfy condition (iii), we show that

$$\lambda \widehat{\Phi}(\omega + 2j\pi) \text{ is zero for all } j \in \mathbb{Z}.$$

Suppose ω is m-cyclic in $\mathbb{T}$ for some integer $m \geq 2$ and suppose $j \in \mathbb{Z}$ is given. Then it follows from equation (2.1) that

$$\begin{aligned}\lambda\widehat{\Phi}(\omega + 2j\pi) &= \lambda\widehat{\Phi}\left(2^{mn}\omega + 2^{mn-k}\nu\pi\right)\\ &= \lambda\mathcal{A}_{mn,k}(\omega)\widehat{\Phi}(2^{k+1}\omega + 2\nu\pi)\\ &= \lambda\mathcal{A}_{mn,k}(\omega)A(2^k\omega + \nu\pi)\widehat{\Phi}(2^k\omega + \nu\pi),\end{aligned}$$

for some $n \in \mathbb{N}$, $k \in \{1, 2, \ldots, m-1\}$, $\nu \in \mathbb{Z}\backslash 2\mathbb{Z}$. So $\lambda\widehat{\Phi}(\omega + 2j\pi)$ is zero if condition (iii) is not satisfied.

This completes the proof of necessity. To prove sufficiency, suppose that the shifts of Φ are dependent. Suppose, moreover, that A satisfies conditions (i) and (ii). We show that then (iii) is violated.

Assuming that the shifts of Φ are dependent, (2.2) implies the existence of $\lambda \in \mathbb{C}^{\Phi}\backslash 0$ and $\vartheta \in \mathbb{T} + i\mathbb{R}$ so that $\lambda U(\vartheta) = 0$. We will show that this implies that ϑ is cyclic.

Since $\lambda U(\vartheta) = 0$, equation (2.3) implies that

$$\lambda A\left(\frac{\vartheta}{2}\right)U\left(\frac{\vartheta}{2}\right) = \lambda U(\vartheta)B\left(\frac{\vartheta}{2}\right) = 0$$

and

$$\lambda A\left(\frac{\vartheta}{2}+\pi\right)U\left(\frac{\vartheta}{2}+\pi\right) = \lambda U(\vartheta)B\left(\frac{\vartheta}{2}+\pi\right) = 0.$$

Condition (ii) then implies that U is singular at either $\frac{\vartheta}{2}$ or $\frac{\vartheta}{2} + \pi$ (if the shifts of Φ are in fact not stable, then $\vartheta \in \mathbb{R}$). Since $\det U$ is a trigonometric polynomial, it has only finitely many zeros in $\mathbb{T} + i\mathbb{R}$, and the arguments used in the proof of [JW, Lemma 1] imply that $2^m\vartheta - \vartheta \in 2\mathbb{Z}\pi$ for some $m \geq 2$. We must still show that $\vartheta \notin 2\mathbb{Z}\pi$.

Suppose $\lambda U(0) = 0$, then $\lambda\widehat{\Phi}(0) = 0$ and (i) implies that $\lambda A^n(0)A(\pi)$ is not zero for some $n \in \mathbb{Z}_+$. Evidently,

$$\lambda A^n(0)A(\pi)U(\pi) = \lambda U(0)B^n(0)B(\pi) = 0.$$

But this is a contradiction, since π is acyclic, so $U(\pi)$ is invertible.

Accepting the existence of the integer $m \geq 2$ and the m-cyclic ϑ for which $\lambda U(\vartheta)$ is zero, we proceed to prove that this violates condition (iii).

Since

$$2^{mn}\omega - \omega = \frac{2^{mn}-1}{2^m-1}(2^m\omega - \omega) \in 2\mathbb{Z}\pi,$$

$\lambda U(\omega) = 0$ implies that $\lambda U\left(2^{mn}\omega\right) = 0$. By (2.5) and the 2π periodicity of U,

$$\begin{aligned}\lambda\mathcal{A}_{mn,k}(\omega)A(2^k\omega+\pi)U(2^k\omega+\pi) &= \lambda\mathcal{A}_{mn,k}(\omega)U(2^{k+1}\omega)B(2^k\omega+\pi)\\ &= \lambda U(2^{mn}\omega)\mathcal{B}_{mn,k}(\omega)B(2^k\omega+\pi)\\ &= 0.\end{aligned}$$

Since, $2^k\omega+\pi$ is acyclic, $U(2^k\omega+\pi)$ is invertible, so

$$\lambda\mathcal{A}_{mn,k}(\omega)A(2^k\omega+\pi) = 0$$

for every $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, m-1\}$. □

Proof of Corollary 2.3. We first point out that conditions (i) and (ii) of Corollary 2.3 clearly imply conditions (i) and (ii) of Theorem 2.1. Note, then, that it is sufficient to prove the corollary as it relates to stability. For, suppose that $\det A$ has no π-periodic zeros in $\mathbb{C}$, then it certainly has no π-periodic zeros in $\mathbb{R}$. If, moreover, condition (i) is satisfied then Theorem 2.1 implies that the sequences (1.1) are independent for all $\xi \in \mathbb{C}\backslash\mathbb{R}$. If condition (iii) is also satisfied, and we accept the stability statement of the corollary, then the shifts of Φ must in fact be independent.

So, suppose that (i) and (ii) are satisfied, and that the shifts of Φ are *not* stable. We will show that condition (iii) is violated.

The proof of Theorem 2.1 implies that the matrix U of (2.2) has at least one singularity, and that all such singularities are at points which are cyclic. Now, we show that for any ω which is cyclic,

$$\det U\left(2\omega\right) = 0 \Longrightarrow \begin{cases}\det U\left(\omega\right) = 0 \text{ and}\\ \det A\left(\omega+\pi\right) = 0.\end{cases} \tag{2.6}$$

First, equation (2.3) implies that

$$\det(A\left(\omega+\pi\right)U\left(\omega+\pi\right)) = \det(U\left(2\omega\right)B\left(\omega+\pi\right)) = 0.$$

And, since $\omega+\pi$ is acyclic, $U\left(\omega+\pi\right)$ is invertible. This implies that $\det A\left(\omega+\pi\right) = 0$. Condition (ii) then implies that

$$\det A\left(\omega\right) \neq 0. \tag{2.7}$$

Next, the definition of B also implies that

$$\det(A\left(\omega\right)U\left(\omega\right)) = \det(U\left(2\omega\right)B\left(\omega\right)) = 0$$

which together with (2.7) implies that $\det U\left(\omega\right) = 0$, proving (2.6).

Now, if U has a singularity at a point ω which is m-cyclic in $\mathbb{T}$, then $\det U(2^m\omega) = \det U(\omega) = 0$. Applying (2.6) repeatedly, shows that $\det A$ has a contaminating zero. □

2.5. Examples

Example 2.6. *This example demonstrates the necessity of the hypothesis* $\operatorname{len} S(\Phi) = \#\Phi$.

Let $v := (\,1 \quad 0\,)^T$ and $A(\omega) := \widetilde{A}(e^{-i\omega})$ where

$$\widetilde{A}(z) := \begin{pmatrix} \frac{z^2-z+2}{2} & -\frac{z+1}{4} \\ -z^2+z & \frac{z+1}{2} \end{pmatrix}.$$

Then

$$A(0) = \widetilde{A}(1) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

satisfies $A(0)v = v$. Now, $\det \widetilde{A}(z) = (z+1)/2$, which means that A is invertible except at π. Therefore, A satisfies the conditions (ii) and (iii) of Corollary 2.3. Moreover,

$$\left[\,\widehat{\Phi}(0) \quad A(\pi)\,\right] = \left[\,v \quad \widetilde{A}(-1)\,\right] = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

has full rank. The mask A satisfies all three conditions of Corollary 2.3. However, there is a solution $\Phi \subset L^2(\mathbb{R})$ which satisfies the refinement equation (1.2) for this A, which does not have independent (or even stable) shifts.

Let

$$\Phi := \begin{pmatrix} \frac{1}{2}\chi_{[0..2)} \\ \chi_{[0..1)} - \chi_{[1..2)} \end{pmatrix}.$$

Then

$$\widehat{\Phi}(\omega) = \begin{pmatrix} \frac{e^{-i2\omega}-1}{-i2\omega} \\ \frac{-e^{-i2\omega}+2e^{-i\omega}-1}{-i\omega} \end{pmatrix}$$

satisfies the refinement equation (1.2). Moreover,

$$\widehat{\Phi}\left((2j+1)\pi\right) = \begin{pmatrix} 0 \\ \frac{4}{(2j+1)\pi i} \end{pmatrix} \quad \text{for every } j \in \mathbb{Z},$$

demonstrating that the sequences (1.1) are linearly dependent for $\xi = \pi$. This implies that Φ does not have stable shifts. In this case, $S(\Phi) = S(\chi_{[0..1)})$. So $\operatorname{len} S(\Phi) < \#\Phi$. □

Example 2.7. *For this example, let* $v := (\,1 \quad 1\,)^T$ *and* $A(\omega) := \widetilde{A}(e^{-i\omega})$ *where*

$$\widetilde{A}(z) := \frac{1}{2}\begin{pmatrix} 1 & z \\ z & z^2 \end{pmatrix}.$$

If, for some $\omega \in \mathbb{C}$, $\lambda A(\omega) = 0$ then λ is a scalar multiple of $(\,e^{-i\omega} \quad -1\,)$. So if $\lambda A(\omega) = \lambda A(\omega+\pi) = 0$, then $\lambda = 0$. This implies that A satisfies condition (ii) of Corollary 2.2. Moreover,

$$\left[\,\widehat{\Phi}(0) \quad A(\pi)\,\right] = \begin{pmatrix} 1 & 1/2 & -1/2 \\ 1 & -1/2 & 1/2 \end{pmatrix}$$

has full rank, so A satisfies condition (i) of Corollary 2.2.

However, the shifts of Φ are not stable, since A does not satisfy condition (iii) of Corollary 2.2. To see this, note that $\frac{2\pi}{3}$ is 2-cyclic in $\mathbb{T}$. Defining $z_0 := e^{-i\pi/3}$, we have

$$A(\tfrac{\pi}{3}) = \tfrac{1}{2}\begin{pmatrix} 1 & z_0 \\ z_0 & z_0^2 \end{pmatrix}, \qquad A(\tfrac{2\pi}{3}) = \tfrac{1}{2}\begin{pmatrix} 1 & z_0^2 \\ z_0^2 & z_0^{-2} \end{pmatrix},$$
$$A(\tfrac{4\pi}{3}) = \tfrac{1}{2}\begin{pmatrix} 1 & z_0^{-2} \\ z_0^{-2} & z_0^2 \end{pmatrix}, \quad A(\tfrac{5\pi}{3}) = \tfrac{1}{2}\begin{pmatrix} 1 & z_0^{-1} \\ z_0^{-1} & z_0^{-2} \end{pmatrix}.$$

We also point out that

$$\mathcal{A}_{2n,k}\left(\frac{2\pi}{3}\right) A\left(2^k\frac{2\pi}{3}+\pi\right) = \begin{cases} M^{n-1}A\left(\frac{4\pi}{3}\right)A\left(\frac{5\pi}{3}\right) & \text{if } k=0, \text{ and} \\ M^{n-1}A\left(\frac{\pi}{3}\right) & \text{if } k=1, \end{cases}$$

where

$$M := A\left(\frac{4\pi}{3}\right)A\left(\frac{2\pi}{3}\right) = \frac{1}{2}\begin{pmatrix} 1 & z_0^2 \\ z_0^{-2} & 1 \end{pmatrix}.$$

Now, setting $\lambda := (\, z_0 \quad -1\,)$,

$$\lambda A\left(\frac{4\pi}{3}\right)A\left(\frac{5\pi}{3}\right) = \lambda A\left(\frac{\pi}{3}\right) = 0.$$

Moreover, $\lambda M = \lambda$. So

$$\lambda\mathcal{A}_{2n,k}\left(\frac{2\pi}{3}\right) A\left(2^k\frac{2\pi}{3}+\pi\right) = 0$$

for $k = 0$, 1 and any $n \in \mathbb{N}$. This violates condition (iii) of Corollary 2.2, so the shifts of Φ are not stable.

For completeness, we point out that

$$\Phi = \frac{2}{3}\begin{pmatrix} \chi_{[0..\frac{3}{2})} \\ \chi_{[\frac{1}{2}..2)} \end{pmatrix}$$

is a solution to the refinement equation (1.2) and indeed its shifts are not stable.

□

The above example, in which $\det A$ is identically zero, should not lead one to believe that this property implies the instability of the shifts of Φ. A counter-example to such a conjecture is provided by

$$\widetilde{A}(z) := \begin{pmatrix} \frac{1+z}{2} & 0 \\ \frac{1-z}{2} & 0 \end{pmatrix}, \quad v := \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

for which

$$\Phi := \begin{pmatrix} \chi_{[0..1)} \\ \chi_{[0..\frac{1}{2})} - \chi_{[\frac{1}{2}..1)} \end{pmatrix},$$

satisfying the refinement equation (1.2), has linearly independent shifts.

Example 2.8. *In this example, we use the results of this section to investigate the refinable functions constructed in [DGHM]. These were actually shown, in [DGHM], to have orthonormal shifts.*

Let $v := (\,\sqrt{2} \quad 1\,)^T$ and $A(\omega) := \widetilde{A}(e^{-i\omega})$ where

$$\widetilde{A}(z) := \frac{\sqrt{2}}{20}\begin{pmatrix} 6\sqrt{2}+6\sqrt{2}z & 16 \\ -1+9z+9z^2-z^3 & -3\sqrt{2}+10\sqrt{2}z-3\sqrt{2}z^2 \end{pmatrix}.$$

Then Φ is defined as the solution to the refinement equation (1.2). It follows that

$$\det \widetilde{A}(z) = -\frac{1}{40}(z+1)^3.$$

So conditions (ii) and (iii) of Corollary 2.3 are satisfied. Moreover, the matrix

$$[\,\widehat{\Phi}(0) \quad A(\pi)\,] = \begin{bmatrix} \sqrt{2} & 0 & 4\sqrt{2}/5 \\ 1 & 0 & -8/5 \end{bmatrix}$$

has full rank. So Corollary 2.3 implies that the shifts are linearly independent (as long as $\operatorname{len} S(\Phi) = 2$, which we will show in Example 3.16). □

3. Notes on length

3.1. Background

A **shift-invariant subspace** of $L^2(\mathbb{R})$ is a closed linear subspace S of $L^2(\mathbb{R})$ which is also **closed under shifts**, i.e., which, for each $\phi \in S$, also contains $\phi(\cdot - j)$ for every integer $j \in \mathbb{Z}$. A **finitely generated shift-invariant**, or **FSI** space, is then a space of the form

$$S(\Phi) := \overline{\operatorname{span}\{\, \phi(\cdot - j) \;:\; \phi \in \Phi,\, j \in \mathbb{Z}\,\}}$$

for some finite set $\Phi \subset L^2(\mathbb{R})$. The cardinality of a smallest generating set for an FSI space S is called the **length** of S. We write

$$\operatorname{len} S := \min\{\, \#\Phi \;:\; S = S(\Phi)\,\}.$$

This terminology and notation is from [BDR].

In Section 2, stability and linear independence of the shifts of a finite set of mutually refinable functions Φ were characterized in terms of the mask under the assumption that $\operatorname{len} S(\Phi) = \#\Phi$, i.e., that the FSI space generated by Φ could not be generated by any set consisting of fewer elements. However, no means for verifying this assumption by investigation of the mask were provided. In this section we give a characterization of this assumption in terms of the mask.

In [BDR], finitely generated shift-invariant subspaces of $L^2(\mathbb{R})$ were characterized as follows. For any finite $\Phi \subset L^2(\mathbb{R})$ and any $f \in L^2(\mathbb{R})$, $f \in S(\Phi)$ if and only if

$$\widehat{f} = \sum_{\phi \in \Phi} \tau_\phi \widehat{\phi} \tag{3.1}$$

for some 2π-periodic functions τ_ϕ. They say that Φ is a **basis** for $S(\Phi)$ if the functions τ_ϕ in (3.1) are uniquely determined by f. In this case, the functions τ_ϕ were shown to be rational trigonometric polynomials. They also prove that every finite set $\Phi \subset L^2(\mathbb{R})$ of *compactly supported* functions contains a basis for $S(\Phi)$. It follows that for any finite set $\Phi \subset L^2(\mathbb{R})$ of compactly supported functions, there exists $\Psi \subset \Phi$ and a $\Phi \times \Psi$ matrix U of rational polynomials, so that $\operatorname{rank} U = \#\Psi$ and

$$\widehat{\Phi}(\omega) = U(e^{-i\omega})\widehat{\Psi}(\omega) \text{ for almost every } \omega \in \mathbb{R}. \tag{3.2}$$

If Φ is a basis for $S(\Phi)$, then $\#\Phi = \operatorname{len} S(\Phi)$. It follows that $\operatorname{rank} U = \#\Psi = \operatorname{len} S(\Phi)$ in (3.2). In fact, for compactly supported Φ, Φ is a basis for $S(\Phi)$ if and only if $\operatorname{len} S(\Phi) = \#\Phi$.

The necessity of the conditions provided in this section will not require that $\Phi \subset L^2(\mathbb{R})$. The background we provide here dealing with this more general setting is primarily from [HC] and [CDP]. As with $\Phi \subset L^2(\mathbb{R})$, we say that the compactly supported finite set $\Phi \subset \mathcal{D}'(\mathbb{R})$ is refinable with mask A if A is a $\Phi \times \Phi$ matrix of trigonometric polynomials, v is a given right 1-eigenvector of $A(0)$, and $\Phi \subset \mathcal{D}'(\mathbb{R})$ is a compactly supported solution to

$$\widehat{\Phi}(2\cdot) = A\widehat{\Phi}; \qquad \widehat{\Phi}(0) = v. \tag{3.3}$$

Evidently,

$$\widehat{\Phi} = \prod_{j=1}^{n} A\left(\frac{\cdot}{2^j}\right) \widehat{\Phi}\left(\frac{\cdot}{2^n}\right)$$

for any $n \in \mathbb{N}$. Now, suppose that $\widehat{\Phi}$ is the only entire function satisfying (3.3) and that the limit

$$\Gamma := \lim_{n\to\infty} \prod_{j=1}^{n} A\left(\frac{\cdot}{2^j}\right) v \tag{3.4}$$

converges uniformly on compact sets. Clearly, $\Gamma(2\cdot) = A\Gamma$ and $\Gamma(0) = \widehat{\Phi}(0)$. Therefore, Γ must be the Fourier-Laplace transform of Φ.

When proving the necessity of the conditions provided in this section, I will assume that $\widehat{\Phi}$ is the only entire function satisfying (3.3). Sufficient conditions for this property are provided in [HC] and [CDP]. The following theorem from [CDP] provides sufficient conditions for the convergence of (3.4).

Result 3.9. *Suppose A is a square matrix of trigonometric polynomials. Suppose that v is a right 1-eigenvector of $A(0)$, i.e., $A(0)v = v \neq 0$. Suppose $\rho < 2$, where*

$$\rho := \rho(A(0)) := \max\{\, |\lambda| \ :\ A(0)u = \lambda u \text{ for some } u \neq 0 \,\}.$$

Then the limit (3.4) converges uniformly on compact sets.

Throughout this section, we use Π to denote the set of all polynomials of one variable with complex coefficients and we use $\mathcal{F}$ to denote the *field* of (univariate) rational polynomials with complex coefficients, i.e.,

$$\mathcal{F} := \{ f/g \ : \ f, g \in \Pi, \ g \neq 0 \}.$$

Then, for any set S, the set

$$\mathcal{F}^S := \{p : S \to \mathcal{F} : s \mapsto p_s\}$$

is a vector space (over the field $\mathcal{F}$).

We interpret any finite subset P of $\mathcal{F}^S$ also as a matrix $P \in \mathcal{F}^{S\times P}$ and *vice versa*; and We identify both of these with the linear map

$$P : \mathcal{F}^P \to \mathcal{F}^S : u \mapsto Pu := \sum_{p\in P} pu_p$$

We make no distinction between these three interpretations of P. For such P and any $z \in \mathbb{C}$, $P(z) \subset \mathbb{C}^S$, equivalently $P(z) \in \mathbb{C}^{S\times P}$, equivalently $P(z) : \mathbb{C}^P \to \mathbb{C}^S$ denotes P evaluated at z.

When $\#S < \infty$, the subspace $P^\perp \subset \mathcal{F}^S$ is defined for $P \subset \mathcal{F}^S$ by

$$P^\perp := \left\{ v \in \mathcal{F}^S \ : \ \langle p, v\rangle := \sum_{s\in S} p_s v_s = 0 \ \forall p \in P \right\}.$$

That is, $P^\perp = \ker P^T$, where

$$P^T : \mathcal{F}^S \to \mathcal{F}^P : v \mapsto P^T v := (\langle p, v\rangle)_{p\in P}\,.$$

Evidently, $\operatorname{rank} P + \dim P^\perp = \dim \mathcal{F}^S = \#S$ whenever $\#P < \infty$; and $P^{\perp\perp} = P$ whenever P is a subspace of $\mathcal{F}^S$.

We denote by σ, the map which takes any function p defined on $\mathbb{C}$ and maps it to the function $z \mapsto p(z^2)$. That is, for any function p with domain $\mathbb{C}$, σp also has domain $\mathbb{C}$ and is defined by the rule

$$(\sigma p)(z) := p(z^2) \quad \text{for all } z \in \mathbb{C}.$$

Moreover, if P is any set of functions on $\mathbb{C}$, then σP is defined by

$$\sigma P := \{ \sigma p \ : \ p \in P \}.$$

Evidently, $\sigma P \subset \Pi$ whenever $P \subset \Pi$ and $\sigma(AB) = (\sigma A)(\sigma B)$ whenever AB is well-defined.

Lastly, for any matrix A of rational trigonometric polynomials, I denote by $\widetilde{A}$ the matrix of rational polynomials defined by

$$\widetilde{A}(e^{-i\omega}) = A(\omega).$$

With these definitions in place, we begin our analysis with a

Lemma 3.10. *Suppose $\#S < \infty$. Suppose X is a subspace of $\mathcal{F}^S$. Then X has a basis $U \subset \Pi^S$. Moreover, such a basis exists with the property that* $\operatorname{rank} U(z) = \#U$ *for every $z \in \mathbb{C}$; equivalently,* $\det(U^T U)$ *is a non-zero constant.*

Proof. Any basis of X consists of finitely many vectors from $\mathcal{F}^S$. Since $\#S < \infty$, a basis from Π^S may be obtained simply by multiplying by a polynomial which is a common multiple of all denominators of a given basis. This proves the first assertion.

Next we show that such a basis $U \subset \Pi^S$, is everywhere of full rank if and only if the polynomial $p := \det(U^T U)$ is a non-zero constant. Clearly, $p(z) \neq 0$ if and only if $U(z)$ is of full rank. Moreover, the polynomial p is a non-zero constant polynomial if and only if it never vanishes, hence if and only if U is everywhere of full rank.

Now, suppose U is a basis for X. Then we can complete U to a basis $V =: [U \quad U']$ for $\mathcal{F}^S$. Clearly, if V is everywhere of full rank, then U is as well.

Suppose V is not everywhere of full rank. Specifically, suppose $\exists r \in \mathbb{C}$ such that $\det V(r) = 0$. Then, $\det V \in (\cdot - r)\Pi$. Moreover, since $V(r)$ is not of full rank, there exists $a \in \mathbb{C}^V \subset \mathcal{F}^V$ such that $V(r)a = 0$. Since $V \subset \Pi^S$ is a basis, $Va \neq 0$ (i.e., Va is not the zero element of $\mathcal{F}^S$). Therefore, $Va \in (\cdot - r)\Pi^S \backslash 0$.

If $a_v \neq 0$ for some $v \in U'$, then we can replace v in U' with the *polynomial* vector $Va/(\cdot - r)$. This obviously has no effect on the basis U for X, while it reduces the degree of $\det V$ by one (indeed, $\det V$ has been multiplied by $a_v/(\cdot - r)$), since $\det V \neq 0$.

If on the other hand, $a|_{U'} = 0$, then there is some $v \in U$ for which $a_v \neq 0$. Again, replacing v in U with $Va/(\cdot - r)$, reduces the degree of $\det V$ by one. Moreover, although we have altered the basis U, the resulting set is still a basis for X, since $a_v \neq 0$, while $a|_{U'} = 0$. □

3.2. Length of an FSI space

In this section we provide conditions on the mask A which are sufficient to conclude that $\operatorname{len} S(\Phi) = \#\Phi$ under the assumption that $\Phi \subset L^2(\mathbb{R})$. These conditions are shown to be also necessary if the mask satisfies the hypotheses of Result 3.9. The assumption $\Phi \subset L^2(\mathbb{R})$ is not required to conclude the necessity of the conditions.

In the satement of the following theorem, we assume that A is a square matrix of trigonometric polynomials; that 1 is an eigenvalue of $A(0)$; and that $v \neq 0$ is a right 1-eigenvector of $A(0)$. We assume moreover, that $\Phi \subset L^2(\mathbb{R})$ is a (compactly supported) solution to the refinement equation (3.3).

Theorem 3.11. *If $\ell := \operatorname{len} S(\Phi) < \#\Phi$, then there exists $P \subset \Pi^\Phi$ with $\#P = \operatorname{rank} P = \#\Phi - \ell$, and such that*
(i) $P^T(1)\widehat{\Phi}(0) = 0$, *while* $\operatorname{rank} P(1) = \#P$; *and*
(ii) $(\sigma P^T)\widetilde{A}P^\perp = 0$.

Proof. Suppose $\ell := \operatorname{len} S(\Phi) < \#\Phi$. Then there exists $\Psi \subset \Phi$, as in (3.2), with $\#\Psi = \ell$, and such that

$$\widehat{\Phi} = U(e^{-i\cdot})\widehat{\Psi}$$

for some $U \in \mathcal{F}^{\Phi \times \Psi}$. Moreover, $\operatorname{rank} U = \ell$.

Let $P \subset \Pi^\Phi$ be a basis for $U^\perp$. Then $\#P = \operatorname{rank} P = \#\Phi - \ell$. And since $P^TU = 0$,

$$P^T(1)\widehat{\Phi}(0) = P^T(1)U(1)\widehat{\Psi}(0) = 0.$$

Moreover, by Lemma 3.10, we may assume that $P^T(1)$ is of full rank.

Also,

$$P^T(e^{-i2\cdot})AU(e^{-i\cdot})\widehat{\Psi} = P^T(e^{-i2\cdot})A\widehat{\Phi} = P^T(e^{-i2\cdot})\widehat{\Phi}(2\cdot) = (P^TU)(e^{-i2\cdot})\widehat{\Psi}(2\cdot) = 0,$$

again since $P^TU = 0$. Now, Ψ is compactly supported and $\operatorname{len} S(\Psi) = \ell = \#\Psi$, so Ψ is a basis for $S(\Psi)$. Therefore $P^T(e^{-i2\cdot})AU(e^{-i\cdot}) = 0$. Equivalently, $(\sigma P)^T\widetilde{A}U = 0$. Since U is of full rank, it is a basis for $U^{\perp\perp} = P^\perp$, so

$$(\sigma P)^T\widetilde{A}P^\perp = 0.$$

□

To deal with necessity, we drop the assumption that $\Phi \subset L^2(\mathbb{R})$. The theorem below is valid for general compactly supported distributions. We will not specify precisely what is meant by $S(\Phi)$ in this case. However, the proof of the theorem below provides, under certain assumptions on the mask A, a set Ψ satisfying

(i) $\#\Psi < \#\Phi$;
(ii) Ψ is a finite linear combination of the shifts of Φ; and
(iii) Φ is a finite linear combination of the shifts of Ψ.

Regardless of the definitions of $S(\Phi)$ and $\operatorname{len} S$, it is reasonable to say that points (i), (ii), and (iii) imply that $\operatorname{len} S(\Phi) \le \#\Psi < \#\Phi$.

In the statement of the following theorem, we assume that A satisfies the hypotheses of Result 3.9. That is, we assume that 1 is an eigenvalue of $A(0)$; that $v \neq 0$ is a right 1-eigenvector; and that the spectral radius $\rho(A(0)) < 2$. We assume moreover that $\widehat{\Phi}$ is the only entire function satisfying (3.3).

Theorem 3.12. *If there exists $P \subset \Pi^\Phi$ with $\operatorname{rank} P = \#P$ and satisfying conditions (i) and (ii) of Theorem 3.11, then $\operatorname{len} S(\Phi) \le \#\Phi - \#P$.*

Proof. We will provide Ψ with $\#\Psi = \#\Phi - \#P$, so that the elements of Φ are finite linear combinations of the shifts of the elements of Ψ and *vice versa.*

By Lemma 3.10, there exists a basis $U \subset \Pi^\Phi$ for $P^\perp$ with the properties that $U(z)$ has full rank everywhere and $\det(U^TU)$ is a constant. This second property implies (say by Cramer's rule) that $(U^TU)^{-1}$, and hence the left inverse

$$U^{-L} := (U^TU)^{-1}U^T,$$

of U, also consist of polynomials. Since $\operatorname{rank} P = \#P$, $\#U = \#\Phi - \#P$.

Now, condition (ii) implies that $\widetilde{A}P^\perp \subset \sigma P^\perp$. And since U is a basis for $P^\perp$, this implies the existence of $\widetilde{B} \in \mathcal{F}^{U\times U}$ for which

$$\widetilde{A}U = (\sigma U)\widetilde{B}.$$

We claim that B also satisfies the hypotheses of Result 3.9.

Evidently, B has trigonometric polynomial entries, since $\widetilde{B} = (\sigma U^{-L})\widetilde{A}U$. Also, $\widehat{\Phi}(0) \in P^{\perp}(1)$ by condition (i), so $\widehat{\Phi}(0) = U(1)v$ for some $v \in \mathbb{C}^{U}$. For this v,

$$U(1)\widetilde{B}(1)v = \widetilde{A}(1)U(1)v = \widetilde{A}(1)\widehat{\Phi}(0) = \widehat{\Phi}(0) = U(1)v.$$

And since $U(1)$ is of full rank, it follows that

$$B(0)v = v.$$

Now suppose $B(0)u = \lambda u$ for some $u \neq 0$. Then $U(1)u \neq 0$, while

$$\widetilde{A}(1)U(1)u = U(1)\widetilde{B}(1)u = \lambda U(1)u.$$

That is, every eigenvalue of $B(0)$ is also an eigenvalue of $A(0)$. Therefore,

$$\rho(B(0)) \leq \rho(A(0)) < 2.$$

By Result 3.9,

$$\Gamma := \lim_{n\to\infty} \prod_{j=1}^{n} B\left(\frac{\cdot}{2^j}\right) v$$

converges uniformly on compact sets. Moreover, Γ is an entire function satisfying $\widehat{\Phi}(0) = U(1)\Gamma(0)$, as well as

$$U(e^{-i2\cdot})\Gamma(2\cdot) = U(e^{-i2\cdot})B\Gamma = AU(e^{-i\cdot})\Gamma.$$

Since we assumed that $\widehat{\Phi}$ was the only entire function satisfying (3.3), we must have

$$\widehat{\Phi} = U(e^{-i\cdot})\Gamma. \tag{3.5}$$

Consequently, $\Gamma = U^{-L}(e^{-i\cdot})\widehat{\Phi}$. Since U^{-L} consists of polynomials, this implies that Γ is the Fourier-Laplace transform of some compactly supported set $\Psi \subset \mathcal{D}'(\mathbb{R})$ which is a finite linear combination of the shifts of $\phi \in \Phi$. Then, (3.5) implies that the elements of Φ are also finite linear combinations of the shifts of the elements of Ψ. Since $\#\Psi = \#U = \#\Phi - \#P$, the proof is complete. $\square$

3.3. Two and three functions

Suppose $\#\Phi = 2$, then the results of Section 3.2 take the following simpler form, in which we also have a bound for $\deg P$. As we are assuming that $\widehat{\Phi}(0)$ is not zero, $\operatorname{len} S(\Phi)$ is either one or two. For $\Phi \subset L^2(\mathbb{R})$, the sufficient conditions then simplify to

Corollary 3.13. *Suppose $\widetilde{A} =: (a_{ij})$ is a two-by-two matrix of polynomials. Suppose $v \neq 0$ is a right 1-eigenvector of $A(0)$. Suppose $\Phi \subset L^2(\mathbb{R})$ is a solution to the refinement equation (3.3). If $\operatorname{len} S(\Phi) = 1$, then there exist polynomials p_1 and p_2 satisfying*

$$\deg p_1 \leq \max\{\deg a_{22}, \frac{\deg a_{21} + \deg p_2}{2}\}$$
$$\deg p_2 \leq \max\{\deg a_{11}, \frac{\deg a_{12} + \deg p_1}{2}\},$$

and such that
(i) $p_1(1)v_1 + p_2(1)v_2 = 0$, while one of $p_1(1)$ or $p_2(1)$ is non-zero; and
(ii)
$$\begin{pmatrix} p_1(z^2) & p_2(z^2) \end{pmatrix} \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} \begin{pmatrix} p_2(z) \\ -p_1(z) \end{pmatrix} = 0.$$

We point out that the bounds on the degrees of p_1 and p_2 could be replaced with the simpler bounds

$$\deg p_1 \leq \max\left\{\deg a_{22}, \frac{\deg a_{11} + \deg a_{21}}{2}, \frac{2\deg a_{21} + \deg a_{12}}{3}\right\}$$
$$\deg p_2 \leq \max\left\{\deg a_{11}, \frac{\deg a_{22} + \deg a_{12}}{2}, \frac{\deg a_{21} + 3\deg a_{12}}{3}\right\}$$

or the even simpler bounds $\deg p_k \leq \max_{i,j}\{\deg a_{ij}\}$ which, though weaker than the bounds in the statement of the theorem, more clearly display that the degrees of p_1 and p_2 can be bounded *a priori*.

Proof. It is enough to provide the bounds for $\deg p_1$ and $\deg p_2$ (the rest follows immediately from Theorem 3.11). We may assume that p_1 and p_2 are relatively prime. Then condition (ii) implies that

$$\sigma p_1 \text{ divides } a_{22}p_1 - a_{21}p_2, \text{ and}$$
$$\sigma p_2 \text{ divides } a_{11}p_2 - a_{12}p_1.$$

The bounds on $\deg p_1$ and $\deg p_2$ follow. □

The results of Section 3.2 take a simple form in the case $\#\Phi = 3$ as well. Now there are two possibilities, $\operatorname{len} S(\Phi) = 1$ or $\operatorname{len} S(\Phi) = 2$. Note that the *three* equations in condition (ii) of the next two corollaries are inherently redundant. However, since we don't know which if any of the p_k might be zero, we have no choice but to consider all three equations. To handle the possibility that $\operatorname{len} S(\Phi) = 1$, we offer the

Corollary 3.14. *Suppose $\widetilde{A} =: (a_{ij})$ is a three-by-three matrix of polynomials. Suppose $v \neq 0$ is a right 1-eigenvector of $A(0)$. Suppose $\Phi \subset L^2(\mathbb{R})$ is a solution to the refinement equation (3.3). If $\operatorname{len} S(\Phi) = 1$, then there exist polynomials p_1, p_2 and p_3 satisfying*

$$\begin{aligned}
\deg p_1 &\leq \max\left\{\deg a_{11}, \frac{\deg a_{12} + \deg p_2}{2}, \frac{\deg a_{13} + \deg p_3}{2}\right\}\\
\deg p_2 &\leq \max\left\{\frac{\deg a_{21} + \deg p_1}{2}, \deg a_{22}, \frac{\deg a_{23} + \deg p_3}{2}\right\}\\
\deg p_3 &\leq \max\left\{\frac{\deg a_{31} + \deg p_1}{2}, \frac{\deg a_{32} + \deg p_2}{2}, \deg a_{33}\right\}
\end{aligned}$$

and such that

(i) $p_2(1)v_3 - p_3(1)v_2 = p_3(1)v_1 - p_1(1)v_3 = p_1(1)v_2 - p_2(1)v_1 = 0$, while not all of $p_1(1)$, $p_2(1)$, and $p_3(1)$ are zero; and

(ii)

$$\begin{pmatrix} 0 & -p_3(z^2) & p_2(z^2) \\ p_3(z^2) & 0 & -p_1(z^2) \\ -p_2(z^2) & p_1(z^2) & 0 \end{pmatrix} \begin{pmatrix} a_{11}(z) & a_{12}(z) & a_{13}(z) \\ a_{21}(z) & a_{22}(z) & a_{23}(z) \\ a_{31}(z) & a_{32}(z) & a_{33}(z) \end{pmatrix} \begin{pmatrix} p_1(z) \\ p_2(z) \\ p_3(z) \end{pmatrix} = 0.$$

Again we point out that the bounds above imply that $\deg p_k \leq \max_{i,j}\{\deg a_{ij}\}$.

Proof. Assuming, as we may, that the polynomials p_1, p_2, and p_3 are relatively prime, condition (ii) implies that

$$\begin{aligned}
&\sigma p_1 \text{ divides } a_{11}p_1 + a_{12}p_2 + a_{13}p_3,\\
&\sigma p_2 \text{ divides } a_{21}p_1 + a_{22}p_2 + a_{23}p_3, \text{ and}\\
&\sigma p_3 \text{ divides } a_{31}p_1 + a_{32}p_2 + 1_{33}p_3.
\end{aligned}$$

The bounds given now follow easily. The rest follows from Theorem 3.11. □

To handle the possibility that $\operatorname{len} S(\Phi) = 2$, we make the additional assumption that $\det A$ is not identically zero. For this case, we offer the

Corollary 3.15. *Suppose $\widetilde{A} =: (a_{ij})$ is a three-by-three matrix of polynomials, for which $\det \widetilde{A}$ is not identically zero. Suppose $v \neq 0$ is a right 1-eigenvector of $A(0)$. Suppose $\Phi \subset L^2(\mathbb{R})$ is a solution to the refinement equation (3.3). If $\operatorname{len} S(\Phi) = 2$, then there exist polynomials p_1, p_2, and p_3 satisfying $\deg p_k \leq 2\max_{i,j} \deg a_{ij}$ for $k = 1$, 2, and 3, and such that*

(i) $p_1(1)v_1 + p_2(1)v_2 + p_3(1)v_3 = 0$, while one of $p_1(1)$, $p_2(1)$, or $p_3(1)$ is non-zero; and

(ii)

$$\begin{pmatrix} p_1(z^2) & p_2(z^2) & p_3(z^2) \end{pmatrix} \begin{pmatrix} a_{11}(z) & a_{12}(z) & a_{13}(z) \\ a_{21}(z) & a_{22}(z) & a_{23}(z) \\ a_{31}(z) & a_{32}(z) & a_{33}(z) \end{pmatrix} \begin{pmatrix} 0 & -p_3(z) & p_2(z) \\ p_3(z) & 0 & -p_1(z) \\ -p_2(z) & p_1(z) & 0 \end{pmatrix} = 0.$$

Proof.
Let $M_A := (m_{ij})$ be the matrix of cofactors of $\widetilde{A}$. That is,

$$m_{ji} := (-1)^{i+j} \det \left(a_{i'j'}\right)_{i' \neq i, j' \neq j} \qquad (i, j = 1, 2, 3).$$

Since $\det \widetilde{A}$ is not identically zero, $\widetilde{A}$ is almost everywhere invertible, and

$$M_A^T = (\det \widetilde{A}) \widetilde{A}^{-1}.$$

Moreover, $\deg m_{ij} \leq 2 \max_{i',j'} \{\deg a_{i'j'}\}$ for $i, j = 1, 2, 3$.

By Theorem 3.11, there exists $P^T =: \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}$ such that

$$\sigma P^T \widetilde{A} = u P^T$$

for some $u =: f/g \in \mathcal{F}$. This in turn implies that

$$\frac{g \det \widetilde{A}}{f} \sigma P^T = P^T M_A.$$

As we may assume that p_1, p_2, and p_3 are relatively prime and as the elements of M_A are polynomials, f must divide $g \det \widetilde{A}$ and, in turn, σP^T divides $P^T M_A$. This then implies that every element of P is of degree less than $\max_{i,j}\{\deg m_{ij}\} \leq 2 \max_{i,j}\{\deg a_{ij}\}$. $\square$

3.4. Example

Example 3.16. *We complete the analysis of the refinable functions from [DGHM] begun in Example 2.8.*

Recall that

$$\widetilde{A}(z) = \frac{\sqrt{2}}{20} \begin{pmatrix} 6\sqrt{2} + 6\sqrt{2}z & 16 \\ -1 + 9z + 9z^2 - z^3 & -3\sqrt{2} + 10\sqrt{2}z - 3\sqrt{2}z^2 \end{pmatrix}, \qquad v = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}.$$

By Corollary 3.13, if $\operatorname{len} S(\Phi) = 1$, then there would be polynomials p_1 and p_2 satisfying

$$\begin{pmatrix} p_1(z^2) & p_2(z^2) \end{pmatrix} \widetilde{A}(z) \begin{pmatrix} p_2(z) \\ -p_1(z) \end{pmatrix} = 0 \tag{3.6}$$

as well as $\sqrt{2}p_1(1) + p_2(1) = 0$. The bounds provided there for the degrees of these polynomials implies that $\deg p_1 \leq 2$ and $\deg p_2 \leq 1$. We use `Maple` to try to find such p_1 and p_2. We let

$$p_1(z) =: a_0 + a_1 z + a_2 z^2 \text{ and } p_2(z) =: b_0 + b_1 z.$$

The following `Maple` code then sets up these two conditions and attempts to solve for the coefficients a_0, a_1, a_2, b_0, b_1.

```
> A:=sqrt(2)/20*matrix([[6*sqrt(2)*(1+z),16],
> [-1+9*z+9*z^2-z^3, sqrt(2)*(-3+10*z-3*z^2)]]);

> v:=matrix([[sqrt(2)],[1]]);

> p1:=a0+a1*z+a2*z^2; p2:=b0+b1*z;

> sPT:=matrix([[subs(z=z^2,p1),subs(z=z^2,p2)]]);

> Pperp:=matrix([[p2],[-p1]]);

> condi:=subs(z=1,evalm(sPT&*v))[1,1];

> exprii:=evalm(sPT&*A&*Pperp)[1,1];

> condii:=coeffs(collect(exprii,z),z);

> solve({condi,condii},{a0,a1,a2,b0,b1});
```

`Maple`'s response is

$$\{b_0 = 0,\ a_2 = 0,\ b_1 = 0,\ a_1 = 0,\ a_0 = 0\}$$

$$\{b_1 = b_1,\ a_2 = \frac{1}{2}\sqrt{2}b_1,\ a_0 = -\frac{1}{2}\sqrt{2},\ a_1 = 0,\ b_0 = -b_1\}$$

Corresponding to $p_1 = p_2 = 0$; and $p_1 = \frac{\sqrt{2}}{2}(z^2 - 1)$, $p_2 = z - 1$, respectively. However, neither of these choices for p_1, p_2 satisfies the condition that $(\,p_1(1) \quad p_2(1)\,)$ be of full rank. Therefore, $\operatorname{len} S(\Phi) \neq 1$.

It is worth noting that the polynomials $p_1(z) = \frac{\sqrt{2}}{2}(1 + z)$ and $p_2(z) = 1$ satisfy condition (3.6). Moreover $P^T(1) = (\,p_1(1) \quad p_2(1)\,) = (\,\sqrt{2} \quad 1\,)$ has full rank. However, $P^T(1)v = 3 \neq 0$. Multiplying p_1 and p_2 by $z - 1$ does not effect (3.6) and clearly makes $P^T(1)v = 0$. But in this case, $P^T(1) = (\,0 \quad 0\,)$ is not of full rank. So this example also illustrates that all three conditions are necessary to conclude that $\operatorname{len} S(\Phi) < \#\Phi$. □

References

[BDR] Boor, C. de, R. DeVore, and A. Ron (1994), "The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$", *J. Funct. Anal.* **119**, 37–78.

[CDP] Cohen, A., I. Daubechies, and G. Plonka (1995), "Regularity of refinable function vectors", preprint.

[DGHM] Donovan, G. C., J. S. Geronimo, D. P. Hardin, and P. R. Massopust (1996), "Construction of orthogonal wavelets using fractal interpolation functions", *SIAM J. Math. Anal.* **27**, 1791–1815.

[GHM] Geronimo, J. S., D. P. Hardin, and P. R. Massopust (1994), "Fractal functions and wavelet expansions based on several scaling functions", *J. Approx. Theory* **78**, 373–401.

[GL] Goodman, T. N. T. and S. L. Lee (1994), "Wavelets of multiplicity r", *Trans. Amer. Math. Soc.* **342**, 307–324.

[GLT] Goodman, T. N. T., S. L. Lee, and W. S. Tang (1993), "Wavelets in wandering subspaces", *Trans. Amer. Math. Soc.* **338**, 639–654.

[H] Hogan, T. (1995), "Stability and independence of the shifts of a multivariate refinable function", CMS. Techn. Rep. 95-07.

[HKM] Hardin, D. P., B. Kessler, and P. R. Massopust (1992), "Multiresolution analyses based on fractal functions", *J. Approx. Theory* **71**, 104–120.

[HC] Heil, C. and D. Colella (1996), "Matrix refinement equations: existence and uniqueness", *J. Fourier Anal. Appl.* **2**, 363–377.

[HSS] Heil, C., G. Strang, and V. Strela (1996), "Approximation by translates of refinable functions", *Numer. Math.* **73**, 75–94.

[JM1] Jia, Rong-Qing and C. A. Micchelli (1992), "On linear independence for integer translates of a finite number of functions", *Proc. Edinburgh Math. Soc.* **36**, 69–85.

[JM2] Jia, Rong-Qing and C. A. Micchelli (1991), "Using the refinement equations for the construction of pre-wavelets II: powers of two", in *Curves and Surfaces* (P.-J. Laurent, A. Le Méhauté, and L. L. Schumaker, eds), Academic Press (New York), 209–246.

[JW] Jia, Rong-Qing and Jianzhong Wang (1993), "Stability and linear independence associated with wavelet decompositions", *Proc. Amer. Math. Soc.* **117**, 1115–1124.

[P] Plonka, G. (to appear), "Approximation order of shift-invariant subspaces of $L^2(\mathbb{R})$ generated by refinable function vectors", *Constr. Approx.* **xxx**, xx–xx.

[SS] Strang, G. and V. Strela (1995), "Finite element multiwavelets", in *Approximation Theory, Wavelets and Applications* (S. P. Singh, Antonio Carbone, and B. Watson, eds), Kluwer (Dordrecht, Netherlands), 485–496.