The multiplicity of a spline zero

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Abstract. The multiplicity of a zero of a (univariate, polynomial) spline is defined in terms of its B-spline coefficients, thus making certain bounds trivial while, at the same time, adhering to the principle that the multiplicity of a zero indicates the number of simple zeros nearby achievable by a nearby element from the same class. In particular, the multiplicity depends on the class to which the function is assumed to belong.

The resulting multiplicity turns out to coincide with that given recently in more traditional terms by T. N. T. Goodman.

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Dedicated to Ted Rivlin, on the occasion of his 70th birthday

The variation-diminishing property of B-splines provides a ready upper bound on the number of zeros of a (univariate) spline in terms of the number of sign changes in the sequence of its B-spline coefficients. Because of the compact support of the B-spline, this bound is even local. Finally, the bound even holds if zeros are counted with certain suitable multiplicities. However, the sharpness of the resulting bound depends on just how one defines the multiplicity of a spline zero.

For arbitrary real-valued functions, smooth or not, on some interval I, zeros have long been classified as **nodal** (aka *simple*, *odd*) or **nonnodal** (aka *double*, *even*) depending on whether or not the function changes sign across the zero. Here, the function f is said **to change sign across** ξ, or, ξ **is a sign change of** f, iff, for all $\varepsilon > 0$, $f(\alpha)f(\omega) < 0$ for some $\xi - \varepsilon \le \alpha < \xi < \omega \le \xi + \varepsilon$.

For a continuous f, a sign change is necessarily also a zero of f, but need not be a zero if f fails to be continuous. Not surprisingly, such a sign change is treated, nevertheless, as a zero whenever zeros are considered in the context of the possible sign changes of the function.

For *smooth* f, this rudimentary multiplicity concept (of simple *vs* double zeros) has been augmented and refined by assigning the multiplicity $m+1$ to any zero of f which is a zero of multiplicity m of Df, the first derivative of f. This refinement is mainly motivated by the desire to exploit Rolle's theorem to the fullest, and is, offhand, of not much help when f is not smooth.

It is one purpose of this note to advocate, instead, the notion (implicit in various definitions of multiplicity of zeros) that the multiplicity of a zero of f depend on the class F of which f is thought to be a member. Because this note is devoted to splines, it is convenient (as noted already in [5]) to define a zero of f to be an *interval*, more precisely any maximal closed interval whose interior lies in the set

$$f^{-1}\{0\} := \{x \in \operatorname{dom} f : f(x) = 0\}.$$

This definition takes account of the fact that a spline need not be continuous but does have right and left limits at every point. By this definition, for every $\zeta \in \operatorname{dom} f$, $[\zeta \,..\, \zeta]$ is a zero of f, but usually a zero of multiplicity 0, in the sense of the following definition.

Definition 1. *We say that z in $\operatorname{dom} F$ is a zero* **of multiplicity** *m of the element f of the collection F of real-valued functions, all defined on the real interval $\operatorname{dom} F$, if z is a zero of f and, for every neighborhood of f and every open interval w containing z, there is some element g of F which has m sign changes in w.*

It is the main purpose of this note to compare the resulting multiplicity of a zero of a spline to some definitions in the literature and to give a very simple characterization of this multiplicity in terms of the B-spline coefficients of the spline. In other words,

$$F = S_{k,t} := \operatorname{span}\{B_i := B_{i,k,t} : i = 1, \ldots, n\},$$

with k the **order** of the splines, with

$$t := (t_1 \le \cdots \le t_{n+k})$$

the **knot sequence**, and $B_i := B(\cdot|t_i, \ldots, t_{i+k})$ the **B-spline with knots** $t_i, \ldots, t_{i+k}$. Since the order, k, is not varied, I will refer to B_i or $B_{i,t}$ rather than $B_{i,k,t}$. It will be assumed throughout that $t_i < t_{i+k}$ for all i, i.e., none of the B_i is trivial. Further, the open interval $(t_1 \,..\, t_{n+k})$ will, at times, be referred to as the **basic interval** of the spline space (and its elements).

[7 : p.184] traces the development of the multiplicity concept of a spline zero, up to 1976, mentioning work by Schoenberg, Johnson, Braess, Karlin, Micchelli, Pence and others, with Schumaker's paper [6] providing a kind of capstone. All these different ways of counting spline zeros according to some multiplicity result in a count $Z^+(f)$ of the zeros of the spline f which satisfies the bound

$$Z^+(f) \;\le\; S^+(a), \tag{2}$$

with $f =: \sum_j a_j B_j$ the B-spline expansion for f, and

$$S^+(a), \qquad S^-(a)$$

the maximal, respectively, minimal, number of sign changes in the coefficient sequence a obtainable by a suitable choice of the signum of any zero entry.

However, the recent paper [2] by T.N.T. Goodman showed that, surprisingly, there was still room for improvement, both in the definition of spline-zero multiplicities and in the bound (2). As simple examples, Goodman offers (i) the quadratic spline $f := B_1 + B_3 - B_5 - B_7$ with simple knots which obviously has just one zero in its basic interval, and this zero is nodal, hence $Z^+(f) = 1$ while $S^+(a) = S^+(1, 0, 1, 0, -1, 0, -1) = 5$; and (ii) the linear spline $f := B_1 - B_4$ on the knot sequence $t = (1, 2, 3, 3, 4, 5)$, which has just one zero in its basic interal, and this zero is of multiplicity 1 by Schumaker's count but is of multiplicity 3 according to the multiplicity count (1), and the latter count makes (2) sharp for this example.

This note offers a slight improvement on Goodman's results and their proofs and may serve as motivation for his definition of spline-zero multiplicities.

Goodman's results

Goodman [2] relies on nothing more than knot insertion, by now a standard tool even in spline theory, as it has been in CAGD for a long time. The basic fact, first pointed out by Boehm in [1], has a very pretty geometric interpretation in terms of the **control polygon** $C_{k,t}f$ of $f := \sum_i a_i B_{i,t} \in S_{k,t}$, which, by definition, is the broken line with vertices

$$v_j := (t_j^* := (t_{j+1} + \cdots + t_{j+k-1})/(k-1),\ a_j) \in \mathbb{R}^2, \quad j = 1, \ldots, n.$$

Here is Boehm's result.

Lemma 3 (Boehm). *If the knot sequence $\hat{t}$ is obtained from the knot sequence t by the insertion of just one point, and $\sum_i a_i B_{i,t} = f = \sum_i \widehat{a}_i B_{i,\hat{t}}$, then, for each i, the vertex $\widehat{v}_i$ of $C_{k,\hat{t}}f$ lies in the segment $[v_{i-1} \,.\,.\, v_i]$ of $C_{k,t}f$.*

More explicitly, if ζ is the point inserted, then $\widehat{v}_i = (1-\alpha_i)v_{i-1} + \alpha_i v_i$, all i, with

$$\alpha_i \in \begin{cases} \{1\}, & \text{if } \zeta \le t_i; \\ (0 \,.\,.\, 1), & \text{if } t_i < \zeta < t_{i+k-1}; \\ \{0\}, & \text{if } t_{i+k-1} \le \zeta. \end{cases}$$

It is immediate that

$$S^-(\widehat{a}) \le S^-(a), \tag{4}$$

hence, by suitable choice of additional knots and induction, that

$$S^-(\sum_i a_i B_i) \le S^-(a), \tag{5}$$

as was first proved this way by Lane and Riesenfeld in [4]. Here, $S^-(f)$ counts the number of sign changes of f.

Goodman and Lee [3] prove that also

$$S^+(\widehat{a}) \le S^+(a) \tag{6a}$$

in case the following

$$\textbf{Condition}(a,t)\text{:}\ \ \forall x \in (t_1 \,.\,.\, t_{n+k})\ \exists\, i \ \text{ s.t. }\ t_i < x < t_{i+k} \text{ and } a_i \ne 0 \tag{6b}$$

holds. Based on this result, Goodman proves in [2] that, under Condition(a,t),

$$Z^+(f) \le S^-(a). \tag{7}$$

Examination of Goodman's proof shows that this striking improvement on (2) relies on nothing more than the following observation: Assume without loss (i.e., without change of f or $S^-(a)$) that $a_1 a_n \ne 0$. If $S^-(a) < S^+(a)$, then there must be some $\mu \le \nu$ for which

$$a_{\mu-1} \ne 0 = a_\mu = \cdots = a_\nu \ne a_{\nu+1}. \tag{8}$$

If $t_{\mu-1+k} \le t_{\nu+1}$, then $a_i = 0$ for all i with $t_i < t_{\nu+1} < t_{i+k}$, a situation excluded by Condition(a,t). Consequently, we may choose some $\zeta \in (t_{\nu+1} \mathinner{\ldotp\ldotp} t_{\mu-1+k})\backslash t$. Let $\hat{t}$ be the knot sequence obtained from t by insertion of any one such ζ, and let $\widehat{a}$ be the corresponding B-spline coefficient sequence for f. By (8) and Lemma 3,

$$a_{\mu-1}\widehat{a}_\mu > 0, \qquad \widehat{a}_{\nu+1}a_{\nu+1} > 0,$$

$$\widehat{a}_\mu \ne 0 = \widehat{a}_{\mu+1} = \cdots = \widehat{a}_\nu \ne \widehat{a}_{\nu+1}.$$

It follows that, while the B-spline coefficient segment with endpoints $a_{\mu-1} = \widehat{a}_{\mu-1}$ and $a_{\nu+1} = \widehat{a}_{\nu+2}$ has been lengthened by one entry, the number of strong sign changes in it has not changed and, more noteworthy, *the number of zero entries in it has decreased*. Also, Condition$(\hat{t}, \widehat{a})$ holds. Therefore, induction provides a refined knot sequence $\widetilde{t}$ for which the corresponding B-spline coefficient sequence $\widetilde{a}$ has *no* zero entries, thus ensuring that $S^+(\widetilde{a}) = S^-(\widetilde{a})$ while, in any case, $S^-(\widetilde{a}) \le S^-(a)$, from (5). Therefore, (7) now follows from (2).

The argument just given suggests, as does Goodman in [2 : p.126], that, for the sharpest possible bound on the number of zeros of the spline $f = \sum_i a_i B_{i,t}$, one should break up f into what I will call here its **connected component**s. By this, I mean the partial sums

$$f_I := \sum_{i \in I} a_i B_i$$

with $I = (\nu, \ldots, \mu)$ for which $\sum_{i \in I} |a_i| B_i > 0$ on its basic interval (i.e., on $(t_\nu \mathinner{\ldotp\ldotp} t_{\mu+k})$) while $f - f_I$ vanishes on that interval. Between any two neighboring such intervals, f is obviously zero, since even

$$f^t := \sum_i |a_i| B_{i,t}$$

is zero there. For this reason, such a zero of f is called 'obvious' below.

However, in contrast to [2], I will carry out the discussion entirely in terms of the B-spline coefficients. This requires the characterization in terms of B-spline coefficients of the multiplicity of a spline zero as defined by Definition 1. This is the subject of the next section.

Definition of spline-zero multiplicity

Since multiplicity is defined in terms of nearby sign changes possible in nearby splines, it is important to understand just how many sign changes an element of $S_{k,t}$ can have 'near' some interval. Denote by

$$S^-(f, w) := \sup\{r : \exists x_0 < \cdots < x_r \text{ in } w \text{ s.t. } S^-(f(x_0), \ldots, f(x_r)) = r\}$$

the number of sign changes of f on the open interval w. From (5), we know that

$$S^-(\sum_i a_i B_i, I(w)) \le S^-(a_{I(w)}), \tag{9}$$

with

$$I(w) := (i : (t_i \mathinner{..} t_{i+k}) \cap w \ne \emptyset).$$

Further, equality is always achievable in (9) by proper choice of a since, by the Schoenberg-Whitney theorem, we can uniquely interpolate from $\mathrm{span}(B_i : i \in I(w))$ to arbitrary data at any strictly increasing sequence $(x_i : i \in I(w))$ as long as $t_i < x_i < t_{i+k}$, all $i \in I(w)$.

With this, consider first what I will call an **obvious** zero of $f \in S_{k,t}$. By this, I mean any zero z of $f = \sum_i a_i B_i$ which satisfies

$$a_{I(z)} = 0.$$

Let

$$a_{\mu,\nu} := (a_\mu, \ldots, a_\nu)$$

be the corresponding **zero of** a, i.e., the maximal sequence of consecutive zero entries of a containing the segment $a_{I(z)}$. It is convenient (and consistent) to include here the possibility that $I(z) = \emptyset$, in which case

$$\mu - 1 := \max\{i : t_{i+k} \le z,\, a_i \ne 0\}, \quad \nu + 1 := \min\{i : z \le t_i,\, a_i \ne 0\}.$$

Now consider the multiplicity to be assigned to z as a(n obvious) zero of f. By (2), for every g in $S_{k,t}$ near f, there is some open interval w properly containing z on which g has at most

$$S^+(a_{\mu-1}, \ldots, a_{\nu+1}) \tag{10}$$

sign changes, with

$$a_0 := a_1, \qquad a_{n+1} := a_n.$$

The next proposition shows this bound to be sharp, hence, with Definition 1, I *define the multiplicity of z as a zero of f to be the number (10).*

Proposition 11. *If $f = \sum_i a_i B_i \in S_{k,t}$ has the obvious zero z, with $a_{\mu,\nu}$ the corresponding zero of a, then, for every open interval w containing z, every neighborhood of f in $S_{k,t}$ contains some g with $S^+(a_{\mu-1}, \ldots, a_{\nu+1})$ sign changes in w.*

Proof. The proof is case-by-case.

Case $\mu > \nu$: Then, necessarily, $I(z) = \emptyset$, and so, necessarily, $z = [\zeta \mathinner{..} \zeta]$ for some $\zeta \in [t_1 \mathinner{..} t_{n+k}]$.

If ζ is one of the endpoints, $\zeta = t_1$ say, then $a_1 \neq 0$, and (10) equals 0, thus there is nothing to prove.

Else, $\zeta \in (t_1 \mathinner{..} t_{n+k})$. Since $I(z) = \emptyset$, ζ is not in $(t_i \mathinner{..} t_{i+k})$ for any i, hence, necessarily, ζ is a knot, of multiplicity k. Further, since $\mu > \nu$,

$$f(\zeta-) = a_{\mu-1}, \quad \mu = \nu + 1, \quad f(\zeta+) = a_\mu,$$

hence (10) is 0 or 1 depending on whether $f(\zeta-)f(\zeta+)$ is positive or negative, thus, $g = f$ will serve.

Case $\mu \leq \nu$: In this case,

$$[z_l \mathinner{..} z_r] := z = [t_{\mu+k-1} \mathinner{..} t_{\nu+1}].$$

Let

$$I(z) =: (\mu', \ldots, \nu').$$

Reduction to the case $I(z) = (\mu, \ldots, \nu)$: Assume that, e.g., $\nu' < \nu$. In this case,

$$f(z_r + s) = a_{\nu+1} s^\kappa + O(s^{\kappa+1}) \quad \text{as } s \downarrow 0,$$

with $k - \kappa$ the multiplicity of z_r as a knot for $B_{\nu+1}$. Since $\nu \notin I(z)$, it follows that $t_\nu = z_r$, hence B_ν has z_r as a knot of multiplicity $k - \kappa + 1$. Therefore, as $\varepsilon \downarrow 0$,

$$a_\nu^\varepsilon := -f(z_r + \varepsilon)/B_\nu(z_r + \varepsilon)$$

goes to zero, while, for every small $\varepsilon > 0$,

$$g_\varepsilon := f + a_\nu^\varepsilon B_\nu$$

has a simple zero at $z_r + \varepsilon$, hence $z_r + \varepsilon$ is a sign-change for g_ε. Since $g_\varepsilon \to f$ as $\varepsilon \to 0$, we can choose ε so small that any sign change of f to the right of z_r gives rise to a sign change in g_ε. Consequently, we can now use induction on $\nu - \nu'$ to conclude, for all small $\varepsilon > 0$, the existence of

$$g_\varepsilon = \sum_i a_i^\varepsilon B_i$$

within ε of f, with z an obvious zero, with $a_i^\varepsilon = a_i$ for all $i \notin \{\nu' + 1, \ldots, \nu\}$, with $\nu - \nu'$ sign changes in $(z_r \mathinner{..} z_r + \varepsilon)$, and with

$$S^-(a_{\nu'+1}^\varepsilon, \ldots, a_\nu^\varepsilon, a_{\nu+1}) = \nu - \nu' = S^+(a_{\nu'+1}, \ldots, a_\nu, a_{\nu+1}).$$

There is an analogous argument in case $\mu < \mu'$. Hence the proposition is proved once we deal with the following case.

Case $I(z) = (\mu, \ldots, \nu)$:

Subcase $z_l < z_r$: In this case, we may choose $x_\mu < \cdots < x_\nu$ in $(z_l \mathrel{..} z_r)$ with $t_i < x_i < t_{i+k}$, all i, and, by the Schoenberg-Whitney theorem, for any such fixed choice and any $\varepsilon > 0$, $S_{\mu,\nu}$ contains an element $h_\varepsilon =: \sum_{i=\mu}^{\nu} a_i^\varepsilon B_i$ of size ε for which

$$S^-(h_\varepsilon(x_i) : i = \mu, \ldots, \nu) = \nu - \mu = S^-(a_i^\varepsilon : i = \mu, \ldots, \nu),$$

while $h_\varepsilon(z_r-)a_{\nu+1} < 0$. Thus, for all sufficiently small $\varepsilon > 0$,

$$g_\varepsilon := f + h_\varepsilon$$

has $\mu - \nu$ sign changes in $(z_l \mathrel{..} z_r)$ and an additional sign change at z_r or to the right of it (in addition to all the sign changes inherited from f) and, finally, an additional sign change at z_l or to the left of it in case $a_{\mu-1}(-1)^{\nu-\mu}a_{\nu+1} > 0$; hence, in either case, g_ε is close to f and has $S^+(a_{\mu-1}, \ldots, a_{\nu+1})$ sign changes near z.

Subcase $z_l = z_r$: Let $\zeta := z_r$. Then the multiplicity of ζ in t is $k-\kappa$, with $\kappa := \nu+1-\mu$, and this is also the multiplicity of ζ as a knot for both $B_{\mu-1}$ and $B_{\nu+1}$, hence

$$f(x) = a_{\mu-1}(\zeta - x)_+^\kappa + a_{\nu+1}(x - \zeta)_+^\kappa + O(|x - \zeta|^{\kappa+1}).$$

For given $\varepsilon > 0$, choose p_ε to be the unique polynomial of degree $< \kappa$ which agrees with f at $z + j\varepsilon$, $j = 1, \ldots, \kappa$. Then $\lim_{\varepsilon\to 0} p_\varepsilon = 0$, hence the unique element, g_ε, of $S_{k,t}$ which agrees with $f - p_\varepsilon$ on $(\zeta \mathrel{..} \zeta + (\kappa + 1)\varepsilon)$ has κ sign changes there and converges to f as $\varepsilon \to 0$. Further, $g_\varepsilon(\zeta+)(-1)^\kappa a_{\nu+1} > 0$ while $g_\varepsilon(\zeta - s)a_{\mu-1} > 0$ for some small positive s. Hence, depending on whether or not $a_{\mu-1}(-1)^\kappa a_{\nu+1} < 0$, g_ε has an additional sign change near, and to the left of, ζ, for a total of $S^+(a_{\mu-1}, \ldots, a_{\nu+1})$ sign changes. □

The remaining zeros of f, if any, I will call **nonobvious**. Any such zero is an interior zero of one of the connected components of f, and necessarily of the form $z = [\zeta \mathrel{..} \zeta]$.

Proposition 12. *Let $z = [\zeta \mathrel{..} \zeta]$ be a nonobvious zero of $f =: \sum_i a_i B_i \in S_{k,t}$ and let $f =: \sum_i \widetilde{a}_i B_{i,\widetilde{t}}$, with $\widetilde{t}$ obtained from t by insertion of ζ just enough times to make z an obvious zero of f as an element of $S_{k,\widetilde{t}}$.*

Then, the multiplicity according to Definition 1, of ζ as a zero of $f \in S_{k,t}$, equals

$$S^+(\widetilde{a}_{\mu-1}, \ldots, \widetilde{a}_{\nu+1}) = \mu + 1 - \nu, \tag{13}$$

with $\widetilde{a}_{\mu,\nu}$ the corresponding zero of $\widetilde{a}$.

Proof. The only issue is the equality (13) and whether $S^+(\widetilde{a}_{\mu-1}, \ldots, \widetilde{a}_{\nu+1})$ is the multiplicity of z as a zero of f *as an element of* $S_{k,t}$.

Let

$$\widetilde{I}(z) := (i : (\widetilde{t}_i \mathrel{..} \widetilde{t}_{i+k}) \cap z \neq \emptyset) =: (j, \ldots, j + r).$$

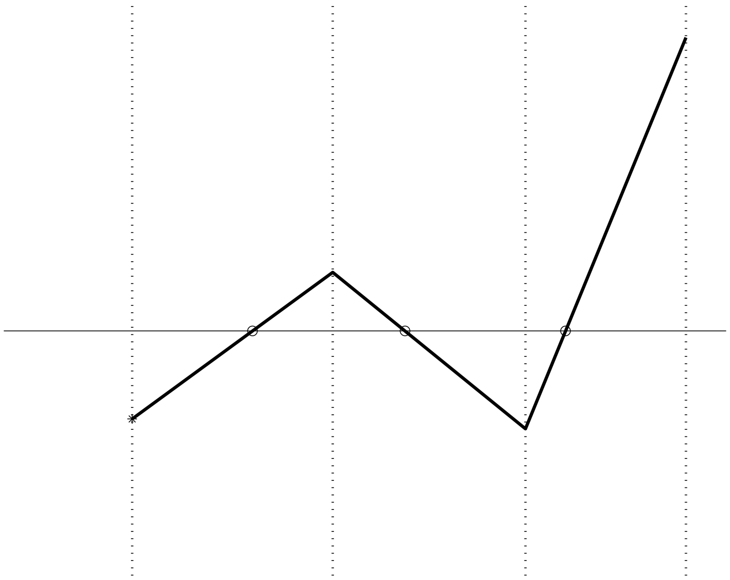

Figure 1. Since the control polygon must have its vertices on the dotted lines and must cross the axis at the circled points strictly between the dotted lines, it maximally oscillates unless it is entirely flat.

Since z is an obvious zero of $f = \sum_i \widetilde{a}_i B_{i,\widetilde{t}}$, $\widetilde{I}(z)$ must be a subsequence of $(\mu, \ldots, \nu)$. I claim that, actually,

$$\widetilde{I}(z) \;=\; (\mu, \ldots, \nu). \tag{14}$$

To see this, let $\hat{t}$ be the knot sequence obtained from $\widetilde{t}$ by removal of one occurrence of ζ. For each $i \in \widetilde{I}(z)$, the most recently inserted knot, ζ, satisfies $\hat{t}_i = \widetilde{t}_i < \zeta < \widetilde{t}_{i+k} = \hat{t}_{i+k-1}$, hence, by the second part of Lemma 3, the corresponding vertex $\widetilde{v}_i = (\widetilde{t}^*_i, 0)$ must lie strictly inside the corresponding segment $[\widehat{v}_{i-1} \mathinner{\ldotp\ldotp} \widehat{v}_i]$ of the control polygon for f as an element of $S_{k,\hat{t}}$. This implies (see Figure 1) that $\widehat{v}_i$, $i \in \widetilde{I}(z)$, is determined once $\widehat{v}_{j-1}$ is known. In particular,

$$S^-(\widehat{a}_{j-1}, \ldots, \widehat{a}_{j+r}) = r + 1, \tag{15}$$

unless all these coefficients are zero. However, this alternative would mean that z is already an obvious zero of f as an element of $S_{k,\hat{t}}$, contrary to construction. Thus, $\mu = j$, $\nu = j + r$ and, correspondingly,

$$\widehat{a}_{j-1} = \widetilde{a}_{\mu-1} \neq 0 = \widetilde{a}_\mu = \cdots = \widetilde{a}_\nu \neq \widetilde{a}_{\nu+1} = \widehat{a}_{j+r},$$

proving (13). In particular, *the multiplicity to be assigned to z equals the number of entries in the corresponding zero $\widetilde{a}_{\mu,\nu}$ of $\widetilde{a}$.*

With (14) now proved, we know that, thinking of f as an element of $S_{k,\widetilde{t}}$, we are in the last case treated in the proof of Proposition 11, hence know from there the existence of a g_ε in $S_{k,t}$ (since g_ε was obtained there by extending a suitable *polynomial* to a spline) close to f and with $S^+(\widetilde{a}_{\mu-1}, \ldots, \widetilde{a}_{\nu+1})$ sign changes close to z. □

With this way of counting the multiplicity of the zeros of f, the following improvement

$$Z^+(f) \le S^-(a_1,\ldots,a_{\mu_1-1}) + S^+(a_{\mu_1-1},\ldots,a_{\nu_1+1}) + S^-(a_{\nu_1+1},\ldots,a_{\mu_2-1}) + \cdots \\ \cdots + S^+(a_{\mu_{J-1}-1},\ldots,a_{\nu_J+1}) + S^-(a_{\nu_J+1},\ldots,a_n) \tag{16}$$

over (2) is immediate, in which a_{μ_j,ν_j}, $j = 1,\ldots,J$, are all the zeros of a corresponding to obvious zeros of f, in order.

Comparison with the multiplicity of Schumaker and of Goodman

Both Schumaker and Goodman distinguish explicitly between *isolated zeros* and *zero intervals*, providing separate discussions of their multiplicities.

So, let z be a zero, interval or isolated. Both Schumaker and Goodman begin with the determination of a certain number $\alpha(z)$, which I will denote here by $\alpha_S(z)$ and $\alpha_G(z)$ if that distinction matters and which, by definition, is the multiplicity of z unless that would lead to an incorrect parity, i.e., when $\alpha(z)$ is odd while z is nonnodal, or $\alpha(z)$ is even while z is nodal; in the latter case, the actual multiplicity is taken to be $\alpha(z)+1$. The multiplicity count I am proposing is certain to assign odd (even) multiplicity to nodal (nonnodal) zeros, without any case distinction being required.

When z is an interval zero, with endpoints z_l, z_r, then $\alpha_S(z) \le \alpha_G(z)$, with strict inequality possible. Precisely, $\alpha_S(z) = k+$ the number of knots strictly inside the interval z, while $\alpha_G(z)$ replaces k by $l+r+p+q-k$, with l the exact order of $f(z_l-)$, p the multiplicity of z_l as a knot in t, and, correspondingly, r the exact order of $f(z_r+)$ and q the multiplicity of z_r in t. It follows that, in the B-representation of the spline, no B-spline with some support in the interior of z can appear nontrivially, nor can any B-spline in which z_l occurs with multiplicity $\ge k-l$ nor any in which z_r occurs with multiplicity $\ge k-r$. This readily shows that the multiplicity I assigned to such a zero agrees with Goodman's.

When z is an isolated zero, obvious or nonobvious, then also $\alpha_S(z) \le \alpha_G(z)$ with strict inequality possible. Precisely, with l (r) the exact order of $f(z-)$ ($f(z+)$), $\alpha_S(z) = \max(l,r)$, and this agrees with $\alpha_G(z)$ except when $l \ge k-s$, with s the multiplicity of z in t, in which case $\alpha_G(z) = l+r+s-k$. In the latter case, all B-splines that have z in the interior of their support have it appear there to multiplicity $s \ge k-l$, hence cannot appear nontrivially in the B-representation for f. In other words, z is an obvious zero. In either case, the multiplicity I assigned to this isolated zero agrees with Goodman's.

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