A smooth and local interpolant with "small" k–th derivative

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1. Introduction. For nondecreasing $\mathbf{t} := (t_i)_1^{n+k}$ and sufficiently smooth f, denote by

$$f|_{\mathbf{t}} := (f_i)$$

the corresponding sequence given by the rule

$$f_i := f^{(j)}(t_i) \quad \text{with} \quad j := j(i) := \max\{m \mid t_{i-m} = t_i\}.$$

We will write "$f = g$ on $\mathbf{t}$", or, "f and g agree on $\mathbf{t}$" in case $f|_{\mathbf{t}} = g|_{\mathbf{t}}$. Assuming that $\operatorname{ran}\mathbf{t} \subseteq [a, b]$ and that $t_i < t_{i+k}$, all i, $f|_{\mathbf{t}}$ is defined for every f in the Sobolev space

$$L_p^{(k)}[a,b] := \{f \in C^{(k-1)}[a,b] \mid f^{(k-1)} \quad \text{abs.cont.}; \quad f^{(k)} \in L_p[a,b]\}.$$

In order to demonstrate that the number

$$K(k) := \sup_{f_0,\mathbf{t}} \frac{\inf\{\|f^{(k)}\|_\infty \mid f \in L_\infty^{(k)}, \quad f|_{\mathbf{t}} = f_0|_{\mathbf{t}}\}}{\max_i k!|[t_i, \ldots, t_{i+k}]f_0|} \tag{1}$$

is finite (with $[t_i, \ldots, t_{i+k}]g$ the k–th divided difference of g at the points $t_i, \ldots, t_{i+k}$), Favard [5] constructs, for each $\mathbf{t}$, a map $P_{\mathbf{t}}$ with the property that $P_{\mathbf{t}}f$ agrees with f on $\mathbf{t}$ while

$$\|(P_{\mathbf{t}}f)^{(k)}\|_\infty \le \text{const}_k \max_i |[t_i, \ldots, t_{i+k}]f|, \quad \text{all } f \in L_\infty^{(k)}$$

for some const_k depending only on k. But, Farvard's $P_{\mathbf{t}}$ can actually be shown to satisfy the following:
(i) $P_{\mathbf{t}} : L_\infty^{(k)} \to L_\infty^{(k)}$ is a linear projector of rank $n + k$ with $P_{\mathbf{t}}f = f$ on $\mathbf{t}$, all f.
(ii) For some constant C_k depending on k but not on $\mathbf{t}$ or n, and for all j,

$$\|(P_{\mathbf{t}}f)^{(k)}\|_{\infty,(t_j,t_{j+1})} \le C_k \max_{i \le j < i+k} k!|[t_i, \ldots, t_{i+k}]f|.$$

Hence, Farvard's construction can be used to demonstrate the finiteness of

$$K_0(k) := \inf\{C_k \mid C_k \text{ satisfies (i) and (ii)}\}. \tag{2}$$

Farvard shows that $K(2) = K_0(2) = 2$, but gives no quantitative information about $K_0(k)$ or $K(k)$ for $k > 2$.

A different construction, in [3], provides the explicit upper bound

$$K_0(k) \le k^2(2k+1)(2k-1)^{k-1} \tag{3}$$

which, already for $k = 5$, gives a uselessly large bound, i.e., $K_0(5) \le 1,804,275$. This is to be compared with the lower bound

$$K_0(k) \ge K(k) \ge \gamma_k := (\pi/2)^{k+1} / \sum_{j=-\infty}^{\infty} (-1/(2j+1))^{k+1}$$

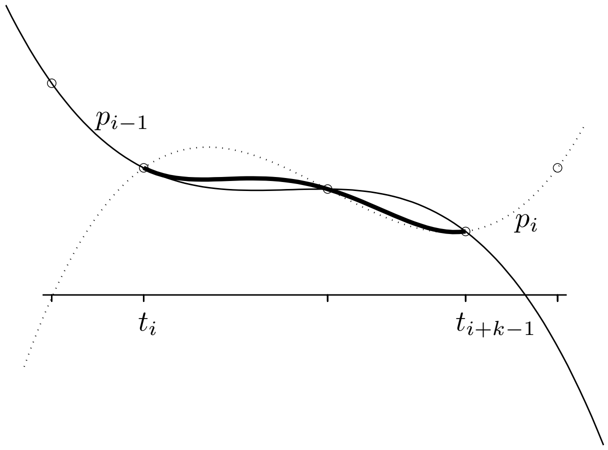
also proved in [3], giving, e.g., the lower bound $K_0(5) \geq 7.5$.

It is relatively easy to estimate Favard's C_k numerically, but the resulting bounds for $K_0(k)$ are not much better than those obtained from (3). A simple modification does improve the estimate somewhat, giving, e.g., $K_0(5) \leq 1,730$. In terms of Farvard's construction as described in [3], the modification consists in choosing, in Step 4, the break points for the piecewise constant function g_i *not* equally spaced but as the zeroes of the appropriate Chebyshev polynomial.

It is the purpose of this note to describe a more effective modification of Farvard's construction, resulting, e.g., in the computed bound $K_0(5) \leq 21.04$. In addition, the construction is described in a simpler way which makes its localness obvious. Finally, following up an idea of D.J.Newman [7], it is then possible to prove that

$$K_0(k) \leq (k-1)9^k. \tag{4}$$

The author's interest in these questions was sparked by work of H.-O. Kreiss reported in these Proceedings [6].



The construction of q from p_{i-1} and p_i for $k = 3$.

2. A modification of Farvard's construction. To recall, with p_i the polynomial of degree $\leq k$ which agrees with f_0 at $t_i, \ldots, t_{i+k}$, Farvard's construction consists in blending the n polynomials $p_1, \ldots, p_n$ together smoothly and without increasing the k–th derivative very much. Farvard carries out the transition from p_{i-1} to p_i over a largest subinterval (t_j, t_{j+1}) in (t_i, t_{i+k-1}). Our modification consists in carrying out this transition from p_{i-1} to p_i over the entire interval (t_i, t_{i+k-1}).

For this, consider the problem of constructing a function $q \in L_\infty^{(k)}$ for which

$$q = \begin{cases} p_{i-1} & \text{on} \quad t < t_i, \\ f_0 (= p_{i-1} = p_i) & \text{on} \quad t_i, \ldots, t_{i+k-1}, \\ p_i & \text{on} \quad t > t_{i+k-1}. \end{cases}$$

Since

$$p_i - p_{i-1} = \alpha_i \psi_i$$

with

$$\psi_i(t) := (t - t_i)\cdots(t - t_{i+k-1}),$$
$$\alpha_i := ([t_i, \ldots, t_{i+k}] - [t_{i-1}, \ldots, t_{i+k-1}])f_0,$$

we can describe q equivalently as being of the form

$$q = p_{i-1} + \alpha_i h_i,$$

where h_i is any particular element of the class H_i consisting of those $h \in L_\infty^{(k)}$ for which

$$h = \begin{cases} 0 & \text{on} \quad t < t_i, \\ 0 = \psi_i & \text{on} \quad t_i, \ldots, t_{i+k-1}, \\ \psi_i & \text{on} \quad t > t_{i+k-1}. \end{cases}$$

For any $h_i \in H_i$, we have

$$([t_j, \ldots, t_{j+k}] - [t_{j-1}, \ldots, t_{j+k-1}])h_i = \delta_{ij}$$

since each such h_i agrees with 0 on $(t_r)_{r<i+k}$ and agrees with the monic k–th degree polynomial ψ_i on $(t_r)_{r\ge i}$. Since h_i agrees with 0 on $t_1, \ldots, t_{k+1}$ (for $i > 1$), the function

$$f := p_1 + \sum_{i=2}^{n} \alpha_i h_i \tag{5}$$

therefore agrees with p_1 on $t_1, \ldots, t_{k+1}$ and has the same k–th divided differences on points of $\mathbf{t}$ as does f_0, hence f and f_0 agree on $\mathbf{t}$. In fact, on (t_j, t_{j+1}),

$$\begin{aligned} f &= p_1 + \sum_{i \le j+1-k} \alpha_i h_i + \sum_{i=j+2-k}^{j} \alpha_i h_i \\ &= p_{\max\{1, j+1-k\}} + \sum_{i \le j < i+k} \alpha_i h_i \end{aligned}$$

since, on (t_j, t_{j+1}), $\alpha_i h_i = \alpha_i \psi_i = p_i - p_{i-1}$ for $i \le j+1-k$ while $\alpha_i h_i = 0$ therefore $i > j$. Hence, f is a *local* interpolant to f_0, with f on (t_j, t_{j+1}) depending only on $p_{j+1-k}, \ldots, p_j$. In particular,

$$\begin{aligned} \|f^{(k)}\|_{\infty,(t_j,t_{j+1})} &\le |p_{j+1-k}^{(k)}| + \sum_{i \le j < i+k} |(p_i^{(k)} - p_{i-1}^{(k)})/k!| \|h_i^{(k)}\|_{\infty,(t_i,t_{i+k-1})} \\ &\le (1 + 2(k-1) \max_i \|h_i^{(k)}\|_{\infty,(t_i,t_{i+k-1})}/k!) \max_{i \le j < i+k} k! |[t_i, \ldots, t_{i+k}]f_0| \end{aligned} \tag{6}$$

We conclude that *each choice of $h_i \in H_i$, $i = 2, \ldots, n$, gives rise via (5) to a map $P : f_0 \mapsto f$ which is a linear projector on $L_\infty^{(k)}$, produces Pf_0 which agrees with f_0 on $\mathbf{t}$, and satisfies*

$$\|(Pf_0)^{(k)}\|_{\infty,(t_j,t_{j+1})} \le C_{k,\mathbf{t},(h_i)} \max_{i \le j < i+k} k! |[t_i, \ldots, t_{i+k}]f_0| \tag{7}$$

all j, *with*

$$C_{k,\mathbf{t},(h_i)} := 1 + 2(k-1) \max_i \|h_i^{(k)}\|_{\infty,(t_i,t_{i+k-1})} / k!. \tag{8}$$

3. The minimization of $C_{k,\mathbf{t},(h_i)}$ with respect to (h_i) is a *local* matter entirely as it involves the minimization of $\|h^{(k)}\|_{\infty,(t_i,t_{i+k-1})}$ over all $h \in H_i$ for each i separately. After a linear change of variables which takes (t_i, t_{i+k-1}) into $(0,1)$, the problem is one of minimizing $\|h^{(k)}\|_\infty/k!$ over all $h \in L_\infty^{(k)}[0,1]$ which satisfy

$$\begin{aligned} &h^{(j)}(0^+) = 0, \quad j = 0,\dots,k-1 \\ &h \quad \text{agrees with} \quad \psi \quad \text{on} \quad \tau_0,\dots,\tau_{k-1} \\ &h^{(j)}(1^-) = \psi^{(j)}(1^-), \quad j = 0,\dots,k-1 \end{aligned} \tag{9}$$

for a certain $0 = \tau_0 \le \cdots \le \tau_{k-1} = 1$ and with

$$\psi(t) := (t-\tau_0)\cdots(t-\tau_{k-1}).$$

Denote the collection of all such h by $H_{\boldsymbol{\tau}}$ and set

$$\text{const}_{\boldsymbol{\tau}} := \inf_{h\in H_{\boldsymbol{\tau}}} \|h^{(k)}\|_\infty/k!.$$

Then, from the previous section,

$$K_0(k) \le 1 + 2(k-1) \sup_{0<\tau_1<\cdots<\tau_{k-2}<1} \text{const}_{\boldsymbol{\tau}}. \tag{10}$$

Let $\boldsymbol{\sigma} := (\sigma_i)_1^{r+k}$ be the smallest extension of $\boldsymbol{\tau}$ to a nondecreasing sequence containing both 0 and 1 exactly k times. Then, since ψ vanishes at $\tau_0,\dots,\tau_{k-1}$, we can describe $H_{\boldsymbol{\tau}}$ more simply as the collection of all $h \in L_\infty^{(k)}[0,1]$ which agree with ψ_+ at $\boldsymbol{\sigma}$, where

$$\psi_+(t) := \psi(t)(t-\widehat{t})_+^0$$

for some (entirely arbitrary) $\widehat{t} \in (0,1)$. Our task then becomes to construct a "best" interpolant h to ψ_+, i.e., to find among the functions agreeing with ψ_+ one which has smallest k–th derivative as measured in the max–norm. As elaborated upon in [4], the normalized k–th derivative $\widehat{g} := \widehat{h}^{(k)}/k!$ of such an interpolant provides (and is provided by) a norm–preserving extension to all of $L_1[0,1]$ for the linear functional λ given on

$$\$_{k,\boldsymbol{\sigma}} := \text{span}(M_{1,k},\dots,M_{r,k}) \subseteq L_1[0,1]$$

by the rule

$$\lambda : \$_{k,\boldsymbol{\sigma}} \to \mathbb{R} : \varphi \mapsto \int_0^1 \varphi(t) h^{(k)}(t)dt/k! \qquad (\text{any } h \in H_{\boldsymbol{\tau}}). \tag{11}$$

Here, $M_{i,k}$ is the B–spline of order k with knots $\sigma_i,\dots,\sigma_{i+k}$, normalized to have unit integral. Equivalently, $M_{i,k}$ represents the k–th divided difference at the points $\sigma_i,\dots,\sigma_{i+k}$ in the same sense that

$$k!\ [\sigma_i,\dots,\sigma_{i+k}]f = \int_0^1 M_{i,k}(t) f^{(k)}(t)dt, \quad \text{all } f \in L_1^{(k)}[0,1].$$

It follows that

$$\text{const}_{\boldsymbol{\tau}} = \|\lambda\| = \sup_{\varphi\in\$_{k,\boldsymbol{\sigma}}} \lambda\varphi/\|\varphi\|_1 \tag{12}$$

while

$$M_{i,k} = [\sigma_i, \ldots, \sigma_{i+k}]\psi_+, \quad \text{all } i.$$

Now let σ_m be the entry of $\boldsymbol{\sigma}$ corresponding to τ_0 when $\boldsymbol{\tau}$ was extended to $\boldsymbol{\sigma}$. If $\tau_0 < \tau_1$, then $m = k$. More generally, m is such that $0 = \tau_0 = \cdots = \tau_{k-m} < \tau_{k-m+1}$. In any event, m is such that ψ_+ agrees with the monic polynomial ψ at $\sigma_i, \ldots, \sigma_{i+k}$ for $i \geq m$ while ψ_+ agrees with 0 at $\sigma_i, \ldots, \sigma_{i+k}$ for $i < m$. Hence

$$\lambda M_{i,k} = \begin{cases} 0, & i < m \\ 1, & i \geq m \end{cases} \tag{13}$$

and therefore

$$|\lambda \sum_i \alpha_i M_{i,k}| = |\sum_{i \geq m} \alpha_i| \leq \sum_i |\alpha_i| \leq D_k \| \sum_i \alpha_i M_{i,k} \|_1,$$

the last inequality valid, by the theorem in [1:Sec. 3], for some constant D_k depending only on k. Consequently, $\|\lambda\| \leq D_k$ and, combining this with (10) and (12), we get

$$K_0(k) \leq 1 + 2(k-1)D_k.$$

Unfortunately, the argument for the theorem in [1:Sec. 3] produces rather pessimistic bounds for D_k, as reflected in (3) above.

By contrast, D.J.Newman [7] gave the following very effective and simple argument for a bound on $K_0(k)$: Let

$$G(t) := \text{const} \int_0^t s^{k-1}(1-s)^{k-1} ds$$

with const $:= \frac{k}{2}\binom{2k}{k}$ so that $G(1) = 1$. Then

$$h(t) := G(t)\psi(t)$$

agrees with ψ_+ at $\boldsymbol{\sigma}$, hence

$$K_0(k) \leq 1 + 2(k-1)\|h^{(k)}\|_\infty / k!.$$

On the other hand, h is a polynomial of degree $3k-1$, hence

$$\|h^{(k)}\|_\infty / k! \leq T_{3k-1}^{(k)}(1)\|h\|_\infty 2^k / k!$$

by Markov's inequality, with T_{3k-1} the Chebyshev polynomial of degree $3k-1$. But

$$\|h\|_\infty \leq 1$$

since $G(t)$ increases monotonely from 0 to 1 as t goes from 0 to 1 while $\psi(t)$ on $[0,1]$ is a product of k factors all ≤ 1 in absolute value. Further,

$$\begin{aligned} T_{3k-1}^{(k)}(1)4^{-k}/k! &\leq \sum_{j=0}^{3k-1} T_{3k-1}^{(j)}(1)4^{-j}/j! \\ &= T_{3k-1}(5/4) = (2^{3k-1} + 2^{-(3k-1)})/2 \end{aligned}$$

therefore

$$\|h^{(k)}\|_\infty / k! < 8^k 2^{3k-1} < 64^k$$

or

$$K_0(k) = 0(64^k),$$

showing $K_0(k)$ to grow only exponentially with k.

Newman's argument can be refined as follows: Choose G, more generally, of the form

$$G(t) := \int_0^t g(s)ds$$

with g any function in $L_\infty^{(k-1)}[0,1]$ having a $(k-1)$ fold zero both at 0 and at 1 and such that $G(1) = 1$. By Leibniz' formula,

$$h^{(k)} = \sum_{i=0}^{k} \binom{k}{i} \psi^{(i)} G^{(k-i)}$$

while

$$\|\psi^{(i)}\|_\infty \leq k(k-1)\cdots(k-i+1)$$

and

$$G^{(k-i)}(t) = \int_0^t (t-s)^{i-1} G^{(k)}(s)ds/(i-1)!.$$

But $G^{(k)} = g^{(k-1)}$ is orthogonal to $\mathcal{P}_{k-1}$ on $[0,1]$ since

$$g^{(j)}(0) = g^{(j)}(1) = 0, \quad j = 0, \ldots, k-2,$$

by choice of g, therefore

$$G^{(k-i)}(t) = \int_0^1 [(t-s)_+^{i-1} - p(t,s)] G^{(k)}(s)ds/(i-1)!$$

with $p(t,.)$ an arbitrary element of $\mathcal{P}_{k-1}$. Choose, in particular, $p(t,.)$ to be the polynomial of degree $< i-1$ which agrees with $(t-.)_+^{i-1}$ at certain points $s_1, \ldots, s_{i-1}$. Then

$$|(t-s)_+^{i-1} - p(t,s)| \leq \prod_{j=1}^{i-1} |s - s_j|$$

while, by [9;2.9.31],

$$\min_{s_1,\ldots,s_{i-1} \in [0,1]} \int_0^1 \prod_{j=1}^{i-1} |s-s_j| ds = 4^{-i+1}.$$

Therefore

$$\|G^{(k-i)}\|_\infty \leq \|G^{(k)}\|_\infty 4^{-(i-1)}/(i-1)!.$$

Finally, by a theorem due to R.Louboutin (see [9; p.8]), among the functions $G \in L^{(k)}[0,1]$ having a k–fold zero at 0 and a k–fold one at 1, $\|G^{(k)}\|_\infty$ is uniquely minimized by the function

$$\widehat{G}(t) := \int_0^t M(s)ds$$

with M the B–spline of order k, normalized to have unit 1–norm and with the $k+1$ knots $(1-\cos(\pi j/k))/2$, $j=0,\ldots,k$. The minimum value is therefore $\|\widehat{G}^{(k)}\|_\infty = 2^{2k-2}(k-1)!$. With this choice $G=\widehat{G}$, we then get

$$\begin{aligned}
\|h^{(k)}\|_\infty/k! &\le \frac{(k-1)!}{k!}\Big(2^{2k-2}+\sum_{i=1}^{k}\binom{k}{i}\frac{k\cdots(k-i+1)}{(i-1)!}2^{2(k-i)}\Big)\\
&= 2^{2k-2}/k+\sum_{i=1}^{k}\binom{k}{i}\binom{k-1}{i-1}2^{2(k-i)}\\
&< 2^{2k-2}/k+\Big[\sum_{i=0}^{k}\binom{k}{i}2^{i}-2^{k}\Big]\sum_{i=1}^{k}\binom{k-1}{i-1}2^{i-1}\\
&= 2^{2k-2}/k+(3^k-2^k)3^{k-1}\\
&< 9^k/3-1/(2k-2).
\end{aligned}$$

Hence, finally we get

$$K_0(k) < (k-1)9^k \tag{4}$$

as mentioned in the introduction.

4. The explicit calculation of $\|\lambda\|$ seems to be the key to more realistic bounds for $K_0(k)$, at least for small k.

To begin with, one might try to compute $\|\lambda\|$ simply by maximizing $\lambda\varphi$ over the unit sphere $\{\varphi\in \$_{k,\boldsymbol{\sigma}} \mid \|\varphi\|_1 = 1\}$ in $\$_{k,\boldsymbol{\sigma}}$. This means, of course, finding an extremal for λ, i.e., a $\chi\in\$_{k,\boldsymbol{\sigma}}$ such that $\|\chi\|_1=1$ and $\lambda\chi=\|\lambda\|$. Unfortunately, the equivalent constrained maximization problem in $\mathbb{R}^r$ "Maximize $\sum_{i\ge m}\alpha_i$ over $S:=\{\boldsymbol{\alpha}\in\mathbb{R}^r \mid \|\sum_i \alpha_i M_{i,k}\|_1=1\}$" is not easily solved by standard techniques since S is only piecewise smooth. In any event, such computations result, strictly speaking, only in *lower* bounds for $\|\lambda\|$.

It seems more appropriate to compute *upper* bounds, by going back to the original problem of finding g with smallest possible sup–form for which $\int g\varphi=\lambda\varphi$, all $\varphi\in\$_{k,\boldsymbol{\sigma}}$, i.e., to the problem of finding norm preserving extensions for λ.

Lemma 1. *There exists exactly one norm preserving extension of λ to a linear functional $\widehat{\lambda}$ on all of $L_1[0,1]$. This extension is given by the rule*

$$\widehat{\lambda}\varphi=\int\widehat{g}\varphi,\quad \text{all }\varphi\in L_1,$$

with

$$\widehat{g}=\|\lambda\|\,\mathrm{signum}\,\chi$$

and χ any extremal for λ. In particular, $\widehat{g}$ is absolutely constant and has fewer than $r=\dim\$_{k,\boldsymbol{\sigma}}$ jumps.

Proof: We claim that

$$\|\lambda\|>1. \tag{14}$$

For, if not, then with $\widehat{g} \in L_\infty[0,1]$ a norm preserving extension of λ to all of $L_1[0,1]$, we would have

$$1 = \lambda M_{m,k} = \int_0^1 \widehat{g} M_{m,k} \le \|\widehat{g}\|_\infty \|M_{m,k}\|_1 = \|\lambda\| \cdot 1 \le 1,$$

therefore equality would hold in Hölder's inequality, hence, as $M_{m,k} > 0$ a.e. on $[0,1]$, $\widehat{g} = 1$ would follow, and so, with (13),

$$0 = \lambda M_{m-1,k} = \int \widehat{g} M_{m-1,k} = \int M_{m-1,k} = 1,$$

a contradiction.

Let $\chi = \sum_i \alpha_i M_{i,k}$ be an extremal for λ, i.e.,

$$\chi \in \$_{k,\boldsymbol{\sigma}}, \quad \|\chi\|_1 = 1, \quad \lambda\chi = \|\lambda\|.$$

Then, from (14),

$$\begin{aligned} \sum_{i<m} \alpha_i &= \sum_i \alpha_i - \lambda\chi \\ &= \int \chi - \|\lambda\| \\ &\le 1 - \|\lambda\| < 0 \end{aligned}$$

therefore $\alpha_i \neq 0$ for some $i < m$. But this implies that

$$\operatorname{supp} \chi = [0,1].$$

For, otherwise χ would vanish on (σ_{i-1}, σ_i) for some $i > k$ with $\sigma_{i-1} < \sigma_i$. Then $\alpha_{i-k} = \cdots = \alpha_{i-1} = 0$ and

$$\|\chi\|_1 = \int_0^{\sigma_{i-1}} |\sum_{j<i-k} \alpha_j M_{j,k}| + \int_{\sigma_i}^1 |\sum_{j \ge i} \alpha_j M_{j,k}|$$

while $\sum_{j<i-k} \alpha_j M_{j,k} \in \ker \lambda$, hence $\sum_{j<i-k} \alpha_j M_{j,k} = 0$ (since otherwise $\|\sum_{j\ge i} \alpha_j M_{j,k}\|_1 < \|\chi\|_1$ while $\lambda \sum_{j \ge i} \alpha_j M_{j,k} = \lambda\chi$, contradicting the fact that χ is an extremal for λ), hence then $\alpha_1 = \cdots = \alpha_{i-1} = 0$ for some $i > k$, contradicting the fact that $\alpha_j \neq 0$ for some $j < m$.

If now $\widehat{g}$ is any norm preserving extension of λ to all of $L_1[0,1]$, – (there exists at least one by the Hahn–Banach theorem), – i.e., if $\widehat{g} \in L_\infty[0,1]$ with $\|\widehat{g}\|_\infty = \|\lambda\|$ and $\lambda\varphi = \int \widehat{g}\varphi$, all $\varphi \in \$_{k,\boldsymbol{\sigma}}$, then, in particular,

$$\|\lambda\| = \lambda\chi = \int \widehat{g}\chi \le \|\widehat{g}\|_\infty \|\chi\|_1 = \|\widehat{g}\|_\infty = \|\lambda\|,$$

hence equality must hold in Hölder's inequality, therefore, as $\operatorname{supp}\chi = [0,1]$,

$$\widehat{g} = \|\widehat{g}\|_\infty \operatorname{signum} \chi$$

follows. This shows that $\widehat{g}$ is uniquely determined by χ. In particular, $\widehat{g}$ is absolutely constant. Further, $\widehat{g}$ changes sign only when χ does, while χ, as a linear combination of r B–splines, can change sign at most $r-1$ times. Q.E.D.

Lemma 1 suggests that we represent λ by a piecewise constant function g in such a way that $|g|$ is constant. If we succeed in constructing such a g, we may have found $\widehat{g}$, and therefore know $\|\lambda\|$. Such a g can only be found as the limit of some iterative process. The next lemma asserts that every iterate in such a process is apt to carry useful information about $\|\lambda\|$.

Lemma 2. *Let g be a piecewise constant function,*

$$g(t) = \beta_j \quad \text{on} \quad (\rho_{j-1}, \rho_j), \quad j = 1, \ldots, u,$$

for some sequence $(\beta_j)_1^u$ and some sequence $(\rho_j)_0^u$ with $0 = \rho_0 < \cdots < \rho_u = 1$. If g represents λ, i.e., if $\int g\varphi = \lambda\varphi$, all $\varphi \in \$_{k,\boldsymbol{\sigma}}$, *and g has fewer than r sign changes, then*

$$\min_j |\beta_j| \le \|\lambda\| \le \max_j |\beta_j|. \tag{15}$$

Proof: Only the first inequality requires proof, and this only in the case when $\min_j |\beta_j| > 1$, since $\|\lambda\| > 1$ by (14). Hence, assume that $\min_j |\beta_j| > 1$ and let $(v_j)_1^{s-1}$ be the points at which g changes sign. Then $s \le r$, by assumption. Further,

$$M_{i,k}(v_i) \ne 0, \quad i = 1, \ldots, m-1. \tag{16}$$

For, if (by way of contradiction) $M_{i,k}(v_i) \ne 0$ for $i = 1, \ldots, j-1$, but $M_{j,k}(v_j) = 0$ for some $j < m$, then one could find a nonzero $\varphi \in \operatorname{span}(M_{1,k}, \ldots, M_{j,k}) \subseteq \ker \lambda$ which changes sign only at $v_1, \ldots, v_{j-1}$, has $\operatorname{signum} \varphi = \operatorname{signum} g$ on $(0, v_1)$ and vanishes for $t \ge v_j$. But then,

$$0 = \lambda\varphi = \int_0^1 g(t)\varphi(t)dt = \int_0^{v_j} g\varphi \ge \min_{i \le j} |\beta_i| \int_0^1 |\varphi| > 0,$$

a contradiction. Further, since

$$\int_0^1 (1-g) \sum_j \alpha_j M_{j,k} = \sum_{j<m} \alpha_j$$

while $1-g$, like g, changes sign only at $(v_j)_1^{s-1}$, – (a consequence of our assumption that $\min_j |\beta_j| > 1$), – it follows similarly that

$$M_{r-i,k}(v_{s-1-i}) \ne 0, \quad i = 0, \ldots, r-m,$$

hence that

$$M_{i,k}(v_i) \ne 0, \quad i = m, \ldots, s-1, \tag{17}$$

since $\operatorname{supp} M_{i,k} \supseteq \operatorname{supp} M_{j,k}$ for $m \le i \le j$. Because of (16) and (17), we can therefore find $\varphi \in \operatorname{span}(M_{1,k}, \ldots, M_{s,k}) \subseteq \$_{k,\boldsymbol{\sigma}}$ which changes sign only at $v_1, \ldots, v_{s-1}$ and has the same sign as g in $(0, v_1)$. But then

$$\lambda\varphi = \int g\varphi \ge \min_j |\beta_j| \int |\varphi|$$

which proves that $\|\lambda\| \ge \min_j |\beta_j|$ since $\|\varphi\|_1 \ne 0$, by construction. Q.E.D.

Corollary. *If g is absolutely constant with fewer than r jumps and represents λ, then $g = \widehat{g}$ and $\|g\|_\infty = \|\lambda\|$.*

Consider now the problem of computing a piecewise constant representer g with s steps (i.e., $s-1$ breakpoints) for λ. For this g to be useful in bracketing $\|\lambda\|$, it should have $< r$ sign changes. This can be insured by choosing $s \le r$. On the other hand, once the $s-1$ breakpoints are picked, we have only s linear parameters available for matching λ on the r–dimensional space $\$_{k,\boldsymbol{\sigma}}$, hence s must be at least as big as r. For these reasons, we choose $s = r$, i.e.,

$$g(t) = \beta_j \quad \text{on} \quad (\rho_{j-1}, \rho_j), \quad j = 1, \ldots, r$$

with $0 = \rho_0 < \cdots < \rho_r = 1$, and determine $\boldsymbol{\beta}$ from the linear system

$$\sum_{j=1}^{r} \beta_j \int_{\rho_{j-1}}^{\rho_j} M_{i,k} = \begin{cases} 0, & i < m \\ 1, & i \geq m \end{cases}, \quad i = 1, \ldots, r, \tag{18}$$

(see (13)).

It turns out to be more convenient to solve a slightly different, equivalent system. Let $N_{i,k+1}$ be a B–spline of order $k+1$, with knots at $\sigma_1, \ldots, \sigma_{i+k+1}$, normalized in a certain way. Explicitly,

$$\begin{aligned} N_{i,k+1}(t) &:= ((\sigma_{i+k+1} - \sigma_i)/(k+1)) M_{i,k+1}(t) \\ &= ([\sigma_{i+1}, \ldots, \sigma_{i+k+1}] - [\sigma_i, \ldots, \sigma_{i+k}])(\cdot - t)_+^k. \end{aligned}$$

Then $N_{i,k+1}^{(1)} = -(M_{i+1,k} - M_{i,k})$, hence

$$\int_{\rho_{j-1}}^{\rho_j} (M_{i,k} - M_{i+1,k}) = N_{i,k+1}(\rho_j) - N_{i,k+1}(\rho_{j-1}).$$

Since

$$\int_{\rho_{j-1}}^{\rho_j} M_{r+1,k} = 0, \quad j = 1, \ldots, r,$$

– here we have added an arbitrary $\sigma_{r+k+1} > 1$ to $\boldsymbol{\sigma}$, – it follows that (18) is equivalent to

$$A\boldsymbol{\beta} = \mathbf{b} \tag{19a}$$

with

$$A := (N_{i,k+1}(\rho_j) - N_{i,k+1}(\rho_{j-1}))_{i,j=1}^r \tag{19b}$$

$$b_i := \begin{cases} -1, & i = m-1 \\ 1, & i = r \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, \ldots, r. \tag{19c}$$

Note that $N_{i,k+1}(\rho_0) = 0$, all i, hence A is column–equivalent to $(N_{i,k+1}(\rho_j))_{i,j=1}^r$, therefore invertible iff $N_{i,k+1}(\rho_i) \neq 0$, all i, i.e., iff $\sigma_i < \rho_i < \sigma_{i+k+1}$, all i, a condition on $\boldsymbol{\rho}$ easily enforced.

This settles the determination of $\boldsymbol{\beta}$. Consider next the question of how to choose $\boldsymbol{\rho}$ so as to make the resulting g absolutely constant.

Lemma 3. *Let $0 = \rho_0 < \cdots < \rho_r = 1$ be such that*

$$\begin{aligned} &N_{i-1,k+1}(\rho_i) \neq 0, \quad \text{i.e.,} \quad \rho_i < \sigma_{i+k}, \quad i = 2, \ldots, m-1, \\ &N_{i,k+1}(\rho_{i-1}) \neq 0, \quad \text{i.e.,} \quad \sigma_i < \rho_{i-1}, \quad i = m, \ldots, r. \end{aligned} \tag{20}$$

Then also $N_{i,k}(\rho_i) \neq 0$, $i = 1, \ldots, r$, hence (19) has a unique solution $\boldsymbol{\beta}$. This solution satisfies

$$(-)^{m+i}(\beta_i - \beta_{i-1}) > 0, \quad i = 2, \ldots, r.$$

Proof: By (19), the r–vector

$$\boldsymbol{\beta}' := (\beta_1 - \beta_2, \beta_2 - \beta_3, \ldots, \beta_{r-1} - \beta_r, \beta_r)$$

is the solution of

$$B\boldsymbol{\beta}' = \mathbf{b}$$

with

$$B := (N_{i,k+1}(\rho_j))_{i,j=1}^r$$

and $\mathbf{b} = (b_i)$ given by (19c). Therefore, $\boldsymbol{\beta}' = -\boldsymbol{\gamma}^{(m-1)} + \boldsymbol{\gamma}^{(r)}$, with $\boldsymbol{\gamma}^{(j)}$ the j–th column of B^{-1}. Further, since $N_{i,k+1}(\rho_r) = \delta_{ir}$, all i, the last column of B, and therefore also $\boldsymbol{\gamma}^{(r)}$, equals the unit vector with r–th entry equal to 1. Consequently,

$$\beta_i - \beta_{i-1} = \gamma_{i-1}^{(m-1)}, \quad i = 2, \ldots, r.$$

But $\gamma_{i-1}^{(m-1)}$, as the $(i-1, m-1)$–entry of B^{-1}, is given by

$$\gamma_{i-1}^{(m-1)} = (-)^{i+m} \det B_{(m-1,i-1)} / \det B,$$

with $B_{(r,s)}$ the matrix obtained from B by deleting row r and column s. Conditions (20) insure that $B_{(m-1,i-1)}$ has all diagonal entries nonzero which, by a slight extension [2;Theorem 2] of the wellknown fact that B is totally positive, implies that $\det B_{(m-1,i-1)} > 0$, $i = 2, \ldots, r$. Q.E.D.

Since $\det B_{(m-1,i-1)} = 0$ iff one of its diagonal entries is zero, it is now possible to describe the exact circumstances under which $\beta_i = \beta_{i-1}$, i.e., under which g has no jump at ρ_i. More importantly, we have the

Corollary 1. *The unique norm preserving extension $\widehat{g}$ for λ has exactly $r-1$ sign changes.*

Proof: Let $(v_j)_1^{s-1}$ be the increasing sequence of points at which $\widehat{g}$ changes sign. Then $s \le r$, by Lemma 1, and, by the proof for Lemma 2, (16) and (17) must hold. We can therefore extend $(v_j)_1^{s-1}$ to an increasing sequence $(\rho_j)_0^r$ with $\rho_0 = 0$ and $\rho_r = 1$ so that (20) holds, while $g = \beta_j$ on (ρ_{j-1}, ρ_j), $j = 1, \ldots, r$, for some absolutely constant $\boldsymbol{\beta}$. But then $\boldsymbol{\beta}$ satisfies (19), hence $\beta_i \ne \beta_{i-1}$, by Lemma 3, showing that $\widehat{g}$ must change sign at ρ_i, $i = 1, \ldots, r-1$. Q.E.D.

It follows that λ has exactly one extremal. Also, for the record,

Corollary 2. *The function $F(h) := \|h^{(k)}\|_\infty / k!$ discussed in Section 3 has exactly one minimum in $H_{\boldsymbol{\tau}}$ (see (9)). The minimum is a perfect spline of order $k+1$ with $r-1$ interior knots.*

Proof: The minimum is the unique $h \in H_{\boldsymbol{\tau}}$ with $h^{(k)} = k!\widehat{g}$. Q.E.D.

It follows that $\widehat{g}$, i.e., $\boldsymbol{\rho}$ and $\boldsymbol{\beta}$ for $\widehat{g}$, is the unique solution of the system (19a–c) together with the equations

$$\text{(19d)} \qquad \beta_i + \beta_{i-1} = 0, \quad i = 2, \ldots, r.$$

For, $\widehat{g}$ certainly solves this system, while any solution to this system must give $\widehat{g}$, by the Corollary to Lemma 2.

We attempt to solve (19a–d) for the unknowns $\boldsymbol{\rho}$ and $\boldsymbol{\beta}$ by Newton's method. With $\boldsymbol{\beta}$ determined from (19a–c) for given $\boldsymbol{\rho}$, we compute the desired changes $\delta\rho_i$, $i = 1, \ldots, r-1$, from the condition that

$$\sum_{j=1}^{r-1} \Big(\frac{\partial A}{\partial \rho_j} \delta\rho_j\Big)\boldsymbol{\beta} = -A(c\boldsymbol{\varepsilon} - \boldsymbol{\beta})$$

where $\boldsymbol{\varepsilon} := (-1, +1, -1, \ldots)$. This gives

$$\text{(21a)} \qquad \begin{aligned} \delta\rho_i &= y_i / (\beta_i - \beta_{i+1}), \quad i = 1, \ldots, r-1, \\ c &= y_r \end{aligned}$$

with $\mathbf{y}$ the solution of the linear system

(21b) $$C\mathbf{y} = \mathbf{b}$$

where

(21c) $$C := \left(N^{(1)}_{i,k+1}(\rho_1) \,\vdots\, \cdots \,\vdots\, N^{(1)}_{i,k+1}(\rho_{r-1}) \,\vdots\, (A\boldsymbol{\varepsilon})_i\right)_{i=1}^{r}.$$

5. The maximization of $\|\lambda\| = \text{const}_{\boldsymbol{\tau}}$ over $\boldsymbol{\tau}$ is our final goal since, by (10) and (12),

$$K_0(k) \le 1 + 2(k-1) \sup_{0<\tau_1<\cdots<\tau_{k-2}<1} \text{const}_{\boldsymbol{\tau}}.$$

For this, we calculated $\text{const}_{\boldsymbol{\tau}}$, – a number between 1 and 37 for $k \le 10$, – to within an absolute error of .005 at a large number of points $(\tau_1, \ldots, \tau_{k-2})$ on

$$T_k := \{(\tau_1, \ldots, \tau_{k-2}) \mid 0 \le \tau_1 \le \cdots \le \tau_{k-2} \le 1\}$$

and for $k = 3, 4, 5$, using Newton's method as described in the previous section.

$\text{const}_{\boldsymbol{\tau}}$ can be shown to be continuous on T_k and $k-1$ times differentiable in the interior of T_k, but does not appear to be convex. In view of the fact that Newton's method is only as good as the initial guess, it seemed most efficient to evaluate $\text{const}_{\boldsymbol{\tau}}$ along rays, starting at the point $\tau_1 = \cdots = \tau_{k-2} = 1/2$ and using the $r = 2k-2$ Chebyshev points as the initial guess for $\rho_1, \ldots, \rho_r$, and then proceeding along the ray towards the boundary, using the previously computed $\boldsymbol{\rho}$ as the initial guess in the next step.

Details of these computations together with the `Fortran` program used can be found in the Mathematics Research Center TSR #1466.

For $k = 3, 4, 5$, we found the maximum of $\text{const}_{\boldsymbol{\tau}}$ to occur at one of the vertices of T_k. Assuming this to be true for all k, we merely maximized $\text{const}_{\boldsymbol{\tau}}$ for $k = 6, \ldots, 10$ over the vertices of T_k (and the rays leading from the midpoint to these vertices). The resulting upper bounds for $K_0(k)$ are listed in the table below together with the lower bounds for $K(k)$ obtained in [3]. The upper bounds seem to behave like c^k for some c slightly larger than 2, while the lower bounds are known to behave like $(\pi/2)^k$.

k		$\le K(k) \le K_0(k) \le$
2	2	3.414
3	3	6.854
4	4.8	11.665
5	7.5	21.036
6	11.8	42.330
7	18.5	79.276
8	29.1	163.344
9	45.7	316.792
10	71.8	664.020

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