Mirror Descent for Metric Learning

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Formulating the Problem

We **incrementally learn a pseudo-metric**, $d_M(\mathbf{x}, \mathbf{z})^2 = (\mathbf{x}-\mathbf{z})' M (\mathbf{x}-\mathbf{z})$ given triplets of the form $(\mathbf{x}_t, \mathbf{z}_t, y_t)_{t=1}^T$. The label $y_t = \pm 1$ indicates that $\mathbf{x}_t$ is similar/dissimilar to $\mathbf{z}_t$. where $M \subseteq \mathbb{S}_+^n$. We can introduce the **margin function** [4]:

$$m(\mathbf{x}_t, \mathbf{z}_t, y_t) = y_t \left(\mu - (\mathbf{x}_t - \mathbf{z}_t)' M (\mathbf{x}_t - \mathbf{z}_t) \right),$$

which allows us to define loss for a sample $(\mathbf{x}_t, \mathbf{z}_t, y_t)$; for instance, the hinge loss: $\ell_t(M, \mu) = \max\{0,\ 1 - m(\mathbf{x}_t, \mathbf{z}_t)\}$. We also add a regularization function $r(M) = |||M|||$, **the trace-norm** of M i.e., the sum of the singular values of M (for some $\rho > 0$) yields sparsity in the singular value spectrum of M, thus minimizing the rank of M:

$$\min_{M \succeq 0, \mu \geq 1} \ \frac{1}{T} \sum_{t=1}^{T} \ell_t(M, \mu) + r(M),$$

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Duchi et al., [2] generalized **mirror descent** to the case where the functions $\phi_t = \ell_t + r$ are composite, consisting of loss and regularization terms. In composite mirror descent (COMID), the ℓ_t is linearized, while r is not. We derive **generalized update rules for a general loss function and Bregman divergence**:

$$\begin{aligned} M_{t+1} &= \arg\min_{M \succeq 0} \ B_\psi(M,\ M_t) + \eta \langle \nabla_M \ell_t(M_t, \mu_t),\ M - M_t \rangle + \eta\, \rho\ |||\, M |||, \\ \mu_{t+1} &= \arg\min_{\mu \geq 1} \ B_\psi(\mu,\ \mu_t) \quad + \eta\, \nabla_\mu \ell_t(M_t, \mu_t)'\ (\mu - \mu_t). \end{aligned}$$

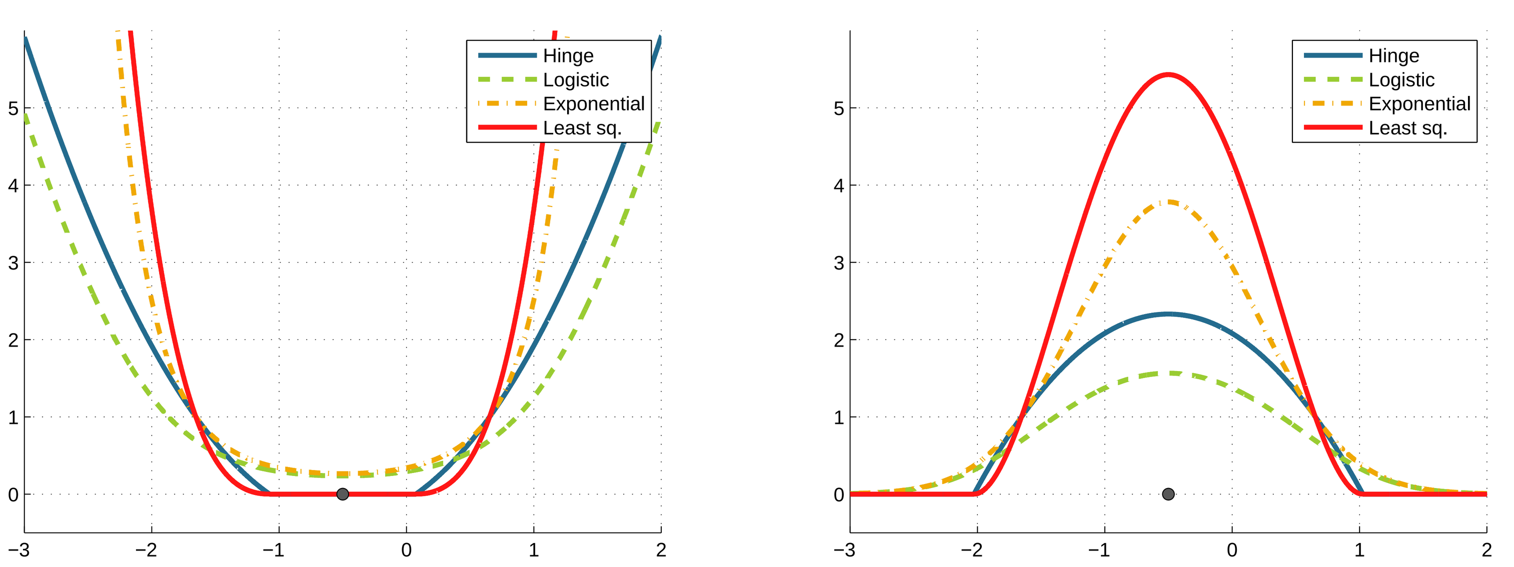
1. **Unifying framework**. Different algorithms arise from various Bregman and loss functions. E.g., using Euclidean distance and relative entropy results in additive and multiplicative updates respectively.
2. **Scalability**. Update rules require rank-one modification of the EVD of $M = V \Lambda V'$; this can be **implemented efficiently** and is **embarrassingly parallel**.
3. **Sparse metric**. The **trace norm** is $|||X||| = \mathbf{e}'|\boldsymbol{\lambda}|$, where $\boldsymbol{\lambda}$ are the EVs of X. Minimizing the trace norm ensures that M is **sparse in its eigenspectrum** i.e., only $r < n$ eigenvalues are used in calculating distances: $\tilde{L} = V_r \sqrt{\Lambda_r}$.
4. **Kernelizable**. The techniques of Chatpatanasiri et al., [1] can be applied here to kernelize it and learn nonlinear metrics.

Bregman Functions and Loss Functions

We consider the following Bregman functions. The squared p-norms $\psi(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_p^2$ are strongly convex and induce the **squared-Frobenius distance** i.e., $B_\psi(X, Z) = \frac{1}{2}\|X - Z\|_F^2$. The function $\psi(\mathbf{x}) = \sum_i x_i \log x_i - x_i$ induces the **von Neumann divergence**, $B_\psi(X, Y) = \mathrm{tr}\,(X \log X - X \log Y - X + Y)$.

The formulation admits several loss functions. If a loss function is Lipschitz, we obtain algorithms that are characterized by $O(\sqrt{T})$ regret. In the tables below, $\mathbf{u}_t = \mathbf{x}_t - \mathbf{z}_t$.

Loss	$\ell_t(M_t,\ \mu_t)$	$\nabla_M\, \ell_t(M_t,\ \mu_t)$
Hinge	$(1 - m_t)_+$	$(1 - m_t)_\star\ (y_t \mathbf{u}_t \mathbf{u}_t')$
Modified Least Sq.	$\frac{1}{2}(1 - m_t)_+^2$	$(1 - m_t)_+\ (y_t \mathbf{u}_t \mathbf{u}_t')$
Logistic	$\log(1 + \exp(-m_t))$	$\frac{\exp(-m_t)}{1 + \exp(-m_t)}\ (y_t \mathbf{u}_t \mathbf{u}_t')$



Different loss functions around $x = -0.5$; **(left)** when $(\mathbf{x}_t, \mathbf{z}_t)$ are similar ($y_t = 1$); **(right)** when $(\mathbf{x}_t, \mathbf{z}_t)$ are dissimilar ($y_t = -1$).

Update rules can be derived in closed-form using the **eigenvalue thresholding/shrinkage operator**: $S_\tau(X) = V\,\mathrm{diag}(\boldsymbol{\lambda}_\tau)\, V'$, where $(\lambda_\tau)_i = \mathrm{sign}(\lambda_i)\, \max\{|\lambda_i| - \tau,\ \}$. The closed-form solutions are:

$$\begin{aligned} \textbf{vonNeumann} \quad M_{t+1} &= \exp\left(S_{\eta\rho}(\log M_t - \eta \nabla_M \ell_t(M_t, \mu_t)) \right), \\ \textbf{Frobenius} \quad M_{t+1} &= S_{\eta\rho}\left(M_t - \eta \nabla_M \ell_t(M_t, \mu_t) \right). \end{aligned}$$

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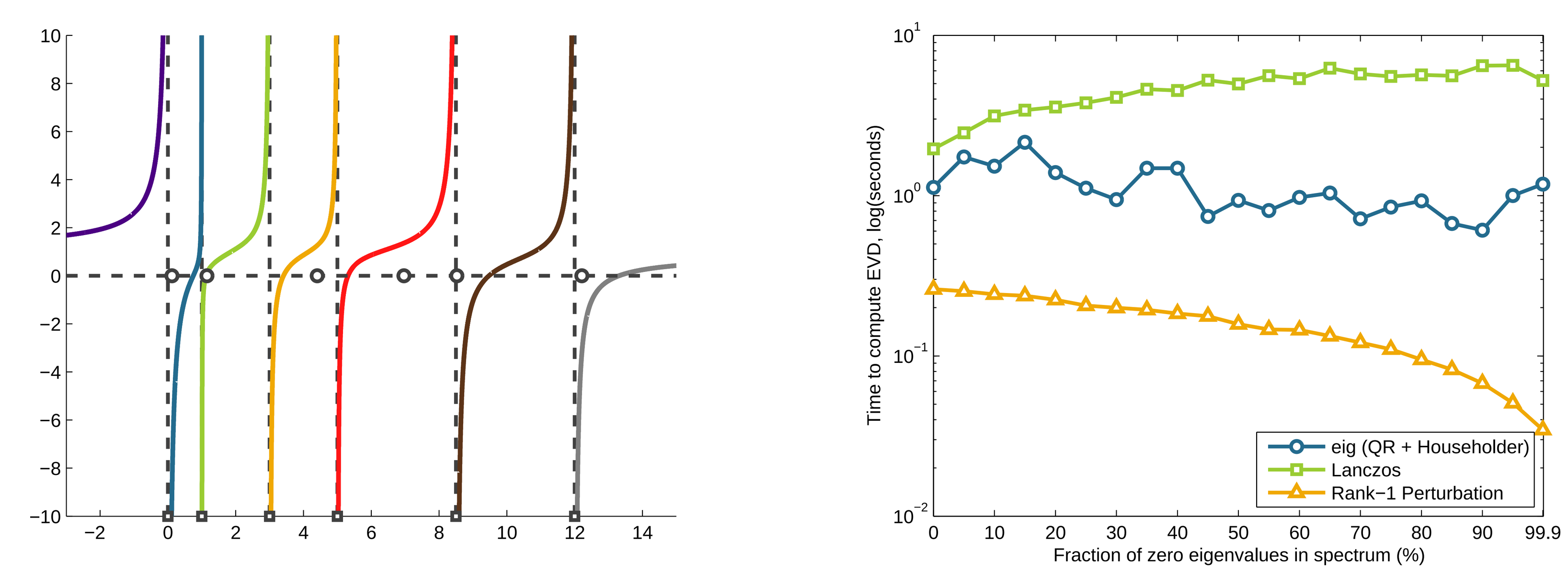
input: data $(\mathbf{x}_t, \mathbf{z}_t, y_t)_{t=1}^T$, parameters ρ, $\eta > 0$
choose: Bregman functions $\psi(M)$; $\psi(\mu)$, loss $\ell(M, \mu)$
initialize: $M_0 = I_n$, $\mu_0 = 1$
for $(\mathbf{x}^t,\ \mathbf{z}_t,\ y_t)$ **do**
  let $\mathbf{u}_t = \mathbf{x}_t - \mathbf{z}_t$, $\quad \eta_t = \eta/\sqrt{t}$
  compute gradients of loss $\nabla_M \ell_t = \alpha_t \mathbf{u}_t \mathbf{u}_t'$ and $\nabla_\mu \ell_t = -\alpha_t$
  write $\nabla\psi(M_t) = V_t \nabla\psi(\Lambda_t) V_t'$
  rank-one update $V_{t+1} \Lambda_{t+1} V_{t+1}' = V_t \nabla\psi(\Lambda_t) V_t' - \alpha \mathbf{u}_t \mathbf{u}_t'$
  shrink the eigenvalues $M_{t+1} = V_{t+1} \nabla\psi^{-1}\left(S_{\eta\rho}(\Lambda_{t+1}) \right) V_{t+1}'$
  margin update $\mu_{t+1} = \max\left(\nabla\psi^{-1}\left(\nabla\psi(\mu_t) - \eta\, \nabla\ell_t(M_t, \mu_t)\right),\ 1 \right)$
end for

Computing EVD Efficiently

We have $M_{t+1} = V_t \nabla\psi(\Lambda_t) V_t' - \alpha \mathbf{u}_t \mathbf{u}_t'$, a *rank-one update* of the EVD at iteration t.

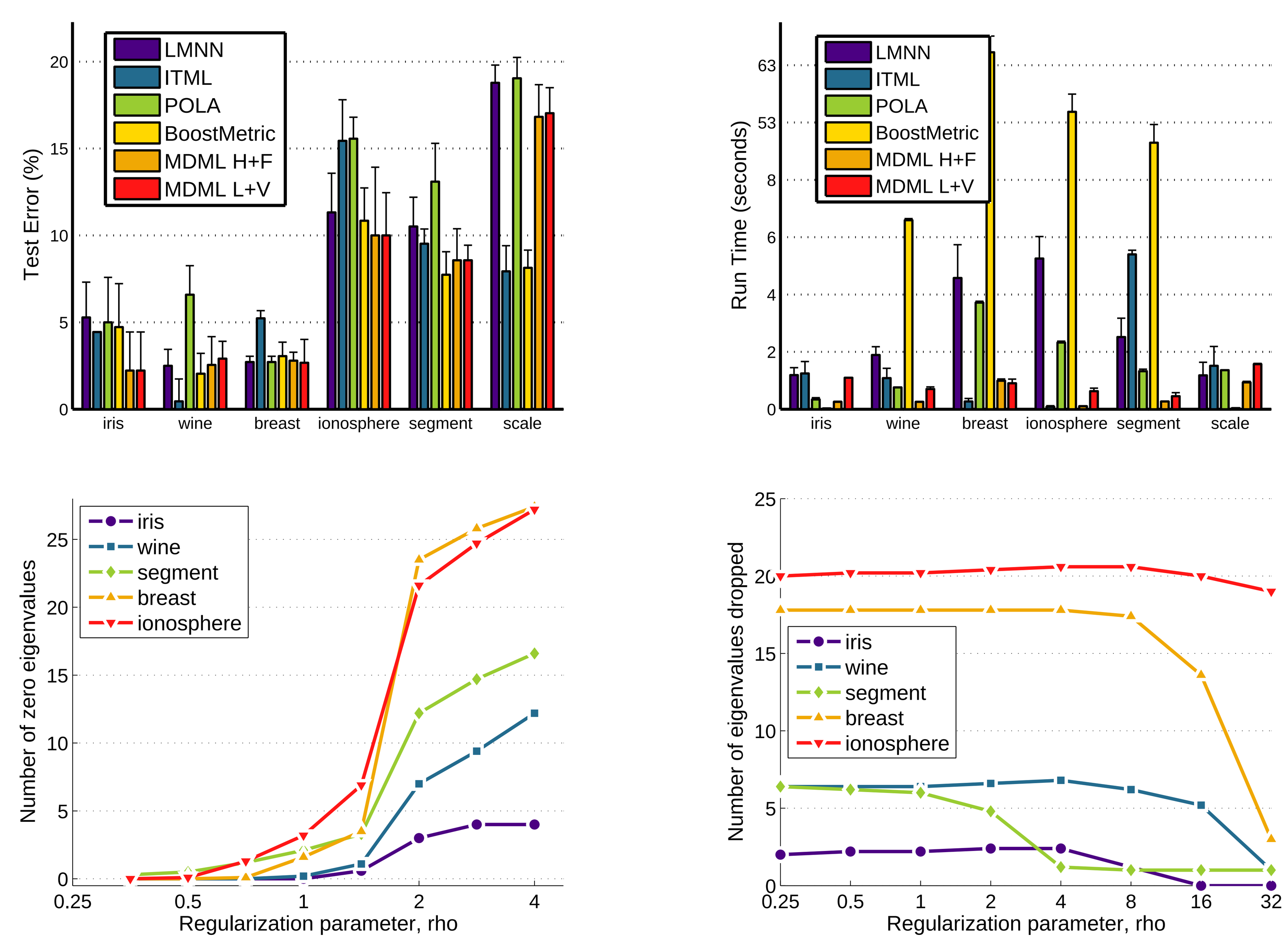
- **Eigenvalue Interlacing**. The EVs of M_t, M_{t+1} interlace; each EV can be computed independently from the secular equation. General root-finding techniques such as Newton may result in non-orthogonal eigenvectors; we adopt the rational interpolation approach of Gu and Eisenstat [3].
- **Learning rate**. An adaptive rate, $\eta_t = \eta/\sqrt{t}$ gives $O(\sqrt{T})$ regret.
- **Low Rank Learning with von Neumann divergence**. This is undefined for low-rank matrices; we update with the *reduced eigendecomposition*, $M_t = \tilde{V}_t \tilde{\Lambda}_t \tilde{V}_t'$. Also, in this case, M_t's smallest EVs are all 1, resulting in full rank; we still perform feature selection by selecting the r largest EVs, similar to PCA.



(left) Interlacing eigenvalues of a matrix and its rank-one perturbation; **(right)** EVD algorithms for randomly generated 500d matrices, over increasing spectrum sparsity.

Experiments: Benchmark Data Sets

We consider two algorithms: an **additive algorithm with hinge loss and Frobenius** (MDML H+F), and a **multiplicative algorithm with logistic loss and von Neumann** (MDML L+V). They are compared to four metric learning approaches: LMNN, ITML, BoostMetric and POLA [4].



References

[1] R. Chatpatanasiri, T. Korsrilabutr, P. Tangchanachaianan, and B. Kijsirikul. On kernelization of supervised mahalanobis distance learners. *Computing Research Repoisitory (CoRR)*, abs/0804.1441, 2008.

[2] J. Duchi, S. Shalev-Shwartz, Y. Singer, and A. Tewari. Composite objective mirror descent. In *COLT*, 2010.

[3] Ming Gu and Stanley C. Eisenstat. A stable and efficient algorithm for the rank-one modification of the symmetric eigenproblem. *SIAM Journal on Matrix Analysis and Applications*, 15(4), 1994.

[4] S. Shalev-Shwartz, Y. Singer, and A. Y. Ng. Online and batch learning of pseudo-metrics. In *ICML'04*, 2004.