QUASIINTERPOLANTS AND APPROXIMATION POWER OF MULTIVARIATE SPLINES

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ABSTRACT. The determination of the approximation power of spaces of multivariate splines with the aid of quasiinterpolants is reviewed. In the process, a streamlined description of the existing quasiinterpolant theory is given.

1. Approximation power of splines

I begin with a brief review of the approximation power of univariate splines since the techniques for its investigation are also those with which people have tried to understand the multivariate setup. (That may in fact be the reason why we know so little about it.) I will then briefly discuss three examples to illustrate some basic limitations of the standard univariate approach.

Let $S := S_{k,\mathbf{t}}$ be the univariate space of splines of **order** k with **knot sequence** $\mathbf{t}$. This means that

$$S := \operatorname{span} \Phi$$

with

$$\Phi := \big(\varphi_i\big)_{i=1}^n,$$

$\varphi_i := M(\cdot|t_i, \ldots, t_{i+k})$ the normalized B-spline for the knots $t_i, \ldots, t_{i+k}$, and $\mathbf{t} := (t_j)_{j=1}^{n+k}$ a nondecreasing real sequence. This definition of a spline space is taylor-made for the consideration of its approximation power, since the B-spline basis

$$\Phi : \ell_\infty(n) \to S : c \mapsto \Phi c := \sum_{i=1}^n \varphi_i \, c(i)$$

is so well-behaved. (I have found it convenient to identify the *sequence* $(\varphi_1, \ldots, \varphi_n)$ with the *map* $c \mapsto \sum_i \varphi_i c(i)$.) We consider specifically approximation from S to $X := C([a,b])$, with

$$[a,b] := [t_k, t_{n+1}]$$

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the interval of interest. In the corresponding norm

$$\|f\| := \{\sup |f(x)| : a \le x \le b\},$$

the basis (map) Φ satisfies

$$\|\Phi\| = \sup_c \|\Phi c\|/\|c\|_\infty = 1$$

(the result of the fact that the φ_i are nonnegative and sum to 1, i.e., form a **partition of unity**). Since any linear map Q on X into S has the form $Q =: \sum_i \varphi_i \lambda_i$ for suitable linear functionals λ_i, it follows that

$$\|Q\| \le \|\lambda\| := \max_i \|\lambda_i\|. \tag{1.1}$$

The φ_i have local support. Therefore, such a map Q is **local** to the extent that the λ_i are local. In [B68$_1$], I chose

$$\operatorname{supp} \lambda_i \subset \operatorname{supp} \varphi_i = [t_i, t_{i+k}],$$

and will take this to be the meaning of the statement 'Q is **local**'. This implies, more precisely than (1.1), that

$$\|Qf\|_{[t_i,t_{i+1}]} \le \|\lambda\| \; \|f\|_{[t_{i+1-k},t_{i+k}]}. \tag{1.2}$$

The only additional feature needed to make Q a useful approximation scheme from S is *positive (polynomial) order* where by the **(polynomial) order** of any such map Q I mean the largest integer m for which

$$Q = 1 \quad \text{on } \pi_{<m} := \operatorname{span}(()^j)_{j<m}.$$

(Strictly speaking, our Q is only defined on $C([a,b])$, hence I should talk about $(\pi_{<m})_{|[a,b]}$ instead of $\pi_{<m}$, but pedantry can be overdone.)

It follows that, for any f and any $p \in \pi_{<m}$ and any $I := [t_i, t_{i+1}]$,

$$(f - Qf)(I) = \big((1-Q)(f-p)\big)(I),$$

hence

$$\|(f - Qf)(I)\| \le \|1 - Q\| \; \|(f-p)_{[t_{i+1-k},t_{i+k}]}\|,$$

with $\|1-Q\|$ boundable by $1 + \|Q\|$, hence by $1 + \|\lambda\|$. Consequently,

$$\operatorname{dist}(f, S) \le (1 + \|Q\|) \max_i \operatorname{dist}(f, \pi_{<m})_{[t_{i+1-k},t_{i+k}]}. \tag{1.3}$$

In particular, for a sufficiently smooth f,

$$\operatorname{dist}(f, \pi_{<m})_{[c,c+h]} = O(\|D^m f\| h^m),$$

therefore

$$\operatorname{dist}(f, S) = O(\|Q\| \; \|D^m f\| \, |\mathbf{t}|^m), \tag{1.4}$$

with

$$|\mathbf{t}| := \max_i \Delta t_i.$$

This formulation is careless. Offhand, the knot sequence $\mathbf{t}$ enters here not only in the **meshsize** $|\mathbf{t}|$ but also in the $\|Q\|$, since the actual construction of suitable Q is bound to involve the knot sequence. There are many ways to construct local Q (choose, e.g., each λ_i to be evaluation at some point in the support of φ_i). There are also many ways to construct Q of any order $\le k$ (choose, e.g., spline interpolation, or least-squares approximation from S). The existence of a Q which is local *and* of order k was first proved, for odd k and for $X = C^k([a,b])$ and without the aid of B-splines, in [Bi67], with a corrected version, which also covered even k, to be found in [B68$_2$]. For the present situation, i.e., for $X = C([a,b])$, such Q were first constructed in [B68$_1$]. The λ_i there were even chosen to be linear combinations of point evaluations. Finally, it was shown there that the λ_i could be so chosen that $\|\lambda\|$ was boundable independent of $\mathbf{t}$ (depending only on the order k of the spline space). The essential part of the argument in [B68$_1$] is the observation that the B-spline basis is locally well conditioned, i.e., that

$$1/d_k := \inf_{\mathbf{t}} \min_i \operatorname{dist}(\varphi_i, \operatorname{span}(\varphi_j)_{j\ne i})_{[t_i,t_{i+k}]} > 0.$$

By Hahn-Banach, this entitles one to believe in the existence of λ_i on X **dual to** Φ, i.e., satisfying $\lambda_i\varphi_j = \delta_{i-j}$, with $\operatorname{supp}\lambda_i \subset [t_i, t_{i+k}]$, and with $\|\lambda\| \le d_k$. The resulting Q is therefore not only of order k, it is actually the identity on all of S, i.e., it is a **linear projector** onto S. Since $\pi_{<k} \subseteq S$, this provides the bound

$$\operatorname{dist}(f,S) = O(d_k\|D^k f\|)|\mathbf{t}|^k, \tag{1.5}$$

in which the order term $O(\cdot)$ is independent of the knot sequence $\mathbf{t}$.

While [B68$_1$] contains only a recipe for the construction of suitable λ_i, [BF73] establishes the formula

$$\lambda_i : f \mapsto \sum_{j<k} (-D)^{k-1-j}\psi_i(\tau_i)(D^j f)(\tau_i) \tag{1.6}$$

$$\psi_i := (t_{i+1} - \cdot)\cdots(t_{i+k-1} - \cdot)/(k-1)!$$

for the **dual functionals** for the normalized B-splines (with $\tau_i \in]t_i, t_{i+k}[$ arbitrary). While one may object to the use of derivative information here, the formula makes it easy to extend the coordinate functionals of the B-spline basis to linear functionals with more desirable characteristics (such as applicability to $\mathbf{L}_p$ functions [B74], or employing only function values [LS75]).

On the other hand, one may choose to ignore the formula (1.6) and choose λ_i explicitly so that $Q = 1$ on $\pi_{<m}$. This requires each λ_i to be an extension from $\pi_{<m}$ only, of the linear functional which associates with each $p \in \pi_{<m}$ its ith B-spline coefficient. If each extension happens to have its support in an interval containing no knots in its interior and if $m = k$, then the resulting Q will be the identity on all of S.

The specifics just discussed provide an instance of the following more abstract situation: We wish to approximate from a given **directed** family (S_h) of function spaces in $C(G)$, for some compact domain $G \subset \mathbb{R}^d$. Specifically, we are interested in

$$\operatorname{dist}(f, S_h)$$

as a function of the parameter h whose corresponding **meshsize** $|h|$ we think of as going to zero (and it is in this sense that we think of (S_h) as 'directed'). We are able to exhibit a **good quasiinterpolant** Q_h **of polynomial order** m, i.e., a linear map Q_h on $C(G)$ into S_h of (polynomial) order m (meaning that $Q_h = 1$ on $\pi_{<m}$) which is **uniformly local** in the sense that

$$|(Q_h f)(x)| \le \mathrm{const}\|f_{|B_{r|h|}(x)}\|$$

for all $x \in G$, for some constants const and r, and with $B_r(x)$ the ball of radius r around x. This implies the error bound

$$\mathrm{dist}(f, S_h) \le \mathrm{const}_f |h|^m$$

valid for all 'sufficiently smooth' f and so establishes the **approximation order from** (S_h) to be (at least) m. Further, we are able to establish m to be the (exact) approximation order from (S_h) by exhibiting a particular 'sufficiently smooth' function f for which

$$\mathrm{dist}(f, S_h) \ne o(|h|^m).$$

Finally, we are able to choose Q_h to be even a projector onto S_h.

This abstraction has motivated much of the work on the approximation power of multivariate splines. But, before starting that discussion here, I want to bring three cautionary examples.

Example 1 This example appears in [DR8x]. It concerns the space S_h spanned by the $h\mathbb{Z}$-translates of the piecewise linear function

$$\varphi_h : x \mapsto \begin{cases} x+1, & 0 \le x < h; \\ 0, & \text{otherwise.} \end{cases}$$

Thus S_h consists of certain piecewise linear functions, with breakpoint sequence $h\mathbb{Z}$, but the only constant function it contains is the constant 0. In particular, it is not possible to construct a quasiinterpolant of positive order for it. Nevertheless, the approximation

$$Q_h f := \sum_{j \in h\mathbb{Z}} \varphi_h(\cdot - j) f(j)$$

has the error

$$f - Q_h f = f - \sum_{j \in h\mathbb{Z}} \chi_h(\cdot - j) f(j) \quad + \quad \sum_{j \in h\mathbb{Z}} (\chi_h - \varphi_h)(\cdot - j) f(j),$$

with χ_h the characteristic function of the interval $[0, h[$, hence $\|\chi_h - \varphi_h\| = h$. Therefore

$$\|f - Q_h f\| \le \omega_f(h) + \|f\| h,$$

with ω_f the modulus of continuity of f. It follows that $Q_h f$ converges to f uniformly in case f is uniformly continuous and bounded.

This example strongly stresses that, in the earlier argument involving good quasiinterpolants, positive order is too strong a condition. In fact, as is evident from the argument's details, it is sufficient to demand that the good quasiinterpolant Q_h be of positive **local** order m, meaning that

$$Q_h f = f + O(\|f\|\,|h|^m)$$

on any ball of radius $O(|h|)$ and for any $f \in \pi_{<m}$. A sufficient condition for this is that $Q_h = 1$ on a D-invariant space F_h of entire functions which is locally close to $\pi_{<m}$ in the sense that its 'limit at the origin', $(F_h)_\downarrow$, contains $\pi_{<m}$. (Here, $F_\downarrow := \operatorname{span}\{f_\downarrow : f \in F\}$, with $f_\downarrow$ the first nonzero term in the expansion $f = f_0 + f_1 + f_2 + \cdots$ of f into homogeneous polynomials f_j of degree j, all j.)

This example also illustrates the limits of the Strang-Fix conditions (see Section 4). For it shows that (S_h) has positive approximation order even though none of the S_h contains π_0. In fact, $\cap_h S_h = \{0\}$.

Example 2 This disturbing example comes from [BH83]. The space $S := \pi^1_{3,\Delta}$ of C^1-cubics on the *three-direction mesh* Δ contains all cubic polynomials. It even contains them *locally* in the sense that any cubic polynomial on one of the triangles of the partition Δ can be extended to an element of S with compact support. However, the expectation raised by this that the approximation order from $(S_h := \pi^1_{3,h\Delta})$ is 4 will be disappointed. There are polynomials in π_4 for which $\operatorname{dist}(f, \pi^1_{3,h\Delta}) \neq o(h^3)$. This indicates that it is not sufficient to find out which polynomials are contained in S_h. One needs to know that they are contained in S_h in a local *and stable* way.

Example 3 In [BHS87], Hermite interpolation to planar curves by parametric piecewise cubic curves is investigated. The curves being piecewise cubic, one would expect an approximation order of 4, i.e., an error of $O(|h|^4)$, with $|h|$ a measure of the spacing of the length of the cubic pieces. But, in fact, the scheme described there is shown to approximate smooth curves (without inflection points) to $O(|h|^6)$. Further, an example is given there of a convex smooth curve with a flat spot to which the scheme approximates only within $O(|h|^4)$. Finally, [S8x] shows that the scheme can be appropriately modified to approximate to smooth curves with simple inflection points to within $O(|h|^6)$.

Now add to this the fact that the approximation scheme used is **nonlinear**. This means that the basic trick of the above argument, viz. the introduction of a local polynomial approximation to f, is not readily applicable. Still, the only ready means for estimating approximation order is the introduction of a local polynomial approximation. Now add to this the additional difficulty that there is no natural way of measuring the distance between curves, other than their Hausdorff distance which is apt to set up a not very smooth map between the points of the two curves.

The essential facts in the proof of the $O(|h|^6)$ error turned out to be that (i) the scheme is local, and (ii) each cubic curve piece matches the given curve in six independent pieces of information. This is sufficient to show that, for sufficiently small $|h|$, the difference between each cubic piece and the given curve, measured in a suitable local direction (i.e., interpreting both as (graphs of) functions in some suitable local coordinate system), is $O(|h|^6)$, with a constant that can be bounded in terms of the local radius of curvature of the given curve.

All in all, it is again the use of a local and stable scheme of some positive order that supplies the approximation order, but the argument is much harder, and is unsatisfactory precisely because it takes recourse to functions. It seems pretty hopeless to try to settle approximations of surfaces by surfaces in this way. Hence, although this is a meeting on computing with curves and surfaces, I will stick to the approximation of multivariate functions.

2. Quasiinterpolant construction for a scale

We now commence the discussion of quasiinterpolants and approximation order for multivariate spline spaces.

The simplest model for a family (S_h) of approximating spaces is that of a **scale**, i.e.,

$$S_h := \sigma_h(S)$$

for some fixed space S, with

$$\sigma_h f : x \mapsto f(x/h)$$

and $h > 0$. The simplest nontrivial model for S investigated is that of the span of the integer translates of a compactly supported function. This means that S is taken to be of the form

$$S(\varphi) := \operatorname{ran} \varphi* := \{\varphi * c : c \in \mathbb{C}^{\mathbb{Z}^d}\}.$$

This description makes use of the convolution

$$\varphi * c := \sum_{\alpha \in \mathbb{Z}^d} \varphi(\cdot - \alpha) c(\alpha)$$

of the compactly supported function φ with the complex-valued sequence or meshfunction c, i.e., $c : \mathbb{Z}^d \to \mathbb{C}$. Since φ is compactly supported, the infinite sums $\varphi * c$ converge trivially uniformly on compact sets, and it is in this sense that I mean to interpret them.

I now consider the problem of constructing a good quasiinterpolant of polynomial order m for $S(\varphi)$. Following [BH82], it has become customary to construct such a quasiinterpolant in the form

$$Q : f \mapsto \sum_{\alpha} \varphi(\cdot - \alpha)\, \lambda f(\cdot + \alpha) \tag{2.1}$$

for some suitable local bounded linear functional λ. With the notation

$$\varphi *' f := \varphi * f_| = \sum_{\alpha} \varphi(\cdot - \alpha) f(\alpha)$$

for the **semi-discrete** convolution, which uses the helpful abbreviation

$$f_| := f_{|\mathbb{Z}^d},$$

and the notation

$$\Lambda f : x \mapsto \lambda f(\cdot + x),$$

we can write (2.1) more simply as

$$Q : f \mapsto \varphi *' \Lambda f.$$

With both λ and φ compactly supported, the corresponding family $Q_h := \sigma_h Q \sigma_{1/h}$ is uniformly local, hence establishes approximation order m for the scale $(S(\varphi)_h)$ provided $Q = 1$ on $\pi_{<m}$.

Assuming that $\pi_{<m} \subset S(\varphi)$, this raises the question of just how one might construct Q of the form (2.1) so that it is of polynomial order m. Consider the following slightly more general problem: Construct λ so that Q of (2.1) is the identity on the finite-dimensional linear space F.

Such a space F must necessarily lie in $S(\varphi)$. Further, since Q is to be the identity on it, we might as well assume that F is E-invariant, meaning that

$$E^\alpha F \subset F \quad \forall(\alpha \in \mathbb{Z}^d),$$

with E the **shift**, i.e.,

$$E^\alpha f : x \mapsto f(x+\alpha).$$

For, the shift commutes with $\varphi *'$ and with Λ, hence with Q, and this implies that $Q = 1$ on the smallest E-invariant space containing F. (This would imply that F is even D-invariant, i.e., invariant under differentation, in case $F \subset \pi$; cf. [B87]).

It follows that F is also $\varphi *'$-invariant, as is any E-invariant subspace of $S(\varphi)$, since

$$\varphi *' f = f *' \varphi \qquad \forall(f = \varphi * c \in S(\varphi)). \tag{2.2}$$

(Here are the details from [B87]: $\varphi *' f = \varphi * (\varphi_| * c) = \varphi * (c * \varphi_|) = (\varphi * c) * \varphi_| = f *' \varphi$). This means that the linear map

$$T := \varphi *'{}_{|F}$$

carries F into itself.

Assume, in addition, that T is 1-1, hence invertible. Then any λ which is an extension of the linear functional

$$\lambda_0 : F \to \mathbb{C} : f \mapsto (T^{-1}f)(0) \tag{2.3}$$

provides $Q = 1$ on F, since for such a λ and for any $f \in F$ and any $\alpha \in \mathbb{Z}^d$,

$$\lambda f(\cdot + \alpha) = (T^{-1}E^\alpha f)(0) = (E^\alpha T^{-1} f)(0) = (T^{-1}f)(\alpha)$$

(using the fact that T commutes with E, hence so does T^{-1}), and therefore $\Lambda f = T^{-1}f$ on $\mathbb{Z}^d$, consequently $Qf = TT^{-1}f = f$.

If the translates of φ are linearly independent, then every $f \in F$ has a unique representation in the form $\varphi * c$, hence $Q = 1$ on F only if λ is an extension of the functional $\lambda_0 = [0]T^{-1}$. Here and below, $[x]$ denotes the linear functional $[x] : f \mapsto f(x)$ of point-evaluation.

Since the action of Q on F is decided entirely by the values the linear functionals $f \mapsto \lambda f(\cdot + \alpha)$ take on F, it is easy to enlarge the class of available quasiinterpolants by considering (as already cited earlier from the univariate literature, but see also [CD88] and [BeR8x]), more generally,

$$Q : f \mapsto \sum_\alpha \varphi(\cdot - \alpha)\, \lambda_\alpha f$$

with $\lambda_\alpha = [0]T^{-1}E^\alpha = [\alpha]T^{-1}$ on F.

This leaves the question of just how one might actually construct suitable λ. I take up this question in Section 5, after a discussion of the Strang-Fix conditions. But it is convenient here to discuss quickly the special case when F is a polynomial space. Since, by (2.2), for any $p \in \pi \cap S(\varphi)$,

$$\varphi *' p = p *' \varphi \in p \sum_\alpha \varphi(\alpha) + \pi_{<\mathrm{degp}},$$

T is 1-1 if and only if $\sum_\alpha \varphi(\alpha) \neq 0$. With this assumption, we may assume without further loss of generality that φ is, in fact, **normalized**, i.e., that

$$\sum_\alpha \varphi(\alpha) = 1. \tag{2.4}$$

Then, the linear functional

$$\tau := [0]\varphi*' = \sum_\alpha \varphi(-\alpha)[\alpha]$$

is a finite weighted sum of function values and takes the value 1 at the constant function 1. This makes it possible to construct numerically its associated Appell sequence (p_α) (see the appendix for complete relevant details on Appell polynomials). This sequence is characterized by the fact that $\forall(\alpha, \beta)\ \tau D^\alpha p_\beta = \delta_{\alpha-\beta}$. This implies that $p_\alpha \in \pi_\alpha$, hence the sequence can be constructed numerically from the numbers $\varphi_|$ by recursion:

$$p_\alpha = [\![]\!]^\alpha - \sum_{\beta<\alpha} \tau([\![]\!]^{\alpha-\beta})\, p_\beta. \tag{2.5}$$

Here, $[\![]\!]^\alpha : x \mapsto x^\alpha/\alpha!$ is the **normalized power function**. The Appell sequence is of interest here since ([B87])

$$\varphi *' p_\alpha = [\![]\!]^\alpha$$

for all $[\![]\!]^\alpha \in S(\varphi)$. This implies that, for all such α,

$$\lambda_0 [\![]\!]^\alpha = p_\alpha(0),$$

thus providing us with the matrix representation of λ_0 with respect to the normalized power basis (if any) for F. In particular, since $[0]D^\alpha [\![]\!]^\beta = \delta_{\alpha-\beta}$, this provides immediately the extension

$$\lambda := \sum_\alpha p_\alpha(0)\, [0]D^\alpha$$

of λ_0.

If F does not have a normalized power basis, or is not known exactly, then it is useful to observe that T is the restriction of $\varphi_|*$ to F, where we now interpret $\varphi_|$ as the **linear functional** $f \mapsto \sum_\alpha \varphi(\alpha) f(\alpha) = \tau f(-\cdot)$, hence $(\varphi_| * f)(x) = \varphi_| f(x-\cdot) = \sum_\alpha f(x-\alpha)\varphi(\alpha)$ and

$$\varphi_| * p_\alpha = [\![]\!]^\alpha \quad \forall(\alpha \in \mathbb{Z}^d).$$

The assumption (2.4) guarantees that $\varphi_|*$ is invertible on any E-invariant polynomial space and, in particular, on any superspace π_k for F. This implies that $\sum_{|\alpha|\le k} p_\alpha(0)\,[0]D^\alpha$ extends $[0](\varphi_|*)|_{\pi_k}^{-1}$, hence also extends λ_0.

This observation also makes apparent the following comment concerning the numbers $p_\alpha(0)$. If, as is certainly the case here, τ is a compactly supported distribution, then (cf. Appendix)

$$p_\alpha = [\![\cdot -iD]\!]^\alpha (1/\widehat{\phi}_\tau)(0) = \sum_{\gamma\le\alpha} [\![]\!]^{\alpha-\gamma}\ [0][\![-iD]\!]^\gamma (1/\widehat{\phi}_\tau). \tag{2.6}$$

This means that

$$p_\alpha(0) = [\![-iD]\!]^\alpha (1/\widehat{\phi}_\tau)(0),$$

where, in our case,

$$\widehat{\phi}_\tau = \widehat{\varphi}_| = \widetilde{\varphi} = \sum_\alpha \varphi(\alpha) e^{-i\alpha()}$$

is a trigonometric polynomial, the **symbol** of φ. In particular cases, it might be easier to compute the first few terms in the Taylor expansion of its reciprocal directly rather than by the recurrence (2.5).

3. Characterization of local approximation order

The Strang-Fix conditions originally served (see [SF73] and references there) to characterize the 'controlled' approximation order from the scale $(S(\varphi)_h)$. The surprising aspect of this result was the claim that having $\pi_{<m} \subset S(\varphi)$ was **necessary** if $(S(\varphi)_h)$ was to have approximation order m. The conclusion drawn from this (perhaps too eagerly) was that we might as well restrict ourselves to piecewise polynomial φ. Yet, as the example from [DR8x] quoted earlier (and, in fact, the results in [DR8x] on the approximation power of $S(\varphi)$ when φ is a piecewise exponential) indicate, there are perfectly reasonable 'directed' families (S_h) with positive approximation power which contain not a single (nontrivial) polynomial.

Following a suggestion by Babuška, [SF73] considered the more general problem when

$$S = S(\Phi) := \sum_{\varphi\in\Phi} S(\varphi)$$

for some finite set Φ of compactly supported functions. The basic, and perhaps surprising, result is that, in effect, nothing new happens: The 'local' approximation order (defined after the statement of the Theorem below) of such a scale is the best that can be had from any $S(\psi)$ with ψ in

$$S(\Phi)^{\mathrm{loc}} := \{\sum_{\varphi} \varphi * c_\varphi : \#\operatorname{supp} c_\varphi < \infty\}.$$

In other words, even for $S(\Phi)$, the 'local' approximation order is realizable by a quasiinterpolant of the simple form $\sum_\alpha E^{-\alpha}\psi\lambda E^\alpha$ discussed in the previous section. But, in contrast to the case when Φ is a singleton, there is at present no computational procedure for the construction of a suitable ψ or for the determination of the approximation order.

The univariate case and the experience with the simple scale $(S(\varphi)_h)$ related in the previous section gave rise to the hope that the approximation order of $(S(\Phi)_h)$ would be the largest m for which $\pi_{<m} \subset S(\Phi)$. Even the determination of such an m would be nontrivial, but less involved than finding a best-possible ψ. Unfortunately, any such hope was dashed in [BH83] where it is shown that the approximation order of the scale obtained from the space of C^1-cubics on the three-direction mesh is only 3 even though its subspace $S(\Phi)$ with Φ consisting of the two box-splines M_{221} and M_{122} contains $\pi_{<4}$.

[SF73] as well as [DM84] speak of 'controlled' $\mathbf{L}_p$-approximation order and mean by that the largest m so that, for all sufficiently smooth functions f,

$$\operatorname{dist}_p(f, S(\Phi)_h^f) \le \operatorname{const}\|D^m f\|_p(B_{rh}(G))\, h^m,$$

with

$$S(\Phi)_h^f := \{\sigma_h \sum_{\varphi\in\Phi} \varphi * c_\varphi : \sum_{\varphi\in\Phi} \|c_\varphi\|_p \le \operatorname{const}\|f\|_p(B_{rh}(G))/h^{d/p}\} \tag{3.1}$$

for some const and r independent of h or f. Here, the distance is to be measured in the p-norm on the underlying domain G. Jia gave an example in [J84] to show that, contrary to the assertions in [SF73], the 'controlled' approximation order defined this way for the case $G = \mathbb{R}^d$ cannot be characterized by the Strang-Fix conditions. Jia's example does not contradict the claims made in [DM84] (in reliance on [SF73]) since there the above 'controlled' approximation order condition is assumed to hold for *all* domains G, with const and r also independent of G. In fact, these claims are verified in [BJ85] where the following corrected version of the Strang-Fix theorem is proved.

(3.2)Theorem. *Let Φ be a finite collection of compactly supported essentially bounded functions on $\mathbb{R}^d$. Then the following conditions are equivalent.*

(i) *For some sequence $(\psi_\alpha)_{|\alpha|<m}$ in $\operatorname{span}\Phi$,*

(ia) $\widehat{\psi}_0(0) = 1,\ \widehat{\psi}_0 = 0$ on $2\pi\mathbb{Z}^d\backslash 0$;

(ib) $\sum_{\beta\le\alpha} [\![-iD]\!]^\beta \widehat{\psi}_{\alpha-\beta} = 0$ on $2\pi\mathbb{Z}^d\backslash 0$ *for* $0 < |\alpha| < m$.

(ii) *For some sequence $(\psi_\alpha)_{|\alpha|<m}$ in $\operatorname{span}\Phi$,*

$$[\![]\!]^\alpha - \sum_{\beta\le\alpha} \psi_{\alpha-\beta} *' [\![]\!]^\beta \in \pi_{<|\alpha|} \qquad \forall(|\alpha| < m). \tag{3.3}$$

(iii) *For some $\psi \in S(\Phi)^{\text{loc}}$,*

$$[\![]\!]^\alpha - \psi *' [\![]\!]^\alpha \ \in\ \pi_{<|\alpha|} \qquad \forall(|\alpha| < m). \tag{3.4}$$

(iv) *For all $p \in [1,\infty]$, Φ provides local $\mathbf{L}_p$-approximation order m.*

(v) *For some $p \in [1,\infty]$, Φ provides local $\mathbf{L}_p$-approximation order m.*

Several comments are in order.

The statement 'Φ provides local $\mathbf{L}_p$-approximation order m' is meant as an abbreviation for the condition that, for all $f \in \mathbf{L}_p^m(\mathbb{R}^d)$,

$$\operatorname{dist}_p(f, S(\Phi)_h^f) \le \text{const}\|D^m f\|_p\, h^m$$

with

$$S(\Phi)_h^f := \{\sigma_h \sum_{\varphi\in\Phi} \varphi * c_\varphi : \operatorname{dist}(jh, \operatorname{supp} f) > r \implies c_\varphi(j) = 0\} \tag{3.5}$$

and for some const and r independent of h or f. Here, all norms are the $\mathbf{L}_p$-norm on $\mathbb{R}^d$. Note that the claimed equivalence between (iv) and (v) makes it possible to drop the qualifier $\mathbf{L}_p$- and talk simply of the **local approximation order** provided by Φ.

The proof of (i)$\Longrightarrow$(ii) uses Poisson's summation formula, in the manner detailed in the next section, the proof of (ii)$\Longrightarrow$(iii) obtains ψ in the form $\sum_{|\beta|<m} \psi_\beta * c_\beta$ with c_β finitely supported sequences for which $[\![]\!]^\gamma * c_\beta = [\![]\!]^\beta$ for all $\beta \le \gamma := (m,\ldots,m)$, while the proof of (iii)$\Longrightarrow$(iv) uses the material detailed in the preceding section. The main point of [BJ85] is a proof of the implication (v)$\Longrightarrow$(i) which had been missing in the earlier literature. The argument uses a smooth, compactly supported function f whose Fourier transform satisfies

$$\widehat{f}(0) = 1, \qquad D^\alpha \widehat{f}(x/h) = O(h^m) \quad \forall(x \in \mathbb{R}^d\backslash 0,\ |\alpha| < m). \tag{3.6}$$

(The specific function f used happens to be the d-fold tensor product of the centered B-spline of order $m+1$, hence its Fourier transform is the d-fold tensor product of Whittaker's sinc function $t \mapsto \sin(t/2)/(t/2)$, but such detail doesn't matter.) Condition (v) provides an approximation $f_h = \sigma_h \sum_\varphi \varphi * c_\varphi^h$ to f which has compact support, uniformly in h, and for which

$$\epsilon_h := f - f_h = O(h^m)$$

in whatever p-norm we happen to use. It follows that its Fourier-Laplace transform $\widehat{\epsilon} : z \mapsto \int_{\mathbb{R}^d} e^{-izx}\epsilon(x)dx$ satisfies

$$|\widehat{\epsilon}(z)| = O(h^m) \quad \text{ uniformly on } \|\operatorname{Im} z\| \le \text{const}$$

for some const, hence Cauchy's formula supplies the estimate

$$\|D^\alpha \widehat{\epsilon}\|_\infty(\mathbb{R}^d) \le \operatorname{const} h^m. \tag{3.7}$$

Consequently, the function

$$\widehat{f}_h =: \sum_\varphi \widehat{\varphi}(h\cdot) v_\varphi^h(h\cdot)$$

satisfies the conditions (3.6) (with $\widehat{f}$ replaced by $\widehat{f}_h$) as $h \to 0$. Here,

$$v_\varphi^h : z \mapsto h^d \sum_j e^{-izj} c_\varphi^h(j)$$

is 2π-periodic, therefore

$$[\![-iD]\!]^\alpha \widehat{f}_h(2\pi j/h) = h^{|\alpha|} \sum_\varphi \sum_\beta [\![-iD]\!]^\beta \widehat{\varphi}(2\pi j) [\![-iD]\!]^{\alpha-\beta} v_\varphi^h(0).$$

Combining (3.6) and (3.7), we conclude that

$$\lim_{h\to 0} \sum_\varphi \widehat{\varphi}(0) v_\varphi^h(0) = 1 \tag{3.8}$$

and, for all $j \in \mathbb{Z}^d \backslash 0$ and all $|\alpha| < m$,

$$\lim_{h\to 0} \sum_\varphi \sum_\beta [\![-iD]\!]^\beta \widehat{\varphi}(2\pi j) [\![-iD]\!]^{\alpha-\beta} v_\varphi^h(0) = 0. \tag{3.9}$$

Define now

$$\psi_\gamma := \sum_\varphi \varphi w_{\varphi,\gamma},$$

with w taken from the orthogonal complement $W\perp$ of the space W of all 'sequences' $w = (w_{\varphi,\gamma})$ for which

$$\lim_{h\to 0} \sum_\varphi \sum_{|\gamma|<m} w_{\varphi,\gamma} [\![-iD]\!]^\gamma v_\varphi^h(0) = 0.$$

We may choose w so that $\widehat{\psi}_0(0) = 1$, since $\widehat{\psi}_0(0) = 0$ for all $w \perp W$ would imply that the sequence $(\varphi, \gamma) \mapsto \widehat{\phi}(0)\delta_\gamma$ is in $W\perp\perp = W$, thus $\lim_{h\to 0} \sum_\varphi \widehat{\varphi}(0) v_\varphi^h(0) = 0$ and this would contradict (3.8). It follows from (3.9) that the sequence $(\varphi, \gamma) \mapsto [\![-iD]\!]^{\alpha-\gamma} \widehat{\varphi}(2\pi j)$ is in W for all $|\alpha| < m$ and all $j \in 2\pi\mathbb{Z}^d \backslash 0$, hence

$$\sum_\beta [\![-iD]\!]^\beta \widehat{\psi}_{\alpha-\beta}(2\pi j) = \sum_\beta [\![-iD]\!]^\beta \sum_\varphi (2\pi j) w_{\varphi,\alpha-\beta} = 0,$$

and this finishes the proof.

The proof (i)$\Longrightarrow$ (ii) $\Longrightarrow$(iii) supports the stronger claims that the displayed functions in (ii) and (iii) are necessarily in $\pi_{<\alpha}$.

The condition actually used in [SF73] instead of (ii) is the existence of some sequence $(\psi_\alpha)_{|\alpha|<m}$ in span Φ for which

$$[\![\,]\!]^\alpha = \sum_{\beta\le\alpha} \psi_{\alpha-\beta} *' [\![\,]\!]^\beta \qquad \forall(|\alpha| < m). \tag{3.10}$$

Although seemingly stronger, this is actually equivalent to (ii), as can be seen directly as follows: There is a unique linear map F on $\pi_{<m}$ which carries $[\![\,]\!]^\alpha$ to $\sum_{\beta\le\alpha} \psi_{\alpha-\beta} *' [\![\,]\!]^\beta$. By (ii), F carries $\pi_{<m}$ into itself and is degree-preserving, hence must be 1-1, hence must be invertible. This is all happening on a finite-dimensional linear space, therefore $p(F) = 1$ for some polynomial p. But, for any polynomial p, $p(F)$ carries $[\![\,]\!]^\alpha$ to $\sum_{\beta\le\alpha} \chi_{\alpha-\beta} *' [\![\,]\!]^\beta$ for certain χ_γ expressible as linear combinations of the ψ_γ.

The theorem characterizes the *local* approximation order from the scale $(S(\Phi)_h))$. The only results available so far on the approximation order itself have been obtained by producing an *upper bound* on the approximation order which happened to coincide with the local approximation order. This is so when Φ consists of a single box spline (cf. [BH82]), and is often so when Φ consists of all the different box splines in $\pi^\rho_{k,\Delta}$ with Δ the three-direction mesh (cf. [J86]).

4. The Strang-Fix conditions

The Strang-Fix conditions arose in the characterization of the approximation power of the scale (S_h) when $S = S(\varphi)$. According to [SF73; Theorem I] and roughly speaking, the scale $(S(\varphi)_h)$ has approximation order m if and only if $\pi_{<m} \subset S(\varphi)$. In the proof, the basic tool is the Fourier transform

$$\widehat{\varphi} : \xi \mapsto \int e^{-i\xi x} \varphi(x) dx,$$

with $\xi x := \langle \xi, x \rangle = \sum_j \xi(j)\overline{x}(j)$ the scalar product. Since φ has compact support, $\widehat{\varphi}$ is an entire function. As shown in [SF73], the following conditions

$$\widehat{\varphi}(0) = 1 \tag{4.1a}$$

$$\forall(\beta \le \alpha) \;\; D^\beta \widehat{\varphi} = 0 \quad \text{on} \quad 2\pi \mathbb{Z}^d \backslash 0, \tag{4.1b}$$

called the **Strang-Fix conditions of index** α these days (cf., e.g., [C88]), imply that

$$\pi_\alpha := \operatorname{span}\big(()^\beta\big)_{\beta \le \alpha} \;\subset\; S(\varphi).$$

[DM83] prove that, more generally, any affinely invariant subspace P of

$$\{p \in \pi : p(D)\widehat{\varphi} = 0 \text{ on } 2\pi\mathbb{Z}^d \backslash 0\} \tag{4.2}$$

is necessarily in $S(\varphi)$. This is indeed a generalization since the polynomial space π_α is, in particular, affinely invariant, i.e., scale- and translation-invariant. Their proof, as does the argument in [SF73], involves the **semidiscrete** convolution

$$\varphi *' f := \varphi * (f_|) = \sum_{\beta \in \mathbb{Z}^d} \varphi(\cdot - \beta) f(\beta),$$

in which

$$f_| := f_{|\mathbb{Z}^d}.$$

It is observed in [B87] that the assumption of affine invariance can be weakened to E-invariance, i.e., to the assumption that

$$\forall(\alpha \in \mathbb{Z}^d) \;\; E^\alpha P \subset P,$$

with E the **shift**, i.e.

$$E^\alpha f : x \mapsto f(x + \alpha).$$

Explicitly, [B87; Prop.2.2] proves that, more generally (and without the assumption that $\widehat{\varphi}(0) \ne 0$),

$$\{p \in \pi : \varphi *' p = p *' \varphi\} \tag{4.3}$$

is the largest E-invariant subspace of

$$\Pi_\varphi := \{p \in \pi : p(-iD)\widehat{\varphi} = 0 \text{ on } 2\pi\mathbb{Z}^d \backslash 0\}.$$

Note that the possible lack of scale-invariance (cf. [R8x] for a φ for which (4.3) fails to be scale-invariant) forces the switch from $p(D)$ in (4.2) to $p(-iD)$ here. Note further that (4.3) is the collection of polynomials in the kernel of the **commutator** of φ, i.e., of the map

$$f \mapsto [\varphi | f] := \varphi *' f - f *' \varphi$$

singled out in [CJW87] as an object of interest.

Finally, [BR8x] prove the following more general and suggestive result, in which

$$\pi_\varphi$$

is, by definition, the largest E-invariant subspace of Π_φ.

(4.4)Proposition. *For any compactly supported (measurable) function φ on $\mathbb{R}^d$ and any $f \in \pi$, the following conditions are equivalent:*

(a) $\quad f \in \pi_\varphi$;

(b) $\quad \varphi *' f = \varphi * f$;

(c) $\quad \varphi *' f \in \pi$.

Proof. [BR8x] use Poisson's summation formula (as did [SF73], [DM83], etc). But since the proposition imposes no continuity requirement on φ, the formula cannot be applied directly. In this sense, the argument for the corresponding Proposition 2.2 in [B87] is incomplete. This technical point is handled in [BR8x] by proving (b) to hold when applied to test functions. This makes the proof valid even when φ is only a compactly supported distribution. Here is the proof, for completeness.

For any compactly supported test function u,

$$(\varphi *' f)(u) = \sum_\alpha f(\alpha)\varphi(\cdot - \alpha)(u) =: \sum_\alpha \psi(\alpha),$$

with $\psi : x \mapsto f(x)\varphi(\cdot - x)(u)$ also a compactly supported test function, and therefore, by Poisson's summation formula,

$$(\varphi *' f)(u) = \sum_\alpha \hat{\psi}(2\pi\alpha), \tag{4.5}$$

with

$$\hat{\psi} = f(-iD)\big(\hat{\varphi}(-\cdot)\hat{u}\big). \tag{4.6}$$

On the other hand,

$$(\varphi * f)(u) = \int\int \varphi(y-x)f(x)dx\,u(y)dy = \hat{\psi}(0). \tag{4.7}$$

If now $f \in \pi_\varphi$, then $D^\beta f \in \Pi_\varphi$ for all β, hence $\hat{\psi}(2\pi\alpha) = 0$ for all $\alpha \in \mathbb{Z}^d \backslash 0$ since, by (4.6) and the Leibniz-Hörmander identity (cf. Appendix),

$$\hat{\psi}(\xi) = \sum_\beta \big(D^\beta f(-iD)\big)\,\hat{\varphi}(-\xi)\ (-iD)^\beta \hat{u}(\xi)/\beta!. \tag{4.8}$$

Therefore, from (4.5) and (4.7),

$$(\varphi *' f)(u) = \hat{\psi}(0) = (\varphi * f)(u),$$

showing that (a) $\Longrightarrow$ (b). The implication (b) $\Longrightarrow$ (c) is trivial since $\varphi * \pi \subset \pi$. Finally, if (c) holds, then the linear functional $\widehat{u} \mapsto (\varphi *' f)(u)$ has support only at the origin. Since the collection of linear functionals $\{u \mapsto (-iD)^\beta \widehat{u}(2\pi\alpha) : \beta, \alpha \in \mathbb{Z}^d\}$ is globally linearly independent over the space of compactly supported test functions, this implies with (4.5) and (4.8) that f and all its derivatives must belong to Π_φ, thus showing that (c) $\Longrightarrow$ (a). ♠

One recovers [B87;Prop.2.2] from (4.4)Proposition with the aid of the observation (already made there) that $\varphi *' f$ and $f *' \varphi$ agree on $\mathbb{Z}^d$ and $f *' \varphi$ is a polynomial in case

$f \in \pi$, hence agrees with $\varphi *' f$ iff $\varphi *' f \in \pi$. More importantly, (4.4)Proposition brings in an important point that simplifies considerably the construction of quasiinterpolants, viz. the equality

$$\varphi *' \; = \; \varphi * \quad \text{ on } \; \pi_\varphi \tag{4.9}$$

which suggests the construction of quasiinterpolants of the form

$$Q_\mu := \varphi *' \mu * : f \mapsto \sum_\alpha \varphi(\cdot - \alpha)\mu f(\alpha - \cdot) \tag{4.10}$$

with the compactly supported distribution μ chosen so that $\mu*$ represents the inverse of $\varphi*$ on π_φ. I pursue this point in the next section.

As pointed out in [BR8x], (4.4)Proposition has a ready extension to exponential f, i.e., to

$$f \in \mathrm{Exp}_\mathrm{T} := \sum_{\theta \in \mathrm{T}} e_\theta \pi$$

for some finite $\mathrm{T} \subset \mathbb{C}^d$, with $e_\theta : x \mapsto e^{\theta x}$. It is natural to consider such exponential f since the essential part of the space

$$H(\varphi) := \{ f \in S(\varphi) : f \text{ entire } \}$$

consists of exponentials. I will use the abbreviation

$$\boldsymbol{\Theta}(\varphi)$$

for the **spectrum** of $H(\varphi)$, i.e., the smallest T for which $H(\varphi) \subset \mathrm{Exp}_\mathrm{T}$. The appropriate generalization of π_φ is the space

$$H_\varphi := \sum_{\theta \in \boldsymbol{\Theta}(\varphi)} e_\theta \pi_\varphi(\theta),$$

with $\pi_\varphi(\theta)$ the largest E-invariant subspace of

$$\Pi_\varphi(\theta) := \{ p \in \pi : p(-iD)\widehat{\varphi} = 0 \text{ on } -i\theta + (2\pi \mathbb{Z}^d \backslash 0) \}.$$

(4.11)Theorem [BR8x]. *Let φ be a compactly supported function, and let $f \in \mathrm{Exp}_\mathrm{T}$. Consider the following conditions:*

(a) $\quad f \in H_\varphi$;
(b) $\quad \varphi *' f = \varphi * f$;
(c) $\quad \varphi *' f \in \mathrm{Exp}_\mathrm{T}$.

Then (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c). If, in addition,

$$(\mathrm{T} - \mathrm{T}) \cap 2\pi i \mathbb{Z}^s = \{0\}, \tag{4.12}$$

then (c) $\Longrightarrow$ (a) as well.

Here is a proof outline: The implication (a)$\Longrightarrow$(b) follows from (4.4)Proposition by shifting in the frequency domain, using the fact that (a) implies that $\mathrm{T} \subset \mathbf{\Theta}(\varphi)$, hence that $f = \sum_{\theta \in \mathbf{\Theta}(\varphi)} f_\theta$ with $f_\theta = e_\theta q_\theta$ for some $q_\theta \in \pi_\varphi(\theta)$. It follows that each such q_θ lies in $\pi_{e_{-\theta}\varphi}(0)$. Therefore, by (4.4)Proposition,

$$\varphi *' f_\theta = e_\theta \left((e_{-\theta}\varphi) *' q_\theta\right) = e_\theta \left((e_{-\theta}\varphi) * q_\theta\right) = \varphi * f_\theta$$

for each $\theta \in \mathbf{\Theta}(\varphi)$, and (b) follows.

For the implication (c) $\Longrightarrow$ (a), decompose f into its separate exponential terms, $f = \sum_{\theta \in \mathrm{T}} f_\theta$, with $f_\theta = e_\theta q_\theta$. Assuming (4.12) and (c), it is possible to provide, for each $\theta \in \mathrm{T}$, a polynomial $p = p_\theta$ so that $p(E)f = f_\theta$ and $p(E)(\varphi *' f) \in e_\theta \pi$. Therefore

$$e_\theta \pi \ni p(E)\,(\varphi *' f) = \varphi *' p(E) f = \varphi *' f_\theta.$$

But this says that $r := e_{-\theta}(\varphi *' f_\theta) \in \pi$, i.e., $(e_{-\theta}\varphi) *' q_\theta = r \in \pi$, therefore $q_\theta \in \pi_{e_{-\theta}\varphi}(0)$ by (4.4)Proposition, i.e., $f_\theta \in H_\varphi$. ♠

The assumption (4.12) is essential here: If $\theta, \vartheta \in \mathrm{T}$ and $\theta - \vartheta \in 2\pi i \mathbb{Z}^d \backslash 0$, then $f := e_\theta q - e_\vartheta q$ vanishes on $\mathbb{Z}^d$ for any $q \in \pi$, hence $\varphi *' f = 0$, yet $f \neq 0$, hence does not belong to H_φ if q has sufficiently high degree.

5. The construction of quasiinterpolants for $S(\varphi)$

This section follows closely the development in [BR8x].

In Section 5, we discussed the construction of F-quasiinterpolants for $S(\varphi)$ in the form

$$\varphi *' \Lambda \tag{5.1}$$

with $\Lambda f : x \mapsto \lambda f(\cdot + x)$ and F an E-invariant, hence $\varphi *'$-invariant subspace of $S(\varphi)$. I pick up on this discussion now in order to give a unified view of the various concrete quasi-interpolants already available in the literature (cf., e.g., [SF73],[BH82], [DM83], [DM84], [DM85], [DM8x], [BJ85], [CJW87], [CD87], [CD88], [CL87],[B87], [DR8x], [R8x], [J88$_1$], [J8x], [BeR8x]. This is made possible by the observation (proved in the preceding section) that

$$\varphi *' = \varphi * \quad \text{ on } H_\varphi. \tag{5.2}$$

This suggests that, in trying to invert $\varphi *'$ on H_φ, we might as well consider the *simpler* problem of inverting the convolution $\varphi *$ on H_φ, as its inverse is given again by a convolution. Thus, we look for quasiinterpolants of the form

$$Q_\mu := \varphi *' \mu * \tag{5.3}$$

(instead of the form (5.1)), with $\mu *$ any convenient convolution which, on $F \subset H_\varphi$, agrees with T^{-1}. Here, as before, it is assumed that

$$T := \varphi *'{}_{|F}$$

is 1-1, hence invertible.

One recovers the earlier formulation (5.1) for the specific choice $\mu : f \mapsto \lambda f(-\cdot)$ (since $\mu * f(x) = \lambda(E^x f)$).

Consider now the problem suggested by the formulation (5.3) in light of (5.2): For given E-invariant $F \subset H_\varphi \cap S(\varphi)$, find distributions μ (of some desirable form) so that $\mu *_{|F} = (\varphi *_{|F})^{-1}$. In fact, there is useful additional freedom here: It is sufficient to construct μ so that $\mu * = (\psi *_{|F})^{-1}$ for some distribution ψ for which $\psi * = \varphi *$ on F.

For example, one might choose to use the particular distribution

$$\varphi_| : f \mapsto \sum_\alpha \varphi(\alpha) f(\alpha)$$

in place of φ. This is legitimate: For, as one computes,

$$(\varphi_| * f)(x) = \varphi_|(f(x - \cdot)) = \sum_\alpha \varphi(\alpha) f(x - \alpha) = f *' \varphi = \varphi *' f,$$

with the last equality holding [B87] for any $f \in S(\varphi)$. Note that the Fourier transform of $\varphi_|$ is the **symbol** of φ, i.e., the trigonometric polynomial

$$(\varphi_|)\widehat{\ } = \widetilde{\varphi} := \sum_\alpha \varphi(\alpha) e_{-i\alpha}, \tag{5.4}$$

which made its appearance already in Section 5.

We now describe some concrete approaches to the construction of suitable μ. While the practical interest (if any) would center on $F \subset \pi_\varphi$, it is no complication to consider, more generally, $F = e_\theta P \subset H_\varphi \cap S(\varphi)$.

(i) **Matching of Fourier transform** In effect, we are looking for a distribution μ so that

$$\psi * \mu * f = f \tag{5.5}$$

for all f in some finite-dimensional exponential space F. By going to Fourier transforms, (5.5) gives

$$\widehat{\psi}\widehat{\mu}\widehat{f} = \widehat{f},$$

which shows that we could use here any μ for which $\widehat{\psi}\widehat{\mu}-1$ vanishes to sufficiently high order on the (necessarily finite) spectrum of the exponential f. We now make this observation precise.

Consider first the case that F is the exponential space $e_\theta P$ for some E-invariant, hence D-invariant, polynomial space P. One computes (cf. the Appendix on Polynomial Identities) that

$$\psi * (e_\theta [\![]\!]^\alpha) = e_\theta \left((\psi e_{-\theta}) * [\![]\!]^\alpha\right) = e_\theta \sum_\gamma [\![]\!]^{\alpha-\gamma} [\![-iD]\!]^\gamma \widehat{\psi}(-i\theta) = e_\theta [\![\cdot -iD]\!]^\alpha \widehat{\psi}(-i\theta). \tag{5.6}$$

Given that P is D-invariant, this implies that, for any polynomial $p \in P$,

$$e_{-\theta}\psi * (e_\theta p) = p(\cdot - iD)\widehat{\psi}(-i\theta) = \sum_\gamma D^\gamma p \,[\![-iD]\!]^\gamma \widehat{\psi}(-i\theta) \;\in\; \widehat{\psi}(-i\theta)p + P \cap \pi_{<\deg p}. \tag{5.7}$$

In particular, $\psi*$ maps $e_\theta P$ into itself, and is invertible on $e_\theta P$ if and only if $\widehat{\psi}(-i\theta) \neq 0$. Further, since $p(\cdot - iD)1 = p$, the first equality in (5.7) (with ψ replaced by $\psi * \mu$) shows that $\psi * \mu * = 1$ on $e_\theta P$ if and only if $p(\cdot - iD)(\widehat{\psi}\widehat{\mu} - 1)(-i\theta) = 0$ for $p \in P$. This last condition is equivalent to $\widehat{\psi}\widehat{\mu} - 1$ having a $P(-i\cdot)$-fold zero at $-i\theta$ (i.e., $p(-iD)(\widehat{\psi}\widehat{\mu} - 1)(-i\theta) = 0$ for all $p \in P$), since

$$p(\cdot - iD) = \sum_\gamma [\![]\!]^\gamma (D^\gamma p)(-iD)$$

and P is D-invariant. This proves the following.

(5.8)Proposition. *If the compactly supported function ψ satisfies $\widehat{\psi}(-i\theta) \neq 0$, and the finite-dimensional polynomial space P is E-invariant, then $\psi*$ maps $e_\theta P$ 1-1 onto itself, and any convolution $\mu*$ with*

$$p(-iD)\widehat{\mu}(-i\theta) = p(-iD)(1/\widehat{\psi})(-i\theta) \qquad \forall (p \in P) \tag{5.9}$$

provides the inverse of $\psi*$ on $e_\theta P$.

In applications, the polynomial space P is often not known precisely, but its degree can be ascertained, i.e., a k with $P \subset \pi_k$ can be found. In that case, one would satisfy (5.9) for π_k rather than P, i.e., one would make certain that all derivatives of order $\le k$ of $\widehat{\mu}$ at $-i\theta$ match those of $1/\widehat{\psi}$ there.

By choosing μ so that (5.9) is satisfied with $P = P_\theta$ for every $\theta \in \mathbf{\Theta}(\varphi)$, one obtains a suitable distribution μ and a quasi-interpolant $\varphi *' \mu *$. For example, if $\mu * = q(iD)$ for some polynomial q, then $\widehat{\mu} = q$, while the choice $\mu * = q(E)$ leads to the 'trigonometric' polynomial $\widehat{\mu} = q(e^{i()})$. In either case, appropriate osculatory interpolation to $1/\widehat{\varphi}$ at $-i\mathbf{\Theta}(\varphi)$ provides an appropriate μ. In the first case, μ is a linear combination of values and derivatives at the origin, while, in the second case, μ employs only function values at some points from $\mathbb{Z}^s$. More generally, one could use μ of the form $f \mapsto \sum_{x\in X} p_x(iD)f(x)$ in which the polynomial p_x is to be chosen so that the exponential $\widehat{\mu} = \sum_{x\in X} p_x()e^{ix()}$ osculates to $1/\widehat{\varphi}$ appropriately at $-i\mathbf{\Theta}(\varphi)$.

If one uses μ of the form $w_|$, then $\mu * = \sum_\alpha w(\alpha)[\alpha]$ is a **difference operator**, hence commutes with $\varphi *'$ and thus

$$\varphi *' (\mu * f) = (\mu * \varphi) *' f.$$

This provides us with a quasi-interpolant of the simple form $\psi *'$, with $\psi \in S(\varphi)$, and with the support of ψ not exceeding the sum of the supports of μ and φ. In fact, it is shown in [R8x] that the support of the difference operator $\mu *$ can be chosen so that

$$\operatorname{diam} \operatorname{supp} \psi \le 2 \operatorname{diam} \operatorname{supp} \varphi,$$

in contrast to the minimal polynomial procedure below in which the inverting difference operator is supported on a relatively large domain.

We summarize the Fourier transform approach in the following

(5.10) Theorem. *Let φ be a compactly supported function whose Fourier transform does not vanish on $-i\mathbf{\Theta}(\varphi)$, let F be an E-invariant subspace of $H_\varphi \cap S(\varphi)$ and μ a compactly supported distribution. Then*

$$Q_\mu := \varphi *' \mu *$$

is an F-quasiinterpolant (i.e., is the identity on F) if and only if

$$p(-iD)\widehat{\mu}(-i\theta) = p(-iD)(1/\widehat{\psi})(-i\theta), \ \text{all}\, e_\theta p \in F,$$

where ψ is any compactly supported distribution for which $\psi *$ coincides with $\varphi *'$ on F. Suitable choices for ψ are $\psi = \varphi$ and $\psi = \varphi_|$.

By comparison with the Fourier transform approach, the remaining two approaches don't seem very efficient. They apply directly only to F of the simple from $e_\theta P$.

(ii) **Minimal polynomial** Here, one would choose an 'easily computable' distribution ψ for which $\psi * = \varphi *$ on F and observe that, since $T = (\varphi *')_{|F}$ is invertible, we can represent T^{-1} as $p(\psi *)$ for some polynomial p (i.e., obtain the inverse of the operator $\psi *$ as a linear combination of powers of $\psi *$). We obtain such a polynomial as a multiple of

$$(m_T(0) - m_T)/()^1,$$

with m_T the **minimal (annihilating) polynomial** for T. It may be hard, in general, to produce this polynomial, particularly if the space $H_\varphi \cap S(\varphi)$ is not known precisely. But, we conclude from (5.7) that, on the exponential space $e_\theta\pi$, $\varphi*$ is **degree-preserving** in the sense that, for any polynomial p,

$$\varphi * e_\theta p \in \widehat{\varphi}(-i\theta)(e_\theta p) + e_\theta \pi_{<\deg p}.$$

In fact, (5.7) implies that

$$\varphi * e_\theta p \in \widehat{\varphi}(-i\theta)(e_\theta p) + e_\theta \pi_{\deg p - k}$$

in case $\widehat{\varphi} - \widehat{\varphi}(-i\theta)$ has a zero of order k at $-i\theta$. For example, if φ is radially symmetric, i.e., $\varphi(-x) = \varphi(x)$, then $\widehat{\varphi}(x) = \widehat{\varphi}(0) + O(|x|^2)$, hence

$$\varphi * p \in \widehat{\varphi}(0)p + \pi_{\deg p - 2} \tag{5.11}$$

in that case, as was already pointed out in [CD87]. In any case, this makes available the linear polynomial $r_\theta := \widehat{\varphi}(-i\theta) - \cdot$, for which $r_\theta(\varphi*)$ is **degree-reducing** on $e_\theta\pi$. It follows that, for any F in $e_\theta\pi$, there is a suitable power $(r_\theta)^n$ of r which annihilates T, yet does not vanish at 0 (since $r_\theta(0) = \widehat{\varphi}(-i\theta) \neq 0$ as T is assumed to be invertible) and therefore is available for the construction of T^{-1} in the form $p(\psi*)$.

(iii) **Recurrence** Equation (5.7) suggests the solution of the equation $\psi*? = f \in e_\theta P$ by backsubstitution, i.e., by recurrence, since it implies that, for $f = e_\theta p \in e_\theta P$,

$$\psi * f = \sum_\gamma (e_\theta D^\gamma p)[\![-iD]\!]^\gamma \widehat{\psi}(-i\theta),$$

therefore (using the invertibility of $\psi*$ on $e_\theta P$)

$$f = \sum_\gamma (\psi*)^{-1}(e_\theta D^\gamma p)[\![-iD]\!]^\gamma \widehat{\psi}(-i\theta),$$

hence

$$(\psi*)^{-1} f = (\psi*)^{-1}(e_\theta p) = \left(f - \sum_{\gamma \neq 0} (\psi*)^{-1}(e_\theta D^\gamma p)[\![-iD]\!]^\gamma \widehat{\psi}(-i\theta) \right) / \widehat{\psi}(-i\theta). \tag{5.12}$$

For a general exponential f, the resulting solution depends of course on the choice of $\psi*$, but necessarily, since $\psi* = \varphi*'$ on F, this solution is independent of ψ in case $f \in F$ (as it then equals $(\varphi*')^{-1} f$). Note that the recurrence does require the numbers

$$D^\gamma \widehat{\psi}(-i\theta)$$

for all γ for which $D^\gamma p \neq 0$. It also requires the generation of the 'tree' of derivatives of p which, during the solution process, would have to be traversed in the order opposite to the order of its generation.

By contrast, the use of Appell polynomials, detailed in Section 5, does not require explicit information about the derivatives of $\widehat{\psi}$. In effect, it computes the solution p_α for the equation $\psi*? = [\![]\!]^\alpha$ for all α needed directly (by simple recurrence), and obtains the solution for general p in the form $\sum_\alpha p_\alpha D^\alpha p(0)$.

6. Projectors from quasiinterpolants

Linear projectors R onto S are particularly desirable approximation maps into S since they provide the estimate

$$|\lambda f - \lambda Rf| \le \operatorname{dist}(\lambda, \operatorname{ran} R')\|1 - R\| \operatorname{dist}(f, S)$$

for any function f and any linear functional λ. (Here, $\operatorname{ran} R'$ consists of the *interpolation conditions* for R, i.e., $\operatorname{ran} R' = \{\lambda : \lambda f = \lambda Rf, \ \forall f\}$.) In particular, if we are able to construct a bounded sequence (R_h) of projectors for the directed set (S_h), then we know that

$$\operatorname{dist}(f, S_h) = O(\|f - R_h f\|),$$

hence can determine the approximation order of (S_h) from the behavior of $\|f - R_h f\|$ for smooth f as $|h| \to 0$. Such behavior will be hard to determine unless R_h is local. Thus we are looking for good quasiinterpolants which are also projectors.

Note that the existence of such a good projector implies that the approximation order is at least m in case $\pi_{<m} \subset S_h$ for all h. Thus we cannot expect to construct such good projectors for spaces like C^1-cubics on the three-direction mesh, not even for its locally spanned subspace $S(\{M_{122}, M_{212}\})$ (cf. Section 2).

Neither can we expect to construct such a good projector for S unless we have available a basis Φ for S which is **locally finite** (i.e., locally finite-dimensional) in the sense that, for all n, at most finitely many $\varphi \in \Phi$ have some support in any particular open ball. With $B_n := B_n(0)$ the ball of radius n around the origin, this is equivalent to saying that, for every n, the set

$$\nu_n := \{\varphi \in \Phi : \operatorname{supp} \varphi \cap B_n \neq \emptyset\}$$

is finite. For any such collection, it makes sense to define the linear map

$$\Phi : \mathbb{R}^\Phi \to F : c \mapsto \Phi c := \sum_{\varphi \in \Phi} \varphi\, c(\varphi),$$

with the (possibly) infinite sum taken pointwise, or, more strongly, uniformly (since trivially) convergent on compact sets. (There seems to be no harm in using the letter Φ both for the set and the map induced by it, much as we might use the same letter A to denote a matrix and the sequence $A = [a_1, \ldots, a_n]$ of its columns.) We say that Φ is **(globally) linearly independent** if the map Φ is 1-1. In that case, Φ provides a basis for its range

$$S := \operatorname{ran} \Phi.$$

This implies that any linear map into S can be written in the form

$$\sum_{\varphi \in \Phi} \varphi \lambda_\varphi$$

for suitable linear functionals λ_φ (and with $\varphi\lambda : f \mapsto \varphi\lambda(f)$). In particular, such a map is a linear projector if and only if, for every $\varphi \in \Phi$, λ_φ extends the coordinate functional

$$\lambda^0_\varphi S \to \mathbb{R} : f \mapsto (\Phi^{-1} f)(\varphi).$$

Consequently, such a projector cannot be local unless each coordinate functional is local, i.e., satisfies

$$f_{|A} = 0 \quad \Longrightarrow \quad \lambda^0_\varphi f = 0$$

for all $f \in S$ and some bounded set A. This makes the following recent result [BeR8x] of Ben-Artzi and Ron particularly interesting.

(6.1)Theorem. *Let Φ be a locally finite, globally linearly independent collection. Then, for every $\varphi \in \Phi$, there exists a ball $B = B_\varphi$ so that $(\Phi c)_{|B} = 0$ implies $c(\varphi) = 0$.*

Proof. Without loss, we assume that (after a suitable translation) φ has the origin in its support. In view of the definition of $\Phi c := \sum_{\varphi\in\Phi} \varphi\, c(\varphi)$ as a pointwise sum,

$$\ker \Phi = \bigcap_n \ker R_n,$$

with

$$R_n : \mathbb{R}^\Phi \to \mathbb{R}^{B_n} : c \mapsto (\sum_{\varphi\in\Phi} \varphi\, c(\varphi))_{|B_n}.$$

We claim that the sequence $\ker R_1 \supset \ker R_2 \supset \cdots$ is decreasing fast enough so that, for some n, $[\varphi] = 0$ on $\ker R_n$, i.e. $c \in \ker R_n$ implies $c(\varphi) = 0$. The proof is by contradiction, i.e., by showing that, in the contrary case, $\ker \Phi = \bigcap_n \ker R_n \neq \emptyset$.

Whether or not the sequence c belongs to $\ker R_n$ depends entirely on its behavior on ν_n, i.e., $c \in \ker R_n$ iff $r_n c \in \ker R_n$, with

$$r_n : c \mapsto \chi_{\nu_n} c$$

the natural projection of $\mathbb{R}^\Phi$ onto the space of coefficient 'sequences' having support only on ν_n. Thus, with

$$\nu_0 := \{\varphi\},$$

which is consistent with the otherwise evident relationship

$$\nu_n \subset \nu_{n+1}, \quad \forall n,$$

since $\varphi(0) \neq 0$ by assumption, we can describe the contrary case as asserting that, for every n, r_0 is not trivial on

$$M_n := \ker R_n \cap \operatorname{ran} r_n,$$

and this linear space is *finite-dimensional* by the local finiteness of Φ. Since $r_n(M_{n+1}) \subset M_n$ for any n, this implies that, for any n and k,

$$M_{n,k} := r_n \cdots r_{n+k-1}(M_{n+k})$$

is a nontrivial subspace of M_n since, by assumption, $\ker R_{n+k}$ contains some c with $c(\varphi) \neq 0$, hence $c' := r_n \cdots r_{n+k} c \in M_n \backslash 0$ since $c'(\varphi) = c(\varphi) \neq 0$. It follows that $M_{n,0}, M_{n,1}, \cdots$ is a decreasing sequence of nontrivial finite-dimensional spaces, hence has a nontrivial limit

$$K_n := \bigcap_k M_{n,k} = \bigcap_{k>1} r_n M_{n+1,k-1} = r_n K_{n+1}.$$

Therefore, starting with some $c_0 \in K_0\backslash 0$, we can find, by induction, the sequence $c_0, c_1, c_2, \ldots$ so that $c_n \in K_n$ and $r_{n-1}c_n = c_{n-1}$ for $n = 0, 1, 2, \ldots$. Its limit, c_∞, provides a nontrivial element of $\ker \Phi = \cap_n \ker R_n$, thus contradicting the linear independence of Φ. ♠

We call A for which $f_{|A} = 0 \implies \lambda f = 0$ for all $f \in S$ a **determining set** for $\lambda \in S'$. Note that a linear functional on a function space on $\mathbb{R}^d$ may have many different determining sets. For example, the coordinate functional of a B-spline has every open subset of the B-spline's support as a determining set.

The theorem implies that each λ^0_φ is local, hence, by the local finitenes of S, sees S as a *finite-dimensional* linear space, i.e., the space

$$S_A := S_{|A}$$

for some bounded A. This implies that each λ^0_φ is **bounded** (in whatever norm we care to impose on S), hence admits extensions to a continuous linear functional with support in A on whatever normed linear superspace X we wish to approximate from $S(\varphi)$.

What is offhand missing is the **uniform boundedness**, i.e., the finiteness of

$$\sup_{\varphi\in\Phi} \|\lambda^0_\varphi\|/\|\varphi\|,$$

or, equivalently, the finite condition of the basis Φ. Such information is needed for the construction of bounded linear projectors to S. We do obtain such boundedness when $\Phi = \{\varphi(\cdot - \alpha) : \alpha \in \mathbb{Z}^d\}$ and, correspondingly,

$$S = S(\varphi)$$

for some compactly supported function φ. For, in that case, the fact that $E^\alpha\varphi * c = \varphi*(E^\alpha c)$ implies that

$$\lambda^0_{\varphi(\cdot-\alpha)} = \lambda^0_\varphi E^\alpha,$$

hence that

$$f = \varphi *' \lambda_0 * f \quad \forall (f \in S(\varphi))$$

with

$$\lambda_0 : f \mapsto \lambda^0_\varphi f(-\cdot)$$

bounded. Each $Q_\lambda := \varphi *' \lambda*$ so obtained is a uniformly local linear projector onto $S(\varphi)$.

Consider now the actual construction of such λ, in the spirit of [J8x], [J88], and [BeR8x], and in imitation of the dual functionals for the univariate B-splines discussed in the first section. This requires us to consider the linear functionals of the form

$$[x]p(D) : f \mapsto (p(D)f)(x)$$

for $x \in \mathbb{R}^d$ and $p \in \pi$. We will write

$$p^* := [0]p(D)_{|S_A}.$$

Specifically, assume that, for some $x \in A$ and some E-invariant polynomial space P, the map

$$P \to (S_A)' : p \mapsto [x]p(D) \tag{6.2}$$

is invertible. Then there exists a unique $p \in P$ so that

$$\lambda_0 = [x]p(D) \quad \text{on} \quad S_A.$$

If now also F is a D-invariant (hence smooth) space for which

$$F_{|A} = S_A = S(\varphi)_{|A},$$

then, for any $y \in \mathbb{R}^d$, $F = E^y F$, hence

$$P \to F' : p \mapsto [y]p(D) \tag{6.3}$$

is invertible for each $y \in \mathbb{R}^d$, thus providing a unique extension of λ_0 of the form

$$\lambda_0 = [x]p_x(D) \quad \text{on} \quad F$$

for every $x \in \mathbb{R}^d$. In order to discover the behavior of p_x as a function of x, take any basis B for F, let $(p_b{}^*)_{b \in B}$ be the corresponding dual basis in $[0]P(D)$, and let μ be an arbitrary linear functional on S_A. Then

$$\mu = \sum_{b \in B} \mu b \; p_b{}^* =: q^*,$$

therefore

$$\mu = (\mu E^{-x})E^x = \sum_b q^* b(\cdot - x) \; [x]p_b(D),$$

using the fact that, for any x and any smooth f and any $p \in \pi$,

$$p^*(E^x f) = [0]p(D)f(\cdot + x) = [x]p(D)f.$$

This extracts the essential features of the argument in [BeR8x] for the following proposition, which generalizes the dual functional formula (1.6) for univariate B-splines.

(6.4)Proposition. *If Φ is globally linearly independent and F is a D-invariant space which agrees with* $\operatorname{ran}\Phi$ *on some determining set A for $\lambda_0 := \lambda_\varphi^0$ for some $\varphi \in \Phi$, then, for an arbitrary basis B of F and for every $x \in \mathbb{R}^d$, the linear functional*

$$\sum_{b \in B} q(D)b(-x) \; [x]p_b(D) \tag{6.5}$$

extends $\lambda_{0|F}$, where (p_b) is the unique 'dual basis' for B in any E-invariant polynomial space P for which the map

$$P \to F' : p \mapsto [0]p(D)_{|F}$$

is invertible, and q is the unique element of P for which $[0]q(D) = \lambda_0$.

Since (6.5) only extends $\lambda_{0|F}$, it extends λ_0 only if x is in some determining set A for λ_0 on which $F = \operatorname{ran}\Phi$. Thus we could recover the formula (1.6) from this proposition (using $F = P = \pi_{<k}$). On the other hand, the fact that any nontrivial knot interval in the support of a univariate B-spline is determining for its coordinate functional is most easily proved with the aid of the dual functional formula (1.6).

The propositon makes no use of any *local* linear independence. (To recall, Φ is locally linearly independent if $\{\varphi_{|A} : \varphi \in \Phi, \varphi_{|A} \neq 0\}$ is linearly independent for any open set A.) While it is an open question whether the linear independence of Φ implies the linear independence of Φ_n for all sufficiently large n (the guess is that it does not), there are simple examples to show that linear independence does not imply local linear independence. For example, the integer translates for the characteristic function of the interval $[0, 1/2]$ are linearly independent but not locally linearly independent.

7. Good projectors into smooth bivariate polynomial spaces

Once we consider spline spaces more general than those generated by the translates of a single compactly supported function, we are at the frontier of research. The only case studied in any depth is that of

$$S = \pi^{\rho}_{k,\Delta}$$

of all functions in C^{ρ} which are piecewise polynomial of degree k at most, with Δ a triangulation of some domain in $\mathbb{R}^2$. The results are as follows.

It has been known for some time (see, e.g., [Z74] and references there) that S has **full approximation order**, i.e.,

$$\operatorname{dist}(f, S) = O(|\Delta|^{k+1})$$

for all smooth functions, in case $k \geq 4\rho + 1$, since it is then possible to construct a good projector by local Hermite interpolation in each triangle. Here, $|\Delta|$ is the largest diameter of a triangle in Δ. (I use here the adjective 'full' since the approximation order from the entirely unconstrained (and larger) space $\pi_{k,\Delta}$ cannot be better than $O(|\Delta|^k)$, by arguments like those we used in the univariate case in the first section.)

It was shown in [BH88] that full approximation power was had by S even if only $k \geq 3\rho + 2$. The argument given is not constructive, but, as is made clear in [B90], contains the essential information for the construction of a stable, local basis, hence of a good projector, for S. [CL88] provide an explicit construction of a good projector onto a subspace of S which contains π_k, hence already achieves the full approximation order. These results are sharp, i.e., already for Δ the three-direction mesh (cf. [BH88] for $\rho \leq 3$, R.-q. Jia for general ρ), the approximation order is less than $k + 1$.

Yet these results have a puzzling aspect. While the local basis constructed consists of 'vertex splines', i.e., of splines with support within the triangles sharing one vertex, the stability of the basis, i.e., its condition number, depends on the mesh Δ. This is to be expected since there should be difficulties with skinny triangles, i.e., triangles with a small angle. What is annoying is that there are also difficulties when Δ has many near-singular vertices, i.e., vertices with just four edges and with opposing edges nearly parallel. Jia has recently established the existence of a local basis for $\pi^{\rho}_{k,\Delta}$ for $k \geq 3\rho + 2$ whose condition is bounded in terms of the smallest angle in the partition only, but some of its elements may fail to be vertex splines.

A good and challenging test case is provided by the space of bivariate C^1-cubics. For Δ the three-direction mesh, its approximation order is known to be 3 rather than 4 (cf. Section 2), but it is not known what its generic approximation order is. Since there are partitions Δ in which some triangles are not in the support of any compactly supported element, hence for which S fails to contain a local partition of unity, my guess is that, generically, bivariate C^1-cubics have approximation order 0. In this regard, my working hypotheses, in order of increasing content, are the following. While they are meant for any 'directed' set (S_h) with

$$S_h := \pi^{\rho}_{k,\Delta_h}$$

and $|h| := |\Delta_h|$, one would try them first for scales, i.e., for $\Delta_h = h\Delta$ for some fixed triangulation Δ for $\mathbb{R}^2$. Also, **local support** means 'support of $O(|\Delta_h|)$', and, correspondingly, **local** partition of unity or **local** basis means a partition of unity or a basis whose elements have local support.

Conjecture 1. *If* (S_h) *has positive approximation order, then* S_h *contains elements with local support.*

Conjecture 2. *If the* scale (S_h) *has positive approximation order, then* S_h *contains a local partition of unity.*

Conjecture 3. *The approximation order of* (S_h) *equals that of* $({S_h}^{\mathrm{loc}})$*, with*

$$S^{\mathrm{loc}} := \mathrm{span}\{f \in S : \mathrm{supp}\, f \text{ compact}\}.$$

Conjecture 4. *The approximation order of* (S_h) *can always be realized by a corresponding set* (Q_h) *of good quasiinterpolants (with* $\mathrm{ran}\, Q_h \subset {S_h}^{\mathrm{loc}}$ *necessarily).*

A first stab at Conjecture 2 was made in [BD85], in a univariate context, with S restricted to be E-invariant. [J88$_2$] extended this to a univariate result which, if true also in the multivariate setting, would give the following positive answer to Conjecture 4.

Conjecture 5. *If* S *is an* E*-invariant, locally finite-dimensional space of functions closed under uniform convergence on compact sets, then the corresponding scale* (S_h) *has approximation order* m *if and only if* S *contains a compactly supported function* ψ *whose Fourier transform satisfies the Strang-Fix conditions of order* m*, i.e.,* $\widehat{\psi}(0) = 1$ *and* $D^\alpha \widehat{\psi} = 0$ *on* $2\pi\mathbb{Z}^d \backslash 0$ *for all* $|\alpha| < m$*.*

Free knot splines

In these lectures, I have failed entirely to cover the multivariate equivalent of *splines with free knots*. This is already a challenging problem in the tensor product case, which, however, doesn't strike me as germane since the meshes there have global effect while an adaptive meshing should have local adaption as its goal. There is an early paper [BRi82] on the local, adaptive refinement of a rectangular grid. While [BRi82] deals only with piecewise polynomials without smoothness constraints, a corresponding development for smooth bivariate piecewise polynomials spanned by simplex splines can be found in [D82]. Both of these papers (and others) deal with adaptive *refinement*. The problem of the construction of an **optimal** partition Δ, i.e., a partition of the given domain G into a given number of triangles, say, so as to minimize $\mathrm{dist}(f, \pi^\rho_{k,\Delta})$ is considerably harder. There is Nadler's result [N85] on the locally optimal shape of such triangles when $k = 1$ and $\rho = 0$. A totally different, and very intriguing, approach has been started by DeVore and Popov [DP87]: They determine a best approximation (in the $\mathbf{L}_p$-norm) from the space of linear combinations of n characteristic functions of shifted (hyper)cubes of arbitrary size. In particular, they do not insist that these cubes not overlap and thus avoid a major difficulty associated with the construction of optimal meshes. Subsequent work by DeVore, Jawerth and Popov (in progress) makes use of the existing and developing theory of multiscale expansions (wavelets) to extend these results to box-splines and other smooth piecewise polynomials. Since this uses dyadic subdivision, it does look more like adaptive refinement than optimal mesh generation.

8. Appendix: Appell polynomials

Traditionally, an **Appell sequence** (p_α) of polynomials is characterized by the fact that $p_0 = 1$ and $D^\beta p_\alpha = p_{\alpha-\beta}$.

Thorne [T45] pointed out the much more useful characterization in terms of a linear functional λ, normalized so that $\lambda()^0 = 1$, with which there is associated uniquely a polynomial sequence (p_α) by the conditions that

$$\text{(i) } p_\alpha \in \pi_\alpha \qquad \text{(ii) } (\lambda D^\beta) p_\alpha = \delta_{\beta-\alpha}. \tag{8.1}$$

Actually, Thorne and others only consider univariate polynomials, but the multivariable extension is clear in view of the now standard multivariable index notation.

Standard examples include

$\lambda = [0]$, giving the **normalized powers** or **Taylor polynomials** $p_\alpha = [\![]\!]^\alpha$;

$\lambda : f \mapsto \int_\Omega f / \int_\Omega 1$, giving the **Bernoulli polynomials for** Ω.

$\lambda = ([0] + [1])/2$ on $C(\mathbb{R})$, giving the **Euler polynomials**.

Existence and uniqueness of the Appell sequence for given λ follows directly from the definition. In fact (as Amos Ron kindly pointed out), (ii) implies (i): If γ is a maximal index with the property that $D^\gamma p_\alpha(0) \neq 0$, then $D^\gamma p_\alpha$ is a nonzero constant, hence $\lambda D^\gamma p_\alpha \neq 0$, which implies that $\gamma = \alpha$. With that,

$$p_\alpha = \sum_{\gamma \le \alpha} [\![]\!]^\gamma a(\gamma),$$

and the condition $\lambda D^\beta p_\alpha = \delta_{\alpha-\beta}$ is equivalent to the linear system

$$\sum_{\gamma \le \alpha} \lambda([\![]\!]^{\gamma-\beta}) a(\gamma) = \delta_{\beta-\alpha}, \quad \forall(\beta \le \alpha),$$

which has a triangular coefficient matrix with unit diagonal (since $\lambda[\![]\!]^0 = 1$ by assumption), hence has exactly one solution, and this solution can be obtained by backsubstitution,

$$p_\alpha = [\![]\!]^\alpha - \sum_{\beta \neq \alpha} p_{\alpha-\beta}\, \lambda [\![]\!]^\beta. \tag{8.2}$$

The correctness of this formula can be verified directly by induction on α:

$$\begin{aligned} \lambda D^\gamma p_\alpha &= \lambda D^\gamma [\![]\!]^\alpha - \textstyle\sum_{\beta \neq \alpha} \lambda D^\gamma p_{\alpha-\beta}\, \lambda [\![]\!]^\beta \\ &= \lambda [\![]\!]^{\alpha-\gamma} - \qquad \lambda [\![]\!]^{\alpha-\gamma} \qquad = \ 0 \end{aligned}$$

for $\gamma < \alpha$, while $\lambda D^\alpha p_\alpha = \lambda D^\alpha [\![]\!]^\alpha = \lambda()^0 = 1$.

The uniqueness implies that

$$D^\beta p_\alpha = p_{\alpha-\beta}, \tag{8.3}$$

hence

$$p_\alpha = \sum_{\gamma \le \alpha} [\![]\!]^{\alpha-\gamma} c(\gamma)$$

for some (as yet mysterious) sequence c.

(8.4)Proposition. *For $\phi := \phi_\lambda : f \mapsto \lambda f(-\cdot)$ and for all α,*

$$p_\alpha = \llbracket \cdot - iD \rrbracket^\alpha (1/\widehat{\phi}_\lambda)(0) = \sum_{\gamma \le \alpha} \llbracket\rrbracket^{\alpha-\gamma}\ [0] \llbracket -iD \rrbracket^\gamma (1/\widehat{\phi}_\lambda). \tag{8.5}$$

Proof. Convolution maps polynomials to polynomials. Precisely, for any $p \in \pi$ and any ϕ,

$$\phi * p = p(\cdot - iD)\widehat{\phi}(0) = \sum_\gamma D^\gamma p \ \llbracket -iD \rrbracket^\gamma \widehat{\phi}(0).$$

Consequently,

$$\phi * p \in \widehat{\phi}(0) p + \pi_{<\deg p}. \tag{8.6}$$

This implies that $\phi*$ maps each π_α into itself, and is invertible on each such π_α iff $\widehat{\phi}(0) \ne 0$. Moreover, on π_α, the action of $\phi*$ is entirely determined by the numbers $D^\beta \widehat{\phi}(0),\ \beta \le \alpha$. In particular, if μ is any function(al), then convolution with the composite $\psi := \phi * \mu$ also carries each π_α into itself; it is the identity on π_α iff the Fourier transform $\widehat{\psi} = \widehat{\phi}\widehat{\mu}$ satisfies

$$D^\gamma \widehat{\psi}(0) = \delta_\gamma \qquad \forall (\gamma \le \alpha). \tag{8.7}$$

In particular, take now convolution with any functional μ for which $\widehat{\phi}_\lambda \widehat{\mu} = 1$ α-fold at 0, i.e., for which

$$D^\beta (\widehat{\phi}_\lambda \widehat{\mu})(0) = \delta_\beta \quad \forall (\beta \le \alpha). \tag{8.8}$$

Then

$$D^\beta \widehat{\mu}(0) = D^\beta (1/\widehat{\phi}_\lambda)(0) \quad \forall (\beta \le \alpha) \tag{8.9}$$

and $\mu*$ represents the inverse of $\phi_\lambda *$ on π_α. In particular,

$$\llbracket\rrbracket^\beta = \phi_\lambda * (\mu * \llbracket\rrbracket^\beta) \quad \forall (\beta \le \alpha). \tag{8.10}$$

Consequently (using the fact that D commutes with convolution),

$$\delta_{\alpha-\beta} = [0] D^\beta \llbracket\rrbracket^\alpha = [0] \phi_\lambda * (D^\beta (\mu * \llbracket\rrbracket^\alpha)),$$

while $[0](\phi_\lambda * f) = \lambda f$. Since $\mu * \llbracket\rrbracket^\alpha \in \pi_\alpha$, it follows that

$$p_\alpha = \mu * \llbracket\rrbracket^\alpha = \sum_\gamma \llbracket\rrbracket^{\alpha-\gamma} \llbracket -iD \rrbracket^\gamma (1/\widehat{\phi}_\lambda)(0).$$

♠

The proof suggests that it would be particularly helpful to identify linear functionals μ satisfying (8.8) since that would obviate use of the recurrence for the construction of the p_β for $\beta \le \alpha$. This is the same question as finding a formula for the linear functional for which

$$q_\alpha := \phi_\lambda * \llbracket\rrbracket^\alpha$$

are the Appell polynomials. That these q_α form an Appell sequence in the traditional sense of the word follows from the fact that, on π, $\phi*$ commutes with differentiation. This suggests an alternative (and simpler) way for generating an Appell sequence from a linear functional. It would be harder, though, to know the interpolation conditions which go with the expansion in terms of the q_α.

9. Appendix: Polynomial identities

(9.1)Definition.

$$[\![\,]\!]^n : \mathbb{R} \to \mathbb{R} : \; x \mapsto x^n/n!$$

$$[\![\,]\!]^\alpha : \mathbb{R}^d \to \mathbb{R} : \; x \mapsto \prod_{j=1}^{d} [\![x(j)]\!]^{\alpha(j)}$$

(9.2)Multinomial identity.

$$[\![x + y + \cdots + z]\!]^\alpha = \sum_{\xi+\upsilon+\cdots+\zeta=\alpha} [\![x]\!]^\xi [\![y]\!]^\upsilon \cdots [\![z]\!]^\zeta$$

Proof. by induction on $|\alpha|$. ♠

(9.3)Taylor. *For any polynomial* p,

$$p(x+y) = \sum_\alpha [\![x]\!]^\alpha D^\alpha p(y).$$

Proof. For the particular polynomial $p = [\![\,]\!]^\beta$,

$$p(x+y) = \sum_\alpha [\![x]\!]^\alpha [\![y]\!]^{\beta-\alpha} = \sum_\alpha [\![x]\!]^\alpha (D^\alpha p)(y).$$

♠

(9.4)Leibniz. *For any functions* $f, g, \ldots, h$ *and any scalar* s,

$$[\![sD]\!]^\alpha (fg\cdots h) = \sum_{\varphi+\gamma+\cdots+\eta=\alpha} [\![sD]\!]^\varphi f \; [\![sD]\!]^\gamma g \; \cdots [\![sD]\!]^\eta h$$

Proof. by induction on $|\alpha|$. ♠

(9.5)Leibniz-Hörmander. *For any polynomial* p, *scalar* s, *and functions* f, g,

$$p(sD)(fg) = \sum_\beta \left(\left([\![D]\!]^\beta p \right) (sD) f \right) [\![sD]\!]^\beta g.$$

Proof.

$$\begin{aligned} p(sD)(fg) &= \sum_\alpha a(\alpha) [\![sD]\!]^\alpha (fg) \\ &= \sum_\alpha a(\alpha) \sum_\beta [\![sD]\!]^{\alpha-\beta} f \; [\![sD]\!]^\beta g \; = \; \sum_\beta \left(\left([\![D]\!]^\beta p \right) (sD) f \right) [\![sD]\!]^\beta g. \end{aligned}$$

♠

Note. The identity is linear in p, f, g, hence verifiable by checking it just for pure powers.

(9.6) Convolution. *For any compactly supported ϕ and any $p \in \pi$,*

$$\phi * p = p(\cdot - iD)\widehat{\phi}(0) = \sum_\gamma [\![]\!]^\gamma D^\gamma p(-iD)\widehat{\phi}(0) = \sum_\gamma D^\gamma p\, [\![-iD]\!]^\gamma \widehat{\phi}(0).$$

Proof.

$$D^\beta \widehat{\phi} = \int \phi(y)(-iy)^\beta e^{-iy()} dy,$$

hence

$$(-iD)^\beta \widehat{\phi}(0) = \int \phi(y)(-y)^\beta dy.$$

So,

$$\begin{aligned}\phi * [\![]\!]^\alpha &= \int \phi(\cdot - y)[\![y]\!]^\alpha dy = \int \phi(y)[\![\cdot - y]\!]^\alpha dy \\ &= \int \phi(y) \sum_\beta [\![]\!]^{\alpha-\beta}[\![-y]\!]^\beta dy = \sum_\beta [\![]\!]^{\alpha-\beta} \int \phi(y)[\![-y]\!]^\beta dy\,. \\ &= \sum_\beta [\![]\!]^{\alpha-\beta}[\![-iD]\!]^\beta \widehat{\phi}(0) = [\![\cdot - iD]\!]^\alpha \widehat{\phi}(0).\end{aligned}$$

♠

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