The Sobolev regularity of refinable functions

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ABSTRACT

Refinable functions underlie the theory and constructions of *wavelet systems* on the one hand, and the theory and convergence analysis of *uniform subdivision algorithms*. The regularity of such functions dictates, in the context of wavelets, the smoothness of the derived wavelet system, and, in the subdivision context, the smoothness of the limiting surface of the iterative process. Since the refinable function is, in many circumstances, not known analytically, the analysis of its regularity must be based on the explicitly known mask. We establish in this paper a formula that computes, for isotropic dilation and in any number of variables, the sharp L_2-regularity of the refinable function ϕ in terms of the spectral radius of the restriction of the associated *transfer operator* to a specific invariant subspace. For a compactly supported refinable function ϕ, the relevant invariant space is proved to be finite dimensional, and is completely characterized in terms of the dependence relations among the shifts of ϕ together with the polynomials that these shifts reproduce. The previously known formula for this compact support case requires the further assumptions that the mask is finitely supported, and that the shifts of ϕ are stable. Adopting a stability assumption (but without assuming the finiteness of the mask), we derive that known formula from our general one. Moreover, we show that in the absence of stability, the lower bound provided by that previously known formula may be abysmal.

Our characterization is further extended to the FSI (i.e., vector) case, to the unisotropic dilation matrix case, and to even more general setups. We also establish corresponding results for refinable distributions.

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1. Introduction

1.1. General

Let ϕ be a function in $L_2(\mathbb{R}^d)$. We say that ϕ is (dyadically) **refinable** if there exists a 2π-periodic function m such that, a.e.,

$$\widehat{\phi}(2\cdot) = m\widehat{\phi}.$$

The function m is usually referred to as the **refinement mask**, and the refinable function is sometimes referred to as a **father wavelet** and/or a **scaling function**. The importance of refinable functions stems from their role in the construction of wavelet systems via the tool of multiresolution analysis (cf. [D2]) and in the analysis of subdivision schemes (cf. [DGL], [DyL] and [RiS]).

Smooth scaling functions are particularly desired. In the context of subdivision, the smoothness of the scaling function dictates the smoothness of the limiting surface of the process. In wavelet constructions, the smoothness of the scaling function is passed on to the wavelets, hence dictates the smoothness of the wavelet system. One should, thus, keep in mind that the scaling function ϕ is, in most circumstances, not known analytically, hence the analysis of the smoothness properties of the refinable ϕ must be based primarily on its mask m. This can be done, although not with ease; for example, a key ingredient in the success of Daubechies' construction of univariate orthonormal systems in [D1] was her ability to prove that the underlying scaling function can be selected to be as smooth as one wishes.

Initiated with the study of the smoothness of Daubechies' scaling functions, the study of smoothness properties of refinable functions via their masks has become one of the cornerstones of wavelet theory. Usually, this study is carried out under one or more of the following conditions (all notions used here will be defined in the sequel):

(1) The spatial dimension is 1.
(2) The number of scaling functions is 1 (aliased the 'scalar case', and/or the 'PSI case', and which is the only case we had described so far).
(3) The function ϕ is of compact support or the mask m is a trigonometric polynomial (the latter implies the former, but not vice versa).
(4) The shifts (i.e., integer translates) of ϕ are orthonormal; alternatively, these shifts are stable (i.e., form a Riesz basis).
(5) The dilation is dyadic, or, at least, isotropic.
(6) ϕ can be factored into $\phi_1 * \phi_2$, with ϕ_1 a smooth, well-understood (refinable) component, and ϕ_2 some distribution.

The analysis of the regularity of Daubechies' scaling functions was first done by inspecting directly the infinite product representation

$$\widehat{\phi} = \prod_{j>0} m(\cdot/2^j),$$

and estimating the decay of the Fourier transform. A treatment along these lines is given in [D1] and [D2], and already this approach establishes the fact that the underlying scaling function can be selected to be as smooth as one wishes, by choosing the B-spline factor to be of high order. (Recall that each Daubechies' scaling function is the convolution of a B-spline with a suitable compactly supported distribution). While this approach yields the asymptotic relation between the smoothness and the approximation order of Daubechies' functions (with the latter being explicitly known), it does not provide sharp estimates for the smoothness of a given fixed scaling function in this class, but only *lower bounds* on that smoothness. Later on, Cohen and Daubechies, [CD], used a similar method for estimating the smoothness of bivariate refinable functions, which are refinable with respect to the dilation matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The use of the *transfer operator* in the analysis of the smoothness of refinable functions appears first in the work of Deslauriers and Dubuc, [DD] Eirola, [E], and Villemoes, [V] (see also the related work [L]). Those studies are concerned with a univariate compactly supported refinable function whose mask is a polynomial, and are based on the factorization of the function into the convolution of a B-spline and 'another' factor. (Indeed, as the general analysis of the present paper shows, it is somewhat easier to estimate *negative* smoothness parameters of refinable elements.) There, a lower bound estimate on the smoothness of the refinable function is provided, and the lower bound is then shown to be sharp under the assumption that the shifts of the scaling function are stable.

The analysis of the smoothness of *compactly supported univariate* refinable functions via factorizations on the Fourier domain was generalized by Cohen, Daubechies and Plonka, to the FSI (i.e., vector) case, [CDP]. They used a factorization technique that was developed in [P], and, upon assuming that the shifts of Φ are linearly independent (an assumption stronger than stability, and which forces the mask to be polynomial), provided lower bounds on the smoothness of Φ. Recently, using factorization techniques on the 'time' domain, Micchelli and Sauer, [MS], studied the smoothness of univariate compactly supported PSI and FSI refinable functions. In [MS], a lower bound estimate on the smoothness of the refinable functions is provided and the lower bound is then shown to be sharp under the assumption that the shifts of the refinable functions are stable. While this work still assumes the mask(s) to be polynomial, estimates are obtained there not only in the L_2-case (aliased in the literature 'Sobolev exponents') but also in the general L_p-space (aliased 'Besov exponents'. The alias 'Hölder exponents' refers to the L_∞-case). More recently, Jia, Riemenschneider and Zhou [JRZ] obtained results similar to those of [MS] by using the subdivision operators and transfer operators and without using any factorization technique. We stress that the results in [CDP], [MS] and [JRZ] are 'global' in the sense that they only estimate the smoothness of the least smooth function in the vector Φ, and cannot determine the smoothness of any other function in that vector.

In contrast with the univariate case, masks of multivariate refinable functions are not guaranteed to be factorable in any convenient way. In the PSI multivariate situation, and under the additional assumption that the refinable function is the convolution of a box spline with another factor, Goodman, Micchelli and Ward [GMW] obtained some estimates of the smoothness. Other results concerning this same problem also obtained by Cohen and Daubechies in [CD1] and by Dahlke, Dahmen and Latour in [DDL]. Aiming at estimating the smoothness of certain multivariate interpolatory refinable functions that were constructed in [RiS], Riemenschneider and Shen provided a method for bounding that smoothness from below without using the factorization of the mask (they also provided simplified estimates for factorable masks). Their technique was generalized by Shen in [S] resulting in lower bounds on the global smoothness in the FSI case. Both [RiS] and [S] deal with the dilation matrix $s = 2I$. Jiang, [Ji], generalized the smoothness result of [S] to a general dilation matrix s.

Most recently, the transfer operator approach was employed (independently) by Jia [J1], and by Cohen, Gröchenig and Villemoes, [CGV]. The results of these two papers are closely related: both show, without assuming any possible factorization, how to provide lower bounds on the smoothness of a single multivariate compactly supported refinable function, and both make fairly minimal assumptions on the dilation matrix (In [J1], the dilation is assumed to be isotropic, i.e., a constant multiple of a unitary transformation; even less is assumed in [CGV], but, alas, they have to measure smoothness, in case the dilation is not isotropic, in non-standard ways). Both articles show that, if the refinable function has stable shifts, then the lower bound estimates are *sharp*, i.e., they characterize the smoothness class of the function. It was the reading of these two articles, and especially the use of 'damping factors' in [CGV], that had led us to the understanding of the regularity problem, thereby to the results reported in the present paper.

Finally, we remark that there are several papers devoted to the estimation of the smoothness of the refinable function directly in the 'time' domain. The interested readers can find details on those approaches in [DL1,2], [MP] and [DLM].

While, as partially detailed above, many important advances on the regularity problem have been made so far, the current literature is far from offering a comprehensive solution to this question. In particular, there are only a handful of results that characterize smoothness without making the stability assumption. (An exception is [LMW], where Lau, Ma and Wang gave a characterization of the smoothness in the univariate PSI case without assuming the stability. One must note, however, that the univariate case is, once again, simpler, due to the availability of factorization techniques).

It is probably worth emphasizing that the smoothness of the scaling function is completely independent of the stability of it shifts. With that in mind, one should desire to have an analysis of the smoothness property that does not rely on that stability assumption. We also add that the currently known *lower* bounds on the smoothness are not sharp in general: indeed, we provide examples in this paper of univariate compactly supported scaling functions which, on the one hand, are as smooth as one wishes, while, on the other hand, the lower bound estimates from the literature (we used those of [J1] and [CGV]) cannot ascribe any amount of smoothness to these functions.

Furthermore, in the FSI setup (i.e., when several scaling functions are involved), the present literature on the regularity problem is even more limited in nature; in particular, we are not aware of any characterization of the smoothness parameter in the multivariate FSI case (i.e., several variables, a vector of scaling functions), let alone we do not know of any result in the literature (even in one variable) that can be applied to estimate separately the smoothness of each individual scaling function.

The discussion in this paper is confined to the development of the theory, and the theory only. It is then natural to question (as a referee of this article did) whether the characterizations of this paper can be implemented. The problem is particularly interesting in the absence of stability, since the characterization provided in this paper then requires input that is not available by a mere inspection of the mask. We would like, thus, to refer to the subsequent article, [RST], where an algorithm for computing the smoothness parameter (for bivariate scaling functions) is developed, and tested.

In the main development of this paper, we assume the scaling function(s) to be of compact support. This is an important assumption, that allows one to provide a characterization based on the action of a linear operator acting on a *finite dimensional* space. In contrast, nowhere in this article we make the stronger assumption that the mask is a trigonometric polynomial. A referee had asked us whether we currently have interesting examples of compactly supported scaling functions whose masks are *not* trigonometric polynomials, and whether, given a non-polynomial mask, we can effectively decide whether the corresponding scaling function(s) has compact support or not. Unfortunately, our current answer to each of these questions is in the negative (we do provide in §2 examples of compactly supported scaling functions whose masks are not polynomial, but we are reluctant to label them as 'interesting'). We avoid the stronger assumption that the underlying mask is a trigonometric polynomial, simply because that assumption does not lead us to any strengthening of our results, nor to any simplification in our arguments.

1.2. An overview of this paper

In the current paper we characterize completely the L_2-regularity of refinable functions, or, more generally, distributions. The analysis is carried out without any restriction on the refinable element: it may be a vector or a singleton, it may be an L_2-function, or a tempered distribution. It may be compactly supported, or it may decay very slowly at ∞, and we do not assume the shifts of the scaling function(s) to satisfy any stability or similar assumption. Upon imposing a compact support assumption on the scaling function(s), we can make crisper, cleaner statements.

The paper is laid out as follows. In the rest of the introduction, we outline the main idea that is invoked this article, and state and prove the two key lemmata that unravel the regularity problem. In §2, we discuss in detail the PSI case, under the assumptions that ϕ is compactly supported, and that the dilation is isotropic. In §3, we present our general setup, and derive the basic results. In §4, we discuss equivalent definitions of standard smoothness spaces.

Here is a short summary of the main finding of this paper. It is described in a general PSI setup, but under the assumption that ϕ is an L_2-function (and not a mere distribution). The treatment of refinable distributions is obtained by modifying slightly the description below. The treatment of the FSI case (when the refinable element is a vector of functions) is obtained by generalizing correctly the observations made below. The description assumes intimate familiarity with wavelet terminology. Of course, notions that are not defined here will be defined in the main body of this paper.

Let $\phi \in L_2(\mathbb{R}^d)$ be arbitrary. Its **autocorrelation** function $\phi^{\#}$ is then defined by

$$\phi^{\#} : x \mapsto \int_{\mathbb{R}^d} \phi \, \overline{\phi(\cdot - x)}.$$

The L_2-regularity of ϕ is completely determined by the smoothness of its autocorrelation function at the origin: let ∇ be a 'suitable' difference operator of order 2ℓ (which means that ∇ is a finite linear combination of integer translations, that it annihilates all polynomials of degree $< 2\ell$, and that its Fourier series is non-negative everywhere while being positive in some punctured origin-neighborhood). Then (as it must be well-known), given any $\alpha < \ell$, $\nabla_h \phi^{\#}(0) = O(h^{2\alpha})$ if and only if ϕ lies in the Besov space $B_\infty^\alpha(L_2(\mathbb{R}^d))$ (which is slightly larger than the Sobolev space $W_2^\alpha(\mathbb{R}^d)$; here ∇_h is the h-scale of ∇). Therefore, if we denote by t_h the discrete Fourier transform of $\nabla_h \phi^{\#}$, the L_2-smoothness of ϕ is completely determined by rate of decay of

$$\nabla_h \phi^{\#}(0) = \int_{\mathbb{T}^d} t_h = \|t_h\|_{L_1(\mathbb{T}^d)},$$

provided ∇ is of sufficiently large order.

Now assume, further, that ϕ is refinable with bounded mask m, and let T be the transfer operator of $\tau := |m|^2$ (this operator is defined in the sequel). The crux of the analysis of this paper is the observation that, if the dilation matrix s is a scalar multiple by $\lambda > 1$ of a unitary matrix, then, for each $h_k := \lambda^{-k}$ we have that

$$\int_{\mathbb{T}^d} t_{h_k} = \int_{\mathbb{T}^d} T^k(t_1).$$

For a more general (i.e., unisotropic) dilations, the above equality does not hold, still, its left hand side can be bounded above and below in terms of the right hand side (for appropriate choices of k).

In conclusion, iterations with the transfer operator determine completely the regularity of ϕ, if we choose correctly the initial seed t_1. One then computes that $t_1 = t\widetilde{\phi}^2$, with t the Fourier series of ∇, and $\widetilde{\phi}^2$ is the discrete Fourier transform of $\phi^{\#}$, the latter is well-known to be an eigenvector of T corresponding to the eigenvalue 1. This lays ground to the belief that t_1 is computable, and that the entire process is feasible.

In all examples of interest (that we are aware of) $\widetilde{\phi}$ is known to be *bounded.* In that event, we obtain that $t_1 \leq \text{const}\, t$, hence iterating with the explicitly known t provides one with *lower bounds* on the sharp regularity parameter. Conversely, if $1/\widetilde{\phi}$ is bounded, then $t_1 \geq \text{const}\, t$, hence iterating with t provides one with *upper bounds* on the sharp regularity parameter. Consequently, if $\widetilde{\phi}$ is bounded above and away from zero, iterations with t are on par with iterating with t_1. The boundedness above and below of $\widetilde{\phi}$ is a property known as the **stability** or the **Riesz basis** property of the shifts of ϕ, and in this way we recover the current literature results on the estimation of regularity under that Riesz basis assumption. In the absence of the stability assumption, a deeper analysis, based on the dependence relations satisfied by the shifts of ϕ, can be made to allow one to choose an alternative good initial seed.

Finally, if ϕ is compactly supported, is refinable with respect to any dilation matrix s, and its mask is bounded (but is not necessarily a polynomial), then we show that after few iterations the function $T^k(t\widetilde{\phi}^2)$ must lie in some well-defined finite dimensional space of trigonometric polynomials. This means that, in the compact support case, a slightly cruder analysis of the regularity of ϕ can be given in terms of the spectral radius ρ of the transfer operator, when restricted to an appropriate finite-dimensional subspace of trigonometric polynomials. That approach leads to a simple formula that connects the regularity parameter $\alpha(\phi)$ of ϕ (i.e., the maximal number such that $\phi \in W_2^\alpha(\mathbb{R}^d)$, for every $\alpha < \alpha(\phi)$) and the above-mentioned ρ.

Remark. Our analysis indicates that in the compact support case, unless $\alpha(\phi)$ above is non-positive, the relation $\phi \in W_2^{\alpha(\phi)}(\mathbb{R}^d)$ never holds. That should come at no surprise: in the case of a B-spline, for example, $\alpha(\phi) = k - 1/2$, with k the order of the B-spline; the B-spline, indeed, does not lie in $W_2^{k-1/2}(\mathbb{R}^d)$; it, nevertheless, lies in the Besov space $B_\infty^{k-1/2}(L_2)$. This slightly weaker relation, i.e., that $\phi \in B_\infty^{\alpha(\phi)}(L_2)$ is possible: it is shown to be related to non-defectiveness of certain eigenvalues of the transfer operator.

During the preparation of this paper, we debated whether to analyse the regularity of the function ϕ is terms of 'plain' Sobolev spaces, or in term of the more accurate Besov spaces $B_\infty^\alpha(L_2(\mathbb{R}^d))$. Since the former presentation, in the PSI case, is conceptually simpler, and since we already flood the reader with fine details, insights and hindsights, we decided to stick with Sobolev space analysis. □

1.3. The two key lemmata

Our approach is largely based on two fairly simple observations, that we list and prove below. First, let us extend the notion of "a refinable function", and introduce the underlying operators employed in the analysis.

Let s be a **dilation matrix**. By that we mean any $d \times d$ integer invertible matrix which is also **expansive**, i.e., its entire spectrum lies outside the closed unit disc. Let ϕ be a tempered distribution whose Fourier transform is a function. We say that ϕ is **refinable with respect to the dilation matrix** s, if there exists an essentially bounded 2π-periodic function m, such that, almost everywhere,

$$\mathcal{D}^{-1}\widehat{\phi} = m\widehat{\phi}, \tag{1.1}$$

with $\mathcal{D}$ the dilation operator

$$\mathcal{D}f : \omega \to f(s^{*-1}\omega).$$

The **subdivision operator** T_m^* associated with m is defined as follows:

$$T_m^* : L_2(\mathbb{T}^d) \to L_2(\mathbb{T}^d) : f \mapsto \sqrt{|\det s|}\, m\, \mathcal{D}^{-1} f. \tag{1.2}$$

Note that, with $\tau := |m|^2$,

$$T^* : f \mapsto |\det s|\, \tau\, \mathcal{D}^{-1} f$$

is, up to a normalization factor, the subdivision operator associated with the autocorrelation $\phi^\#$ (say, in case $\phi \in L_2(\mathbb{R}^d)$).

The adjoint of the subdivision operator T^* is the **transfer** or **transition operator** $T := T_\tau$. To define this operator, let

$$\Gamma$$

be any representer set of the quotient group $2\pi(\mathbb{Z}^d/s^*\mathbb{Z}^d)$. Then T is defined as follows:

$$T : L_2(\mathbb{T}^d) \mapsto L_2(\mathbb{T}^d) : f \mapsto \mathcal{D}\big(\sum_{\gamma\in\Gamma}(\tau f)(\cdot + s^{*-1}\gamma)\big). \tag{1.3}$$

For example, in the case of dyadic dilations in one dimension, Γ can be chosen as $\{0, 2\pi\}$, and T becomes

$$(Tf)(\omega) = (\tau f)(\frac{\omega}{2}) + (\tau f)(\frac{\omega}{2} + \pi).$$

Now, let ν be a compactly supported distribution (not necessarily refinable) for which

$$\eta := \phi * \nu$$

is known to be in L_2. Set

$$\widetilde{\eta}^2 := \sum_{j\in 2\pi\mathbb{Z}^d} |\widehat{\eta}(\cdot + j)|^2.$$

It is well-known that the series converges in L_1 on

$$C := [-\pi, \pi]^d$$

to the discrete Fourier transform of the autocorrelation function $\eta^\#$.

Lemma 1.4. *Let t be a bounded 2π-periodic function. Then, with ϕ, ν and $\widetilde{\eta}^2$ as above, we have for every $k = 0, 1, 2, \ldots$:*

$$\|T^k(|t|^2\widetilde{\eta}^2)\|_{L_1(\mathbb{T}^d)} = \|t\widetilde{\eta}\, T_m^{*\,k} 1\|^2_{L_2(\mathbb{T}^d)} = \int_{\mathbb{R}^d} |\widehat{\phi}|^2\, \mathcal{D}^k(|t\widehat{\nu}|^2). \tag{1.5}$$

Proof: Upon changing variables, we get from the right-most expression, after using k times the refinement equation:

$$|\det s|^k \int_{\mathbb{R}^d} |t\widehat{\nu}|^2 |\widehat{\phi}|^2 \prod_{j=0}^{k-1} \tau(s^{*j}\cdot) = \int_{\mathbb{R}^d} |t\widehat{\eta}|^2 |T_m^{*\,k} 1|^2.$$

Writing $\mathbb{R}^d$ as the disjoint union of integer shifts of $C = [-\pi, \pi]^d$, and using the 2π-periodicity of t and $T_m^{*\,k}1$, we obtain the right equality in (1.5). The left-most equality follows from the fact that $|T_m^{*\,k}1|^2 = |T^{*k}1|$, together with the fact that T is the adjoint of T^*. □

An identical argument to that used in the proof of the above lemma shows that, under the same conditions, and for every $f \in L_2(\mathrm{T}^d)$,

$$\langle T^k(t\widetilde{\eta}^2), f\rangle_{L_2(\mathrm{T}^d)} = \int_{\mathbb{R}^d} \overline{f}\, |\widehat{\phi}|^2\, \mathcal{D}^k(t|\widehat{\nu}|^2). \tag{1.6}$$

Note that this identity explicitly identifies $T^k(t\widetilde{\eta}^2)$ as the 2π-periodization of $|\widehat{\phi}|^2\, \mathcal{D}^k(t|\widehat{\nu}|^2)$.

Next, assume that ϕ is compactly supported, let

$$\Omega_\phi$$

be the convex hull of $\operatorname{supp}\phi - \operatorname{supp}\phi$ and set

$$\mathcal{Z}_\phi := \Omega_\phi \cap \mathbb{Z}^d.$$

Also, let

$$H_\phi \tag{1.7}$$

be the space of all trigonometric polynomials with spectrum in $\mathcal{Z}_\phi$, i.e.,

$$f \in H_\phi \iff f(\omega) = \sum_{j\in\mathcal{Z}_\phi} c(j) e^{ij\cdot\omega}.$$

If m is a trigonometric polynomial, it is easy to see then that for (at least) certain dilation matrices, and for any trigonometric polynomial t, $T^k t \in H_\phi$ for all sufficiently large k. Our next lemma rigorously establishes a slightly different assertion, that for *certain* polynomials t, for *any* dilation matrix (as defined in the beginning of the present subsection), and without assuming the polynomiality of the mask, the relation $T^k t \in H_\phi$ holds for all large k.

Lemma 1.8. *Let ϕ, ν, η, be as above, and assume that ϕ is compactly supported. Let $q : \mathbb{Z}^d \to \mathbb{C}$ be any finitely supported sequence. Then, there exists an integer k_0, that depends on* $\operatorname{supp}\phi$, $\operatorname{supp}\nu$, *and* $\operatorname{supp} q$, *as well as on the dilation matrix s (but on nothing else), such that, for $k \ge k_0$, $T^k(\widehat{q}\widetilde{\eta}^2) \in H_\phi$.*

Proof: Set $f^\vee$ for the inverse Fourier transform of f. Let ε be the distance between Ω_ϕ and $\mathbb{Z}^d\backslash\mathcal{Z}_\phi$. Since s is expansive, we can find a sufficiently large k_0 such that $\operatorname{supp}((\mathcal{D}^{k_0}(\widehat{q}|\widehat{\nu}|^2))^\vee)$ lies in a ball centered at the origin with radius ε. For each $k \ge k_0$, this implies that function $g_k := (|\widehat{\phi}|^2\, \mathcal{D}^k(\widehat{q}|\widehat{\nu}|^2))^\vee$ is supported in a region that is disjoint of $\mathbb{Z}^d\backslash\mathcal{Z}_\phi$ (since it is the convolution product of the function $\phi^\#$ which is supported in Ω_ϕ with a distribution that is supported in an ε-ball). Thus, choosing $t := \widehat{q}$ and $f := e_j$ in (1.6), with $e_j : \omega \mapsto e^{ij\cdot\omega}$ the exponential with frequency j, we get that

$$\langle T^k(\widehat{q}\widetilde{\eta}^2), e_j\rangle_{L_2(\mathrm{T}^d)} = g_k(-j) = 0,$$

for $j \in \mathbb{Z}^d\backslash\mathcal{Z}_\phi$, and for $k > k_0$. □

2. The PSI case (under simplifying assumptions)

2.1. Postmortem analysis, main result, some examples

We assume in this subsection that ϕ is a compactly supported refinable distribution with a *bounded* mask m (cf. (1.1)). We further assume, in this section only, that the dilation is **isotropic**, i.e.,

$$s^* s = \lambda I, \tag{2.1}$$

for some $\lambda > 1$.

Note that, as anywhere else in this article, the refinable distribution is assumed to be compactly supported, and its mask is assumed to be bounded. The next example provides an abundance of compactly supported refinable functions with bounded non-polynomial masks.

Example: Let ϕ be an arbitrary refinable function with a trigonometric polynomial mask m. Let h be an arbitrary trigonometric polynomial with $h(\omega) \ne 0$, $\omega \in \mathbb{R}^d$. Suppose that $m(\omega)h(s^*\omega)/h(\omega)$, is not a trigonometric polynomial. Define

$$\widehat{f} = h\widehat{\phi}.$$

Then, f is a compactly supported refinable function with a bounded (non-polynomial) mask $m(\omega)h(s^*\omega)/h(\omega)$. □

We define the L_2**-regularity parameter** $\alpha(\phi)$ of ϕ to be the maximal number α for which $\phi \in W_2^{\alpha'}(\mathbb{R}^d)$ for every $\alpha' < \alpha$. Note that $\alpha(\phi)$ may be negative.

The relevance of T (cf. (1.3)) and H_ϕ (cf. (1.7)) to the present context is due to the following result.

Theorem 2.2. *Let ϕ be a compactly supported refinable distribution and let T be its associated transfer operator. Then there exists an eigenpair (μ, f_μ) of T such that $f_\mu \in H_\phi$ and such that, with $\rho := |\mu|$,*

$$\alpha(\phi) = -\frac{\log_\lambda \rho}{2}.$$

Also, though we do not formally prove it, our analysis strongly indicates that $f_\mu \geq 0$, hence that $\mu > 0$. In any event, such a result is not extremely useful if one cannot find the "correct" pair (μ, f_μ), and we therefore characterize in the next result the eigenpair of Theorem 2.2. For simplicity, we first assume that ϕ is in L_2. The characterization utilizes the ideal $I_\phi \subset L_\infty(\mathbb{T}^d)$ which is defined below. There, as elsewhere in this paper,

$$\Pi$$

stands for the space of all d-variate (algebraic) polynomials.

Definition 2.3: the ideal I_ϕ. Let ϕ be a compactly supported L_2-function (not necessarily refinable), and assume that $\widehat{\phi}(0) \neq 0$. Let $\phi^\#$ be its autocorrelation. Let

$$\Pi_\phi$$

be the space of all polynomials reproduced by the shifts of $\phi^\#$, i.e., $p \in \Pi_\phi$ if and only if $\sum_{j\in\mathbb{Z}^d} p(j)\phi^\#(\cdot - j) \in \Pi$. Then I_ϕ is the collection of all $L_\infty(\mathbb{T}^d)$-functions f that are smooth at the origin and satisfy:

(i) $f/\widetilde{\phi}^2 \in L_\infty(\mathbb{T}^d)$.

(ii) f is annihilated by Π_ϕ in the sense that $p(-iD)f(0) = 0$, for all $p \in \Pi_\phi$.

Example: I_ϕ under a stability assumption. The shifts of ϕ are stable if $\widetilde{\phi}$ vanishes nowhere (in $\mathbb{R}^d$). In this case, the first condition in the definition of I_ϕ is vacuous, and hence I_ϕ is then the space of all $L_\infty(\mathbb{T}^d)$-functions which vanish at the origin to "a sufficiently high" degree. It is worthwhile noting that the above stability property (of refinable functions) can also be checked via the corresponding transfer opreators (see e.g. [L], [LLS1] and [S]). □

Lemma 1.4 (or, more precisely the remark after (1.6)) can be used to show that I_ϕ is an invariant subspace of the transfer operator T. Our eventual proof of Theorem 2.2 is based on inspecting the iterations $T^k(t\widetilde{\phi}^2)$, for a suitable trigonometric polynomial t. By Lemma 1.8, such iterations must bring us into H_ϕ. At the same, if t vanishes to a high order at the origin, $t\widetilde{\phi}^2 \in I_\phi$, and are going to stay in I_ϕ, hence to enter $H_\phi \cap I_\phi$. In fact, we have the following result which is proved at the end of §2.2.

Theorem 2.4. *For a compactly supported refinable L_2-function ϕ with bounded mask and non-zero mean value, the eigenvector f_μ of Theorem 2.2 lies in $I_\phi \cap H_\phi$. Moreover, the magnitude $\rho = |\mu|$ of the eigenvalue μ in Theorem 2.2 is the spectral radius of the restriction of T to the largest T-invariant subspace of $I_\phi \cap H_\phi$.*

Discussion. The first condition in the definition of I_ϕ is intimately related to the dependence relations satisfied by the shifts of $\phi^\#$. Indeed, one can show that f satisfies that first condition if and only if the condition

$$\sum_{j\in\mathbb{Z}^d} e^{i\theta\cdot j} p(j)\phi^\#(\cdot - j) = 0,$$

for some $\theta \in \mathbb{R}^d$ and $p \in \Pi$, implies that $p(-iD)f(\theta) = 0$.

Indeed, various eigenvalues of the restriction of T to H_ϕ are there due to dependence relations among the shifts of $\phi^\#$, or because of certain polynomials that these shifts reproduce; for example, T may have various eigenvectors that are of the form $\widetilde{\eta}^2$, with $\eta = p(D)\phi$, for a suitable differential operator $p(D)$. None of these eigenvectors is the one specified in Theorem 2.2. Fortunately, the eigenvectors of T^* (conceived as an operator on H_ϕ^*) that are related to those eigenvalues are supported on the zero set of $\widetilde{\phi}^2$ when augmented by the origin. This, in fact, is the heuristic explanation to Theorem 2.4. □

Remark. As said before, the underlying assumption in our analysis is that the refinable function is not given explicitly, and the only readily available information is its mask. The mask clearly suffices in order to define and iterate with the transfer operator. However, an attempt to implement the characterization of Theorem 2.4 (say, via a suitable eigensolver) requires more: it requires 'an access' to the space $H_\phi \cap I_\phi$ (such as an algorithm that, given the mask m, constructs a basis for that space). It is beyond the scope of this paper to discuss these (important) algorithmic details, and code implementation. The forthcoming article [RST] contains a comprehensive discussion of that topic (in one and two dimensions). In brief, the approach there is based on the introduction of a set of projection operators which project vectors into $I_\phi \cap H_\phi$. This, together with the Arnoldi method as the eigensolver form the backbone of a rebust algorithm for computing the regularity parameter. One must keep in mind that the main challenge in the [RST] algorithm is the possible lack of stability. When the shifts of ϕ *are* stable, a basis for the space $I_\phi \cap H_\phi$ can be computed directly from the mask. In fact, [RiS] and [HJ] computed regularity parameters of several multivariate interpolatory refinable functions (whose shifts are stable). □

Remark. As we have just explained, the space H_ϕ contains various eigenvectors that should be excluded when determining the smoothness of ϕ. However, the space $I_\phi \cap H_\phi$ avoids various 'harmless' eigenvalues, i.e., eigenvalues that are smaller in magnitude than the value ρ of Theorems 2.2 and 2.4. This means that Theorem 2.4 remains true if we replace I_ϕ there by carefully selected superspaces of it.

□

Example: B-splines. Only in very rare situations the information provided in Theorem 2.4 enables one to easily find ρ. One such situation is that of the *kth*-order univariate B-spline. Here, H_ϕ is the collection of all trigonometric polynomials with spectrum in $\{-k,\ldots,k\}$. The function $\phi^\#$ is now the centered B-spline of order $2k$, whose shifts reproduce all polynomials of degree $< 2k$. Since the shifts of ϕ are stable, this implies that the trigonometric polynomials with $2k$-fold zero at the origin and spectrum in $\{-k,\ldots,k\}$ comprise $I_\phi \cap H_\phi$. From that, it is easy to conclude that $H_\phi \cap I_\phi$ is the 1-dimensional space spanned by $f_\mu(\omega) := \sin^{2k}(\omega/2)$. So, it must be that this function is an eigenvector of the corresponding transfer operator, whose eigenvalue is our desired ρ, and that conclusion must hold true regardless of the choice of the dilation (recall that the B-spline is totally refinable, i.e., refinable with respect to all integer dilations). Indeed, for, say, dyadic dilations, $\tau(\omega)$ is $\cos^{2k}(\omega/2)$ and one immediately finds that $\rho = 2^{1-2k}$, recovering thereby the fact that the L_2-regularity parameter of the B-spline is $k - 1/2$. □

Example: the support function of $[0,3]$**.** In this case $\alpha(\phi) = \frac{1}{2}$, and ρ of Theorem 2.2 should, thus, be $\frac{1}{2}$. The function ϕ is dyadically refinable with $\tau(\omega) = \cos^2(3\omega/2)$. The space H_ϕ consists of the trigonometric polynomials with spectrum in $\{-3,\ldots,3\}$. Already the smaller space H_1 of trigonometric polynomials with spectrum in $\{-2,\ldots,2\}$ is invariant under T. The spectrum of the restriction of T to H_1 is $(1,1,-1,\frac{1}{2},-\frac{1}{2})$, but all these eigenvalues are related to polynomial reproduction properties or linear dependence properties of the shifts of $\phi^\#$ and none is indicative of $\alpha(\phi)$ (the appearance of the 'right' value $\frac{1}{2}$ is accidental). The ideal I_ϕ contains all polynomials with double zeros at each of 0, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$. One finds that $\dim(H_\phi \cap I_\phi) = 1$, and that this space is spanned by $f(\omega) = 1 - \cos 3\omega$. Thus, according to our theory, the pair $(\frac{1}{2}, f)$ must be an eigenpair of T, and, indeed, it is. □

Example: bivariate box splines. Let Ξ be a set of bivariate integer vectors, such that each pair of them is linearly independent. A bivariate box spline is the compactly supported function defined by

$$\widehat{\phi}(\omega) = \prod_{\xi\in\Xi}\left(\frac{e^{-i\xi\cdot\omega}-1}{-i\xi\cdot\omega}\right)^{n_\xi}, \quad n_\xi \in \mathbb{Z}_+.$$

The box spline is piecewise-polynomial of degree $n-2$, with $n := \sum_{\xi\in\Xi} n_\xi$, and is dyadically refinable with mask

$$\tau(\omega) = \prod_{\xi\in\Xi}\cos^{2n_\xi}(\xi\cdot\omega/2).$$

The space $H_\phi \cap I_\phi$ is spanned here by functions of the form

$$f_{\xi'}(\omega) \prod_{\xi \in \Xi \backslash \xi'} \sin^{2n_\xi}(\xi \cdot \omega/2), \quad \xi' \in \Xi,$$

where f_ξ is any trigonometric polynomial with spectrum in $\{-n_\xi \xi, \ldots, n_\xi \xi\}$. The eigenpair of Theorem 2.4 can be computed analytically: the eigenvector is the polynomial

$$f(\omega) := \widetilde{B}^2(\xi' \cdot \omega) \prod_{\xi \in \Xi \backslash \xi'} \sin^{2n_\xi}(\xi \cdot \omega/2),$$

with $\xi' \in \Xi$ the direction with the highest multiplicity $n_{\xi'}$ (which, of course, may not be unique), and with B the univariate B-spline of order $n_{\xi'}$ (the description above of the eigenvector holds only if $\xi' \notin 2\mathbb{Z}^2$; the eigenvalue below, nonetheless, is correct even without this assumption). One then computes the critical eigenvalue to be

$$\rho = 2^{1+2n_{\xi'}-2n}.$$

By Theorem 2.2, the regularity parameter of ϕ is then

$$\alpha(\phi) = n - n_{\xi'} - 1/2.$$

Note that it is well-known that $\phi \in C^{n-n_{\xi'}-2} \backslash C^{n-n_{\xi'}-1}$. □

2.2. Finding the regularity parameter by iterations

While it seems hard to find the pair (μ, f_μ) analytically, one can instead try to use the power method for estimating $\rho = |\mu|$, i.e., for a generic $f \in H_\phi \cap I_\phi$, we will have

$$\rho = \lim_{k\to\infty} \|T^k f\|^{1/k}. \tag{2.5}$$

Moreover, we do not have to start the iterations with $f \in I_\phi \cap H_\phi$: it suffices to choose a suitable $f \in I_\phi$, and to let the iterations bring us into H_ϕ.

We now embark on the actual (theoretical) computations of the regularity parameter. For that, we first make the following definition:

Definition 2.6. Let U be a finite collection of non-negative trigonometric polynomials. We say that U is a **complete system of order** ℓ if $\sum_{u\in U} u$ has an isolated zero of order 2ℓ at the origin. □

Now, with λ as in (2.1), let V_k, $k = 0, 1, \ldots$ be the rings

$$V_k = \{\omega \in \mathbb{R}^d : \ \lambda^{k-1} K \le |\omega| \le \lambda^k K\},$$

with K some positive number. It is well-known that ϕ lies in the space W_2^α if and only if the sequence

$$k \mapsto \lambda^{\alpha k} \|\widehat{\phi}\|_{L_2(V_k)}$$

is square-summable. From that one immediately concludes that the regularity parameter of ϕ is

$$\alpha(\phi) = -\limsup_{k\to\infty} \frac{\log_\lambda(\|\widehat{\phi}\|_{L_2(V_k)})}{k}. \tag{2.7}$$

Lemma 1.4 tells us how to compute the norms in (2.7): assuming $\phi \in L_2$, one may choose t there to be the 2π-periodization of the support function of V_0 (for a small enough K), and conclude from the lemma that, given any $\alpha > 0$,

$$\|\widehat{\phi}\|_{L_2(V_k)} = O(\lambda^{-\alpha k}) \quad \iff \quad \|T^k(t\widetilde{\phi}^2)\|^{1/2}_{L_1(\mathbb{T}^d)} = O(\lambda^{-\alpha k})$$

(as $k \to \infty$).

This allows us, in our search for $\alpha(\phi)$, to iterate with T, starting with the initial seed $t\widetilde{\phi}^2$. A possible snag here is that the selected t is not a polynomial, and we do not, thus, benefit from Lemma 1.8, i.e., the iterations do not stay inside a well-prescribed finite dimensional space. For that reason, we approximate the support function of V_0 by non-negative trigonometric polynomials, which is exactly the role played by systems of order ℓ that were introduced before. Indeed, the following is true (and known; it is related to the very basic definition of smoothness spaces in terms of finite differencing):

Lemma 2.8. *Let U be a complete system of order ℓ, and let $\phi \in L_2(\mathbb{R}^d)$. Let $\alpha(U,\phi)$ be the supremum of all α for which, for every $u \in U$, the sequence*

$$k \mapsto \lambda^{\alpha k} \, \|\widehat{\phi}^2 \, \mathcal{D}^k u\|^{1/2}_{L_1(\mathbb{R}^d)}$$

is bounded. Let $\alpha(\phi)$ be a regularity parameter of ϕ. Then either $\alpha(\phi) = \alpha(U,\phi)$, or $\alpha(\phi) \ge \alpha(U,\phi) \ge \ell$.

We prove this elementary lemma in §4. Here, we combine this lemma with Lemma 1.4 (with $|t|^2$ there being our u here), to conclude:

Theorem 2.9. *Let ϕ be a compactly supported L_2-function with bounded mask m and a transfer operator T. Let U be a complete system of order ℓ. Set*

$$\alpha(u,\phi) := -\limsup_{k\to\infty} \frac{\log_\lambda \|T^k(u\widetilde{\phi}^2)\|^{1/2}_{L_1(\mathbb{T}^d)}}{k},$$

and $\alpha(U,\phi) := \min_{u\in U} \alpha(u,\phi)$. Then the regularity parameter $\alpha(\phi)$ of ϕ is $\ge \alpha(U,\phi)$. Moreover, $\alpha(\phi) = \alpha(U,\phi)$, in case $\alpha(U,\phi) < \ell$.

Theorem 2.2 (for a *function* ϕ) now easily follows from the above theorem when combined with Lemma 1.8.

We stated Theorem 2.9 in terms of the transfer operator iterations. In view of Lemma 1.4, we could also state it in terms of the subdivision iterations:

$$\|t\widetilde{\phi}\, {T_m^*}^k 1\|^{1/2}_{L_2(\mathbb{T}^d)}, \quad t := \sqrt{u},$$

as is discussed below. In any event, the function $\widetilde{\phi}^2$ that is involved in the above estimation may not be known, hence may be wished to be avoided. Clearly, if f is *any* function, then

$$|f| \le \widetilde{\phi} \implies \|tf\, {T_m^*}^k 1\|^{1/2}_{L_2(\mathbb{T}^d)} \le \|t\widetilde{\phi}\, {T_m^*}^k 1\|^{1/2}_{L_2(\mathbb{T}^d)},$$

$$|f| \ge \widetilde{\phi} \implies \|tf\, {T_m^*}^k 1\|^{1/2}_{L_2(\mathbb{T}^d)} \ge \|t\widetilde{\phi}\, {T_m^*}^k 1\|^{1/2}_{L_2(\mathbb{T}^d)}.$$

This allows us to obtain *upper bounds* and *lower bounds* on the regularity parameter by iterating with suitable initial seeds f. We switch now back to the transfer operator language, and set, for any non-negative function g:

$$\alpha_g(u,\phi) := -\limsup_{k\to\infty} \frac{\log_\lambda \|T^k(ug)\|^{1/2}_{L_1(\mathbb{T}^d)}}{k}.$$

The above discussion when combined with Theorem 2.9 implies the following result. Under the additional assumption that m is a polynomial, parts (c,d) of that result below are due to Jia [J1], and Cohen, Gröchenig, Villemoes, [CGV].

Corollary 2.10. *Let ϕ be a compactly supported L_2-function with bounded mask m, associated with a transfer operator T. Let U be a complete system of order ℓ, and let g be some non-negative $L_\infty(\mathbb{T}^d)$-function. Let $\alpha_g(u,\phi)$ be defined as above. Then:*

(a) If $\widetilde{\phi}^2 \le \text{const}\, g$, then $\alpha(\phi) \ge \alpha_g(U,\phi) := \min_{u\in U} \alpha_g(u,\phi)$.
(b) If $\widetilde{\phi}^2 \ge \text{const}\, g$, and $\alpha_g(U,\phi) < \ell$, then $\alpha(\phi) \le \alpha_g(U,\phi)$.
(c) We always have $\alpha(\phi) \ge \alpha_1(U,\phi)$.
(d) If the shifts of ϕ are stable, and $\alpha_1(U,\phi) < \ell$, then $\alpha(\phi) = \alpha_1(U,\phi)$.

Proof: From the discussion preceding the corollary, we conclude that, under the assumption in (a), $\alpha_g(U,\phi) \le \alpha(U,\phi)$, hence (a) follows form Theorem 2.9, and (b) is proved similarly. Item (c) is obtained by observing that $\widetilde{\phi}^2$ is a polynomial (since we assume ϕ to have compact support) hence bounded, and so (a) is certainly satisfied for the choice $g = 1$. Finally, the stability assumption in (d) tells us that $\widetilde{\phi}^2 \ge \text{const}$, so under this assumption, we can apply (b) with respect to $g = 1$. $\square$

Parts (c,d) of Corollary 2.10 suggest a simpler lower bound on the regularity parameter $\alpha(\phi)$, and show that this bound is sharp under a stability assumption. It must be understood that, in the absence of stability, these lower bounds, not only that may not be sharp, but may simply be pitiful. This observation is implicit in one of our previous examples and is generalized in the next one.

Example: iterating with initial seeds that are not divisible by $\widetilde{\phi}^2$ may be a waste of time. Let ϕ_0 be any univariate dyadically refinable function with mask m_0 and set $\phi_{n+1} = \phi_n + \phi_n(\cdot + 1) + \phi_n(\cdot + 2)$, $n = 1, 2, 3, \ldots$. Then ϕ_n is refinable with mask

$$m_n(\omega) = \left(\frac{e^{3i\omega} + 1}{e^{i\omega} + 1}\right)^n m_0(\omega).$$

Let T_n^* be the subdivision operator associated with $|m_n|^2$. One then observes that for $j = 1, 2$, and with δ_j the linear functional of point-evaluation at $2\pi j/3$

$$T_n^* \delta_j = (4^n |m_0(2\pi \frac{3-j}{3})|^2)\, \delta_{3-j}.$$

This implies that $\mu_n := 4^n |m_0(\frac{2\pi}{3}) m_0(\frac{4\pi}{3})|$ is an eigenvalue of T_n^* with eigenvector $\delta := |m_0(\frac{2\pi}{3})|\delta_1 + |m_0(\frac{4\pi}{3})|\delta_2$, hence also that μ_n is an eigenvalue of T_n. Assuming that $m_0(2\pi \frac{j}{3}) \neq 0$, $j = 1, 2$, we may choose n so that to make μ_n as large as we wish (hence, in particular, to ensure that it is > 1). Then, if we iterate with T_n with the initial seed f satisfying $\langle f, \delta\rangle \neq 0$, the iterations will not avoid the eigenvalue μ_n, and the smoothness estimates so obtained may not even grant us the conclusion that $\phi_n \in L_2$. Note that ϕ_n here is (at least) as smooth as ϕ_0 is, and ϕ_0 can be chosen to be as smooth as one wishes.

Specifically, if the shifts of the original ϕ were known to be stable, the iterations with T_0 may start with $\sin^{2\ell}(\omega/2)$, for a sufficiently large ℓ. This initial seed is faulty if we iterate with T_1. Instead, we may take $\sin^{2\ell}(3\omega/2)$; a more efficient choice is $\sin^2(3\omega/2)\sin^{2\ell-2}(\omega/2)$.

The example incidentally shows that the spectral radius of transfer operators of refinable compactly supported L_2-functions can be as large as one wishes. □

Remark. The above example shows also that the convergence of cascade algorithm associated with a given mask m is not implied, in general, by the smoothness of the underlying refinable function. Indeed, in the above we have generated smooth refinable functions whose transfer operator have arbitrarily large spectral radius. At the same time, if the mask is a polynomial, the corresponding cascade algorithm induced by the mask converges in the L_2-norm only if all the eigenvalues with trignometric polynomial eigenvectors of the transfer operator are in the closed unit disc (cf. e.g. [LLS2], [S]; the result, by the way, does not require any special assumption on the dilation matrix).

The subdivision approach. The prevailing system U in the subdivision literature is $U = (u_j)_{j=1}^d$, where

$$u_j(\omega) := \sin^{2\ell}(\omega_j/2).$$

This system is certainly complete of order ℓ. Note that each u_j is of the form $|t_j|^2$, with $t_j(\omega) = (e^{i\omega_j} - 1)^\ell$, which is the Fourier series of the ℓ-fold forward difference in the jth direction. Thus, the middle expression in Lemma 1.4 (for the current choice of t and with $\eta := \phi$) tells us that we can compute $\alpha(u, \phi)$ (hence, eventually, the regularity parameter) by, starting with $f = 1$, iterating sufficiently many times with the subdivision operator T_m^*, and then applying an ℓ-fold difference to the so obtained function. This is, indeed, what the subdivision literature mostly suggests in this regard, with one critical difference: one still needs to mask the resulting expression against $\widetilde{\phi}$ in order to obtain the correct expression $\|t\widetilde{\phi}{T_m^*}^k 1\|$ (cf. Lemma 1.4). While, as we observed above, this can be *sometimes* avoided, our results inflict a blow to this 'plain' subdivision approach: for a compactly supported ϕ, $\widetilde{\phi}^2$ is a polynomial, while $\widetilde{\phi}$ may not be so (unless we are in one dimension)! The alternative expression, $\|T^k(|t|^2\widetilde{\phi}^2)\|$, avoids that trap.

We presented in the above discussion one example of a complete system of order ℓ. Another example is the singleton

$$u(\omega) = (\sum_{j=1}^d \sin^2(\omega_j/2))^\ell.$$

This function was used in [CGV] and [RiS].

We conclude this section with a proof of Theorem 2.4.

Proof of Theorem 2.4. We first assume that I_ϕ is T-invariant, and use that to prove the theorem. Then, we prove this invariance assertion.

Let $U = \{u\}$ be a complete system of order ℓ (made of the singleton $\{u\}$). By choosing a sufficiently large ℓ, we can ensure that $u\widetilde{\phi}^2 \in I_\phi$. Since I_ϕ is T-invariant, Lemma 1.8 implies that $T^k(u\widetilde{\phi}^2)$ lies in $H_\phi \cap I_\phi$, for all sufficiently large k. Let F be the largest T-invariant subspace of $I_\phi \cap H_\phi$, and let ρ be the spectral radius of the restriction of T to F. Since almost all the orbit of $u\widetilde{\phi}^2$ lies in $H_\phi \cap I_\phi$, it follows that $u\widetilde{\phi}^2 = f_1 + f_2$, with $f_1 \in H_\phi \cap I_\phi$, and f_2 T-nilpotent. From that one concludes, with $\alpha(u, \phi)$ as in Theorem 2.9, that $\alpha(u, \phi)$ must coincide with $-\frac{\log_\lambda \rho_1}{2}$, with ρ_1 the spectral radius of the restriction of T to F_1, the latter being the smallest T-invariant space that contains f_1. Since $F_1 \subset F$, we have that $\rho_1 \le \rho$, hence that $\alpha(u, \phi) \ge -\frac{\log_\lambda \rho}{2}$. Finally, Theorem 2.9 tells us that $\alpha(\phi) \ge \alpha(u, \phi)$, hence we conclude that $\alpha(\phi) \ge -\frac{\log_\lambda \rho}{2}$.

In order to prove the converse inequality, we let (μ, f) be a dominant eigenvector of the restriction of T to F, with F as above. Then, $\rho = |\mu|$. Let ℓ be the minimal integer which is $\ge \alpha(\phi)$. Since $\ell - \alpha(\phi) < 1$, and since we also assume that $\widehat{\phi}(0) \ne 0$, we conclude (cf. [R3], [J2]) that the shifts of ϕ provide approximation order ℓ, and standard approximation theory techniques can then be used to show that Π_ϕ contains all polynomials of degree $< 2\ell$, hence that all functions in I_ϕ (including the above f) has a zero of order 2ℓ at the origin. Let $u(\omega) := (\sum_{j=1}^d \sin^2(\omega_j/2))^\ell$, and note that $\{u\}$ is complete of order ℓ. The discussion so far yields the factorization

$$|f| = tu\widetilde{\phi}^2,$$

with t *bounded* and non-negative. Choosing $g := t\widetilde{\phi}^2$ in Corollary 2.10 we get from (b) there that $\alpha(\phi) \le \alpha_g(u, \phi)$. However, one notes that

$$T^k(ug) = T^k(|f|) \ge |T^k(f)| = \rho^k |f|,$$

hence that $\alpha_g(u, \phi) \le -\frac{\log_\lambda \rho}{2}$, proving thereby the the desired converse inequality.

It remains to prove that I_ϕ is T-invariant. Let $f \in I_\phi$. We then write $f = t\widetilde{\phi}^2$, $t \in L_\infty(\mathbb{T}^d)$. Lemma 1.4 then identifies $T(t\widetilde{\phi}^2)$ as the 2π-periodization of $|\widehat{\phi}|^2 \mathcal{D}t$. Periodizing the inequality

$$|\widehat{\phi}|^2 \mathcal{D}t \le \text{const}\, |\widehat{\phi}|^2,$$

we obtain that $Tf \le \text{const}\, \widetilde{\phi}^2$, hence Tf satisfies the first condition in the definition of I_ϕ.

As to the second condition, it suffices to prove that, for each $p \in \Pi_\phi$, and for each $j \in 2\pi\mathbb{Z}^d$,

$$p(-iD)(|\widehat{\phi}|^2 \mathcal{D}t)(j) = 0.$$

For $j \ne 0$, this easily follows from the fact that, since $\sum_{n \in \mathbb{Z}^d} p(n)\phi^\#(\cdot - n) \in \Pi$, we must have that $q(-iD)(|\widehat{\phi}|^2)(j) = 0$, for every q obtained from p by any differentiation (cf. e.g., [BR]). At the origin, we know that $p(-iD)t(0) = 0$, by the definition of I_ϕ. Furthermore, the fact that Π_ϕ is invariant under differentiations (cf. [BR]), when combined with the refinability assumption on ϕ easily implies that Π_ϕ is invariant under dilations by s. Consequently, $p(-iD)(\mathcal{D}t)(0) = 0$, as well. This completes the proof that $Tf \in I_\phi$, hence that I_ϕ is T-invariant. □

2.3. Factorization; regularity of refinable distributions

When the *negative* regularity parameter of a compactly supported distribution is sought for, we do not need the 'damping' effect of the system U. We simply should convolve ϕ with (say, compactly supported) mollifier ν, and examine the asymptotic growth of the L_2-norm of $\nu_h * \phi$, with ν_h the normalized h-dilate of ν.

Definition. Let ν be a compactly supported function. We say that ν is **of order -ℓ** if $\widehat{\nu}(0) \ne 0$, and $|\widehat{\nu}|^2$ has a zero of order 2ℓ at ∞, i.e.,

$$|\widehat{\nu}(\omega)|^2 = O(|\omega|^{-2\ell})$$

for large ω.

Lemma 2.11. *Let $\phi \neq 0$ be a tempered distribution whose Fourier transform is locally square integrable (for example, a compactly supported distribution). Let ν be a compactly supported function of order $-\ell$. Let $\beta(\nu, \phi)$ be the supremum of all α for which*

$$k \mapsto \lambda^{\alpha k} \, \|\widehat{\phi}\, \mathcal{D}^k \widehat{\nu}\|_{L_2(\mathbb{R}^d)}$$

is bounded. Then $\beta(\nu, \phi) \leq 0$, and the following is true:
(a) If $-\ell < \beta(\nu, \phi) < 0$, then $\alpha(\phi) = \beta(\nu, \phi)$.
(b) If $\beta(\nu, \phi) \leq -\ell$, then $\beta(\nu, \phi) \leq \alpha(\phi) \leq -\ell$.
(c) $\beta(\nu, \phi) = 0$ if and only if $\alpha(\phi) \geq 0$.

The above lemma, that we prove in section 4, allows us to invoke Lemma 1.4 once again, only that this time we take there $t = 1$, and ν as above. The initial seed $u\widetilde{\phi}^2$ from the function case is replaced by the initial seed $\widetilde{\eta}^2$, and Lemma 1.8 grants us that the iterations will enter H_ϕ, thereby proving Theorem 2.2 for the distribution case.

However, the initial seed $\widetilde{\eta}^2$ is quite obscure, especially since, in contrast with $\widetilde{\phi}^2$ of the function case, it may not be an eigenvector of T. We can still replace the initial seed $\widetilde{\eta}^2$ by other seeds in order to get upper/lower bound estimates on the regularity parameter. The following summerizes some of the present counterparts of the theorems proved in the case of a function ϕ. The definition of *pre-stability* follows the corollary.

Corollary 2.12. *Let ϕ be a compactly supported refinable distribution with bounded mask m and transfer operator T. Let ν be of order $-\ell$, and $\eta := \phi * \nu \in L_2$.*
(a) Define

$$\beta(\nu, \phi) := -\limsup_{k \to \infty} \frac{\log_\lambda \|T^k(\widetilde{\eta}^2)\|^{1/2}_{L_1(\mathbb{T}^d)}}{k}.$$

Then $\alpha(\phi) \geq \beta(\nu, \phi)$. Moreover, $-\ell < \alpha(\phi) < 0$ if and only if $-\ell < \beta(\nu, \phi) < 0$ and in that situation $\alpha(\phi) = \beta(\nu, \phi)$.
(b) Set

$$\beta_1(\phi) := -\limsup_{k \to \infty} \frac{\log_\lambda \|T^k 1\|^{1/2}_{L_1(\mathbb{T}^d)}}{k}.$$

Then $\alpha(\phi) \geq \beta_1(\phi)$.
(c) If the shifts of ϕ are pre-stable, and $\beta_1(\phi) < 0$, then $\alpha(\phi) = \beta_1(\phi)$.

Definition: pre-stability. We say that the shifts of the tempered distribution ϕ are pre-stable if there exists a compactly supported ν such that $\eta := \phi * \nu$ is an L_2-function and has stable shifts, i.e., such that $\widetilde{\eta}$ is bounded above and below by positive constants. □

For a compactly supported distribution ϕ, as here, pre-stability is equivalent to $\widehat{\phi}$ having no (real) 2π-periodic zeros. For an L_2-function ϕ (compactly supported or not), the notion coincides with the notion of stability. We note that, if the shifts of ϕ are pre-stable, and if ν is any compactly supported function for which $\widehat{\nu}(0) \neq 0$, then, for all sufficiently large k, $\widehat{\phi}\,\mathcal{D}^k \widehat{\nu}$ does not have 2π-periodic zeros. This fact (that follows from the continuity of $\widehat{\phi}$ with the aid of an elementary compactness argument) is required in the proof of (c) in Corollary 2.12.

The connection of these results to the existing literature is as follows: suppose that ϕ is a refinable function that can be factored into $\phi = \phi_1 * \phi_2$. Suppose that the smoothness of ϕ_1 is known (e.g., ϕ_1 is a B-spline), and that ϕ_2 lacks any smoothness, i.e., is an 'honest' distribution. Since convolution with a B-spline of order r increases the smoothness exponent of the refinable element *exactly* by r, one may concentrate on analysing the (negative) smoothness parameter of ϕ_2, an analysis that does not require the 'damping' factor u (of Lemma 2.8). If the shifts of ϕ are known to be stable, then, a fortiori, the shifts of ϕ_2 are pre-stable, hence we can simply iterate with the initial seed 1 (the radius ρ is then the spectral radius of the restriction of T to H_ϕ).

There are four major advantages to the factorization approach. First, the mask of ϕ_2 is smaller than the mask of ϕ. Second, the shifts of ϕ_2 can be pre-stable, while the shifts of ϕ are not stable (something that is expected to happen especially in multivariate setups). Third, the use of the additional polynomial u can be avoided (in fact, damping factors were invented in order to circumvent the difficulties in implementing the idea of factorization). Fourth, the factorization diminishes significantly the suppression of the critical eigenvalue by larger irrelevant eigenvalues. However (and unfortunately) factorization does not work in more than one variable: first, in more than one variable there is no guarantee of any simple factorization. Second, one cannot estimate sharply the smoothness exponent of ϕ from those of its factors.

Trivial factorizations. We tie together various aspects of the analysis presented in this subsection, by exploring a trivial type of factorization. We assume here, for simplicity, that the dilation is dyadic.

To recall, one of the objectives of factorizations was to reduce the smoothness of the underlying refinable function, so that the use of the damping factor u can be avoided, or its order can be reduced.

Suppose, therefore, that we let $p(D)$ be a homogeneous differential operator with constant coefficients of degree k. If we can compute the smoothness of $p(D)\phi$, then, by varying p (and k, if necessary), we will eventually identify correctly the smoothness of ϕ. At the same time, if ϕ is refinable with mask m then $p(D)\phi$ is refinable with mask $m_k := 2^k m$. This, seemingly, may suggest that results like (b) and (c) of Corollary 2.12, that were marked as 'useful for estimating negative smoothness parameters', should be useful for estimating positive smoothness parameters. After all, we have just shown that, up to a multiplicative constant, the mask m is also the mask of a non-smooth function. Of course, this is a groundless hope: if $\phi \in L_2(\mathbb{R}^d)$, the transfer operator has $(1, \widetilde{\phi}^2)$ as an eigenpair, and iterating with the initial seed 1 will not avoid (in general) that eigenpair.

Indeed, Corollary 2.12 provides sharp estimates on the smoothness of $p(D)\phi$ only if the shifts of that distribution are pre-stable. However, these shifts are *never* pre-stable (if $\phi \in L_2(\mathbb{R}^d)$, and not a mere distribution): the Fourier transform of $p(D)\phi$ is guaranteed to vanish on $2\pi\mathbb{Z}^d$. This forces the initial seed $\widetilde{\eta}^2$ of Corollary 2.12 to vanish at the origin, and we should then take initial seeds that vanish at the origin, i.e., that we should use damping factors.

In summary, the current discussion shows that a comprehensive understanding of the connection between negative smoothness parameters of refinable distributions and iterations with their transfer operator, leads naturally to the use of damping factors for the analysis of positive smoothness parameters. □

2.4. Regularity of univariate refinable functions

Let ϕ be a compactly supported univariate function. Theorem 3.7 of [R1] then allows us to write ϕ as the convolution $\phi = \phi_1 * \phi_2 * \phi_3$, with ϕ_1 a measure finitely supported on the integers, with ϕ_2 a suitable B-spline, and with ϕ_3 a function/distribution of compact support whose shifts are linearly independent (a property which is by far stronger than pre-stability) on the one hand, and reproduce no polynomials on the other hand. If ϕ is refinable (with respect to dilation by the integer s) then it is easy to prove that ϕ_3 is refinable in the sense that

$$\widehat{\phi}_3(s\cdot) = m_3\widehat{\phi}_3,$$

for some 2π-periodic m_3. Moreover, the linear independence assumption on the shifts of ϕ_3 then implies that m_3 above must be a polynomial (cf. [BAR], [JM], [BDR]). From that one easily concludes that, denoting by m the refinement mask of ϕ, we have a relation of the form

$$m = \frac{t(s\cdot)}{t} m_2 m_3,$$

for some polynomial t (which is the Fourier transform of the measure ϕ_1). This approach, indeed, was already put into good use in [R2], in the analysis of the stability and linear independence properties of univariate refinable functions.

In the context of regularity, the above factorization is truly ideal: if we can find the mask m_3 of ϕ_3, and if we know the order of the B-spline ϕ_2, then, in order to determine $\alpha(\phi)$ we only need to find $\alpha(\phi_3)$, and that, thanks to Corollary 2.12 (parts (b,c)), can be found by iterating with the trivial initial seed $f = 1$.

The last section of [R2] contains an algorithm that, based on the factorization of m into linear factors, finds the mask $m_2 m_3$ (from which the extraction of m_3 is immediate).

3. Smoothness of refinable functions: the general treatment

In contrast with the flavour of the previous section, where the fine details and wrinkles of the regularity problem were discussed, but under simplifying assumptions, we strive in this section at generality and brevity.

Another important difference will be noticed in the presentation: in the previous section, we targeted persistently the 'magic' eigenpair of Theorem 2.2. Even if we decided to maintain here the compact support assumption from the previous section, results in terms of eigenpairs of the transfer operator (though possible) turn out to be too weak in the present context: they can only be used to characterize the common smoothness of all the refinable functions in the vector Φ. In contrast, we would like to characterize separately the smoothness of each individual $\phi \in \Phi$.

In the previous section, we made the point that the regularity of ϕ can be studied by either iterating with the subdivision operator of m, or with the transfer operator of $\tau := |m|^2$. The choice of playing either the subdivision card or the transfer operator card exists here as well. However, the two operators are, though still closely related, much different in nature. For example, the subdivision operator acts on *vectors* while the transfer operator acts on *matrices*. Since the latter approach is the more general one, we derive the results using the transfer operator approach. Near the end of this section, we discuss briefly the alternative subdivision approach. Our setup here includes the FSI one as a special case.

Let G_0 be any $n \times n'$ matrix with $L_1(\mathbb{R}^d)$-entries. We would like to study, one by one, the decay rates at ∞ of the entries of G_0 (much the same as we studied the decay rate of $|\widehat{\phi}|^2$ of the PSI case). The matrix G_0 is assumed to be **refinable** in the sense that there exist two square matrices M and N of orders n' and n respectively, with bounded measurable 2π-periodic entries such that, almost everywhere,

$$\mathcal{D}^{-1} G_0 = N G_0 M^*. \tag{3.1}$$

Here, s is any (but fixed) dilation matrix.

The motivation behind the above setup is the following:

(3.2) The FSI setup. An important special case is as follows: suppose $\Phi \subset L_2(\mathbb{R}^d)$ is a finite vector of $L_2(\mathbb{R}^d)$-functions that are refinable in the sense that

$$\mathcal{D}^{-1}\widehat{\Phi} = M\widehat{\Phi}.$$

for some square matrix M whose entries are 2π-periodic and bounded, and whose rows and columns are indexed by Φ. Then, defining

$$G_0 := \widehat{\Phi}\widehat{\Phi}^*,$$

we obtain that

$$\mathcal{D}^{-1} G_0 = M G_0 M^*.$$

In the FSI case the entries of G_0 of interest are the diagonal ones: $|\widehat{\phi}|^2$, $\phi \in \Phi$. □

Before we embark on further technical details, we would like to provide the reader with the gist of this section. Given G_0 as above, we assume that some entry $(G_0)_{ij}$ of it is of the form $|\widehat{\phi}|^2$, for some function/distribution ϕ, and would like to study the smoothness of that ϕ, in terms of the matrices N and M (in the FSI case, thus, the entries of interest are the diagonal ones). To that end, we first associate the pair (M, N) with a suitable transfer operator T. The role of the function $\widetilde{\phi}^2$ is now played by the **Gramian matrix**

$$G := \sum_{j \in 2\pi\mathbb{Z}^d} G_0(\cdot + j). \tag{3.3}$$

As in the scalar case, G is a 1-eigenvector of T.

Our first result in this direction, Proposition 3.9, allows us to to characterize $\alpha(\phi)$ (with $|\widehat{\phi}|^2$ being the (i,j)-entry of G_0) as follows. Definning

$$u(\omega) := (\sum_{j=1}^{d} \sin^2(\omega_j/2))^{\ell},$$

and denoting by f_{ij}^k the (i,j)-entry of $T^k(uG)$, the proposition yields that, for an istropic dilation and provided that ℓ is sufficiently large,

$$\alpha(\phi) = -\limsup_{k\to\infty} \frac{\log_\lambda \|f_{ij}^k\|_{L_1(\mathbb{T}^d)}^{1/2}}{k},$$

much in the same spirit of the PSI case. For the special FSI case, this result is restated in Theorem 3.15 (as part (a) there).

The implementation of the results that were alluded to above relies on the ability to compute the matrix G. Parts (b,c) in Theorem 3.15 study the alternative of starting the iterations with uI (instead of uG). It is showed there that the smoothness estimate obtained in this way always bounds below (say, if ϕ is compactly supported) the sharp parameter $\alpha(\phi)$, and that those alternative iterations recover completey $\alpha(\phi)$ under a stability assumption on the shifts of (the vector) Φ. Finally, Theorem 3.17 is an analog of Theorem 3.15 where *negative* smoothness parameters are studied.

Let H be the Hilbert space of all $n\times n'$ $L_2(\mathbb{T}^d)$-valued matrices, equipped with the usual inner product, i.e.,

$$\|f\|_H^2 = \sum_{i,j} \|f_{i,j}\|_{L_2(\mathbb{T}^d)}^2.$$

We also need here the space H' of all $n\times n'$ $L_2(\mathbb{R}^d)$-valued matrices. The transfer operator $T := T_{M,N}$ is now defined by

$$T : H \to H : f \mapsto \mathcal{D}\big(\sum_{\gamma\in\Gamma} (NfM^*)(\cdot + s^{*-1}\gamma)\big),$$

and its adjoint is the operator

$$T^* : H \to H : f \mapsto |\det s|\, N^*(\mathcal{D}^{-1}f)M.$$

Checking that, indeed, T^* is the adjoint of T requires the following identity of independent use in the sequel:

$$\langle NgM^*, f\rangle_H = \langle g, N^*fM\rangle_H, \quad \forall f,g\in H. \tag{3.4}$$

The same identity, of course, holds in H':

$$\langle NgM^*, f\rangle_{H'} = \langle g, N^*fM\rangle_{H'}, \quad \forall f,g\in H'. \tag{3.5}$$

Let u be now any scalar function in $L_\infty(\mathbb{T}^d)$, and let $f\in H$. Then, due to the 2π-periodicity of u and f, we may use (3.5), invoke k times the refinability of G_0 and obtain

$$\langle T^k(uG), f\rangle_H = \langle uG_0, T^{*k}f\rangle_{H'} = |\det s|^k \sum_{i,j}\int_{\mathbb{R}^d} u\,\mathcal{D}^{-k}(G_0(i,j)f_{i,j}) = \sum_{i,j}\int_{\mathbb{R}^d} (\mathcal{D}^k u)\, G_0(i,j) f_{i,j}, \tag{3.6}$$

with $G_0(i,j)$ being the (i,j)-entry of G_0. Note that, analogously to (1.6), the identity (3.6) identifies $T^k(uG)$ with the 2π-periodization of $(\mathcal{D}^k u)G_0$:

$$T^k(uG) = \sum_{j\in 2\pi\mathbb{Z}^d} ((\mathcal{D}^k u)G_0)(\cdot + j). \tag{3.7}$$

In particular, choosing $\mathbf{1}^{ij}$ to be the matrix whose (i,j)-entry is 1 and the others are 0, and letting f_0 be any scalar function in $L_2(\mathbb{T}^d)$, we arrive at the following generalization of (1.6):

$$\langle T^k(uG), f_0\mathbf{1}^{ij}\rangle_H = \int_{\mathbb{R}^d} G_0(i,j)\overline{f_0}\,\mathcal{D}^k u. \tag{3.8}$$

Proposition 3.9. *Assume that some (i,j)-entry, $G_0(i,j)$, of G_0 is of the form $|\widehat{\phi}|^2$ for some L_2-function ϕ. Set, for every integer k, and some bounded u,*

$$a^k(u,\phi) := \int_{\mathbb{R}^d} |\widehat{\phi}|^2\,\mathcal{D}^k u.$$

Then, for every k,

$$a^k(u,\phi) = \langle T^k(uG), \mathbf{1}^{ij}\rangle_H,$$

with $\mathbf{1}^{ij}$ a matrix whose (i,j)-entry is 1 and whose other entries are 0.

Discussion. In the FSI case, as presented before, the functions $|\widehat{\phi}|^2$, $\phi\in\Phi$, comprise the diagonal of G_0. Thus, in that case the previous proposition says that the matrix $T^k(uG)$ has diagonal entries whose integrals measure the smoothness of the corresponding ϕ's, provided of course that the numbers $(a^k(u,\phi))_k$ are suitable for measuring that smoothness. The appropriateness of the sequence $(a^k(u,\phi))_k$ for measuring the smoothness of ϕ is discussed in the next section. □

In the PSI case, and under a compact support assumption, we observed that the transfer operator iterations enter a certain finite dimensional subspace. This observation extends to the current situation:

Corollary 3.10. *If some entry $G_0(i,j)$ of G_0 is of the form $|\widehat{\phi}|^2$ for a compactly supported ϕ, then, given any trigonometric polynomial u, the (i,j)-entry of $T^k(uG)$ lies in H_ϕ, for all sufficiently large k.*

Proof: Choose an exponential e_θ, $\theta\in\mathbb{Z}^d$, for f_0 in (3.8), and repeat the argument used in Lemma 1.8 (with u here being being $\widehat{q}$ there, and with ν there taken to be the Dirac δ). □

The result can be easily applied to show that in the FSI setup, if Φ is compactly supported, then, for any given polynomial u, the iterations $T^k(uG)$ will enter a finite dimensional space of H. However, this observation does not seem to be as useful as its PSI counterpart is and we will not pursue it further.

Even though the matrix G is an eigenvector of the transfer operator (with eigenvalue 1), that matrix is, quite likely, very hard to compute. Therefore, it is useful to seek smoothness estimates that do not exploit this matrix. One should be warned that in general there might be different matrices G_0 that satisfy the refinability assumption, and the (i,j)-entry of a solution G_0 may represent a function of different smoothness properties than the (i,j)-entry of another solution. Thus, an attempt to characterize the decay of the (i,j)-entry of G_0, without information on the particular G_0 chosen, may doom to fail.

In the sequel we gradually impose additional conditions on M, N, and G_0. First, we assume here and hereafter that:

(3.11) Assumptions and conventions.
(a) $M=N$ (in particular, that $n=n'$).
(b) $G(\omega)$ is symmetric non-negative definite for almost every $\omega\in\mathbb{R}^d$.
(c) Some *diagonal* entry $G_0(i,i)$ of G_0 is of the form $|\widehat{\phi}|^2$, for some $\phi\in L_2$. □

Here, we note that the condition (c) above is satisfied for the FSI case.

Adopting these assumptions, we set

$$\Lambda_+(\omega)\quad(\Lambda_-(\omega)) \tag{3.12}$$

for the largest (smallest) eigenvalue of $G(\omega)$.

Note that the assumed conditions are certainly satisfied for the case of primary interest, i.e., the FSI case as discussed in (3.2).

Our approach for deriving Gramian-free estimates, is based on the next lemma:

Lemma 3.13. *Adopting assumptions (3.11), we have for any non-negative 2π-periodic (bounded) u,*

$$\langle T^k(u\Lambda_- I), \mathbf{1}^{ii}\rangle_H \le \langle T^k(uG), \mathbf{1}^{ii}\rangle_H \le \langle T^k(u\Lambda_+ I), \mathbf{1}^{ii}\rangle_H.$$

Proof: Let A be any of the matrices G, $\Lambda_+ I$, $\Lambda_- I$. By (3.4),

$$\langle T^k(uA), \mathbf{1}^{ii}\rangle_H = \langle uA, T^{*k}\mathbf{1}^{ii}\rangle_H = |\det s|^k \langle uM_k A M_k^*, \mathbf{1}^{ii}\rangle_H,$$

with

$$M_k := (\mathcal{D}^{-(k-1)}M)\ldots(\mathcal{D}^{-1}M)M.$$

Therefore, with $m_i(k)$ the ith row of M_k, the integrand in the above inner product is

$$|\det s|^k \, u \, A(m_i(k)),$$

with $A(m_i(k))$ the (pointwise) value of the quadratic form A at the vector $m_i(k)$. By our assumption here, the inequalities

$$\Lambda_-(m_i(k)) \le G(m_i(k)) \le \Lambda_+(m_i(k))$$

are valid pointwise almost everywhere, whence the result. □

The function Λ_+ is essentially bounded if and only if each of the entries of G is so.

Corollary 3.14. *If all the entries of G are bounded, and if we adopt the assumptions in (3.11), then for every non-negative u,*

$$a^k(u,\phi) \le \text{const}\, \langle T^k(uI), \mathbf{1}^{ii}\rangle.$$

Here, $a^k(u,\phi)$ is as in Proposition 3.9.

In the FSI case, the boundedness of the entries of G is implied by a mild decay assumption on Φ. E.g. if each $\phi \in \Phi$ decays at ∞ at a rate $-d-\varepsilon$, for some $\varepsilon > 0$.

We say that the shifts of Φ are **stable** if the functions Λ_+ and $1/\Lambda_-$ of the Gramian of G are essentially bounded (this is certainly a non-standard definition of stability, but is equivalent to the standard definition; cf. [JM], [BDR], [RS]. See also [S] where stability of refinable Φ's is characterized in terms of the transfer operator). Under such a stability assumption, we can invoke the last result to conclude (for example) the following one, in which we use, for any given 2π-periodic bounded u, the notation

$$a^k(u,\phi)$$

for the $L_1(\mathbb{T}^d)$-norm of the (ϕ,ϕ)-entry of $T^k(uG)$, and

$$a_I^k(u,\phi)$$

for the $L_1(\mathbb{T}^d)$-norm of the (ϕ,ϕ)-entry of $T^k(uI)$. The proof of the result invokes Lemma 3.13 and Lemma 2.8.

Theorem 3.15. *Let $\Phi \subset L_2(\mathbb{R}^d)$ be a refinable vector with bounded mask M, and with Gramian G. Assume that the dilation is isotropic. Let T be the associated transfer operator, and let U be a complete system of order ℓ. Set*

$$\alpha(U,\phi) := -\max_{u\in U} \limsup_{k\to\infty} \frac{\log_\lambda a^k(u,\phi)}{2k},$$

$$\alpha_I(U,\phi) := -\max_{u\in U} \limsup_{k\to\infty} \frac{\log_\lambda a_I^k(u,\phi)}{2k}.$$

Then:

(a) If $\alpha(U,\phi) \ge \ell$, so is the regularity parameter $\alpha(\phi)$. Otherwise, $\alpha(\phi) = \alpha(U,\phi)$.

(b) If the entries of G are bounded, then $\alpha_I(U,\phi) \le \alpha(\phi)$.

(c) If the shifts of Φ are stable, then $\alpha(U,\phi) = \alpha_I(U,\phi)$.

The subdivision approach. The subdivision operator provides an alternative route for computing the matrices $T^k(uI)$, and/or $T^k(uG)$, in a way that the computation is significantly simplified.

First, we define that operator. Let Φ be a refinable vector with bounded mask M. Let H_0 be the Hilbert space $L_2(\mathbb{T}^d \times \Phi)$. The subdivision operator T_M^* is then defined as

$$T_M^* : H_0 \to H_0 : g \mapsto \sqrt{|\det s|}\, M^* \mathcal{D}^{-1} g.$$

The relation between T_M^* and the adjoint T^* of the transfer operator T is neat:

$$T^*(g_1 g_2^*) = (T_M^* g_1)(T_M^* g_2)^*,$$

which is the analog of the relation $T^*|f|^2 = |T_m^* f|^2$ of the PSI case.

Now, let $1_\phi \in \mathbb{C}^\Phi$ be the vector whose ϕ-entry is 1 and other entries 0. Then, the number $a^k(u,\phi)$ from Theorem 3.15 can be written as

$$a^k(u,\phi) = \langle T^k(uG), 1_\phi 1_\phi^* \rangle_H = \langle uG, T^{*k}(1_\phi 1_\phi^*)\rangle_H = \int_{\mathbb{T}^d} u\, G({T_M^*}^k 1_\phi),$$

with $G(v)$ the (pointwise) value of the quadratic form G at v, i.e.,

$$G(v) = v^* G v.$$

Thus, we may, in lieu of iterating with $T^k(uG)$, compute the vector ${T_M^*}^k 1_\phi$, and then apply to that vector the quadratic form uG, and compute the integral. The computational saving is huge; further, the guaranteed numerical instability of the process (due to the fact that the quadratic form uG must suppress the 'wrong' eigenvectors of T^*) is postponed to the last step. $\square$

We finally discuss briefly the regularity of refinable distributions. The approach, in principle, is identical to that used in the PSI case: we take the refinable Φ and convolve it with a compactly supported scalar ν to obtain $\mathbf{\Phi} := \nu * \Phi \subset L_2$. We then let G_ν be the Gramian of $\mathbf{\Phi}$. By an argument analogous to (3.6) we obtain the following proposition:

Proposition 3.16. *Let Φ be a refinable vector of distributions with bounded mask M, and assume that the Fourier transform of each of these distributions is locally square integrable. For each $\phi \in \Phi$, let f_ϕ be the $\Phi \times \Phi$-matrix whose (ϕ,ϕ)-entry is 1 and its other entries are 0. Let ν be a compactly supported distribution, so that, with ν_h the h-dilate of ν, $\nu_h * \Phi \subset L_2$, all h. Let G_ν be the Gramian of $\nu * \Phi$. Then*

$$b^k(\nu,\phi) := \int_{\mathbb{R}^d} |\widehat{\phi}|^2 \mathcal{D}^k(|\widehat{\nu}|^2) = \langle T^k(G_\nu), f_\phi\rangle_H.$$

We have already discussed in §2.3 the relevance of the numbers $(b^k(\nu,\phi))_k$ to the identification of the smoothness of ϕ. In the present context, the most important conclusions seem to be those that avoid the computation of G_ν. The next theorem is the main result in this direction.

Theorem 3.17. *Let Φ be a refinable vector of tempered distributions in $\mathbb{R}^d$, with bounded mask M and transfer operator T. Assume that $\widehat{\phi}$ is locally in L_2, for every $\phi \in \Phi$. For each $\phi \in \Phi$, let $b_I^k(\phi)$ be the $L_1(\mathbb{T}^d)$-norm of (ϕ,ϕ)-diagonal entry of $T^k(I)$. Assume that the dilation is isotropic (i.e., $ss^* = \lambda^2 I$), and set*

$$\beta_I(\phi) := -\limsup_{k\to\infty} \frac{\log_\lambda b_I^k(\phi)}{2k}.$$

Then,

(a) If there exists a compactly supported function ν of some order $-\ell \le 0$ such that the Gramian G_ν of $\nu * \Phi$ is well-defined and its entries are essentially bounded, then $\alpha(\phi) \ge \beta_I(\phi)$.

(b) If, in addition, the shifts of $\nu * \Phi$ are stable, and $-\ell < \beta_I(\phi) < 0$, then $\alpha(\phi) = \beta_I(\phi)$.

Proof: Let Λ_+ be the eigenvalue function of G_ν. Since we assume that the entries of G_ν are bounded, we have that $\Lambda_+ \in L_\infty(\mathbb{T}^d)$, and we obtain from Lemma 3.13 and Proposition 3.16, in the notations of Proposition 3.16, that

$$b^k(\nu, \phi) = \langle T^k(G_\nu), f_\phi \rangle_H \le c \langle T^k(I), f_\phi \rangle_H = c\, b_I^k(\phi).$$

From that, it follows that $\beta_I(\phi) \le \beta(\nu, \phi)$, with $\beta(\nu, \phi)$ as in Lemma 2.11, and, hence, by Lemma 2.11, $\beta_I(\phi) \le \beta(\nu, \phi) \le \alpha(\phi)$, which proves (a).

Assuming further that $\nu * \Phi$ has stable shifts, we conclude from Lemma 3.13 that, with Λ_- the eigenvalue function of G_ν, and for some other positive constant c,

$$b^k(\nu, \phi) = \langle T^k(G_\nu), f_\phi \rangle_H \ge c \langle T^k(I), f_\phi \rangle_H = c\, b_I^k(\phi).$$

This, together with the argument in the previous paragraph, shows that $\beta_I(\phi) = \beta(\nu, \phi)$. Assertion (b) then follows from Lemma 2.11. $\square$

Discussion. The formulation of the conditions in the last theorem in terms of properties of the mollifier ν was done for sake of convenience and generality. It is not hard to rephrase those assumptions in terms of intrinsic properties of Φ. For example, for as long as Φ are distributions of *finite* order, we can always find a sufficiently smooth ν to ensure that $\nu * \Phi \subset L_2$. The additional assumption in (a) above, i.e., that G_ν has bounded entries, is actually a condition on Φ; that boundedness is obtained, e.g., if each ϕ satisfies some mild decay condition as ∞. In particular, no reference to ν is required in (a) above if we know that Φ is compactly supported.

Similarly, the stability assumption of (b) can also be connected directly to properties of Φ. For example, if Φ is compactly supported (and in many other cases as well), the stability of the shifts of $\nu * \Phi$ amounts to **pre-stability** of the shifts of Φ, i.e., the linear independence, for every fixed $\theta \in \mathbb{R}^d$, of the sequences

$$y_{\phi,\theta} : \mathbb{Z}^d \to \mathbb{C} : j \mapsto \widehat{\phi}(\theta + 2\pi j), \quad \phi \in \Phi.$$

Indeed, under this assumption, it is easy to construct a compactly supported ν of arbitrarily small order $-\ell$ such that $\nu * \Phi$ has stable shifts: one only needs to ensure that $\widehat{\nu}$ does not vanish on a sufficiently large ball centered at the origin, and that ν is sufficiently smooth. This means that we have, e.g., the following:

Corollary 3.18. *Let Φ be a compactly supported refinable vector of tempered distributions on $\mathbb{R}^d$, with bounded mask M and transfer operator T. For each $\phi \in \Phi$, let $b_I^k(\phi)$ be the $L_1(\mathbb{T}^d)$-norm of (ϕ, ϕ)-diagonal entry of $T^k(I)$. Assume that the dilation is isotropic (i.e., $ss^* = \lambda^2 I$), and set*

$$\beta_I(\phi) := -\limsup_{k\to\infty} \frac{\log_\lambda b_I^k(\phi)}{2k}.$$

Then,
(a) $\alpha(\phi) \ge \beta_I(\phi)$.
(b) If, $\beta_I(\phi) < 0$, and, in addition, the shifts of Φ are pre-stable then $\alpha(\phi) = \beta_I(\phi)$.

Final Discussion. Note that we were reluctant to translate the results here to assertions about the spectral radius of the transfer operator restricted to certain spaces. We could still identify, in the compact support case, a finite dimensional subspace $H_\Phi \subset L_2(\mathbb{T}^d \times \Phi)$ of trigonometric polynomials, and an eigenpair (ρ, f_ρ) so that, with $\alpha(\Phi) := -\frac{\log_\lambda \rho}{2}$, and analogously to Theorem 2.2, each $\phi \in \Phi$ lies in $W_2^\alpha(\mathbb{R}^d)$ for every $\alpha < \alpha(\Phi)$. The converse, however, does not hold: if $\alpha > \alpha(\Phi)$, one can only conclude that *some* $\phi \in \Phi$ does not lie in $W_2^\alpha(\mathbb{R}^d)$, i.e., the approach of identifying a critical eigenvalue of the transfer operator allows the identification of the *common* smoothness of the functions in Φ. In contrast, the results in this section allow a separate estimation of the smoothness of each $\phi \in \Phi$.

4. Measuring smoothness

We prove here Lemmata 2.8, 2.11 and an additional lemma (Lemma 4.3). That additional lemma allows us to apply the results of the previous section in the estimation of smoothness in the unisotropic case.

Let ϕ be a tempered distribution whose Fourier transform $\widehat{\phi}$ can be identified with a function in $L_{2,\mathrm{loc}}(\mathbb{R}^d)$. Let $\lambda > 1$ be given. One of the equivalent definitions of the Sobolev space $W_2^\alpha(\mathbb{R}^d)$ goes as follows. Let

$$V_k := \{\omega \in \mathbb{R}^d : K\lambda^{k-1} \leq |\omega| \leq K\lambda^k\}, \tag{4.1}$$

with K some fixed positive number. Let $\alpha \in \mathbb{R}$. Then $\phi \in W_2^\alpha(\mathbb{R}^d)$ if and only if the sequence

$$c_\alpha : \mathbb{N} \to \mathbb{R}_+ : k \mapsto \lambda^{\alpha k} \, \|\widehat{\phi}\|_{L_2(V_k)} \tag{4.2}$$

lies in $\ell_2(\mathbb{N})$. This means that the critical smoothness $\alpha(\phi)$ (defined after (2.1)) can be alternatively defined by

$$\alpha(\phi) := \sup\{\alpha \in \mathbb{R} : c_\alpha \in \ell_\infty(\mathbb{N})\}.$$

(In fact, the relation $c_\alpha \in \ell_\infty$ is one of the definitions of the Besov space $B_\infty^\alpha(L_2(\mathbb{R}^d))$.)

Proof of Lemma 2.8. In view of the above, we need to show that, given a complete system U of order ℓ, and given any α, the sequence c_α of (4.2) is bounded if and, in case $\alpha < \ell$, only if the sequence

$$c_\alpha^u : k \mapsto \lambda^{\alpha k} \, \|\widehat{\phi}^2 \, \mathcal{D}^k u\|_{L_1(\mathbb{R}^d)}^{1/2}$$

is bounded, for every $u \in U$. Clearly, we may assume without loss that U is the singleton $\{u\}$ (otherwise, we may define a new complete system $U' := \{\sum_{u \in U} u\}$). The crux of the proof is that, since the dilation is isotropic, and if we denote by χ the support function of V_0, then $\mathcal{D}^j \chi$ is the support function of V_j. Therefore, for any integers k, j, $\|\mathcal{D}^k f\|_{L_\infty(V_j)} = \|f\|_{L_\infty(V_{j-k})}$. We use that fact in the sequel without further comment.

Now, since u has an *isolated* zero at the origin, then, assuming K is small enough, u is bounded below on V_0 by some positive constant C^2. This mean that

$$\|\widehat{\phi}^2 \, \mathcal{D}^k u\|_{L_1(\mathbb{R}^d)}^{1/2} \geq C \, \|\widehat{\phi}\|_{L_2(V_k)}.$$

Thus, if c_α^u is bounded, so is c_α.

Note that the sequence c_0^u is certainly bounded (since u is bounded and $\phi \in L_2$), hence that we may prove the converse only for $\alpha > 0$. For that, we write $\mathbb{R}^d$ as the union of $\{B_j\}_{j \geq 0}$, with B_0 the ball of radius K, and $B_j = V_j$, for $j \geq 1$. Since u has a zero at the origin of order 2ℓ, and since we assume c_α to be bounded, we can estimate

$$\|\widehat{\phi}^2 \, \mathcal{D}^k u\|_{L_1(B_j)}^{1/2} \leq \|\mathcal{D}^k u\|_{L_\infty(B_j)}^{1/2} \, \|\widehat{\phi}\|_{L_2(B_j)} \leq \mathrm{const} \begin{cases} \lambda^{\ell(j-k)} \lambda^{-\alpha j}, & j \leq k, \\ \lambda^{-\alpha j}, & \text{otherwise.} \end{cases}$$

Summing over $j = 0, 1, \ldots,$, and invoking the assumption $\alpha < \ell$, we obtain that

$$\|\widehat{\phi}^2 \, \mathcal{D}^k u\|_{L_1(\mathbb{R}^d)}^{1/2} = O(\lambda^{-\alpha k}),$$

hence that the sequence c_α^u is bounded. □

Lemma 2.8 requires the assumption that the dilation is isotropic. That assumption is crucial for the sharp estimation of $\|\mathcal{D}^k u\|_{L_\infty(V_j)}$ that appears in the proof. If the dilation is not isotropic, then one has two options to pursue. The first, as was essentially done in [CGV], is to *define* smoothness in terms of the decay rates of

$$\|\widehat{\phi}^2 \mathcal{D}^k u\|.$$

In this case, we immediately get extensions of all the main results of this article to the unisotropic case, as well, only that "smoothness" is now a non-standard notion.

We prefer, instead, to provide in this case lower and upper bounds on the standard regularity parameter (something that already appears in [CGV], too).

Lemma 4.3. *Let s be a dilation matrix whose spectral radius is λ_+, and whose inverse has a spectral radius $1/\lambda_-$. Let U be a complete system of order ℓ. Set*

$$a^k(u,\phi) := \|\widehat{\phi}^2\, \mathcal{D}^k u\|^{1/2}_{L_1(\mathbb{R}^d)},$$

and define

$$\alpha^+(U,\phi) := -\max_{u\in U} \limsup_{k\to\infty} \frac{\log_{\lambda_+} a^k(u,\phi)}{k},$$

and

$$\alpha^-(U,\phi) := -\max_{u\in U} \limsup_{k\to\infty} \frac{\log_{\lambda_-} a^k(u,\phi)}{k}.$$

Then $\alpha^+(U,\phi) \le \alpha(\phi)$, and, if $\alpha(\phi) < \ell$, then $\alpha(\phi) \le \alpha^-(U,\phi)$.

Proof: Let $u \in U$. Note that, since u has a zero of order 2ℓ at the origin,

$$|\mathcal{D}^k u(\omega)| = O(|s^{*-k}\omega|^{2\ell}) = \lambda_-^{-2\ell k} O(|\omega|^{2\ell}).$$

We choose λ in (4.1) to be our present λ_-. Then the above yields the estimate

$$\|\mathcal{D}^k u\|^{1/2}_{L_\infty(V_j)} = O(\lambda_-^{\ell(j-k)}),$$

and with that in hand, the second part of the proof of Lemma 2.8 applies *verbatim* to yield that, if $\alpha(\phi) < \ell$, then $\alpha(\phi) \le \alpha^-(U,\phi)$.

Now, let $\alpha < \alpha^+(U,\phi)$. This implies that the sequence $k \mapsto \lambda_+^{\alpha k}\|\widehat{\phi}^2 \mathcal{D}^k u\|^{1/2}_{L_1(\mathbb{R}^d)}$ is square-summable, for every $u \in U$. Hence, with $u := \sum_{u\in U} u$,

$$\widehat{\phi}^2 \sum_{k=0}^{\infty} \lambda_+^{2k\alpha} \mathcal{D}^k u \tag{4.4}$$

is integrable. Let $(V_k)_k$ be a system of the type (4.1) with respect to $\lambda := \lambda_+$, and let χ_k be the support function of V_k. We need to show that $\phi \in W_2^\alpha(\mathbb{R}^d)$, which is equivalent to the integrability of

$$\widehat{\phi}^2 \sum_{k=0}^{\infty} \lambda_+^{2k\alpha} \chi_k.$$

In view of the integrability of (4.4), this will follow once we show that

$$\sum_{k=0}^{\infty} \lambda_+^{2k\alpha} \mathcal{D}^k u \ge C \sum_{k=0}^{\infty} \lambda_+^{2k\alpha} \chi_k.$$

For that, we take Ω to be a relatively compact neighborhood of the origin, and set $V := \Omega\backslash(s^{*-1}\Omega)$. Then V is disjoint of some neighborhood of the origin. Upon replacing V by some dilate $s^{*-k}V$ of it, if necessary, we can assume without loss of generality that V is disjoint of each V_k, $k = 1, 2, \ldots$, and that $1/u$ is bounded on V (the latter can be assumed since u has an isolated zero at the origin). Note that the s^*-dilations of V fill all $\mathbb{R}^d\backslash 0$. With χ the support function of V, we first see that, since $1/u$ is bounded on V, $\mathcal{D}^k u \geq C\mathcal{D}^k\chi$, hence

$$\sum_{k=0}^{\infty} \lambda_+^{2k\alpha} \mathcal{D}^k u \geq C \sum_{k=0}^{\infty} \lambda_+^{2k\alpha} \mathcal{D}^k \chi.$$

On the other hand, since λ_+ is the spectral radius of s^*, it is easy to see that, whenever

$$\chi_k(\omega) = \mathcal{D}^{k'}\chi(\omega) = 1,$$

we must have that $k' \geq k$. Therefore,

$$\sum_{k=0}^{\infty} \lambda_+^{2k\alpha} \mathcal{D}^k \chi \geq \sum_{k=0}^{\infty} \lambda_+^{2k\alpha} \chi_k.$$

Consequently, $\alpha \leq \alpha(\phi)$, and hence $\alpha^+(U, \phi) \leq \alpha(\phi)$. □

Finally, we prove Lemma 2.11.

Proof of Lemma 2.11: Let B_k be the ball of radius $\lambda^k K$ which is centered at the origin. It is easy to prove that, for any $\alpha < 0$, $\phi \in W_2^\alpha(\mathbb{R}^d)$ if and only if the sequence

$$k \mapsto \lambda^{k\alpha}\, \|\widehat{\phi}\|_{L_2(B_k)}$$

is square-summable. Let a be the supremum of all α for which this sequence is bounded. Clearly, $a \leq 0$, $a = 0$ if and only if $\alpha(\phi) \geq 0$, and otherwise $a = \alpha(\phi)$.

Since we assume $\widehat{\nu}(0) \neq 0$, we can choose K sufficiently small to ensure that $|\widehat{\nu}|^2 \geq C > 0$ on B_0, which is equivalent to the inequality $|\mathcal{D}^k\widehat{\nu}|^2 \geq C$ on B_k. Thus, $n_k := \|\widehat{\phi}\mathcal{D}^k\widehat{\nu}\|_{L_2(\mathbb{R}^d)} \geq C\|\widehat{\phi}\|_{L_2(B_k)}$. Thus, since $\beta(\nu, \phi)$ is the supremum of α that keeps $k \mapsto \lambda^{\alpha k} n_k$ bounded, we see that $\alpha(\phi) \geq a \geq \beta(\nu, \phi)$. Since $a \leq 0$, so must be $\beta(\nu, \phi)$. Also, if $\beta(\nu, \phi) = 0$, then $a = 0$, and hence $\alpha(\phi) \geq 0$. Moreover, if $\alpha(\phi) > 0$, then $(n_k)_k$ is bounded, hence $\beta(\nu, \phi)$ cannot be negative, hence must be 0.

Now, assume $0 \geq \alpha(\phi) > -\ell$; then $a = \alpha(\phi)$. Let $-\ell < \alpha \leq \alpha(\phi)$, and let $V_k := B_k\backslash B_{k-1}$. Then, since $\widehat{\nu}$ vanishes at ∞ to order 2ℓ, we have

$$\begin{aligned}
\|\widehat{\phi}\,\mathcal{D}^k\widehat{\nu}\|^2_{L_2(\mathbb{R}^d\backslash B_k)} &= \sum_{j=k+1}^{\infty} \|\widehat{\phi}\,\mathcal{D}^k\widehat{\nu}\|^2_{L_2(V_j)} \\
&\leq c \sum_{j=k+1}^{\infty} \lambda^{2\ell(k-j)}\|\widehat{\phi}\|^2_{L_2(V_j)} \\
&\leq c \sum_{j=k+1}^{\infty} \lambda^{2\alpha(j-k)}\|\widehat{\phi}\|^2_{L_2(V_j)} = O(\lambda^{-2\alpha k}).
\end{aligned}$$

Thus,

$$\lambda^{2\alpha k}\|\widehat{\phi}\,\mathcal{D}^k\widehat{\nu}\|^2_{L_2(\mathbb{R}^d)} \leq \lambda^{2\alpha k}\|\widehat{\phi}\|^2_{L_2(B_k)} + O(1) = O(1).$$

It follows thus that, in this case, $\beta(\nu, \phi) \geq \alpha(\phi)$, hence that equality holds. □

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